DISEI - Università degli Studi di Firenze

Working Papers – Mathematical Economics

Convergence rate of the Truncated Realized Covariance when prices have infinite variation jumps

Cecilia Mancini

Working Paper N. 2014-03

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze, Italia
www.disei.unifi.it

Convergence rate of the Truncated Realized Covariance when prices have infinite variation jumps

Cecilia Mancini *

May 30, 2014

Abstract

In this paper we consider two processes driven by Brownian motions plus drift and jumps with infinite activity. Given discrete observations on a finite time horizon, we study the truncated (threshold) realized covariance $\hat{IC}$ to estimate the integrated covariation IC between the two Brownian parts and we establish how fast $\hat{IC}$ converges when the small jumps of the processes are Lévy. We find that the speed is heavily influenced by the small jumps dependence structure other than by their jump activity indices. This work follows Mancini and Gobbi (2011) and Jacod (2008), where the asymptotic normality of $\hat{IC}$ was obtained when the jump components have finite activity or finite variation. Separating the sources of covariation (IC and co-jumps) of two financial assets has important applications in portfolio risk management.

Keywords: Brownian correlation coefficient, integrated covariance, co-jumps, stable Lévy jumps, threshold estimator.

JEL classification: C1, C3

1 Introduction

We consider two state variables evolving as follows

$$\begin{aligned} dX_t^{(1)} &= a_t^{(1)}dt + \sigma_t^{(1)}dW_t^{(1)} + dZ_t^{(1)}, \\ dX_t^{(2)} &= a_t^{(2)}dt + \sigma_t^{(2)}dW_t^{(2)} + dZ_t^{(2)}, \end{aligned} \tag{1}$$

for $t \in [0,T]$, T fixed, where $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1-\rho_t^2}W_t^{(3)}$; $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(3)} = (W_t^{(3)})_{t\in[0,T]}$ are independent Wiener processes. $Z^{(1)}$ and $Z^{(2)}$ are correlated pure jump processes. Given discrete equally spaced observations $X_{t_i}^{(1)}, X_{t_i}^{(2)},\ i = 1..n$, in the interval $[0,T]$, with $t_i = ih, h = \frac{T}{n}$, we are interested in the identification of the dependence amount between the two Brownian parts, namely the *integrated covariation* $IC \doteq \int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt$. It is well known that as the observation step h tends to 0 the *Realized Covariance* $\sum_{i=1}^n \Delta_i X^{(1)}\Delta_i X^{(2)}$, where $\Delta_i X^{(m)} \doteq X_{t_i}^{(m)} - X_{t_{i-1}}^{(m)}$, converges to the global quadratic covariation $[X^{(1)}, X^{(2)}]_T = \int_0^T \rho_t\sigma_t^{(1)}\sigma_t^{(2)}dt + \sum_{0\le t\le T}\Delta Z_t^{(1)}\Delta Z_t^{(2)}$, where $\Delta Z_t^{(m)} = Z_t^{(m)} - Z_{t-}^{(m)}$, containing also the *co-jumps* $\Delta Z_t^{(1)}\Delta Z_t^{(2)}$, i.e. the simultaneous jumps of $X^{(1)}$ and $X^{(2)}$. It is also well known that the *Threshold Realized Covariance*

$$\hat{IC} = \sum_{i=1}^n \Delta_i X^{(1)} I_{\{|\Delta_i X^{(1)}|\le r_h\}}\Delta_i X^{(2)} I_{\{|\Delta_i X^{(2)}|\le r_h\}},$$

*Department of management and economics, University of Firenze; cecilia.mancini@unifi.it

with e.g. $r_h = h^u$, and $u \in (0, 1/2)$, is consistent to IC ([16], [9]). [1] Further, a CLT for $\hat{IC}$ has been established when the jumps processes have *finite jump activity* (FA), i.e. only a finite number of jumps can occur, along each path, in each finite time interval, (see [15]) or when the jumps processes have infinite activity (IA) but *finite variation*, i.e. $\sum_{s \le T} |\Delta X_s^{(m)}| < \infty$, $m = 1, 2$, (see [8], Thm 7.4), meaning that the jumps activity of the processes is moderate. Namely, the estimator is asymptotically mixed Gaussian and converges with speed $\sqrt{h}$. It is the unique estimator of IC proposed in the literature in the presence of IA jumps, and it has been shown ([16]) that it is the most efficient in simple cases when the jumps have FA. Here we investigate the speed in the case of infinite activity jumps where at least one component has infinite variation, and we find that the rate crucially depends on the small co-jumps and is determined not only by the jump activity of each one of the two components, but also on the dependence degree of their small jumps. Further, differently on what we had in the univariate case ([14]), in some cases a mixed term containing the Brownian increments and the jumps of the most active $Z^{(m)}$ comes in. In the univariate case such a speed reduces to the one found in [14].
Estimation of IC is of strong interest in financial econometrics (see e.g. [3]) and in the framework of portfolio risk and hedge funds management ([6]).

An outline of the paper is as follows. In section 2 we illustrate the framework; in section 3 we review the known results on $\hat{IC}$ when the jumps of the vector X have finite activity. In section 4 we establish the exact convergence rate when both the $Z^{(m)}$ have IA and at least one has infinite variation. More precisely, we assume that the bivariate small jumps are the small jumps of a Lévy process with stable marginal laws and joint law characterized by a Lévy copula ranging in the class of the convex combinations of $C_\perp$ (independence copula) and $C_\parallel$ (complete dependence copula).

2 The framework

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, let $X^{(1)} = (X_t^{(1)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ be two real processes defined by (1) and $X_0 = (0, 0)$, where

A1. we take two independent Wiener processes $W^{(1)}$ and $W^{(3)}$ and construct

$$W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1 - \rho_t^2}\; W_t^{(3)}, \tag{2}$$

A2. the coefficients $\sigma^{(m)} = (\sigma_t^{(m)})_{t\in[0,T]}$, $a^{(m)} = (a_t^{(m)})_{t\in[0,T]}$, $m = 1, 2$, and $\rho = (\rho_t)_{t\in[0,T]}$ are adapted càdlàg processes,

It turns out that $J^{(m)}$ are finite activity jump processes accounting for the jumps with size bigger in absolute value than 1. They can also be represented as

$$J_t^{(m)} = \int_0^t \gamma_s^{(m)} dN_s^{(m)} = \sum_{k=1}^{N_t^{(m)}} \gamma_{\tau_k^{(m)}}^{(m)}, \quad m = 1, 2,$$

where $N^{(m)} = (N_t^{(m)})_{t\in[0,T]}$ are counting processes with $E[N_T^{(m)}] < \infty$; $\{\tau_k^{(m)},\ k = 1, ..., N_T^{(m)}\}$ denote the instants of jump of $J^{(m)}$ and $\gamma_{\tau_k^{(m)}}^{(m)}$ denote the sizes of the jumps occurred at $\tau_k^{(m)}$. In this representation we intend that $\forall t \in [0, T]$, $P\{\Delta N_t^{(m)} \neq 0, \gamma_{N_t}^{(m)} = 0\} = 0$, i.e. if on (ω, t) we have $\gamma_t^{(m)}(\omega) = 0$,

[1] For the literature on non parametric inference for the IC of stochastic processes driven by Brownian motions plus jumps, see [16].

A3. for $m = 1, 2$

$$Z^{(m)} = J^{(m)} + M^{(m)}$$

are jump processes, with

$$J^{(m)} \doteq \int_0^\cdot \int_{\{|\gamma^{(m)}(s,\omega,x)|>1\}} \gamma^{(m)}(s,\omega,x)\mu^{(m)}(\omega,dx,ds),\, M^{(m)} \doteq \int_0^\cdot \int_{\{|\gamma^{(m)}(s,\omega,x)|\le 1\}} \gamma^{(m)}(s,\omega,x)\tilde{\mu}^{(m)}(\omega,dx,ds),$$

where, for each $m = 1, 2$, $\mu^{(m)}$ is the Poisson random measure counting the jumps of $Z^{(m)}$, $\tilde{\mu}^{(m)}(dx, ds) \doteq \mu^{(m)}(dx, ds) - \nu^{(m)}(dx)ds$ is its compensated measure (see [8]).

then time t has not to be considered a jump time for path ω. On the contrary $M^{(m)}$ are generally infinite activity pure jump processes. The property $\nu^{(m)}(\omega, \mathbb{R} - \{0\}) = \infty$ characterizes the fact that the path ω of $M^{(m)}$ jumps infinitely many times on any finite time intervals. $M^{(m)}$ are compensated sums of jumps which have size bounded in absolute value by 1. Substantially $J^{(m)}$ accounts for the (usually big) rare jumps of $X^{(m)}$, while $M^{(m)}$ accounts for the very frequent and small jumps.

Remark 2.1. If $Z^{(m)}$ is a pure jump Lévy process, assumption **A3** is satisfied, since $Z^{(m)}$ has the same representation as above with x in place of $\gamma^{(m)}(s, \omega, x)$ (see [6]) and $J^{(m)}$ are compound Poisson processes. □

We observe $X^{(1)}, X^{(2)}$ discretely and synchronously. Let, for each n, $\pi_n^{[0,T]} = \{0 = t_{0,n} < t_{1.n} < \cdots < t_{n,n} = T\}$ be a partition of $[0, T]$. We assume equally spaced subdivisions, i.e. $h_n := t_{i,n} - t_{i-1,n} = \frac{T}{n}$ for every $n = 1, 2, ...$. Hence $h_n \to 0$ iff $n \to \infty$. To simplify the notation we write h in place of h_n, t_i in place of $t_{i,n}$ and $\Delta_i X^{(m)}$ in place of $\Delta_{i,n} X^{(m)} \doteq X^{(m)}_{t_i} - X^{(m)}_{t_{i-1}}$.

A4. We choose a deterministic function r_h of h, called *threshold*, satisfying

$$\lim_{h\to 0} r_h = 0, \quad \lim_{h\to 0} \frac{h log \frac{1}{h}}{r_h} = 0.$$

Denote, for each $m = 1, 2$, by

$$D^{(m)}_t = \int_0^t a^{(m)}_s ds + \int_0^t \sigma^{(m)}_s dW^{(m)}_s,$$

the Brownian semimartingale part (abbreviated with BSM) of $X^{(m)}$, and by

$$Y^{(m)}_t = D^{(m)}_t + J^{(m)}_t$$

the BSM part of $X^{(m)}$ plus the finite jump activity component.

3 Preliminary results

The truncated realized covariation is able to separately capture IC because it excludes from $\sum_{i=1}^n \Delta_i X^{(1)} \Delta_i X^{(2)}$ those increments where jumps bigger than the threshold occurred, so when $h \to 0$ all the jumps are excluded. The key point to understand when an increment $\Delta_i X^{(m)}$ is likely to contain some jumps is the Paul Lévy law for the modulus of continuity of the Brownian motion paths (see [12]), telling us that the increments of each $D^{(m)}$ tend to zero at speed $\sqrt{h \ln \frac{1}{h}}$. In fact we have (see [13])

Remark 3.1. *Under* ***A2*** *we have a.s.*

$$\sup_{1\le j\le n} \frac{|\Delta_j D^{(m)}|}{\sqrt{2hlog\frac{1}{h}}} \le K_m(\omega) < \infty, \quad m = 1, 2,$$

where $K_m \doteq \sup_{s\in[0,T]} |a|_s + \sup_{s\in[0,T]} |\sigma|_s + 1$ *are finite random variables.*

In fact, if for small h we have $(\Delta_i X^{(m)})^2 > r_h > \sqrt{2h \ln \frac{1}{h}}$ then either some jumps of $J^{(m)}$ occurred or some jumps of $M^{(m)}$ larger than $2\sqrt{r_h}$ occurred ([13]). $\square$

Define

$$A\hat{V}ar = h^{1-\frac{r+l}{2}} \sum_{i=1}^{n} \prod_{m=1}^{2} (\Delta_i X^{(m)})^2 I_{\{|\Delta_i X^{(m)}|\le\sqrt{r_h}\}}$$

$$-h^{-1} \sum_{i=1}^{n-1} \prod_{j=0}^{1} \Delta_{i+j} X^{(1)} I_{\{|\Delta_{i+j} X^{(1)}|\le\sqrt{r_h}\}} \prod_{j=0}^{1} \Delta_{i+j} X^{(2)} I_{\{|\Delta_{i+j} X^{(2)}|\le\sqrt{r_h}\}} :$$

this is a truncated version of an analogous statistic defined and used in [4] when estimating IC for a continuous bivariate process X. When only finite activity jumps can occur then the speed of convergence of $\hat{IC}$ is $\sqrt{h}$. More precisely we have

Theorem 3.2. *(CLT for* $\hat{IC}$ *when jumps have finite activity, [16]) If* $M^{(m)} \equiv 0$, *for* $m = 1, 2$, *then under the assumptions* ***A1-A4*** *we have, as* $h \to 0$,

$$\mathcal{NB}_h := \frac{\hat{IC} - \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt}{\sqrt{h}\sqrt{A\hat{V}ar}} \stackrel{st}{\to} \mathcal{N},$$

where $\mathcal{N}$ *is a rv with standard Gaussian law* $\mathcal{N}(0, 1)$ *and* $\stackrel{st}{\to}$ *denotes stable convergence in law.*

Theorem 3.3. *(Asymptotic variance, [7]) Under* ***A1-A4*** *we have, as* $h \to 0$,

$$A\hat{V}ar \stackrel{P}{\to} \int_0^T (1 + \rho_t^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt.$$

Remarks. i) In [8] it is shown that the convergence speed to IC of a slightly different version of the threshold estimator is still $\sqrt{h}$ under the restriction that also σ is an Ito semimartingale (SM) but under the more general condition that both the processes $Z^{(m)}$ are semimartingales with, substantially, finite variation. More precisely, the jump sizes are assumed to be substantially bounded by a deterministic strictly positive function $\gamma(x)$ in the following way: $\sup_{x\in\mathbb{R}} \frac{|\gamma^{(m)}(\omega,t,x)|}{\gamma(x)}$ is locally bounded, for $m = 1, 2$. The condition required to ensure a CLT is that $\int_{\{\gamma\le1\}} \gamma^r(x)dx < \infty$ for some $r < 1$. Under our framework such a condition translates into requiring that both the indices α_m defined below are less than one. We therefore are interested here in studying the cases where $\alpha_m \ge 1$ for at least one index.
ii) Note that [8] constructs the estimator of IC imposing a threshold condition of type $(\Delta_i X^{(1)})^2 + (\Delta_i X^{(2)})^2 \le \phi^2 h^{2\varpi}$ with $\phi \in \mathbb{R}$ and $\varpi = u$, $u \in (0, 1/2)$, while we separately ask for each $(\Delta_i X^{(m)})^2$ being dominated by the same r_h. Since the norms $||.||_1$ and $||.||_2$ in $\mathbb{R}^2$ are equivalent, asymptotically the two choices make no difference, meaning that also our $\hat{IC}$ has convergence speed $\sqrt{h}$ when both $\alpha_m < 1$. However it is possible that in finite samples one of the two choices is better than the other.

4 Main results

We find here the speed of convergence of $\hat{IC} - IC$ to 0 when both $M^{(m)} \neq 0$ and at least one of them has infinite variation. We specialize our analysis to the case where the small jumps of each $X^{(m)}$ are of α-stable type, i.e. $M_t^{(m)} = L_t^{(m)} - z^{(m)}t - \sum_{s \le t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}|>1\}}$, where $L^{(m)}$ are α_m-stable Lévy processes with characteristic triplets $(z^{(m)}, 0, \nu^{(m)}(dx))$, with $\nu^{(m)}$ given below. Further we assume that the occurrence of the joint jumps of $L^{(1)}$ and $L^{(2)}$ is characterized by a Lévy copula C ranging in a given class. We have $\alpha_m \in]0, 2[$ for each $m = 1, 2$ and assume (w.l.g.) $\alpha_1 \le \alpha_2$. As said, we are interested in the case where at least one of the two indices is greater or equal than 1, we thus assume $\alpha_2 \ge 1$, and $\alpha_1 \in (0, \alpha_2]$. Further, for simplicity, but w.l.g., we develop our proofs for the case where each $L^{(m)}$ is one sided, i.e. has only jumps with positive sizes.

A5 Take $\alpha_2 \ge 1$, and $\alpha_1 \in (0, \alpha_2]$. The jumps of each $L^{(m)}$ have Lévy density

$$\nu^{(m)}(dx_m) = c_m x_m^{-1-\alpha_m} I_{\{x_m>0\}} dx_m,$$

which has support $\mathbb{R}_+$, where $c_m > 0$.

We denote, for each $m = 1, 2$, by

$$U_m(x_m) := \nu^{(m)}\big([x_m, +\infty[\big) = c_m \frac{x_m^{-\alpha_m}}{\alpha_m}, \quad x_m > 0 \tag{3}$$

the tail integral of the marginal Lévy measure $\nu^{(m)}$ of the jumps of $L^{(m)}$.

Note that α_m is the *Blumenthal-Getoor index* of $L^{(m)}$, of $M^{(m)}$ and of $X^{(m)}$. We now make use of Lévy copulas, because, due to the stationarity of the Lévy processes increments, the Lévy copulas allow to separate the time component in the law of a bivariate pure jump Lévy process L from the jump sizes component and allow to describe the dependence between $L^{(1)}$ and $L^{(2)}$ through only the dependence of their jump sizes. Lévy copulas were introduced in [17], further studied in [11] and their properties are well summarized in [6].

A6 For any t the joint jumps occurrence of $\big(L_t^{(1)}, L_t^{(2)}\big)$ is described by the following tail integrals

$$U(x_1, x_2) = \nu_\gamma([x_1, +\infty) \times [x_2, +\infty)) = C_\gamma(U_1(x_1), U_2(x_2))$$

where $C_\gamma(u, v)$ is a Lévy copula of the form

$$C_\gamma(u, v) = \gamma C_\perp(u, v) + (1 - \gamma) C_\parallel(u, v),$$

where $C_\perp(u, v) = u I_{\{v=\infty\}} + v I_{\{u=\infty\}}$ is the independence copula, $C_\parallel(u, v) = u \wedge v$ is the total positive dependence copula and γ ranges in $[0, 1]$.

A6 means that, at any t, $(L^{(1)}, L^{(2)})$ can only have two basically different classes of jumps:
i) the disjoint jumps, when only one of the two components jumps, meaning that L_t jumps with size either $(0, x_2)$ or $(x_1, 0)$. This type of jumps is regulated only by $C_\perp$;
ii) the joint jumps, when the two components $L^{(m)}$ jump together, meaning that L jumps with size falling into a point (x_1, x_2) with both $x_m \neq 0$. This type of jumps is regulated only by $C_\parallel$.
$C_\parallel$ characterizes a bivariate jump Lévy process $\bar{L}$ whose marginals $\bar{L}^{(m)}$ are Lévy and only make joint

jumps which are completely positively monotonic, i.e. there exists a strictly increasing, strictly positive function f: $\forall s > 0$, $\Delta\bar{L}_s^{(2)} = f(\Delta\bar{L}_s^{(1)})$. In fact the sizes (x_1, x_2) realized by the jumps of $\bar{L}_s$ turn out to be supported by the graph of $f(x_1) = \bar{U}_2^{-1}(\bar{U}_1(x_1))$, which in our case of one sided α-stable marginals is given by $f(x_1) = \left(\frac{c_1}{\alpha_1} \cdot \frac{\alpha_2}{c_2}\right)^{-\frac{1}{\alpha_2}} x_1^{\frac{\alpha_1}{\alpha_2}}$.

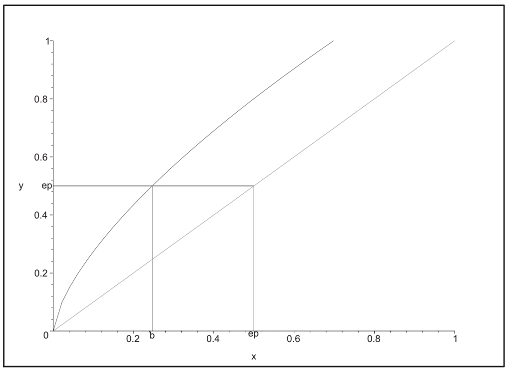

Figure 1. Graph of $f(x_1)$, the support of the jump sizes of $\bar{L}_s$, at any s. Case $\alpha_1 = 0.8; \alpha_2 = 1.2; c_1 = 1; c_2 = 2$. The 45 degrees line is jointly represented, $b = f^{-1}(\varepsilon)$, ep=ε.

Thus our assumption that L has Lévy measure ν_γ means that the jumps of L are supported on the set given by the union of the graph of f and the positive sides of the cartesian axes. Each marginal $\mu^{(m)}$ counts the projection on axis x_m of *all* the jumps of L. Such marginals have α_m-stable law. However when a jump is realized so that $x_2 = f(x_1)$ then this is interpreted as a jump of the parallel component $\bar{L}$ of L. Any other type of jumps of the marginals $L^{(m)}$ are interpreted as being associated to a zero complementary component, i.e. as being the projection of a disjoint jump. By changing γ we keep the same marginals $L^{(m)}$ and the same joint or disjoint jumps, but we change the weight given to the different classes of jumps by the underlying probability measure.

It turns out that $\bar{L}$ has joint Lévy measure $\nu_\parallel([x_1, +\infty) \times [x_2, +\infty)) = I_{\{x_1 \neq 0, x_2 \neq 0\}} \nu^{(1)}([x_1 \vee f^{-1}(x_2), +\infty))$, so ν_γ within **A6** is equivalently writable as

$$\nu_\gamma([x_1, +\infty) \times [x_2, +\infty)) = \gamma I_{\{x_2=0\}} \nu^{(1)}([x_1, +\infty)) + \gamma I_{\{x_1=0\}} \nu^{(2)}([x_2, +\infty)) + \tag{4}$$

$$+(1-\gamma) I_{\{x_1 \neq 0, x_2 \neq 0\}} \nu^{(1)}([x_1 \vee f^{-1}(x_2), +\infty)).$$

The processes we chose to deal with are quite representative since in fact many commonly used models in finance (Variance Gamma model, CGMY model, NIG model, etc.) have Lévy measures related to the ones in assumption **A5** in the sense that they are tempered stable processes where the order of magnitude of the tail integrals as $x_m \to 0$ is as in (3). Moreover C_γ allows to range from a framework of independent components to a framework where the components are completely positively monotonic.

Remark. **A6** is in particular satisfied in a *factor model* for the jump components, where

$$J^{(1)} = V^{(1)}, J^{(2)} = aV^{(1)} + bV^{(2)}$$

with $V^{(1)}, V^{(2)}$ independent pure jump Lévy processes, and a, b real constants. In such a case $f(x) = ax$.

The speed of convergence of $\hat{IC} - IC$ is strictly related to the speed of convergence to zero of the small co-increments of the two $M^{(m)}$ (see [14] for the univariate case), and the small increments of each $X^{(m)}$

substantially behave like the small jumps ([1], Lemma 5). Thus the small co-increments are strictly related to the small co-jumps $\sum_{s\leq T}\Delta X_s^{(1)} I_{|\Delta X_s^{(1)}|\leq\varepsilon}\Delta X_s^{(2)} I_{|\Delta X_s^{(2)}|\leq\varepsilon}$, whose expectation is $\int_{0\leq x,y\leq\varepsilon} xy d\nu_\gamma(x,y)$. Note that the jumps of the bivariate processes M and L coincide in restriction to the subset of $\mathbb{R}^2$ where the jump sizes (x,y) are in $(0,\varepsilon]\times(0,\varepsilon]$, if $\varepsilon<1$, thus for instance $\int_{0\leq x,y\leq\varepsilon} x^k y^m \nu_\gamma(dx,dy)$ is the same for M and L, for any integers m,k.

Remark 4.1. *We need assumption* **A6** *in order to control the speed of convergence to zero of integrals like* $\int_{0\leq x,y\leq\varepsilon} x^k y^m d\nu_\gamma(x,y)$, *for* $\varepsilon>0$ *and integers* k,m. *Note that for the independence part* $C_\perp$ *of the copula when* $k,m\geq 1$ *the integral above is zero, because the independent components of* L *have no common jumps. It follows that under assumption* **A6**, *for both* $k\geq 1$ *and* $m\geq 1$

$$\int_{0\leq x,y\leq\varepsilon} x^k y^m \nu_\gamma(dx,dy) = (1-\gamma)\int_{0\leq x,y\leq\varepsilon} x^k y^m dC_\parallel(U_1(x),U_2(y)).$$

Lemma 4.2. *i) Given the expression of* $C_\parallel$ *and (3), for* $\alpha_1\leq\alpha_2, 0<c_1\leq c_2$, *if* $\varepsilon<e^{-\frac{1}{\alpha_1}}$ *then for any Borel function* g *s.t.* $g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}},\left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right)$ *is Lebesgue-integrable we have*

$$\int_{0\leq x_1,x_2\leq\varepsilon} g(x_1,x_2)\nu_\parallel(dx_1,dx_2) = \int_{\frac{c_2\varepsilon^{-\alpha_2}}{\alpha_2}}^{+\infty} g\left(\left(\frac{\alpha_1 u}{c_1}\right)^{-\frac{1}{\alpha_1}},\left(\frac{\alpha_2 u}{c_2}\right)^{-\frac{1}{\alpha_2}}\right) du$$

ii) for $m,\ k\geq 1$ *note that* $\frac{k}{\alpha_1}+\frac{m}{\alpha_2}-1>0$, *and in particular*

$$\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k x_2^m \nu_\parallel(dx_1,dx_2) = C(k,m)\varepsilon^{m+k\frac{\alpha_2}{\alpha_1}-\alpha_2},$$

with

$$C(k,m)\doteq c_2\left(\frac{\alpha_2 c_1}{\alpha_1 c_2}\right)^{\frac{k}{\alpha_1}}\frac{1}{m+\frac{\alpha_2}{\alpha_1}k-\alpha_2}>0;$$

iii) for $m,k\geq 2$, *for any* $m=1,2$ *we have*

$$\int_{0<x_m\leq\varepsilon} x_m^k \nu_\perp(dx_1,dx_2) = \int_{0<x_m\leq\varepsilon} x_m^k \nu^{(m)}(dx_m) = C_m(k)\varepsilon^{k-\alpha_m},$$

with $C_m(k)=\frac{c_m}{k-\alpha_m}$; *while*

$$\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k \nu_\parallel(dx_1,dx_2) = C(k,0)\varepsilon^{\frac{\alpha_2}{\alpha_1}k-\alpha_2},$$

$$\int_{0\leq x_1,x_2\leq\varepsilon} x_2^m \nu_\parallel(dx_1,dx_2) = C(0,m)\varepsilon^{m-\alpha_2};$$

iv) for $m=1,2$

$$A_m^\varepsilon \doteq \int_{\varepsilon\leq x_m\leq 1} x_m \nu^{(m)}(dx_m) = c_{A_m}\left[(1-\varepsilon^{1-\alpha_m})I_{\alpha_m\neq 1}+\ln\frac{1}{\varepsilon}I_{\alpha_m=1}\right],$$

where $c_{A_m}\doteq\frac{c_m}{1-\alpha_m}I_{\alpha_m\neq 1}+c_m I_{\alpha_m=1}$. *Note that for* $\varepsilon<1$, $c_{A_m}(1-\varepsilon^{1-\alpha_m})>0$ *for any* $\alpha_m\in]0,2[$.

Note that $C(0,m)=\frac{c_2}{m-\alpha_2}=C_2(m)$. The reason why $\int_{0\leq x_1,x_2\leq\varepsilon} x_1^k\nu_\parallel(dx_1,dx_2)$ depends also on α_2 is that the jump sizes of the parallel component of M are connected by $x_2=f(x_1)$. If $\alpha_1\leq\alpha_2$ and $0<c_1\leq c_2$ then for sufficiently small ε we have $U_1(\varepsilon)\leq U_2(\varepsilon)$, thus $\varepsilon\geq U_1^{-1}(U_2(\varepsilon))=f^{-1}(\varepsilon)$. It follows

that by binding both $x_1 \le \varepsilon$ and $x_2 = f^{-1}(x_1) \le \varepsilon$ we impose that $x_1 \le f^{-1}(\varepsilon) \wedge \varepsilon = f^{-1}(\varepsilon)$, so in fact we impose to x_1 a bound depending on α_2.

For $m = 1, 2$ define

$$M_t'^{(m)} \doteq L_t^{(m)} - z^{(m)} t - \sum_{s \le t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}| > \varepsilon\}} = M_t^{(m)} - \sum_{s \le t} \Delta M_s^{(m)} I_{\{|\Delta M_s^{(m)}| > \varepsilon\}}$$

$$= \int_0^t \int_{\{0 < x \le \varepsilon\}} x \tilde{\mu}^{(m)}(dx, ds) - t \int_{\{\varepsilon < x \le 1\}} x \nu^{(m)}(dx)$$

then set

$$\xi_i = \xi_i^\varepsilon \doteq \Delta_i M'^{(1)} \Delta_i M'^{(2)}, \quad \tilde{\xi}_i \doteq \frac{\xi_i - nE[\xi_1]}{\sqrt{nVar(\xi_1)}}.$$

We know that $\sum_{i=1}^n \tilde{\xi}_i$ is always a tight sequence. In the next theorem we computed more explicitly the leading terms of $nE[\xi_1]$ and $\sqrt{nVar(\xi_1)}$.

Theorem 4.3. *Assume* **A1-A3**, **A6-A6**, $0 < \alpha_1 \le \alpha_2 < 2, \alpha_2 \ge 1, 0 < c_1 \le c_2$. *Take* $\varepsilon = h^u$, *any* $u \in]0, \frac{1}{2}[$. *Then as* $\varepsilon \to 0$ *the following quotients are tight:*
i) if $\gamma \in (0, 1)$:

$$\frac{\sum_i \xi_i - T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1} c_{A_2} F_\gamma(\varepsilon)}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h\varepsilon^{2-\alpha_1}\gamma C_1(2)C(0,2) I_{\{\alpha_1 \le x_\star\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}(1-\gamma)C(2,2) I_{\{\alpha_1 \ge x_\star\}}}} \tag{5}$$

where

$$F_\gamma(\varepsilon) = -\varepsilon^{1-\alpha_2} I_{\{\alpha_1 \le \alpha_2 u, \alpha_2 > 1\}} + \log \frac{1}{\varepsilon} I_{\{\alpha_1 \le \alpha_2 u, \alpha_2 = 1\}},$$

$$x_\star \doteq \frac{1 + 2u - \sqrt{-4(2\alpha_2 - 1)u^2 + 4u + 1}}{2u} \in (\alpha_2 u, \alpha_2).$$

ii) If $\gamma = 1$:

$$\frac{\sum_i \xi_i - Thc_{A_1} c_{A_2} F_1(\varepsilon)}{\sqrt{T}\sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}\sqrt{C_1(2)C_2(2)}}, \tag{6}$$

where

$$F_1(\varepsilon) = -\varepsilon^{1-\alpha_2} I_{\{\alpha_1 < 1 < \alpha_2\}} + \log \frac{1}{\varepsilon} I_{\{\alpha_1 < 1 = \alpha_2\}} - \varepsilon^{1-\alpha_2} \log \frac{1}{\varepsilon} I_{\{\alpha_1 = 1 < \alpha_2\}} + \log^2 \frac{1}{\varepsilon} I_{\{\alpha_1 = \alpha_2 = 1\}} + \varepsilon^{2-\alpha_1-\alpha_2} I_{\{1 < \alpha_1 \le \alpha_2\}}.$$

iii) If $\gamma = 0$:

$$\frac{\sum_i \xi_i - TC(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} - Thc_{A_1} c_{A_2}\Big[-\varepsilon^{1-\alpha_2} I_{\{\alpha_1 \le \alpha_2 u, \alpha_2 > 1\}} + \log\frac{1}{\varepsilon} I_{\{\alpha_1 \le \alpha_2 u, \alpha_2 = 1\}}\Big]}{\sqrt{T}\varepsilon^{1-\alpha_2/2}\sqrt{h^2 c_{A_1}^2 C(0,2) I_{\{\alpha_1 < \alpha_2 u\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}\Big[C(2,2) I_{\{\alpha_1 > \alpha_2 u\}} + \big[C(2,2) - 2c_{A_1}C(1,2) + c_{A_1}^2 C(0,2)\big] I_{\{\alpha_1 = \alpha_2 u\}}\Big]}}. \tag{7}$$

Remarks on the Theorem statement.

- The term $-4(2\alpha_2 - 1)u^2 + 4u + 1$ within $x_\star$ turns out to be strictly positive for all $u \in (0, \frac{1}{2})$ and all $\alpha_2 < 2$.
- Recall that $m + k\frac{\alpha_2}{\alpha_1} - \alpha_2 > 0$ for all $m, k \ge 1$, so in (5) and (7) $1 + \frac{\alpha_2}{\alpha_1} - \alpha_2 > 0$.
- The numerator in each quotient is always the difference of $\sum_i \xi_i$ with the leading terms of its (tending to zero) mean. When the sets indicated at the numerator or at the denominator in i) and iii) intersect then $E[\sum_i \xi_i]$ or $\sqrt{nVar(\xi_1)}$ have two asymptotically equivalent leading terms.

- As for the denominator in i), the case $\alpha_1 = \alpha_2$ falls within the region $\alpha_1 \geq x_\star$.
- Note again that, at the numerator of each quotient, if $\alpha_2 > 1$ then $\varepsilon^{1-\alpha_2} \to +\infty$, while $c_{A_2} < 0$ so that the term $-Thc_{m_1}c_{m_2}\varepsilon^{1-\alpha_2} I_{\{\alpha_1 \leq \alpha_2 u, \alpha_2 > 1\}}$ keeps positive (and tends to zero).

Given two (possibly random) sequences U_n and V_n, we say that $U_n = O_P(V_n)$ if there exists $\bar{n}$: for all $n \geq \bar{n}$ we have that for any $\epsilon > 0$, there exists a constant $\eta > 0$ such that $\mathcal{P}(|U_n| > \eta |V_n|) < \epsilon$. When $|V_n|$ are a.s. positive, then $U_n = O_P(V_n)$ means that, for sufficiently large n, U_n/V_n is a sequence bounded in probability (i.e. tight).

Given two sequences U_n, V_n of r.v.s, let us denote by $U_n \sim V_n$ when as $n \to \infty$ we have both $U_n = O_P(V_n)$ and $V_n = O_P(U_n)$.

We now show that in all cases but one, if we take u sufficiently close to $\frac{1}{2}$, we have

$$\sum_i \xi_i \sim nE[\xi_1]$$

$$\begin{cases} \sim T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1 > \alpha_2 u\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}} + Thc_{A_1}c_{A_2}F_\gamma(\varepsilon) & \text{if } \gamma \in [0,1) \\ Thc_{A_1}c_{A_2}F_1(\varepsilon) & \text{if } \gamma = 1. \end{cases} \tag{8}$$

Proposition 4.4. *Assume $0 < \alpha_1 \leq \alpha_2 < 2$ and $\alpha_2 \geq 1$, and take $u \in (0, \frac{1}{2})$. As $h \to 0$ we have $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ in the following cases:*
i) for $\gamma \in [0,1)$: for any choice of α_1, α_2 and u as in the assumptions;
ii) for $\gamma = 1$: on $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1 < \alpha_2\}$ iff $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$; on $\{1 < \alpha_1 \leq \alpha_2\}$ iff $u \in (\frac{1}{\alpha_1+\alpha_2}, \frac{1}{2})$.
We have $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to +\infty$ in the following case:
iii) for $\gamma = 1$: on $\{\alpha_1 = \alpha_2 = 1\}$, any $u \in (0, \frac{1}{2})$.

Remarks.
1. If $\alpha < 1$ or $\alpha = 1 < \alpha_2$ then $\alpha_1 < \alpha_2$ and requiring that $u > 1/(2 + \alpha_2 - \alpha_1)$ is possible because $1/(2 + \alpha_2 - \alpha_1) < 1/2$. On the contrary, the set $\{1 < \alpha_1 \leq \alpha_2\}$ contains the case $\alpha_1 = \alpha_2$ and $u > 1/(2+\alpha_2-\alpha_1) = 1/2$ is not admissible. Note that condition $u > 1/(2+\alpha_2-\alpha_1)$ implies $u > 1/(\alpha_1+\alpha_2)$ when $\alpha_2 > \alpha_1 > 1$.
2. When $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ then the tightness of $\sum_{i=1}^n \tilde{\xi}_i$ implies that

$$\frac{\sum_{i=1}^n \xi_i}{nE[\xi_1]} \xrightarrow{P} 1.$$

Otherwise, if $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to \infty$, the tightness of of $\sum_{i=1}^n \tilde{\xi}_i$ only allows us to say that $\forall \eta > 0 \ \exists K_\eta$: with probability larger than $1-\eta$ we have $|\sum_{i=1}^n \xi_i - nE[\xi_1]| \leq K_\eta \sqrt{nVar(\xi_1)}$, i.e. $|\sum_{i=1}^n \xi_i| \leq \tilde{K}_\eta \sqrt{nVar(\xi_1)}$, but $\sum_{i=1}^n \xi_i$ could tend to 0 faster than $\sqrt{nVar(\xi_1)}$. However the following CLT gives us the exact asymptotic behavior of $\sum_{i=1}^n \xi_i$.

Theorem 4.5. *When $\gamma = 1 = \alpha_1 = \alpha_2$, for any $u \in (0, \frac{1}{2})$ we have*

$$\frac{\sum_{i=1}^n \xi_i - nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \xrightarrow{d} \mathcal{N},$$

where $\xrightarrow{d}$ denotes convergence in distribution.

Further, a CLT holds also in the case of completely dependent small jumps.

Proposition 4.6 (CLT when $\gamma = 0$, see [7], Thm 4.4). *If $\gamma = 0$ then as $\varepsilon \to 0$ (7) converges in distribution to a standard Gaussian r.v. .*

Remarks: implications of the Theorems and the Proposition.

i) The rate of convergence of $\sum_i \xi_i$ is determined not only by each one of the jump activity indices α_1, α_2 but also on the degree γ of dependence of the two small jumps components of Z.

ii) We have that $\sum_{i=1}^n \xi_i$ tends to zero much faster when $\gamma = 1$ than when $\gamma \in [0, 1)$ (we obtain that by using Proposition 4.4 and comparing $nE[\xi_1]$ or $\sqrt{nVar(\xi_1)}$ in (6) with $nE[\xi_1]$ in (5), while matching all the sets of (α_1, α_2)), i.e. the speed at which the co-increments ξ_i due to the small jumps tend to zero is much faster when $M^{(1)}, M^{(2)}$ are independent, in fact the sum of the co-jumps in the independent case is zero.

iii) Comparing the rate of $\sum_i \xi_i$ with $\sqrt{h}$, we reach that $\sum_i \xi_i << \sqrt{h}$ substantially when α_1 is sufficiently small (and still $\alpha_2 \geq 1$). In other words, when α_1 is sufficiently small, the co-increments of the small jumps components are negligible wrt the Browinan components. More precisely, using Proposition 4.4, (8) and (24) below, defined

$$\alpha_1^{\star} \doteq \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1), \quad \alpha_1^{\star\star} \doteq \frac{1 + 2u(2 - \alpha_2)}{2u} > \frac{1}{2u} > 1,$$

we reach (see the proof in Appendix) that:

$$\begin{cases} \text{if } \gamma \in [0, 1): & \sum_i \xi_i \sim nE[\xi_1] << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{\star}; \\ \text{if } \gamma = 1: & \sum_i \xi_i << \sqrt{h} \quad \text{iff } \alpha_1 < \alpha_1^{\star\star}. \end{cases} \tag{9}$$

In the light of Theorem 4.7 below, since $\alpha_1^{\star} < 1 < \alpha_1^{\star\star}$, the above result means that when the two small jumps components $M^{(m)}$ are independent, then the impact of their co-increments on the convergence speed of $\hat{IC} - IC$ is negligible, wrt the impact $\sqrt{h}$ of the Browinan components, for a wider range of values α_1. □

We now check the speed of $\hat{IC} - IC$. Recall that each α-stable process $L^{(m)}$ at time t can be written as $L_t^{(m)} = z^{(m)} h + \sum_{s \leq t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}| > 1\}} + \int_0^t \int_{|x| \leq 1} x \tilde{\mu}^{(m)}(dx, ds)$, so that

$$X_t^{(m)} = \int_0^t a_s^{(m)} ds + \int_0^t \sigma^{(m)} dW_s^{(m)} + J_t^{(m)} + M_t^{(m)}$$

$$= \int_0^t a_s^{(m)} ds + \int_0^t \sigma^{(m)} dW_s^{(m)} + J_t^{(m)} + L_t^{(m)} - z^{(m)} t - \sum_{s \leq t} \Delta L_s^{(m)} I_{|\Delta L_s^{(m)}| > 1}$$

$$= \tilde{D}_t^{(m)} + \tilde{J}_t^{(m)} + L_t^{(m)},$$

where $\tilde{D}_t^{(m)} = D_t^{(m)} - z^{(m)} t$, $\tilde{J}_t^{(m)} = J_t^{(m)} - \sum_{s \leq t} \Delta L_s^{(m)} I_{|\Delta L_s^{(m)}| > 1}$ and we already know from the literature that the BSM parts $\tilde{D}^{(m)}$ and the FA jumps parts $\tilde{J}^{(m)}$ contribute to the speed of $\hat{IC}$ as $\sqrt{h}$. So, in fact, we could consider the IA jumps part of $X^{(m)}$ as being an α_m-stable process.

Theorem 4.7. *If $\rho \not\equiv 0$ and $\sigma^{(m)}$ are s.t. when $h \to 0$,*

$$\forall s \geq t: \; s - t \leq h, \; \textit{then } E[|\sigma_s^{(m)} - \sigma_t^{(m)}|^2] \leq c(s - t), \tag{10}$$

then, as $h \to 0$, *with* $\varepsilon = \sqrt{r_h} = h^u$ *with*

$$1/2 > u > \begin{cases} \frac{1}{2+\alpha_2-\alpha_1} \vee \frac{1}{3-\frac{\alpha_2}{2}} & \textit{if } \alpha_1 < \alpha_2 \\ \frac{1}{3-\frac{\alpha_2}{2}} & \textit{if } \alpha_1 = \alpha_2 \end{cases} \tag{11}$$

and U_h *a sequence of r.v.s converging stably in law to a mixed Gaussian r.v., we have*

$$\hat{IC} - IC \sim \sqrt{h}U_h + \sum_{i=1}^{n} \xi_i + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta M_s^{(2)}|>\sqrt{r_h}\}} \geq 1\}} \tag{12}$$

$$\sim \sqrt{h} + (1-\gamma)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} + h\varepsilon^{-\alpha_2}$$

$$\begin{aligned} \sim \quad & \sqrt{h}\; I_{\{\alpha_2\in[1,\frac{1}{2u})\}} \left[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1),\alpha_1\leq\alpha_1^\star\}}\right] \\ & +(1-\gamma)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\gamma\in[0,1)\}} \left[I_{\{\alpha_2\in[1,\frac{1}{2u}),\ \alpha_1>\alpha_1^\star\}} + I_{\{\alpha_2\geq\frac{1}{2u}\}} I_{\{\alpha_2=\alpha_1\}\cup\{\alpha_2>\alpha_1,\ u<\frac{\alpha_1}{\alpha_2+\alpha_1}\}}\right] \\ & +h\varepsilon^{-\alpha_2}\; I_{\{\alpha_2\geq\frac{1}{2u}\}} \left[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1)\}} I_{\{\alpha_1\leq\alpha_2 u\}\cup\{\alpha_2>\alpha_1>\alpha_2 u,\ u\geq\frac{\alpha_1}{\alpha_2+\alpha_1}\}}\right]. \end{aligned} \tag{13}$$

Remarks on (13).

i) We did not include condition $u \geq \frac{\alpha_1}{\alpha_2+\alpha_1}$ among the ones in (11), because such conditions are required for the convergence of some terms of I_4 (defined within the proof of the Theorem) in $\hat{IC} - IC$, while $\frac{\alpha_1}{\alpha_2+\alpha_1}$ is only a separator to establish whether the leading term is θ_2 or $\sum_{i=1}^n \xi_i$. There is another proof for the convergence of some of the cited terms of I_4, which avoids conditions (11), but it is much longer than the one given in the appendix.

ii) If $\alpha_1 < \alpha_2$, (11) implies that $u > 1/4$. Also, $u < \frac{\alpha_1}{\alpha_2+\alpha_1}$ implies $\alpha_2 u < \alpha_1$.

iii) As for $\sum_{i=1}^n \xi_i$, also the convergence rate of $\hat{IC} - IC$ not only depends, as in the univariate case, on the jump activity indices, but, surprisingly, also on the dependence structure of the small jumps.
This implies that $\hat{IC}$ contains information that we could exploit to estimate the dependence degree among the small jumps of two processes.

Note that when the dependence degree increases (γ decreases) then the leading term of $\sum_{i=1}^n \xi_i$ also increases ($\sum_i E[\xi_i]$ increases and $\sqrt{nVar(\xi_1)} < \sum_i E[\xi_i]$), and the estimation error $\hat{IC} - IC$ increases. An higher leading term of $\sum_i \xi_i$ means that the average weight of the small jumps is higher so that the disturbing noise when estimating the Brownian feature IC is higher. That is: the higher the dependence degree, the higher the disturbing noise.

iv) Basically, when u is very close to 1/2 ($u \geq \frac{\alpha_1}{\alpha_2+\alpha_1}$), the rate is $\sqrt{h}$ when α_1, α_2 are small (note that when $\alpha_2 < 1/(2u)$ then $\alpha_1 < \alpha_1^{\star\star}$); it is $\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ when the jumps are dependent and either the indices have intermediate values within $(\alpha_1^\star, 1/(2u))$ or they coincide and assume the largest possible values; the rate is $h\varepsilon^{-\alpha_2}$ when α_2 is large and the indices are different if $\gamma \in [0,1)$, any the indices are if $\gamma = 1$.

v) When the leading term is $\sum_{i=1}^n \xi_i$ the speed $\sqrt{nVar(\xi_1)} = \sqrt{h}\varepsilon^{2-\alpha_1/2-\alpha_2/2}$ never appears, because in the cases where the asymptotic behavior of $\sum_{i=1}^n \xi_i$ is $\sqrt{nVar(\xi_1)}$ (e.g. the case $\gamma = \alpha_1 = \alpha_2 = 1$; the case $\gamma = 1$ and $1 < \alpha_1 \leq \alpha_2 < 1/(2u)$, where we have $u < 1/(2\alpha_2) \leq 1/(\alpha_1+\alpha_2)$) it holds that $\sqrt{nVar(\xi_1)}/\sqrt{h} \to 0$, so $\sum_{i=1}^n \xi_i$ is dominated by $\sqrt{h}$.

vi) For $\gamma = 0$ or $\gamma \in (0,1)$ we have the same cases: in the presence of the parallel component, the independent component does not modify the rate of convergence. More precisely, recalling that $\varepsilon = h^u$, with $u < 1/2$ but u close to 1/2 (i.e. verifying conditions (11) and $u \geq \alpha_1/(\alpha_1+\alpha_2)$) and noting that

$\alpha_1^\star < 1 \le 1/(2u)$, we have the following rates when $\gamma \in [0,1)$:

$$\begin{array}{ll} \sqrt{h} & \text{if } \alpha_1 \le \alpha_1^\star < 1 \le \alpha_2 < \frac{1}{2u} \\ \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} & \text{if } \alpha_1^\star < \alpha_1 \le \alpha_2 < \frac{1}{2u} \\ \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} & \text{if } \alpha_1 = \alpha_2 \ge \frac{1}{2u} \\ h\varepsilon^{-\alpha_2} & \text{if } \alpha_1 \ne \alpha_2, \alpha_2 \ge \frac{1}{2u}. \end{array}$$

On the contrary, in the presence of the independent component, the parallel component does worsen the rate of convergence. In fact, for $\gamma = 1$ (complete independence of the small jumps) we have the following rates:

$$\begin{array}{ll} \sqrt{h} & \text{if } \alpha_2 < \frac{1}{2u}, \\ h\varepsilon^{-\alpha_2} & \text{if } \alpha_2 \ge \frac{1}{2u}. \end{array}$$

vii) The rate is $\sqrt{h}$ even in some cases with $\alpha_2 > 1$ (but $\alpha_2 < 1/(2u)$): any α_1 is, if $\gamma = 1$; for α_1 sufficiently small ($\alpha_1 \le \alpha_1^\star$) if the parallel component is present.

viii) When $\alpha_1 = \alpha_2 \doteq \alpha \ge 1$ the jump activity indices α_1 and α_2 coincide but the two jump components are not necessarily completely monotonic and we reach the following speeds of convergence to zero of $\hat{IC} - IC$:

$$\begin{array}{lll} \text{if } \gamma \in [0,1) & \text{rate } (1-\gamma)\varepsilon^{2-\alpha} & \\ \text{if } \gamma = 1 & \text{rate } \sqrt{h} & \text{if } \alpha < \frac{1}{2u} \text{ (note that } \alpha < \alpha_1^\star < 1 \text{ is not in our ossumptions)} \\ \text{if } \gamma = 1 & \text{rate } h\varepsilon^{-\alpha_2} & \text{if } \alpha \ge \frac{1}{2u}. \end{array}$$

ix) The univariate case is when $\alpha_1 = \alpha_2$ and $\gamma = 0$. In that case the rate is $\varepsilon^{2-\alpha} = r_h^{1-\alpha/2}$, for any $\alpha \ge 1$, consistently with [14], for the component of the error $\hat{IV} - IV$, in the estimation of the *Integrated Variance*, due to the infinite activity infinite variation jump part.

x) The convergence speed is a function $s(\gamma, \alpha_1, \alpha_2, u)$ of our parameters. Such a function is smooth most of the times, however it has some singularities. In fact when $u \ge \alpha_1/(\alpha_1+\alpha_2)$ and $\gamma \in [0,1)$: if $\alpha_1 \ne \alpha_2$ but the two indices are close and above $1/(2u)$, then $s = h\varepsilon^{-\alpha_2} = h^{1-\alpha_2 u}$ while at $\alpha_1 = \alpha_2$ the function s jumps to $\varepsilon^{2-\alpha_2} = h^{2u-\alpha_2 u}$. The jump would disappear if it was $u = 1/2$.
On the contrary we have smoothness at $\alpha_1 = \alpha_1^\star$ if $\alpha_2 < 1/(2u)$: in fact if $\alpha_1 << \alpha_2$ (case $\alpha_1 \le \alpha_1^\star < 1 \le \alpha_2 < \frac{1}{2u}$) then $s = h^{1/2}$; for α_1 at $\alpha_1^\star$ we have $s = \sqrt{h} = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, and with $\alpha_1 \in (\alpha_1^\star, \alpha_2]$ still is $s = \varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$.
When $\gamma = 1$ we have smoothness at $\alpha_2 = 1/(2u)$: in fact when $\alpha_2 = 1/(2u)$ we have $\sqrt{h} = h\varepsilon^{-\alpha_2}$. □

Remark. *When $\alpha_2 < 1/(2u)$ and either $\gamma = 1$ or $\gamma \in [0,1), \alpha_1 < \alpha_1^\star$, we have a CLT for $\hat{IC} - IC$.*
Note in fact that $\sqrt{h}$ only comes from the BSM parts $Y^{(m)}$ of the processes $X^{(m)}$: when $\alpha_2 < 1/(2u)$ and either $\gamma = 1$ or $\gamma \in [0,1), \alpha_1 < \alpha_1^\star$, $\sqrt{h}$ is the only leading term of $\hat{IC} - IC$ and the presence of $M^{(1)}$ and $M^{(2)}$ is not influential. Further we know that, even in the presence of $M^{(1)}$ and $M^{(2)}$, $A\hat{V}ar \xrightarrow{P} \int_0^T (1+\rho_t^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt$, for all $\alpha_1, \alpha_2 \in (0,2)$ (Theorem 3.3), thus by Theorem 3.2 we have

$$\frac{\hat{IC} - IC}{\sqrt{h}\sqrt{A\hat{V}ar}} \xrightarrow{st} \mathcal{N}.$$

□

Aknowledgements. I thank Peter Tankov and Jean Jacod for their very important consulting.

5 Appendix

Note that by using a localization procedure similar to the one in [8] (sec. 3.6.3) we can assume wlg that the coefficients $a^{(m)}, \sigma^{(m)}, \rho$ expressing the dynamics of X are bounded. In particular in Remark 3.1 we can take K a constant (in fact, in [13] it is shown that $K(\omega) \le \sup_{s\in[0,T]} |a| + \sup_{s\in[0,T]} |\sigma| + 1$).

In the following denote, for $m = 1, 2$,

$$\tilde{N}_t^{(m)} = \sum_{s\le t} I_{\{|\Delta X_s^{(m)}| > \sqrt{r_h}\}}, \quad \tilde{V}_t^{(m)} = \sum_{s\le t} I_{\{|\Delta M_s^{(m)}| > \sqrt{r_h}\}}, \quad \theta_m = h r_h^{-\frac{\alpha_m}{2}}.$$

c and K are mute names for any positive constants: they keep the same name passing from one side to the other of an inequality/equality even when the constant changes.

Remark 5.1.

1. *(Lemma A.3 part (1) in [16]) Under the assumptions* **A1**, **A2** *and* **A5** *we have, a.s., for small* h, $I_{\{(\Delta_j D^{(m)})^2 > r_h\}} = 0$*, uniformly in* j*;*

2. *(Lemma A.4 in [16]) Let us consider the sequence* $\hat{IC}^{(n)}, n \in \mathbb{N}$*. As long as* $M^{(m)}$ *is a semimartingale, we can find a subsequence* n_k *for which a.s., for any* $\delta > 0$*, for large* k*, for all* $j = 1..n_k$*, on* $\{(\Delta_i M^{(m)})^2 \le 4r(h_k)\}$ *we have that*

$$(\Delta M_{2,s}^{(m)})^2 \le \delta + 4r(h_k), \quad \forall s \in]t_{i-1}, t_i].$$

3. *(Lemma 2 in [2]) If* $\tilde{L}$ *is a symmetric stable process with* $\tilde{\tilde{N}}_t = \sum_{s\le t} \Delta I_{\{|\Delta \tilde{L}_s| > \varepsilon\}}$ *and Lévy density* $F(dx) = \frac{c}{|x|^{1+\alpha}} dx$*, if* $\tilde{\theta} = h\varepsilon^{-\alpha}$*, then:*

$$P\left\{\left|\Delta_i \tilde{L} - \sum_{s\in]t_{i-1},t_i]} \Delta \tilde{L}_s I_{\{|\Delta \tilde{L}_s| > \varepsilon\}}\right| > \varepsilon\right\} \le K\tilde{\theta}^{4/3};$$

$$P\{|\Delta_i \tilde{L}| > \varepsilon, \Delta_i \tilde{\tilde{N}} = 0\} \le K\tilde{\theta}^{4/3};$$
$$P\{|\Delta_i \tilde{L}| \le \varepsilon, \Delta_i \tilde{\tilde{N}} = 1\} \le K\tilde{\theta}^{4/3}.$$

4. *(Lemma 6 in [2]) For any semimartingale* X *with the same form as one of the components in (1) we have*

$$P(|\Delta_i X| > c\varepsilon) \le K\tilde{\theta}_m.$$

5. *([6], ch.3, Prop. 3.7) For any Lévy process* V *with Lévy measure* ν*, then* $\sum_{s\le t} I_{\{|\Delta V_s| > \varepsilon\}}$ *is a Poisson process with parameter* $t\nu\{|x| > \varepsilon\} = tU(\varepsilon)$*, where* $U(x)$ *gives the tail of the jumps sizes measure;*
   it follows that if $\nu(dx) = a|x|^{-1-\alpha} I_{x<0} + bx^{-1-\alpha} I_{x>0}$ *with* $a, b > 0$*, then with* $p \in (0, 1)$*,*

$$P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta V_s| > \varepsilon\}} = 1\} \sim K\tilde{\theta},$$

$$P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta V_s| > \varepsilon\}} \ge 2\} \sim K\tilde{\theta}^2,$$

$$P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta V_s| \in (\varepsilon(1-p), \varepsilon]\}} = 1\} \le K\tilde{\theta}((1-p)^{-\alpha} - 1).$$

Only last point needs to be commented: we have

$$P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta V_s|\in(\varepsilon(1-p),\varepsilon]\}}=1\}=P(\mu\{]t_{i-1},t_i]\times(\varepsilon(1-p),\varepsilon)\}=1)=e^{-\lambda h}\lambda h,$$

where $\lambda=\nu\{(\varepsilon(1-p),\varepsilon)\}=U(\varepsilon(1-p))-U(\varepsilon)$, thus the above display

$$\sim Kh(U(\varepsilon(1-p))-U(\varepsilon))\le Kh\varepsilon^{-\alpha}((1-p)^{-\alpha}-1).$$

Since for small p we have $(1-p)^{-\alpha}\sim(1+\alpha p)$, the thesis follows. □

Let us recall that in our framework the small jumps parts $M^{(m)}$ are the small jumps of *one-sided* stable processes $L^{(m)}$.

Lemma 5.2.

1. *For each $m=1,2$ for any $i=1..n$, we have*

$$P\{\Delta_i N^{(m)}\neq 0,(\Delta_i M^{(m)})^2>r_h\}\le K\frac{h^2}{r_h}.$$

2. *If L is a one-sided α-stable process with characteristic triplet $(z,0,c\cdot I_{\{x>0\}}x^{1-\alpha}dx)$, if we take an $\varepsilon=\varepsilon(h)$ s.t. $\varepsilon(h)/h\to 0$ then, for any constant $p\in(0,1)$ s.t. $p>|z|\varepsilon/h$ then $\forall\ q\in(0,1-p)$ we have*

$$P\{|\Delta_i L|>\varepsilon,\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>\varepsilon\}}=0\}\le K\tilde\theta^{4/3}+K\tilde\theta(q^{-\alpha}-1).$$

3. *For any p,q as above we have*

$$P\{|\Delta_i M^{(m)}|>\sqrt{r_h}(1-p),\Delta_i\tilde V^{(m)}=0\}\le K\theta_m^{4/3}+K\theta_m(q^{-\alpha}-1).$$

4. *Let $H_1\doteq(L_t-zt)_t$, and take p as above, then*

$$P\{|\Delta_i H_1|\le\varepsilon(1+p),\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}}=1\}\le K\tilde\theta_m^{4/3}+K\tilde\theta(1-(1+2p)^{-\alpha})$$

5. *We have*

$$P\{|\Delta_i L|\le\varepsilon,\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>\varepsilon\}}=1\}\le K\tilde\theta_m^{4/3}+K\tilde\theta(1-(1+2p)^{-\alpha})$$

6. *For any fixed $p\in(0,1)$ s.t. $p>|z|\sqrt{r_h}/h$ we have*

$$P\{|\Delta_i M^{(m)}|\le\sqrt{r_h}(1+p),\Delta_i\tilde V^{(m)}\ge 1\}\le K\theta_m^{4/3}+K\theta_m(1-(1+2p)^{-\alpha}).$$

7. $P(|\Delta_i M^{(m)}|>c\sqrt{r_h})\le K\theta_m$

Proof As for point 1, since N and M are independent, using also the Markov inequality we trivially have $P\{\Delta_i N^{(m)}\neq 0,(\Delta_i M^{(m)})^2>r_h\}=P\{\Delta_i N^{(m)}\ge 1\}P\{(\Delta_i M^{(m)})^2>r_h\}\le Kh\frac{E[(\Delta_i M^{(m)})^2]}{r_h}=Kh\frac{h\int_0^1 x^2\nu^{(m)}(dx)}{r_h}=K\frac{h^2}{r_h}$.

Point 2: the idea here is to look at $H_1\doteq\Big(L_t-zt\Big)_t$ as half of a symmetric stable process. More precisely, take an independent and identically distributed copy H_2 of H_1, then $\tilde L=H_1-H_2$ is a symmetric α-stable process because its Lévy measure is $c\cdot|x|^{-1-\alpha}dx$, so Remark 5.1, point 3, holds true for $\tilde L$. Let us now fix any $p\in(0,1)$ s.t. $|z|h/\varepsilon<p$ and call $\tilde L'$, and H'_ℓ the processes $\tilde L$, H_ℓ deprived of their

jumps bigger than ε, e.g. $H'_{\ell t} = H_{\ell t} - \sum_{s\leq t} \Delta H_{1s} I_{\{|\Delta H_{1s}|>\varepsilon\}}$. Note that if $|\Delta_i L| > \varepsilon$ then we have $|\Delta_i H_1| = |\Delta_i L - zh| > |\Delta_i L| - |z|h > \varepsilon - |z|h > \varepsilon(1-p)$, and also that the jumps of L and H_1 are the same and are positive, thus

$$P\left\{|\Delta_i L| > \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>\varepsilon\}} = 0\right\} \leq P\left\{|\Delta_i H_1| > \varepsilon(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\varepsilon\}} = 0\right\}$$

[i salti di L ed H_1 sono gli stessi]

$$= P\left\{|\Delta_i H'_1| > \varepsilon(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\varepsilon\}} = 0\right\} \leq P\{|\Delta_i H'_1| > \varepsilon(1-p)\} \tag{14}$$

$$= P\{|\Delta_i H'_1| > \varepsilon(1-p), \Delta_i H'_2 \leq \varepsilon(1-p-q)\} + P\{|\Delta_i H'_1| > \varepsilon(1-p), \Delta_i H'_2 > \varepsilon(1-p-q)\}.$$

Now within the first set of the last display we have $|\Delta_i \tilde{L}'| = |\Delta_i H'_1 - \Delta_i H'_2| > |\Delta_i H'_1| - |\Delta_i H'_2| > \varepsilon(1-p) - \varepsilon(1-p-q) = \varepsilon q$, while the probability of the second set, by the independence of the processes H'_ℓ, is $P\{|\Delta_i H'_1| > \varepsilon(1-p)\}P\{|\Delta_i H'_2| > \varepsilon(1-p-q)\}$ which is dominated by $K\tilde{\theta}^2$ by Lemma 5.1, point 4, taking $a \equiv \sigma \equiv J \equiv 0$, and $M_t = \int_0^t \int_0^\varepsilon x\tilde{\nu}(dx) - t\int_\varepsilon^1 x\nu(dx)$. $H_1 \neq M$, ma H'_ℓ sarebbe M' del teo 4.3 It follows that (14) is dominated by

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q\} + K\tilde{\theta}^2 : \tag{15}$$

note that

$$P\{|\Delta_i \tilde{L}'| > \varepsilon q\} = P\left\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{\tilde{N}} = 0\right\} + P\left\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{\tilde{N}} \geq 1\right\} :$$

by the independence of $\Delta_i \tilde{L}'$ on $\Delta_i \tilde{\tilde{N}}$ and using Remark 5.1, point 3, we have

$$P\left\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{\tilde{N}} \geq 1\right\} = P\{|\Delta_i \tilde{L}'| > \varepsilon q\}P\left\{\Delta_i \tilde{\tilde{N}} \geq 1\right\} \leq K\tilde{\theta}^{4/3}\tilde{\theta}.$$

Therefore (15) is dominated by

$$K\tilde{\theta}^{7/3} + P\{|\Delta_i \tilde{L}'| > \varepsilon q, \Delta_i \tilde{\tilde{N}} = 0\} + K\tilde{\theta}^2;$$

[ci sono solo salti $\leq \varepsilon <-> $ i salti $\leq \varepsilon p$ + i salti con sizes $\in (\varepsilon p, \varepsilon]$] noting that $\tilde{\theta}^{7/3} << \tilde{\theta}^2$, the last display is dominated by

$$P\left\{|\Delta_i \tilde{L}'| > \varepsilon q, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta \tilde{L}_s>\varepsilon q\}} = 0\right\} + P\left\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta \tilde{L}_s\in(\varepsilon q,\varepsilon]\}} \geq 1\right\} + K\tilde{\theta}^2 : \tag{16}$$

using Remark 5.1 point 5, we have

$$P\left\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta \tilde{L}_s\in(\varepsilon q,\varepsilon]\}} \geq 1\right\} \sim \tilde{\theta}(q^{-\alpha} - 1), \tag{17}$$

using also Remark 5.1, point 3, with εq in place of ε, and the fact that $\tilde{\theta}^2 << \tilde{\theta}^{4/3}$, (16) is dominated by

$$K\tilde{\theta}^{4/3} + K\tilde{\theta}(q^{-\alpha} - 1),$$

which is our thesis.

Point 3 is a consequence of point 2. Let us denote $L^{(m)}$ with L. We are going to compare $\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\sqrt{r_h}\}} = 0\}$ and $\{\Delta_i \tilde{V}^{(m)} = 0\}$. Note that since $M_t = L_t - zt - \sum_{s\leq t} \Delta L_s I_{\{\Delta L_s>1\}}$, the jumps of M are the ones less than 1 of L. The set $\{\Delta_i \tilde{V}^{(m)} = 0\}$ is where, among the jumps less than 1 of L, only those less than $\sqrt{r_h}$ arrived; however some jumps of L bigger than 1 could be happened. On the contrary on $\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\sqrt{r_h}\}} = 0\} = \{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>\sqrt{r_h}\}} = 0\}$ no jumps of L bigger than $\sqrt{r_h}$ arrived, thus not even the jumps bigger than 1. It follows that, with $\varepsilon = \sqrt{r_h}$,

$$P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0\} = P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} = 0\}$$

$$+P\{|\Delta_i M^{(m)}| > \sqrt{r_h}(1-p), \Delta_i \tilde{V}^{(m)} = 0, \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} \geq 1\}$$

$$\leq P\{|\Delta_i H_1'| > \sqrt{r_h}(1-p), \sum_{s\in]t_{i-1},t_i]} I_{\{\Delta H_{1s}>\sqrt{r_h}\}} = 0\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{\Delta L_s>1\}} \geq 1\}$$

the first term is the one in (14) while the second one involves the Poisson process, counting the jumps of L bigger than 1 within $]t_{i-1}, t_i]$, which has parameter $hU(1)$, thus the display above is dominated by

$$K\theta^{4/3} + K\theta(q^{-\alpha} - 1) + Kh \leq K\theta^{4/3} + K\theta(q^{-\alpha} - 1).$$

Point 4. With the same notations as at Point 2, we have

$$P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\} = P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1, |\Delta_i H_2| > \varepsilon p\}$$

$$+P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1, |\Delta_i H_2| \leq \varepsilon p\} : \quad (18)$$

the first term of the rhs is dominated by $P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1, |\Delta_i H_2| > \varepsilon p\}$, which by the independence between the processes H_ℓ coincides with $P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\}P\{|\Delta_i H_2| > \varepsilon p\}$ which is in turn dominated by $K\tilde{\theta}^2$ by Remark 5.1 points 4 and 5. As for the second term, on $\{|\Delta_i H_1| \leq \varepsilon(1+p), |\Delta_i H_2| \leq \varepsilon p\}$ we have $|\Delta_i \tilde{L}| = |\Delta_i H_1 - \Delta_i H_2| \leq |\Delta_i H_1| + |\Delta_i H_2| \leq \varepsilon(1+p) + \varepsilon p = \varepsilon(1+2p)$. Moreover the independence between the processes H_ℓ implies that they have no contemporary jumps, thus if at time s the process H_1 makes 1 jump larger than ε, then also $\tilde{L}$ does. In particular, if H_1 makes only 1 jump larger than ε within the time interval $]t_{i-1}, t_i]$, then: if H_2 doesn't do then $\tilde{L}$ only has 1 jump larger than ε within $]t_{i-1}, t_i]$, while if H_2 does then $\tilde{L}$ has more than 1 jump larger than ε within $]t_{i-1}, t_i]$. That means that $\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1 \Rightarrow \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta \tilde{L}_s|>\varepsilon\}} \geq 1$. However we know that $P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta \tilde{L}_s|>\varepsilon\}} \geq 2\} \leq K\tilde{\theta}^2$, it follows that (18) is dominated by

$$K\tilde{\theta}^2 + P\{|\Delta_i \tilde{L}| \leq \varepsilon(1+2p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta \tilde{L}_s|>\varepsilon\}} = 1\}$$

$$\leq K\tilde{\theta}^2 + P\{|\Delta_i \tilde{L}| \leq \varepsilon(1+2p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta \tilde{L}_s|>\varepsilon(1+2p)\}} = 1\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta \tilde{L}_s|\in(\varepsilon,\varepsilon(1+2p))\}} = 1\} :$$

by Remark 5.1 point 5, the last term is dominated by $Kh[U(\varepsilon) - U(\varepsilon(1+2p))] = H\tilde{\theta}(1 - (1+2p)^{-\alpha})$; using also point 3, with $\varepsilon(1+2p)$ in place of ε, and the fact that $\tilde{\theta}^2 << \tilde{\theta}^{4/3}$ the thesis follows.

Point 5 follows from point 4. In fact if $|\Delta_i L| \leq \varepsilon$ then $|\Delta_i H_1| = |\Delta_i L - zh| \leq |\Delta_i L| + |z|h < \varepsilon(1+p)$, further the jumps of L are exactly the ones of H_1, thus

$$P\{|\Delta_i L| \leq \varepsilon, \sum_{s\in]t_{i-1},t_i]} I_{\{|L_s|>\varepsilon\}} = 1\} \leq P\{|\Delta_i H_1| \leq \varepsilon(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|H_{1s}|>\varepsilon\}} = 1\}.$$

Point 6 also follows from point 4. Let us denote $L^{(m)}$ with L. We have

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1\} = \tag{19}$$

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} = 0\} +$$

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} \ge 1\} :$$

the second term of the rhs is bounded by Kh, as at Point 3, thus (19) is dominated by

$$P\{|\Delta_i M^{(m)}| \le \sqrt{r_h}(1+p), \Delta_i \tilde{V}^{(m)} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} = 0\} + Kh.$$

Now, if L does not make jumps bigger than 1, then the jumps of M coincide with the jumps of L, which in turn are the same as for H_1; further the very M coincides with H_1. Thus the probability of the last display coincides with

$$P\{|\Delta_i H_1| \le \sqrt{r_h}(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} \ge 1, \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta L_s|>1\}} = 0\} + Kh \le$$

$$P\{|\Delta_i H_1| \le \sqrt{r_h}(1+p), \sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} = 1\} + P\{\sum_{s\in]t_{i-1},t_i]} I_{\{|\Delta H_{1s}|>\varepsilon\}} \ge 2\} + Kh$$

$$\le K\theta_m^{4/3} + K\theta_m(1-(1+2p)^{-\alpha}) + K\theta_m^2 \le K\theta_m^{4/3} + K\theta_m(1-(1+2p)^{-\alpha}).$$

Finally, point 7 is a trivial consequence of Remark 5.1, point 4, as $M^{(m)}$ is a semimartingale following the same mode as component $X^{(m)}$ in 1 with $a \equiv \sigma \equiv J^{(m)} \equiv 0$. □

Proof of Theorem 4.3. Define

$$X_m^\varepsilon \doteq \int_0^h \int_{|x|\le\varepsilon} x\tilde{\mu}^{(m)}(dx, dt)$$

and recall that $A_m^\varepsilon = \int_\varepsilon^1 x\nu^{(m)}(dx) = c_{A_m}\Big[(1-\varepsilon^{1-\alpha_m})I_{\alpha_m\neq 1} + \ln\frac{1}{\varepsilon}I_{\alpha_m=1}\Big]$, so that each $\xi_i,\ i=1..n$, has the same law as $(X_1^\varepsilon - hA_1^\varepsilon)(X_2^\varepsilon - hA_2^\varepsilon)$. For simplicity we write A_m in place of A_1^ε. We are going to compute $E[\sum_{i=1}^n \xi_i]$ and $Var[\sum_{i=1}^n \xi_i]$ so that we are sure that the centered and normalized ξ_i given by

$$\tilde{\xi}_i \doteq \frac{\xi_i - E[\xi_i]}{\sqrt{nVar(\xi_i)}}$$

are such that $\sum_{i=1}^n \tilde{\xi}_i$ keeps tight. We thus need to compute the moments $E[(X_1^\varepsilon)^k(X_2^\varepsilon)^m]$, with $k = 2,1,0, m = 2,1,0$. The bivariate process $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon)$ is Lévy with Lévy measure $\nu_\varepsilon(dx_1, dx_2) = I_{\{0\le x_1,x_2\le\varepsilon\}}\nu_\gamma(dx_1, dx_2)$, and note that, for small ε, if $0 \le x_1, x_2 \le \varepsilon$ then also is $x_1^2 + x_2^2 \le 1$, so we reach the desired moments by differentiating the characteristic function

$$\varphi(u_1, u_2) = E[e^{iu_1X_1^\varepsilon + iu_2X_2^\varepsilon}] = \exp\{h\int(e^{iu_1x_1+iu_2x_2} - 1 - iu_1x_1 - iu_2x_2)\ \nu_\varepsilon(dx_1, dx_2)\},$$

then evaluating at (0,0), recalling the expression of ν_γ and using lemma 4.2. In particular we have:

$$E\Big[X_1^\varepsilon\Big] = E\Big[X_2^\varepsilon\Big] = 0$$

$$E\Big[\Big(X_1^\varepsilon\Big)^2\Big] = h\int_{\mathbb{R}^2} x_1^2\nu_\varepsilon(dx_1, dx_2) = \gamma C_1(2)h\varepsilon^{2-\alpha_1} + (1-\gamma)C(2,0)h\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}.$$

Note that if $\gamma \in (0,1)$ then as $\varepsilon \to 0$ we have

$$E\Big[\Big(X_1^\varepsilon\Big)^2\Big] = \gamma C_1(2) h\varepsilon^{2-\alpha_1} + (1-\gamma) C(2,0) h\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2} \sim h\varepsilon^{2-\alpha_1}\mathcal{A},$$

where $\mathcal{A} = \gamma C_1(2) I_{\{\alpha_1 \le \alpha_2\}} + (1-\gamma) C(2,0) I_{\{\alpha_1=\alpha_2\}}$. In fact, with $\phi \doteq \frac{\alpha_2}{\alpha_1} \in [1,+\infty)$, the quotient

$$\varepsilon^{2\frac{\alpha_2}{\alpha_1}-\alpha_2}/\varepsilon^{2-\alpha_1} = \varepsilon^{2\phi-\alpha_1\phi-2+\alpha_1} = \varepsilon^{(2-\alpha_1)(\phi-1)}$$

has an exponent which is non-negative for all $\alpha_1, \alpha_2 \in (0,2)$, and zero for $\alpha_1 = \alpha_2$.

$$E\Big[\Big(X_2^\varepsilon\Big)^2\Big] = h\int_{\mathbb{R}^2} x_2^2 \nu_\varepsilon(dx_1, dx_2) = hC(0,2)\varepsilon^{2-\alpha_2}$$

$$E\Big[X_1^\varepsilon X_2^\varepsilon\Big] = h\int_{\mathbb{R}^2} x_1 x_2 \nu_\varepsilon(dx_1, dx_2) = h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} C(1,1)(1-\gamma)$$

$$E\Big[(X_1^\varepsilon)^2 X_2^\varepsilon\Big] = h\int_{\mathbb{R}^2} x_1^2 x_2 \nu_\varepsilon(dx_1, dx_2) = h\varepsilon^{1+2\frac{\alpha_2}{\alpha_1}-\alpha_2} C(2,1)(1-\gamma)$$

$$E\Big[X_1^\varepsilon (X_2^\varepsilon)^2\Big] = h\int_{\mathbb{R}^2} x_1 x_2^2 \nu_\varepsilon(dx_1, dx_2) = h\varepsilon^{2+\frac{\alpha_2}{\alpha_1}-\alpha_2} C(1,2)(1-\gamma)$$

$$E\Big[(X_1^\varepsilon)^2 (X_2^\varepsilon)^2\Big] = 2E^2\Big[X_1^\varepsilon X_2^\varepsilon\Big] + h\int_{\mathbb{R}^2} x_1^2 x_2^2 \nu_\varepsilon(dx_1, dx_2) + h^2\int_{\mathbb{R}^2} x_1^2 \nu_\varepsilon(dx_1, dx_2)\int_{\mathbb{R}^2} x_2^2 \nu_\varepsilon(dx_1, dx_2)$$
$$\sim (1-\gamma) h\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2} C(2,2) + hC(0,2)\varepsilon^{2-\alpha_2} E\Big[\Big(X_1^\varepsilon\Big)^2\Big]$$

Let us first concentrate on $E[\sum_i \xi_i]$. From the above we reach that

$$E[\xi_i] = E[X_1^\varepsilon X_2^\varepsilon] + h^2 A_1 A_2$$
$$= (1-\gamma) C(1,1) h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} + c_{A_1} c_{A_2} h^2 \Big[(1-\varepsilon^{1-\alpha_1})(1-\varepsilon^{1-\alpha_2}) I_{\alpha_1,\alpha_2 \neq 1} + \ln\tfrac{1}{\varepsilon}(1-\varepsilon^{1-\alpha_2}) I_{\alpha_1=1<\alpha_2} + (1-\varepsilon^{1-\alpha_1})\log\tfrac{1}{\varepsilon} I_{\{\alpha_1<\alpha_2=1\}} + \ln^2\tfrac{1}{\varepsilon} I_{\alpha_1=\alpha_2=1}\Big]. \quad (20)$$

Note that since $\varepsilon = h^u$, as $h \to 0$ we have

$$E[\xi_i] \to 0.$$

i) and iii). If $\gamma \in [0,1)$, then the following terms in the expression of $E[\xi_i]$ are negligible when $h \to 0$: when both $\alpha_m = 1$, for sufficiently small h we have $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} >> h^2 \ln^2\frac{1}{\varepsilon}$ so the leading term is $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$, coming from $E[X_1^\varepsilon X_2^\varepsilon]$;
when $\alpha_1 = 1 < \alpha_2$, then the leading term is still $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$;
when $\alpha_1 < \alpha_2 = 1$, the leading term is $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ when $u < \alpha_1$, while is $h^2(1-\varepsilon^{1-\alpha_1})\log\frac{1}{\varepsilon} \sim h^2\log\frac{1}{\varepsilon}$, coming from $h^2 A_1 A_2$, otherwise;
when both $\alpha_m \neq 1$ then under our framework we necessarily have $\alpha_2 > 1$. Note that $\alpha_2 u < 1$. If $\alpha_1 > 1$ then the leading term turns out to be $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$. If $\alpha_1 < 1 : h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ is the only leading term only if $\alpha_2 u < \alpha_1$; when $\alpha_2 u = \alpha_1$ (and still $\alpha_2 > 1$) then $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} \sim h^2(1-\varepsilon^{1-\alpha_1})(1-\varepsilon^{1-\alpha_2}) \sim -h^2\varepsilon^{1-\alpha_2}$; when $\alpha_2 u > \alpha_1$ then the leading term is $-h^2\varepsilon^{1-\alpha_2}$. Thus

$$E[\xi_i] \sim (1-\gamma) C(1,1) h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2} I_{\{\alpha_1=\alpha_2=1\}\cup\{\alpha_1=1<\alpha_2\}\cup\{u<\alpha_1<\alpha_2=1\}\cup\{\alpha_1\neq 1,\alpha_2>1,\alpha_2 u\le\alpha_1\}}$$
$$+ c_{A_1} c_{A_2} h^2 \log\frac{1}{\varepsilon} I_{\{\alpha_1\le u<\alpha_2=1\}} - c_{A_1} c_{A_2} h^2 \varepsilon^{1-\alpha_2} I_{\{\alpha_1\le\alpha_2 u<1<\alpha_2\}}.$$

However $\{\alpha_1=\alpha_2=1\}\cup\{\alpha_1=1<\alpha_2\}\cup\{u<\alpha_1<\alpha_2=1\}\cup\{\alpha_1\neq 1,\alpha_2>1,\alpha_2 u<\alpha_1\}=\{\alpha_1>\alpha_2 u\}$ and here is where the only leading term is the factor $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$ within $E\Big[X_1^\varepsilon X_2^\varepsilon\Big]$;
$\{\alpha_1<u<1=\alpha_2\}\cup\{\alpha_1<\alpha_2 u<1<\alpha_2\}=\{\alpha_1<\alpha_2 u\}$ and here the only leading term is

$$h^2A_1A_2\sim\begin{cases} h^2\varepsilon^{1-\alpha_2} & \text{if } \alpha_2>1\\ h^2\log\frac{1}{\varepsilon} & \text{if } \alpha_2=1;\end{cases}$$

$\{\alpha_1=u,\alpha_2=1\}\cup\{\alpha_1=\alpha_2 u<1<\alpha_2\}=\{\alpha_1=\alpha_2 u\}$ and here: if $\alpha_2>1$ then $E\Big[X_1^\varepsilon X_2^\varepsilon\Big]$ and $h^2A_1A_2$ have the same speed $h\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}$; if $\alpha_2=1$ then only $h^2A_1A_2\sim h^2\log\frac{1}{\varepsilon}$ is leading. Thus

$$E[\sum_i\xi_i]\sim T(1-\gamma)C(1,1)\varepsilon^{1+\frac{\alpha_2}{\alpha_1}-\alpha_2}I_{\{\alpha_1>\alpha_2 u\}\cup\{\alpha_1=\alpha_2 u,\alpha_2>1\}}+ \tag{21}$$

$$Thc_{m_1}c_{A_2}\Big[-\varepsilon^{1-\alpha_2}I_{\{\alpha_1\le\alpha_2 u,\alpha_2>1\}}+\log\frac{1}{\varepsilon}I_{\{\alpha_1\le\alpha_2 u,\alpha_2=1\}}\Big].$$

ii) If $\gamma=1$, then $nE[\xi_1]=nh^2A_1A_2$, and again the leading term is different for different choices of α_1,α_2. We have

$$nE[\xi_1]=Thc_{m_1}c_{A_2}\Big[-\varepsilon^{1-\alpha_2}I_{\{\alpha_1<1<\alpha_2\}}+\log\frac{1}{\varepsilon}I_{\{\alpha_1<1=\alpha_2\}}+\log\frac{1}{\varepsilon}I_{\{\alpha_1=1<\alpha_2\}}+\log^2\frac{1}{\varepsilon}I_{\{\alpha_1=\alpha_2=1\}}+ \tag{22}$$

$$\varepsilon^{2-\alpha_1-\alpha_2}I_{\{1<\alpha_1\le\alpha_2\}}\Big].$$

As for $Var(\xi_i)$, let us come back to the general case $\gamma\in[0,1]$. Writing X_m for X_m^ε, we have

$$Var(\xi_i)=E[X_1^2X_2^2]-2hA_2E[X_1^2X_2]-2hA_1E[X_1X_2^2]$$

$$+h^2A_2^2E[X_1^2]+h^2A_1^2E[X_2^2]+2h^2A_1A_2E[X_1X_2]-E^2[X_1X_2]$$

$$\begin{aligned}&=h^2\textstyle\int_{0\le x_1,x_2\le\varepsilon}x_1^2d\nu\int_{0\le x_1,x_2\le\varepsilon}x_2^2d\nu+E^2[X_1X_2]+h\int_{0\le x_1,x_2\le\varepsilon}x_1^2x_2^2d\nu+\\ &-2hA_2E[X_1^2X_2]-2hA_1E[X_1X_2^2]+h^2A_2^2E[X_1^2]+h^2A_1^2E[X_2^2]+2h^2A_1A_2E[X_1X_2]\doteq\textstyle\sum_{\ell=1}^8V_\ell.\end{aligned} \tag{23}$$

where

$$V_1=h^2\textstyle\int_{0\le x_1,x_2\le\varepsilon}x_1^2d\nu\int_{0\le x_1,x_2\le\varepsilon}x_2^2d\nu;\ V_2\doteq E^2[X_1X_2];\ V_3\doteq h\int_{0\le x_1,x_2\le\varepsilon}x_1^2x_2^2d\nu;$$

$$V_4\doteq-2hA_2E[X_1^2X_2];\ V_5\doteq-2hA_1E[X_1X_2^2];\ V_6\doteq h^2A_2^2E[X_1^2];\ V_7\doteq h^2A_1^2E[X_2^2];\ V_8\doteq 2h^2A_1A_2E[X_1X_2].$$

As $\varepsilon\to0$ all these terms tend to zero: we now establish which terms are leading ones and we only keep them.

i) If $\gamma\in(0,1)$, we have the following properties. Firstly,

$$V_1=h^2\int_{0\le x_1,x_2\le\varepsilon}x_1^2d\nu\int_{0\le x_1,x_2\le\varepsilon}x_2^2d\nu\sim h^2\varepsilon^{4-\alpha_1-\alpha_2}\mathcal{A}C(0,2),$$

$$V_6=h^2A_2^2E[X_1^2]\sim h^3c_{A_2}^2[(1-\varepsilon^{1-\alpha_2})^2I_{\alpha_2\neq1}+\ln^2\frac{1}{\varepsilon}\ I_{\alpha_2=1}]\mathcal{A}\varepsilon^{2-\alpha_1}$$

However, the displayed leading part of V_6 is negligible wrt the displayed leading part of V_1.
Secondly,

$$V_2=E^2[X_1X_2]=(1-\gamma)^2C^2(1,1)h^2\varepsilon^{2(\frac{\alpha_2}{\alpha_1}+1-\alpha_2)},$$

and

$$V_4 = -2hA_2E[X_1^2X_2] = -2(1-\gamma)h^2c_{A_2}[(1-\varepsilon^{1-\alpha_2})I_{\alpha_2\neq 1} + \ln\frac{1}{\varepsilon}\ I_{\alpha_2=1}]C(2,1)\varepsilon^{2\frac{\alpha_2}{\alpha_1}+1-\alpha_2}$$

are negligible wrt

$$V_3 = h\int_{0\leq x_1,x_2\leq\varepsilon} x_1^2x_2^2 d\nu = (1-\gamma)C(2,2)h\varepsilon^{2\frac{\alpha_2}{\alpha_1}+2-\alpha_2}.$$

Thirdly, recalling that we chose $\alpha_1 \leq \alpha_2$ and we only are interested in the case where at least $\alpha_2 \geq 1$, then we have that

$$V_8 = 2h^2A_1A_2E[X_1X_2] = 2(1-\gamma)C(1,1)c_{A_1}c_{A_2}h^3\varepsilon^{\frac{\alpha_2}{\alpha_1}+1-\alpha_2}$$

$$\cdot\Big[(1-\varepsilon^{1-\alpha_1})(1-\varepsilon^{1-\alpha_2})I_{\alpha_1,\alpha_2\neq 1} + (1-\varepsilon^{1-\alpha_2})\ln\frac{1}{\varepsilon}I_{\alpha_1=1<\alpha_2} + (1-\varepsilon^{1-\alpha_1})\ln\frac{1}{\varepsilon}I_{\alpha_1<1=\alpha_2} + \ln^2\frac{1}{\varepsilon}\ I_{\alpha_1=\alpha_2=1}\Big]$$

is negligible wrt

$$V_5 = -2hA_1E[X_1X_2^2] = -2h^2c_{A_1}[(1-\varepsilon^{1-\alpha_1})I_{\alpha_1\neq 1} + \ln\frac{1}{\varepsilon}\ I_{\alpha_1=1}](1-\gamma)C(1,2)\varepsilon^{\frac{\alpha_2}{\alpha_1}+2-\alpha_2}.$$

Note that since the terms V_2 and V_8 are both negligible, contrarily to what we did for the mean $E[\xi_i]$, here we do not need to distinguish which is the leading term within $V_2+V_8 = E[X_1X_2]\Big(E[X_1X_2]+2h^2A_1A_2\Big)$. Finally

$$V_7 = h^2A_1^2E[X_2^2] = h^3c_{A_1}^2[(1-\varepsilon^{1-\alpha_1})^2I_{\alpha_1\neq 1} + \ln^2\frac{1}{\varepsilon}\ I_{\alpha_1=1}]C(0,2)\varepsilon^{2-\alpha_2}$$

is negligible wrt V_1, so we are left with

$$Var(\xi_j) = V_1 + V_3 + V_5 \sim h^2\mathcal{A}C(0,2)\varepsilon^{4-\alpha_1-\alpha_2} + (1-\gamma)C(2,2)h\varepsilon^{2\frac{\alpha_2}{\alpha_1}+2-\alpha_2}$$

$$-2h^2c_{A_1}[(1-\varepsilon^{1-\alpha_1})I_{\alpha_1\neq 1} + \ln\frac{1}{\varepsilon}\ I_{\alpha_1=1}](1-\gamma)C(1,2)\varepsilon^{\frac{\alpha_2}{\alpha_1}+2-\alpha_2}.$$

Now, as $h\to 0$, we have:

$$V_1/V_5 \to \begin{cases} 0 & \text{if } \alpha_2=\alpha_1=1 \\ c & \text{if } \alpha_2=\alpha_1>1 \\ \infty & \text{on } \{\alpha_1<1\leq\alpha_2\}\cup\{\alpha_1=1<\alpha_2\}\cup\{1<\alpha_1<\alpha_2\} \end{cases}$$

$$V_3/V_5 \to \begin{cases} 0 & \text{if } \alpha_1<\alpha_2u \\ c & \text{if } \alpha_1=\alpha_2u \\ \infty & \text{if } \alpha_1>\alpha_2u \end{cases}$$

$$V_1/V_3 \to \begin{cases} 0 & \text{if } \alpha_1\in(x_\star,2) \\ c & \text{if } \alpha_1=x_\star \\ \infty & \text{if } \alpha_1\in(0,x_\star) \end{cases}$$

By considering the different regions $\alpha_1<\alpha_2; \alpha_1=\alpha_2u; \alpha_1\in(\alpha_2u,x_\star); \alpha_1=5_5; \alpha_1\in(x_\star,2)$; we find that V_5 is never the leading term in $V_1+V_3+V_5$, V_1 is the only leading term for $\alpha_1\in(0,x_\star)$; $V_1\sim V_3$ are leading for $\alpha_1=x_\star$; and V_3 is the only leading term for $\alpha_1\in(x_\star,2)$. Thus

$$Var(\xi_i)\sim V_1I_{\{\alpha_1\leq x_\star\}} + V_3I_{\{\alpha_1\geq x_\star\}} = h^2\mathcal{A}C(0,2)\varepsilon^{4-\alpha_1-\alpha_2}I_{\{\alpha_1\leq x_\star\}} + h(1-\gamma)C(2,2)\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2}I_{\{\alpha_1\geq x_\star\}}$$

however if $\alpha_1\leq x_\star$ then necessarily $\alpha_1<\alpha_2$ so $\mathcal{A}$ becomes $\gamma C_1(2)$ and

$$Var(\xi_i)\sim h^2\gamma C_1(2)C(0,2)\varepsilon^{4-\alpha_1-\alpha_2}I_{\{\alpha_1\leq x_\star\}} + h(1-\gamma)C(2,2)\varepsilon^{2+2\frac{\alpha_2}{\alpha_1}-\alpha_2}I_{\{\alpha_1\geq x_\star\}}$$

$$= h\varepsilon^{2-\alpha_2}\left[h\varepsilon^{2-\alpha_1}\gamma C_1(2)C(0,2)I_{\{\alpha_1\le x_\star\}} + \varepsilon^{2\frac{\alpha_2}{\alpha_1}}(1-\gamma)C(2,2)I_{\{\alpha_1\ge x_\star\}}\right]$$

so that, recalling (21), the sum of the centered and normalized terms $\tilde{\xi}_i$ has the same asymptotic behavior as given in (5), in fact since $\sum_i E[\tilde{\xi}_i] = 0$ and $\sum_i Var(\tilde{\xi}_i) = 1$, then $\sum_i \tilde{\xi}_i$ is tight ([10]), i.e. the convergence speed of $\sum_i \xi_i - \sum_i E[\xi_i]$ to zero is $\sqrt{n}\sqrt{Var(\xi_i)} \to 0$.

ii) If $\gamma = 1$, then it turns out that

$$Var(\xi_1) = V_1 + V_6 + V_7 \sim V_1 = E[X_1^2 X_2^2] = h^2 \int_{0<x_1\le\varepsilon} x_1^2 \nu_\perp(dx_1) \int_{0<x_1\le\varepsilon} x_1^2 \nu_\perp(dx_1)$$

$$= h^2 \varepsilon^{4-\alpha_1-\alpha_2} C_1(2) C_2(2), \tag{24}$$

and thus, recalling (22), (6) is verified.

iii) If $\gamma = 0$ then $Var(\xi_1) \sim V_3 + V_5 + V_7$ and it turns out that

$$Var(\xi_1) \sim \begin{cases} V_3 & \text{if } \alpha_1 > \alpha_2 u \\ V_3 \sim V_5 \sim V_7 & \text{if } \alpha_1 = \alpha_2 u \\ V_7 & \text{if } \alpha_1 < \alpha_2 u, \end{cases}$$

and, recalling (21), (7) follows. □

Proof of Proposition 4.4 i) Case $\gamma \in (0,1)$. We compute $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]}$ by using the information (rate of $nE[\xi_1]$ and of $\sqrt{nVar(\xi_1)}$) summarized in (5) in the four different cases 1) $\alpha_1 \in (0, \alpha_2 u], \alpha_2 > 1$; 2) $\alpha_1 \in (0, \alpha_2 u], \alpha_2 = 1$; 3) $\alpha_1 \in (\alpha_2 u, x_\star]$; 4)$\alpha_1 \in (x_\star, \alpha_2]$. In the cases 1), 2), 3) we have $\alpha_1 \le x_\star < \alpha_2$, thus $\alpha_1 \ne \alpha_2$, and we reach that a sufficient condition for $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ is $u \in (\frac{1}{2+\alpha_2-\alpha_1}, \frac{1}{2})$. However $x_\star < 2 + \alpha_2 - 1/u$, thus if $\alpha_1 \le x_\star$, then $\alpha_1 < 2 + \alpha_2 - 1/u$, which is equivalent to $u > \frac{1}{2+\alpha_2-\alpha_1}$. On the other hand, in the case 4) we reach $\frac{\sqrt{nVar(\xi_1)}}{nE[\xi_1]} \to 0$ for any $u \in (0, 1/2)$.
Case $\gamma = 0$. We now look at (7). Here we separately study the regions $\{\alpha_1 > \alpha_2 u\}$; $\{\alpha_1 = \alpha_2 u\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 > 1\}$; $\{\alpha_1 < \alpha_2 u, \alpha_2 = 1\}$ and conclude.
ii) and iii). For $\gamma = 1$ we look at (6) and we separately study the regions $\{\alpha_1 < 1 < \alpha_2\}$; $\{\alpha_1 < 1 = \alpha_2\}$; $\{\alpha_1 = 1 < \alpha_2\}$; $\{\alpha_1 = \alpha_2 = 1\}$; and $\{1 < \alpha_1 \le \alpha_2\}$ and reach the results. □

Proof of Theorem 4.5. Under $\gamma = 1 = \alpha_1 = \alpha_2$ we have that $M^{(1)}$ and $M^{(2)}$ are independent, and $nVar(\xi_1) = h\varepsilon^2$. By the Lindeberg-Feller Theorem, recalling that

$$\tilde{\xi}_i = \frac{\xi_i - E[\xi_1]}{\sqrt{nVar(\xi_1)}},$$

it is sufficient to show that for all $\delta > 0$ we have $nE[\tilde{\xi}_1^2 I_{\{|\tilde{\xi}_1|>\delta\}}] \to 0$. We begin evaluating $P\{|\tilde{\xi}_1| > \delta\}$: by using that when $\gamma = 1 = \alpha_1 = \alpha_2$ we have $\frac{nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \to 0$, $hA_1 = hA_2$ and $X_1 = X_1^\varepsilon$ has the same law as $X_2 = X_2^\varepsilon$, we obtain

$$P\{|\tilde{\xi}_1| > \delta\} \le P\{|\xi_1| > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\} = P\{|M_h'^{(1)}||M_h'^{(2)}| > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\}$$

$$\le P\{|X_1||X_2| + hA_2|X_1| + hA_1|X_2| + h^2 A_1 A_2 > \frac{\delta}{2}\sqrt{nVar(\xi_1)}\} \le$$

$$P\{|X_1||X_2| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\} + 2P\{hA_2|X_1| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\} + P\{h^2 A_1 A_2 > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\}. \tag{25}$$

Now, for sufficiently small h last term is 0, because

$$\frac{h^2 A_1 A_2}{\sqrt{nVar(\xi_1)}} = \frac{h^2 \log^2 \frac{1}{\varepsilon}}{\sqrt{h}\varepsilon} = h^{\frac{3}{2}-u} \log^2 \frac{1}{\varepsilon} \to 0.$$

We now evaluate the other 2 probabilities in (25) to establish their magnitude orders:

$$P\{|X_1||X_2| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\} \leq \frac{E\big[|X_1||X_2|\big]}{K\sqrt{h}\varepsilon} \leq \frac{||X_1||_2||X_2||_2}{K\sqrt{h}\varepsilon}$$

$$= \frac{E\Big[(\int_{t_{i-1}}^{t_i}\int_{0<x\leq\varepsilon} x^2\nu^{(1)})^2\Big]}{K\sqrt{h}\varepsilon} = \frac{h\varepsilon^{2-\alpha_1}}{K\sqrt{h}\varepsilon} = K\sqrt{h};$$

and

$$P\{hA_2|X_1| > \frac{\delta}{8}\sqrt{nVar(\xi_1)}\} \leq \frac{hA_2E\big[|X_1|\big]}{K\sqrt{nVar(\xi_1)}} \leq \frac{hA_2\sqrt{h\varepsilon^{2-\alpha_1}}}{K\sqrt{h}\varepsilon} = h^{1-\frac{u}{2}}\log\frac{1}{\varepsilon}.$$

Noting that $\frac{h^{1-\frac{u}{2}}\log\frac{1}{\varepsilon}}{\sqrt{h}} \to 0$, it follows that

$$P\{|\tilde{\xi}_1| > \delta\} \leq K\sqrt{h}.$$

Now, for any conjugate exponents p, q,

$$nE[\tilde{\xi}_1^2 I_{\{|\tilde{\xi}_1|>\delta\}}] \leq nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]P^{\frac{1}{q}}\{|\tilde{\xi}_1| > \delta\} \leq nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{2q}}.$$

We now evaluate

$$E[\tilde{\xi}_1^{2p}] = E\Big[\Big(\frac{\xi_1}{\sqrt{nVar(\xi_1)}} - \frac{E[\xi_1]}{\sqrt{nVar(\xi_1)}}\Big)^{2p}\Big] \leq KE\Big[\Big(\frac{\xi_1}{\sqrt{nVar(\xi_1)}}\Big)^{2p}\Big] + K\Big(\frac{E[\xi_1]}{\sqrt{nVar(\xi_1)}}\Big)^{2p}.$$

Recall that we are under the conditions for which $\frac{nE[\xi_1]}{\sqrt{nVar(\xi_1)}} \to 0$, thus at least $\frac{E[\xi_1]}{\sqrt{nVar(\xi_1)}} \leq Kh$. On the other hand

$$E\Big[\frac{\xi_1^{2p}}{(nVar(\xi_1))^p}\Big] \leq K\Big(\frac{E[(X_1X_2)^{2p}]}{(nVar(\xi_1))^p} + 2\frac{E[(hA_2)^{2p}X_1^{2p}]}{(nVar(\xi_1))^p} + \frac{E[(h^2A_1A_2)^{2p}]}{(nVar(\xi_1))^p}\Big):$$

the last term contributes with $\Big(h^{3/2-u}\log^2\frac{1}{\varepsilon}\Big)^{2p}$; the second term, by the Burkholder-Davis-Gundi inequality, is dominated by

$$K\Big(\frac{h^2\log^2\frac{1}{\varepsilon}\int_{t_{i-1}}^{t_i}\int_0^{\varepsilon} x^2\nu^{(1)}(dx)}{h\varepsilon^2}\Big)^p = \Big(h^{2-u}\log^2\frac{1}{\varepsilon}\Big)^p;$$

and the first term is

$$\frac{E[X_1^{2p}X_2^{2p}]}{(h\varepsilon^2)^p} = \frac{E^2[X_1^{2p}]}{(h\varepsilon^2)^p} \leq K\frac{(h\varepsilon)^{2p}}{(h\varepsilon^2)^p} = Kh^p.$$

Thus

$$E[\tilde{\xi}_1^{2p}] \leq K\Big(h^{2p} + \Big(h^{3/2-u}\log^2\frac{1}{\varepsilon}\Big)^{2p} + \Big(h^{2-u}\log^2\frac{1}{\varepsilon}\Big)^p + h^p\Big) \sim h^p.$$

It follows that

$$nE^{\frac{1}{p}}[\tilde{\xi}_1^{2p}]h^{\frac{1}{2q}} \leq Kn\Big(h^p\Big)^{\frac{1}{p}}h^{\frac{1}{2q}} = Kh^{\frac{1}{2q}} \to 0,$$

as we announced. □

Proof of (9). In the case $\gamma \in [0, 1)$ we have $\sum_{i=1}^n \xi \sim nE[\xi_1]$. Using (8) we have that on $\{\alpha_1 \leq \alpha_2 u, \alpha_2 = 1\} \cup \{\alpha_1 \leq \alpha_2 u, \alpha_2 > 1\} \cup \{\alpha_1 = \alpha_2 u, \alpha_2 > 1\}$ both $\alpha_1 < \alpha_1^\star$ and $nE[\xi_1]/\sqrt{h} \to 0$. On $\{\alpha_1 > \alpha_2 u\}$ then $nE[\xi_1]/\sqrt{h} \to 0$ iff $\alpha_1 < \alpha_1^\star$.

In the case $\gamma = 1$ then on $\{\alpha_1 < 1, \alpha_2 \geq 1\} \cup \{\alpha_1 = 1, \alpha_2 > 1\}$ we have $\alpha_1 < \alpha_1^{\star\star}$ and $\sum_{i=1}^n \xi \sim nE[\xi_1]$ iff $u > 1/(2 + \alpha_2 - \alpha_1)$. If $u > 1/(2 + \alpha_2 - \alpha_1)$ then $nE[\xi_1]/\sqrt{h} \to 0$, if $u < 1/(2 + \alpha_2 - \alpha_1)$ then $\sum_{i=1}^n \xi_i/\sqrt{h} \sim \sqrt{nVar(\xi_1)}/\sqrt{h} \to 0$; if $u = 1/(2 + \alpha_2 - \alpha_1)$ then $nE[\xi_1] \sim \sqrt{nVar(\xi_1)} << \sqrt{h}$ and the

tightness result implies that still $\sum_{i=1}^n \xi_i/\sqrt{h} \to 0$.
On $\{1 < \alpha_1 \le \alpha_2\}$: if $u > \frac{1}{\alpha_1+\alpha_2}$ then $\sum_{i=1}^n \xi \sim nE[\xi_1]$ and $nE[\xi_1]/\sqrt{h} \to 0$ iff $\alpha_1 < \alpha_1^{\star\star}$. On the other hand $u \le \frac{1}{\alpha_1+\alpha_2}$ is equivalent to $\alpha_1 \le 1/u - \alpha_2$, which is less than $\alpha_1^{\star\star}$, and if $u \le \frac{1}{\alpha_1+\alpha_2}$ then $\sum_{i=1}^n \xi \sim \sqrt{nVar(\xi_1)}$ and $\sqrt{nVar(\xi_1)}/\sqrt{h} \to 0$. □

Lemma 5.3. *Let, for i=1..n, $A_i \subset \Omega$ be independent on $W^{(1)}$ and $W^{(2)}$ and s.t. $P(A_i) \le \theta_2$. If $\sigma^{(m)}$ satisfy (10), then*

i) $\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{A_i} \sim \frac{1}{\theta_2}\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}.$

ii) *Any $P(A_i)$ be, $E[|\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}|] \le cP(A_i)$.*

Proof i) Denote $\sigma_{t_j} = \sigma_j$. We have $\sigma_s = \sigma_{i-1} + (\sigma_s - \sigma_{i-1})$, thus

$$\frac{1}{\theta_2}\sum_{i=1}^n \Big[\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} - \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)}\Big] I_{A_i} = \tag{26}$$

$$\frac{1}{\theta_2}\sum_{i=1}^n \Big[\sigma_{i-1}^{(1)} \Delta_i W^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)} + \int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \sigma_{i-1}^{(1)} \Delta_i W^{(1)} +$$

$$\int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)} \int_{t_{i-1}}^{t_i} (\sigma_s^{(2)} - \sigma_{i-1}^{(2)}) dW_s^{(2)}\Big] I_{A_i} :$$

$E\Big[\Big(\int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)}) dW_s^{(1)}\Big)^2\Big] = E\Big[\int_{t_{i-1}}^{t_i} (\sigma_s^{(1)} - \sigma_{i-1}^{(1)})^2 ds\Big] = \int_{t_{i-1}}^{t_i} E[(\sigma_s^{(1)} - \sigma_{i-1}^{(1)})^2] ds$, and by assumption (10) this is dominated by ch^2, while by the boundedness of $\sigma^{(m)}$ we have $E\Big[\big(\sigma_{i-1}^{(m)} \Delta_i W^{(m)}\big)^2\Big] \le ch$. Now write each term of the rhs of (26) as

$$\frac{1}{\theta_2}\sum_{i=1}^n H_i^{(1)} H_i^{(2)} I_{A_i} :$$

using the independence assumption, its 1-norm is dominated by

$$\frac{1}{\theta_2}\sum_{i=1}^n ||H_i^{(1)}||_2 ||H_i^{(2)}||_2 P(A_i) \le cnh^{\frac{3}{2}} \to 0.$$

ii) Similarly, $E[|\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{A_i}|] \le \sum_{i=1}^n E[|\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)}|] P(A_i) \le cP(A_i)$. □

Lemma 5.4. *We have*

$$\frac{1}{\theta_m}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \ge 1\}} \stackrel{ucp}{\to} IC.$$

Proof By the independence of each $W^{(j)}$ on $\tilde{N}^{(m)}$, using Lemma 5.3 and Lemma 5.1 we have

$$\frac{1}{\theta_m}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \ge 1\}}$$

$$\sim \frac{1}{\theta_m}\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{\Delta_i \tilde{V}^{(m)} \ge 1\}} \doteq \sum_{i=1}^n \eta_i.$$

Further we have

$$\sum_{i=1}^n E_{i-1}[\eta_i] = \frac{1}{\theta_m}\sum_{i=1}^n \sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}} E_{i-1}[\Delta_i W^{(1)}\Delta_i W^{(2)}] P\{\Delta_i \tilde V^{(m)} \geq 1\}.$$

Now $E_{i-1}[\Delta_i W^{(1)}\Delta_i W^{(2)}] = E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds]$, and $P\{\Delta_i \tilde V^{(m)} \geq 1\} = 1 - e^{-\lambda_m h}$ with $\lambda_m = \nu_m(\sqrt{r_h}, +\infty)$ $= U_m(\sqrt{r_h}) = c_m \frac{r_h^{-\frac{\alpha_m}{2}}}{\alpha_m}$, thus

$$\sum_{i=1}^n E_{i-1}[\eta_i] = \frac{1}{\theta_m}\sum_{i=1}^n \sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}} E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds](1 - e^{-\lambda_m h}).$$

However, by the fact that $|1 - e^{-\lambda_m h} - \lambda_m h| \leq K h^2$, the last display has the same limit in probability as

$$\frac{1}{\theta_m}\sum_{i=1}^n \sigma^{(1)}_{i-1}\sigma^{(2)}_{i-1} E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds]\lambda_m h. \tag{27}$$

Further,

$$\frac{1}{\theta_m} E\Big[\sum_{i=1}^n |\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}}| \Big| E_{i-1}[\int_{t_{i-1}}^{t_i} \rho_s ds] - \rho_{t_{i-1}}\Big| \lambda_m h\Big] \leq$$

$$\frac{1}{\theta_m} E\Big[\sum_{i=1}^n |\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}}| E_{i-1}[\int_{t_{i-1}}^{t_i} |\rho_s - \rho_{t_{i-1}}| ds]\lambda_m h\Big] \leq \frac{Knh^2}{\theta_m} \leq K r_h^{\frac{\alpha_m}{2}} \to 0,$$

and this implies that (27) has the same limit in probability as

$$\frac{1}{\theta_m}\sum_{i=1}^n \sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}} \rho_{t_{i-1}} h \frac{c_m}{\alpha_m}\theta_m \xrightarrow{P} \frac{c_m}{\alpha_m} IC.$$

However by separating $\sigma^{(1)}_{i-1}\sigma^{(2)}_{i-1}\rho_{i-1} = (\sigma^{(1)}_{i-1}\sigma^{(2)}_{i-1}\rho_{i-1})^+ - (\sigma^{(1)}_{i-1}\sigma^{(2)}_{i-1}\rho_{i-1})^-$ and applying the reasoning indicated in [8], just before (3.5), we reach that such a convergence is also ucp. Further

$$\sum_{i=1}^n E_{i-1}[\eta_i^2] = \frac{1}{\theta_m^2}\sum_{i=1}^n (\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}})^2 E_{i-1}[(\Delta_i W^{(1)})^2(\Delta_i W^{(2)})^2] P\{\Delta_i \tilde V^{(m)} \geq 1\} :$$

by using (2) we find that $E_{i-1}[(\Delta_i W^{(1)})^2(\Delta_i W^{(2)})^2] \leq K h^2$, thus

$$\sum_{i=1}^n E_{i-1}[\eta_i^2] \leq K \frac{1}{\theta_m^2} h\theta_m \sum_{i=1}^n (\sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}})^2 h \sim \frac{h}{\theta_m}\int_0^T (\sigma^{(1)}_s\sigma^{(2)}_s)^2 ds \sim \varepsilon^{\alpha_m} \to 0.$$

Thus, by Lemma 4.2 in [8], the thesis follows. □

For two random sequences U_n and V_n with $V_n \neq 0, \forall n$, and a a constant, let us denote $U_n \approx a V_n$ when as $n \to \infty$ we have $U_n/V_n \to a$ in probability.

Lemma 5.5. *We have*

$$\frac{1}{\theta_1}\sum_{i=1}^n \sigma^{(1)}_{t_{i-1}}\Delta_i W^{(1)}\sigma^{(2)}_{t_{i-1}}\Delta_i W^{(2)} I_{\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\}} \xrightarrow{ucp} (1-\gamma)\frac{c_1}{\alpha_1} IC \cdot I_{\{\gamma \in [0,1)\}}.$$

Proof Let us start by proving that

$$P\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\} \approx (1-\gamma)\theta_1 \frac{c_1}{\alpha_1} I_{\{\gamma\in[0,1)\}} + \theta_1\theta_2 \frac{c_1}{\alpha_1}\frac{c_2}{\alpha_2} I_{\{\gamma = 1\}}. \tag{28}$$

In fact, with $\varepsilon = \sqrt{r_h}$,

$$\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\} = \Big\{\mu(]t_{i-1}, t_i] \times (\varepsilon + \infty) \times [0, +\infty)) \geq 1, \mu(]t_{i-1}, t_i] \times [0, +\infty) \times (\varepsilon + \infty)) \geq 1\Big\},$$

this is the set where during the time interval $]t_{i-1}, t_i]$ the bivariate jump sizes fall within the disjoint union $D_1 \cup D_2 \cup D_3$, with

$$D_1 = (\varepsilon, +\infty) \times [0, \varepsilon], D_2 = (\varepsilon, +\infty) \times (\varepsilon, +\infty), D_3 = [0, \varepsilon] \times (\varepsilon, +\infty),$$

and it coincides with the disjoint union

$$\Big\{\mu(]t_{i-1}, t_i] \times D_2) \geq 1\Big\} \cup \Big\{\mu(]t_{i-1}, t_i] \times D_2) = 0, \mu(]t_{i-1}, t_i] \times D_1) \geq 1, \mu(]t_{i-1}, t_i] \times D_3) \geq 1\Big\}.$$

Thus

$$P\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\} = 1 - e^{-\lambda_{D_2} h} +$$
$$P\Big\{\mu(]t_{i-1}, t_i] \times D_2) = 0\Big\} P\Big\{\mu(]t_{i-1}, t_i] \times D_1) \geq 1\Big\}\Big\{\mu(]t_{i-1}, t_i] \times D_3) \geq 1\Big\}$$
$$= 1 - e^{-\lambda_{D_2} h} + e^{-\lambda_{D_2} h}(1 - e^{-\lambda_{D_1} h})(1 - e^{-\lambda_{D_3} h}) \sim \lambda_{D_2} h + \lambda_{D_1}\lambda_{D_3} h^2,$$

where $\lambda_{D_j} = \nu_\gamma(D_j)$. In view of (4) and of the shape of $f(x)$ (see figure 1) due to our choice of the parameters, we have

$$\lambda_{D_1} = \gamma \int_{(\varepsilon, +\infty)} \nu^{(1)}(dx_1) = \gamma U_1(\varepsilon) = \gamma c_1 \frac{\varepsilon^{-\alpha_1}}{\alpha_1},$$

$$\lambda_{D_2} = (1-\gamma) \int_{(\varepsilon, +\infty[\times(\varepsilon,+\infty[} 1 \nu_{\|}(dx_1, dx_2):$$

$\nu_{\|}$ only weights the points (x_1, x_2) with $x_2 = f(x_1)$, and $f(x_1) > \varepsilon$ means that $x_1 > f^{-1}(\varepsilon)$, and recall that $f^{-1}(\varepsilon) < \varepsilon$, thus

$$\lambda_{D_2} = (1-\gamma)\nu_1(\varepsilon, +\infty) = (1-\gamma)U_1(\varepsilon) = (1-\gamma)c_1 \frac{\varepsilon^{-\alpha_1}}{\alpha_1};$$

$$\lambda_{D_3} = \gamma \int_{(\varepsilon, +\infty)} \nu^{(2)}(dx_2) + (1-\gamma) \int_{x_1 \in (0, \varepsilon], x_2 = f(x_1) > \varepsilon} \nu^{(1)}(dx_1):$$

recall that $f = U_2^{-1} \circ U_1$, so we have

$$\lambda_{D_3} = \gamma U_2(\varepsilon) + (1-\gamma) \int_{f^{-1}(\varepsilon)}^{\varepsilon} \nu^{(1)}(dx_1) = \gamma U_2(\varepsilon) + (1-\gamma)(U_1(f^{-1}(\varepsilon)) - U_1(\varepsilon))$$

$$= \gamma U_2(\varepsilon) + (1-\gamma)U_2(\varepsilon) - (1-\gamma)U_1(\varepsilon) = c_2 \frac{\varepsilon^{-\alpha_2}}{\alpha_2} - (1-\gamma)c_1 \frac{\varepsilon^{-\alpha_1}}{\alpha_1}.$$

However if $\gamma \neq 1$ then $\lambda_{D_2} h = (1-\gamma)\theta_1 \frac{c_1}{\alpha_1}$ is the leading term within $P\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\}$, because

$$\frac{\lambda_{D_1}\lambda_{D_3} h^2}{\lambda_{D_2} h} = \frac{\gamma}{1-\gamma} \varepsilon^{-\alpha_2} h \Big(\frac{c_2}{\alpha_2} - (1-\gamma)\frac{c_1}{\alpha_1}\varepsilon^{\alpha_2 - \alpha_1}\Big)$$

which either tends to 0 or is 0.
If $\gamma = 1$ then $\lambda_{D_2} = 0$ and $P\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\} = \lambda_{D_1}\lambda_{D_3} h^2 = \theta_1\theta_2 \frac{c_1}{\alpha_1}\frac{c_2}{\alpha_2}$. Thus (28) is verified.

Let us now define

$$\sum_{i=1}^n \frac{1}{\theta_1} \sigma^{(1)}_{t_{i-1}} \Delta_i W^{(1)} \sigma^{(2)}_{t_{i-1}} \Delta_i W^{(2)} I_{\{\Delta_i \tilde V^{(1)} \geq 1, \Delta_i \tilde V^{(2)} \geq 1\}} = \sum_{i=1}^n \chi_i:$$

by the independence of each $W^{(m)}$ on each $\tilde V^{(\ell)}$, we have

$$\sum_{i=1}^n E_{i-1}[\chi_i] = \sum_{i=1}^n \sigma^{(1)}_{t_{i-1}}\sigma^{(2)}_{t_{i-1}}\int_{t_{i-1}}^{t_i}\rho_s ds\Big[(1-\gamma)\frac{c_1}{\alpha_1}I_{\{\gamma\in[0,1)\}} + \theta_2\frac{c_1}{\alpha_1}\frac{c_2}{\alpha_2}I_{\{\gamma=1\}}\Big] \stackrel{P}{\to} (1-\gamma)\frac{c_1}{\alpha_1}IC\cdot I_{\{\gamma\in[0,1)\}};$$

as in the previous Lemma, we reach that such a convergence is also ucp. Further

$$\sum_{i=1}^n E_{i-1}[\chi_i^2] \le K\frac{h\theta_1}{\theta_1^2} \le K\varepsilon^{\alpha_1} \to 0,$$

so, by Lemma 4.2 in [8], the thesis follows. □

Proof of Theorem 4.7.

We can write

$$\hat{IC} - IC = \sum_{k=1}^4 I_k, \tag{29}$$

where

$$I_1 = \Big[\sum_{i=1}^n \Delta_i Y^{(1)}\Delta_i Y^{(2)} I_{\{|\Delta_i Y^{(1)}|\le 2\sqrt{r_h}\}} I_{\{|\Delta_i Y^{(2)}|\le 2\sqrt{r_h}\}} - IC\Big],$$

$$I_2 = \sum_{i=1}^n \Delta_i Y^{(1)}\Delta_i Y^{(2)}\Big(I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}\}}I_{\{|\Delta_i X^{(2)}|\le\sqrt{r_h}\}} - I_{\{|\Delta_i Y^{(1)}|\le 2\sqrt{r_h}\}}I_{\{|\Delta_i Y^{(2)}|\le 2\sqrt{r_h}\}}\Big),$$

$$I_3 = \sum_{i=1}^n (\Delta_i Y^{(1)}\Delta_i M^{(2)} + \Delta_i Y^{(2)}\Delta_i M^{(1)}) I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}\}}I_{\{|\Delta_i X^{(2)}|\le\sqrt{r_h}\}},$$

$$I_4 = \sum_{i=1}^n \Delta_i M^{(1)}\Delta_i M^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}\}}I_{\{|\Delta_i X^{(2)}|\le\sqrt{r_h}\}}.$$

We know that $I_1/\sqrt{h} \stackrel{st}{\longrightarrow} U$, with U mixed Gaussian r.v. ([16]). We are now going to show that:

$$I_2 \sim \sum_{i=1}^n \int_{t_{i-1}}^{t_i}\sigma^{(1)}_s dW^{(1)}_s \int_{t_{i-1}}^{t_i}\sigma^{(2)}_s dW^{(2)}_s I_{\{|\Delta_i\tilde V^{(2)}|>\sqrt{r_h}\}} \sim \theta_2 = h r_h^{-\frac{\alpha_2}{2}}, \quad I_3 << \sqrt{h}$$

and I_4 is the sum of $\sum_{i=1}^n \xi_i$ with some other terms which however are negligible wrt one of the terms $\sqrt{h}$, θ_2 or $\sum_{i=1}^n \xi_i$. That will prove (12). It then turns out that none of the terms appearing in (12) is always negligible, while depending on the combination of the parameters $\gamma, \alpha_1, \alpha_2$ the leading term is different, and we show (13).

Let us start dealing with I_2 : we have

$$I_2 = \sum_{i=1}^n \Delta_i Y^{(1)}\Delta_i Y^{(2)}\Big(I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}\cap\{|\Delta_i Y^{(1)}|\le 2\sqrt{r_h}, |\Delta_i Y^{(2)}|\le 2\sqrt{r_h}\}^c}$$

$$-I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c\cap\{|\Delta_i Y^{(1)}|\le 2\sqrt{r_h}, |\Delta_i Y^{(2)}|\le 2\sqrt{r_h}\}}\Big).$$

We first show that the first rhs term above

$$I_{2,1} \doteq \sum_{i=1}^n \Delta_i Y^{(1)}\Delta_i Y^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}\cap\{|\Delta_i Y^{(1)}|\le 2\sqrt{r_h}, |\Delta_i Y^{(2)}|\le 2\sqrt{r_h}\}^c}$$

is negligible wrt θ_2. In fact, on $\{|\Delta_i X^{(1)}| \le \sqrt{r_h}, |\Delta_i X^{(2)}| \le \sqrt{r_h}\}\cap\{|\Delta_i Y^{(1)}| \le 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \le 2\sqrt{r_h}\}^c$ we have $|\Delta_i Y^{(m)}| > 2\sqrt{r_h}$ for at least one $m \in \{1, 2\}$, and, using Remark 3.1, $|\Delta_i J^{(m)}| + K\sqrt{h\ln\frac{1}{h}} \ge$

$|\Delta_i D^{(m)}+\Delta_i J^{(m)}| = |\Delta_i Y^{(m)}| > 2\sqrt{r_h}$ implies that, for any $p \in (0,1)$, for sufficiently small h, $|\Delta_i J^{(m)}| \geq 2\sqrt{r_h}(1-p)$, thus $|\Delta_i J^{(m)}| \neq 0$. However $|\Delta_i X^{(m)}| \leq \sqrt{r_h}$, and so $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i X^{(m)}| \leq \sqrt{r_h}$ implies on one hand that $|\Delta_i J^{(m)} + \Delta_i M^{(m)}| < \sqrt{r_h}(1+p)$, and on the other hand that, considering a sufficiently small h, $1-|\Delta_i M^{(m)}| < |\Delta_i J^{(m)}|-|\Delta_i M^{(m)}| < |\Delta_i J^{(m)}+\Delta_i M^{(m)}| < \sqrt{r_h}(1+p)$, and thus, for sufficiently small h, $|\Delta_i M^{(m)}| > 1-\sqrt{r_h}(1+p) > \sqrt{r_h}$. It follows that $\forall i = 1..n$ there is an index m_i s.t. $\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\} \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\}^c \subset \{\Delta_i N^{(m_i)} \neq 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\}$, thus, using Lemma 5.2 point 1,

$$P\left\{\frac{I_{2,1}}{\theta_2} \neq 0\right\} \leq \sum_{i=1}^n P\{\Delta_i N^{(m_i)} \neq 0, \Delta_i M^{(m_i)} > \sqrt{r_h}\} \leq c\frac{h}{r_h} \to 0,$$

which implies that $\frac{I_{2,1}}{\theta_2} \xrightarrow{P} 0$, as we stated.

We now deal with the second term of I_2 : on $\{|\Delta_i Y^{(m)}| \leq 2\sqrt{r_h}\}$ we have $|\Delta_i J^{(m)}| - |\Delta_i D^{(m)}| < |\Delta_i Y^{(m)}| \leq 2\sqrt{r_h}$ and thus, with p as above, a.s., for sufficiently small h, $|\Delta_i J^{(m)}| < 2\sqrt{r_h}(1+p) < 1$, which implies that $\Delta_i J^{(m)} = 0$, i.e. $\Delta_i Y^{(m)} = \Delta_i D^{(m)}$. Thus, calling

$$\mathcal{B}_i = \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}^c \cap \{|\Delta_i Y^{(1)}| \leq 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \leq 2\sqrt{r_h}\},$$

we have

$$\sum_{i=1}^n \Delta_i Y^{(1)} \Delta_i Y^{(2)} I_{\mathcal{B}_i} = \sum_{k=2}^4 I_{2,k},$$

where

$$I_{2,2} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} a_s^{(2)} ds \; I_{\mathcal{B}_i},$$

$$I_{2,3} = \sum_{i=1}^n \Big(\int_{t_{i-1}}^{t_i} a_s^{(2)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} + \int_{t_{i-1}}^{t_i} a_s^{(1)} ds \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}\Big) I_{\mathcal{B}_i},$$

$$I_{2,4} = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\mathcal{B}_i}.$$

We show that $I_{2,4}$ is the leading term and it asymptotically behaves as θ_2.

As for $I_{2,2}$, by the boundedness of each $a^{(m)}$ we have $E[|\frac{I_{2,2}|}{\theta_2}] \leq \frac{h}{\theta_2} \to 0$.

As for $I_{2,3}$, note that on $\{|\Delta_i X^{(m)}| > \sqrt{r_h}, |\Delta_i J^{(m)}| = 0\}$ we have $|\Delta_i M^{(m)}| + K\sqrt{h \ln \frac{1}{h}} > |\Delta_i M^{(m)}| + |\Delta_i D^{(m)}| \geq |\Delta_i D^{(m)} + \Delta_i M^{(m)}| = |\Delta_i X^{(m)}| > \sqrt{r_h}$ thus $|\Delta_i M^{(m)}| > \sqrt{r_h} - K\sqrt{h \ln \frac{1}{h}} > \sqrt{r_h}(1-p)$ (p as above, h sufficiently small). Using also Lemma 5.2 point 7 and noting that $\theta_1 \leq \theta_2$, it follows that

$$\frac{E[|I_{2,3}|]}{\theta_2} \leq \frac{1}{\theta_2} c \sum_{i=1}^n h\sqrt{h \ln \frac{1}{h}} \Big(P\{|\Delta_i M^{(1)}| > c\sqrt{r_h}\} + P\{|\Delta_i M^{(2)}| > c\sqrt{r_h}\}\Big) \leq$$

$$c\frac{\sqrt{h \ln \frac{1}{h}}}{\theta_2}(\theta_1 + \theta_2) \leq c\sqrt{h \ln \frac{1}{h}} \to 0.$$

As for $I_{2,4}$, firstly we show that

$$\frac{I_{2,4}}{\theta_2} \sim \frac{1}{\theta_2} \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \Big(I_{\{|\Delta_i X^{(1)}| > \sqrt{r_h}\}} + I_{\{|\Delta_i X^{(2)}| > \sqrt{r_h}\}}\Big) : \tag{30}$$

in fact, let us begin showing that

$$\frac{I_{2,4}}{\theta_2} \sim \frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c} :$$

$$\frac{1}{\theta_2} E\Big[\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)}||\int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}| I_{\{|\Delta_i X^{(m)}|\le\sqrt{r_h}, m=1,2\}^c \cap \{|\Delta_i Y^{(m)}|\le 2\sqrt{r_h}, m=1,2\}^c}\Big] \le$$

$$\frac{1}{\theta_2} E\Big[\sum_{i=1}^n |\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)}||\int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}| \big(I_{\{\Delta_i N^{(1)}\neq 0\}} + I_{\{\Delta_i N^{(2)}\neq 0\}}\big)\Big], \tag{31}$$

and

$$\frac{\sum_{i=1}^n E\Big[\Big|\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}\Big| I_{\{\Delta_i N^{(m)}\neq 0\}}\Big]}{\theta_2} \sim \frac{\sum_{i=1}^n E[|\sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)}| I_{\{\Delta_i N^{(m)}\neq 0\}}]}{\theta_2}$$

$$\le \frac{1}{\theta_2}\sum_{i=1}^n ||\sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)}||_p ||\sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)}||_q P\{\Delta_i N^{(m)} \ge 1\} \sim \frac{nh^2}{\theta_2} = \varepsilon^{\alpha_2} \to 0.$$

Now we are left with

$$\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h}, |\Delta_i X^{(2)}|\le\sqrt{r_h}\}^c},$$

where we see indicators of type $I_{(A\cap B)^c} = I_{A^c} + I_{B^c} - I_{A^c\cap B^c}$. If we show that

$$\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{A^c\cap B^c} \to 0$$

then (30) is proved.

So we deal with

$$\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h}, |\Delta_i X^{(2)}|>\sqrt{r_h}\}} :$$

this is asymptotically equivalent to

$$\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i \tilde V^{(1)}|\ge 1, |\Delta_i \tilde V^{(2)}|\ge 1\}}] \tag{32}$$

because

$$E\left[\left|\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} \left(I_{\{|\Delta_i X^{(1)}|>\sqrt{r_h}, |\Delta_i X^{(2)}|>\sqrt{r_h}\}} - I_{\{\Delta_i \tilde V^{(1)}\ge 1, \Delta_i \tilde V^{(2)}\ge 1\}}\right)\right|\right] \le$$

$$E\left[\frac{1}{\theta_2}\sum_{i=1}^n \left|\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}\right| \Big(I_{\{|\Delta_i X^{(m)}|>\sqrt{r_h}, m=1,2, \text{ but } \Delta_i \tilde V^{(\ell)}=0 \text{ for at least one index } \ell_i\}}\right.$$
$$\left. + I_{\{\Delta_i \tilde V^{(m)}\ge 1, m=1,2, \text{ but } |\Delta_i X^{(\ell)}|\le\sqrt{r_h} \text{ for at least one index } \ell_i\}}\Big)\right] \tag{33}$$

and on $\{|\Delta_i X^{(\ell_i)}| > \sqrt{r_h}, \Delta_i \tilde V^{(\ell_i)} = 0\}$ either $\Delta_i J^{(\ell_i)} \neq 0$ or $\Delta_i J^{(\ell_i)} = 0$. In this last case we have no jumps of $X^{(\ell_i)}$ bigger than $\sqrt{r_h}$, and $|\Delta_i M^{(\ell_i)}| + |\Delta_i D^{(\ell_i)}| \ge |\Delta_i X^{(\ell_i)}| = |\Delta_i D^{(\ell_i)} + \Delta_i M^{(\ell_i)}| > \sqrt{r_h}$ implies that, for any fixed small $p > 0$, for sufficiently small h, $|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p)$; further also on $\{\Delta_i \tilde V^{(\ell_i)} \geq 1, |\Delta_i X^{(\ell_i)}| \leq \sqrt{r_h}\}$, either $\Delta_i J^{(\ell_i)} \neq 0$ or $\Delta_i J^{(\ell_i)} = 0$, and in this last case we have $|\Delta_i M^{(\ell_i)}| = |\Delta_i X^{(\ell_i)} - \Delta_i D^{(\ell_i)}| \leq |\Delta_i X^{(\ell_i)}| + |\Delta_i D^{(\ell_i)}| \leq \sqrt{r_h}(1+p)$. Thus (33) is dominated by

$$E\left[\frac{1}{\theta_2}\sum_{i=1}^n \left|\int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)}\right| \Big(I_{\{|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p), \Delta_i \tilde V^{(\ell_i)} = 0\}} + \right.$$

$$\left. I_{\{\Delta_i \tilde V^{(\ell_i)} \geq 1, |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+p)\}} + I_{\{\Delta_i J^{(\ell_i)} \neq 0\}} \Big)\right].$$

Now, using the independence, the Hölder inequality, Lemma 5.2, points 3 and 6, and recalling that $\theta_1 \leq \theta_2$, then $\forall q \in (0, 1-p)$ the previous display is dominated by

$$\frac{ch}{\theta_2}\sum_{i=1}^n \Big(P\{|\Delta_i M^{(\ell_i)}| > \sqrt{r_h}(1-p), \Delta_i \tilde V^{\ell_i} = 0\} + P\{\Delta_i \tilde V^{\ell_i} \geq 1, |\Delta_i M^{(\ell_i)}| \leq \sqrt{r_h}(1+p)\}$$

$$+ P\{\Delta_i J^{(\ell_i)} \neq 0\} \Big)$$

$$\leq K \frac{(\theta_2^{\frac{4}{3}} + \theta_2(q^{-\alpha} - 1) + \theta_2(1 - (1+2p)^{-\alpha}) + h)}{\theta_2} \to K((q^{-\alpha} - 1) + (1 - (1+2p)^{-\alpha}).$$

However that holds for any $q \in (0, 1-p)$ and any $p \in (0,1)$: we take $p \to 0$ and $q \to 1$ and we reach that the limit in probability of (33) is 0 and (32) is true.

Now, by Lemma 5.3 and Lemma 5.4, (32) has the same rate as

$$\frac{1}{\theta_2}\sum_{i=1}^n \sigma_{t_{i-1}}^{(1)} \Delta_i W^{(1)} \sigma_{t_{i-1}}^{(2)} \Delta_i W^{(2)} I_{\{|\Delta_i \tilde V^{(1)}| \geq 1, |\Delta_i \tilde V^{(2)}| \geq 1\}}] \sim \frac{\theta_1}{\theta_2},$$

and thus is negligible if $\alpha_1 < \alpha_2$, otherwise, if $\alpha_1 = \alpha_2$, it contributes to $I_{2,4}$ by adding some constants in the limit.

Secondly, by reasoning exactly as for (32) we have

$$\frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{|\Delta_i X^{(m)}| > \sqrt{r_h}\}} \sim \frac{1}{\theta_2}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde V^{(m)} \geq 1\}},$$

which, by Lemma 5.4, has rate θ_m. However $\theta_1 \leq \theta_2$, thus

$$I_2 \sim I_{2,4} \sim \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(1)} dW_s^{(1)} \int_{t_{i-1}}^{t_i} \sigma_s^{(2)} dW_s^{(2)} I_{\{\Delta_i \tilde V^{(2)} \geq 1\}} \sim \theta_2.$$

We now show that I_3 in (29) is negligible wrt $\sqrt{h}$. Here we adjust to the bivariate case the proof given in [5] for the univariate case. $I_3/\sqrt{h}$ is the sum of two terms of type

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i Y^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}\}} I_{\{|\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} \tag{34}$$

with $(m, \ell) \in \{(1,2), (2,1)\}$, and we can treat both the terms at the same time. The expression in (34) equals

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}} + \frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i J^{(m)} \Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h}\}}. \tag{35}$$

As for the second term, as already commented just after the definition of $I_{2,1}$, on $\{|\Delta_i X^{(m)}| \le \sqrt{r_h}, \Delta_i J^{(m)} \neq 0\}$ we have $\{|\Delta_i M^{(m)}| > \sqrt{r_h}\}$, thus, by Lemma 5.2 point 1,

$$P\Big\{\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i J^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h}\}} \neq 0\Big\} \le \sum_{i=1}^n \{\Delta_i J^{(m)} \neq 0, |\Delta_i M^{(m)}| > \sqrt{r_h}\} \to 0,$$

thus the second term of (35) tends to 0 in probability.

As for the first term, on $\{|\Delta_i X^{(\ell)}| \le \sqrt{r_h}\}$ we have $|\Delta_i X^{(\ell)}| > |\Delta_i Z^{(\ell)}| - |\Delta_i D^{(\ell)}|$ then $|\Delta_i Z^{(\ell)}| < |\Delta_i X^{(\ell)}| + |\Delta_i D^{(\ell)}| \le \sqrt{r_h} + \sqrt{h \ln \frac{1}{h}} \le 2\sqrt{r_h}$, thus

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h}\}} \le$$

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i Z^{(\ell)}|\le 2\sqrt{r_h}\}}:$$

the terms where $\Delta_i J^{(\ell)} \neq 0$ are negligible, in fact as above we have

$$P\Big\{\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(\ell)}|\le\sqrt{r_h},|\Delta_i Z^{(\ell)}|\le 2\sqrt{r_h},\Delta_i J^{(\ell)}\neq 0\}} \neq 0\} \to 0.$$

We are left with

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i Z^{(\ell)}|\le 2\sqrt{r_h},\Delta_i J^{(\ell)}=0\}} =$$

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i M^{(\ell)}|\le 2\sqrt{r_h},\Delta_i J^{(\ell)}=0\}},$$

however again

$$P\Big\{\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(\ell)}|\le\sqrt{r_h},|\Delta_i M^{(\ell)}|\le 2\sqrt{r_h},\Delta_i J^{(\ell)}\neq 0\}} \neq 0\} \to 0,$$

because on $\{|\Delta_i X^{(\ell)}| \le \sqrt{r_h}, |\Delta_i M^{(\ell)}| \le 2\sqrt{r_h}, \Delta_i J^{(\ell)} \neq 0\}$ we still have $|\Delta_i M^{(\ell)}| > \sqrt{r_h}$, and we remain with

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \Delta_i D^{(m)}\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i M^{(\ell)}|\le 2\sqrt{r_h}\}}. \quad (36)$$

Now, by Lemma 3.1 in [5] we know that on $|\Delta_i M^{(\ell)}| \le 2\sqrt{r_h}$ we have

$$\Delta_i M^{(\ell)} = \Delta_i M^{(\ell)h} - h\int_{2v_h}^1 x\nu^{(\ell)}(dx),$$

where $\Delta_i M^{(\ell)h} = \int_{t_{i-1}}^{t_i}\int_{0<x\le 2v_h} x\tilde{\mu}^\ell(dx,ds)$, and v_h is a given sequence satisfying $0 < v_h \le r_h^{1/4}$, and recall that $\Delta_i D^{(m)} = \int_{t_{i-1}}^{t_i} a_s^{(m)}ds + \int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}$. As a consequence, exactly as in (43) of [5], the component

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} a_s^{(m)}ds\Delta_i M^{(\ell)} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i M^{(\ell)}|\le 2\sqrt{r_h}\}}$$

of (36) tends to zero in probability. Now we show the negligibility of

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \sigma_s^{(m)}dW_s^{(m)}\Delta_i M^{(\ell)h} I_{\{|\Delta_i X^{(1)}|\le\sqrt{r_h},|\Delta_i X^{(2)}|\le\sqrt{r_h},|\Delta_i M^{(\ell)}|\le 2\sqrt{r_h}\}}:$$

in fact, by the independence of $W^{(m)}$ on $\tilde\mu^{(\ell)}$, the squared $L^2(P,\Omega)$-norm of the last display is dominated by

$$\frac{1}{h}E\Big[\Big(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Delta_i M^{(\ell)h}\Big)^2\Big]=\frac{1}{h}\sum_{i=1}^n E\Big[\Big(\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Big)^2\Big(\Delta_i M^{(\ell)h}\Big)^2\Big]$$

$$\leq\frac{K}{h}nh\cdot h\int_0^{r_h^{1/4}}x^2\nu^{(\ell)}(dx)\leq Kr_h^{\frac{2-\alpha_\ell}{4}}\to 0.$$

Finally we show the negligibility also of

$$\frac{1}{\sqrt h}\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}h\int_{2v_h}^1 x\nu^{(\ell)}(dx)I_{\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h},|\Delta_iM^{(\ell)}|\leq 2\sqrt{r_h}\}}:$$

in fact recall that $h\int_{2v_h}^1 x\nu^{(\ell)}(dx)=c_{A_\ell}\Big[(1-\varepsilon^{1-\alpha_\ell})I_{\alpha_\ell\neq 1}+\ln\frac{1}{\varepsilon}I_{\alpha_\ell=1}\Big]$ is positive for all the values of $\alpha_\ell\in(0,2)$, so the $L^1(P,\Omega)$-norm of the last display is dominated by

$$\sqrt{h}c_{A_m}\Big[(1-\varepsilon^{1-\alpha_\ell})I_{\alpha_\ell\neq 1}+\ln\frac{1}{\varepsilon}I_{\alpha_\ell=1}\Big]E\Big[\Big|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Big|\Big]\tag{37}$$

and noting that if $i\neq j$ then $E[\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\int_{t_{j-1}}^{t_j}\sigma_s^{(m)}dW_s^{(m)}]=E[\int\sigma_s^{(m)}I_{s\in]t_{i-1},t_i]}\sigma_s^{(m)}I_{s\in]t_{j-1},t_j]}ds]$ $=0$, we have

$$E\Big[\Big|\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}\Big|\Big]\leq||\sum_{i=1}^n\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)}||_2=\sqrt{E[\sum_{i=1}^n(\int_{t_{i-1}}^{t_i}\sigma_s^{(m)}dW_s^{(m)})^2]}=O(1).$$

It follows that (37) is dominated by $K\sqrt{h}\Big[|1-\varepsilon^{1-\alpha_\ell}|I_{\alpha_\ell\neq 1}+\ln\frac{1}{\varepsilon}I_{\alpha_\ell=1}\Big]\to 0.$

We now deal with I_4 of (29). We have

$$I_4=\sum_{i=1}^n\Delta_iM^{(1)}\Delta_iM^{(2)}I_{\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}}=$$

$$\sum_{i=1}^n\Delta_iM^{(1)}\Delta_iM^{(2)}\Big[I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0,|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}}+I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c\cap\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}}\Big]$$

$$=\sum_{i=1}^n\Delta_iM^{(1)}\Delta_iM^{(2)}\Big[I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}}-I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}\cap\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}^c}+$$

$$I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c\cap\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}}\Big].$$

However where both $\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0$, we have $\Delta_iM^{(1)}\Delta_iM^{(2)}=\xi_i$, thus

$$I_4=\sum_{k=1}^4 I_{4,k}$$

where

$$I_{4,1}=\sum_{i=1}^n\xi_i,\quad I_{4,2}=-\sum_{i=1}^n\xi_iI_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c},\quad I_{4,3}=-\sum_{i=1}^n\xi_iI_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}\cap\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}^c}$$

$$I_{4,4}=\sum_{i=1}^n\Delta_iM^{(1)}\Delta_iM^{(2)}I_{\{\Delta_i\tilde N^{(1)}=0,\Delta_i\tilde N^{(2)}=0\}^c\cap\{|\Delta_iX^{(1)}|\leq\sqrt{r_h},|\Delta_iX^{(2)}|\leq\sqrt{r_h}\}}.$$

We are going to show that all the terms $I_{4,k}$ with $k = 1, 2, 3$ are negligible wrt to θ_2.

As for $I_{4,2}$, using again that $I_{A\cup B} = I_A + I_B - I_{A\cap B}$, it is sufficient to show that both $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde N^{(\ell)}\geq 1\}} << \theta_2$, for $\ell = 1, 2$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde N^{(1)}\geq 1, \Delta_i \tilde N^{(2)}\geq 1\}} << \theta_2$. Using the independence of ξ_i on $\Delta_i \tilde N^{(\ell)}$, we reach that

$$E_{i-1}[\xi_i I_{\{\Delta_i \tilde N^{(\ell)}\geq 1\}}] \sim E_{i-1}[\xi_i]\theta_\ell$$
$$E_{i-1}[\xi_i^2 I_{\{\Delta_i \tilde N^{(\ell)}\geq 1\}}] \leq E[\xi_i^2]\theta_\ell.$$

Thus, if, for any ℓ, we call

$$\sum_{i=1}^n \frac{\xi_i I_{\{\Delta_i \tilde N^{(\ell)}\geq 1\}}}{\theta_2} \doteq \sum_{i=1}^n \chi_i,$$

we have that $\forall t \geq 0$, $\sum_{t_i\leq t} E_{i-1}[\chi_i] \leq \sum_{t_i\leq t} E_{i-1}[\xi_i] = \left[\frac{t}{h}\right] E[\xi_1] \leq nE[\xi_1]$, which, by looking at Theorem 4.3, tends to zero in all the cases $\gamma \in [0, 1]$. Further, $\sum_{t_i\leq t} E_{i-1}[\chi_i]$ is positive for all t and increasing in t, thus the convergence is also ucp. Moreover $\forall t \geq 0$, $\sum_{t_i\leq t} E_{i-1}[\chi_i^2] \leq nE[\xi_1^2]/\theta_2$: recalling that, with the notations in (23), we have

$$Var(\xi_i) \sim \begin{cases} V_1 + V_3 & \text{if } \gamma \in (0, 1) \\ V_1 & \text{if } \gamma = 1 \\ V_3 + V_5 + V_7 & \text{if } \gamma = 0, \end{cases}$$

and $E^2[\xi_i] = V_2 + V_8 + h^4 A_1^2 A_2^2$, and noting that $V_2 << V_3$, $V_8 << V_5$, $h^4 A_1^2 A_2^2 << V_7$, it follows that $E^2[\xi_i] << Var(\xi_i)$ in all the cases, and thus $E^2[\xi_i] << E[\xi_i^2]$, so $E[\xi_1^2] \sim Var(\xi_1)$ and we can directly use the expressions at the denominators of (5), (6), (7) to verify that under our assumptions $nVar(\xi_1)/\theta_2 \to 0$ in all the cases $\gamma \in [0, 1]$. We remark that for the case $\gamma \in (0, 1)$ and $\alpha_1 \geq x_\star >$ condition $u > 1/[2(1 + \alpha_2/\alpha_1)]$ is needed, however it is implied by our assumption (**??**). It follows that $\sum_{i=1}^n \chi_i \stackrel{ucp}{\to} 0$, that is $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde N^{(\ell)}\geq 1\}} << \theta_2$.

If we now call $P\{\Delta_i \tilde N^{(1)} \geq 1, \Delta_i \tilde N^{(2)} \geq 1\} \doteq \theta_{1,2} \leq \theta_2$, and

$$\sum_{i=1}^n \chi_i \doteq \sum_{i=1}^n \frac{\xi_i I_{\{\Delta_i \tilde N^{(1)}\geq 1, \Delta_i \tilde N^{(2)}\geq 1\}}}{\theta_2},$$

we have $\sum_{t_i\leq t} E_{i-1}[\chi_i] = \left[\frac{t}{h}\right] E[\xi_1]\frac{\theta_{1,2}}{\theta_2} \leq \left[\frac{t}{h}\right] E[\xi_1] \stackrel{ucp}{\to} 0$, and $\sum_{t_i\leq t} E_{i-1}[\chi_i^2] \leq KnVar(\xi_1)/\theta_2 \to 0$, so again $\sum_{i=1}^n \chi_i \stackrel{ucp}{\to} 0$ and $\sum_{i=1}^n \xi_i I_{\{\Delta_i \tilde N^{(1)}\geq 1, \Delta_i \tilde N^{(2)}\geq 1\}} << \theta_2$.

We now show that also $I_{4,3}$ is negligible wrt θ_2. Each term of the sum is counted only if both $\Delta_i \tilde N^{(j)} = 0, j = 1, 2$ but $|\Delta_i X^{(\ell)}| > \sqrt{r_h}$ for at least one index ℓ. Note that if $\Delta_i \tilde N^{(\ell)} = 0$ then $\Delta_i J^{(\ell)} = 0$. However, as commented for $I_{2,3}$, we have $\{|\Delta_i X^{(\ell)}| > \sqrt{r_h}, \Delta_i J^{(\ell)} = 0\} \subset \{|\Delta_i M^{(\ell)}| > \sqrt{r_h}(1 - p)\}$, and we know that $P\{\Delta_i \tilde N^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1 - p)\} \leq K\theta_\ell^{4/3} \leq K\theta_2^{4/3}$. It follows that

$$E\Big[\frac{|I_{4,3}|}{\theta_2}\Big] \leq \frac{\sum_{i=1}^n ||\xi_i||_2 \sqrt{P\{\Delta_i \tilde N^{(\ell)} = 0, |\Delta_i M^{(\ell)}| > \sqrt{r_h}(1 - p)\}}}{\theta_2} \leq K\sqrt{n}\theta_2^{-\frac{1}{3}}\sqrt{nVar(\xi_1)}. \quad (38)$$

Using again the expressions of $\sqrt{nVar(\xi_1)}$ at the denominators of (5), (6), (7) for the different choices of γ, we have that last expression in (38) tends to zero under the following conditions:
for $\gamma \in (0, 1)$ and $\alpha_1 \leq x_\star$, iff

$$(a) \quad u > \frac{1}{6 - \alpha_2/2 - 3\alpha_1/2};$$

for $\gamma \in (0, 1)$ and $\alpha_1 > x_\star$, iff

$$(b) \quad u > \frac{5}{6 - 6\alpha_2/\alpha_1 - \alpha_2};$$

for $\gamma = 1$ under (a);
for $\gamma = 0$ and $\alpha_1 < \alpha_2 u$, under (a), because $h < \sqrt{h}\varepsilon^{1-\alpha_1/2}$;
for $\gamma = 0$ and $\alpha_1 \geq \alpha_2 u$, under (b).
However, under our assumption (**??**) both the conditions (a), (b) are ensured.

Finally we show that $I_{4,4}$ is negligible wrt to θ_2. We have what follows:

if $\Delta_i \tilde{N}^{(2)} = 0$ and $\Delta_i \tilde{N}^{(1)} \geq 1$, then the terms with $\Delta_i J^{(1)} \neq 0$ do not contribute to $I_{4,4}/\theta_2$, since $|\Delta_i X^{(1)}| \leq \sqrt{r_h}$ then for any $p \in (0,1)$ and sufficiently small h we have $|\Delta_i M^{(1)}| > \sqrt{r_h}(1-p)$ and then

$$P\Big\{\frac{1}{\theta_2}\sum_{i=1}^n \Delta_i M^{(1)} \Delta_i M^{(2)} I_{\{\Delta_i J^{(1)} \neq 0, |\Delta_i M^{(1)}| > \sqrt{r_h}(1-p)\}} \neq 0\Big\} \to 0.$$

We then remain with the terms where $\Delta_i J^{(1)} = 0$, but on $\{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, \Delta_i J^{(1)} = 0\}$ we have $|\Delta_i M^{(1)}| \leq \sqrt{r_h}(1+p)$. On the other hand on $\{\Delta_i \tilde{N}^{(2)} = 0\}$ we have $\Delta_i J^{(2)} = 0$, and thus also $|\Delta_i M^{(2)}| \leq \sqrt{r_h}(1+p)$. It follows that

$$E\Big[\frac{1}{\theta_2}\sum_{i=1}^n |\Delta_i M^{(1)} \Delta_i M^{(2)}| I_{\{\Delta_i \tilde{N}^{(2)}=0, \Delta_i \tilde{N}^{(1)} \geq 1\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h},\}}\Big]$$

$$\leq K\frac{r_h}{\theta_2}\sum_{i=1}^n P\{\Delta_i \tilde{N}^{(1)} \geq 1, |\Delta_i M^{(1)}| \leq \sqrt{r_h}(1+p)\} \leq K\frac{n r_h \theta_1^{4/3}}{\theta_2} \leq K n r_h \theta_2^{1/3} \to 0,$$

since assumption (**??**) implies $u > 1/(3-\alpha_2/2)$.

If $\Delta_i \tilde{N}^{(2)} \geq 1$ and $\Delta_i \tilde{N}^{(1)} = 0$, we reason similarly as above and obtain that

$$E\Big[\frac{1}{\theta_2}\sum_{i=1}^n |\Delta_i M^{(1)} \Delta_i M^{(2)}| I_{\{\Delta_i \tilde{N}^{(2)}=0, \Delta_i \tilde{N}^{(1)} \geq 1\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h},\}}\Big]$$

$$\leq K\frac{r_h}{\theta_2}\sum_{i=1}^n P\{\Delta_i \tilde{N}^{(2)} \geq 1, |\Delta_i M^{(2)}| \leq \sqrt{r_h}(1+p)\} \leq K\frac{n r_h \theta_2^{4/3}}{\theta_2} \to 0.$$

If both $\Delta_i \tilde{N}^{(1)} \geq 1, \Delta_i \tilde{N}^{(2)} \geq 1$, then the terms with one $\Delta_i J^{(\ell)} \neq 0$, are negligible and we remain with the terms where both $\Delta_i J^{(\ell)} = 0$, thus we reach that both $|\Delta_i M^{(\ell)}| \leq \sqrt{r_h}(1+p)$ and

$$E\Big[\frac{1}{\theta_2}\sum_{i=1}^n |\Delta_i M^{(1)} \Delta_i M^{(2)}| I_{\{\Delta_i \tilde{N}^{(2)} \geq 1, \Delta_i \tilde{N}^{(1)} \geq 1\} \cap \{|\Delta_i X^{(1)}| \leq \sqrt{r_h}, |\Delta_i X^{(2)}| \leq \sqrt{r_h},\}}\Big]$$

$$\leq K\frac{r_h}{\theta_2}\sum_{i=1}^n P\{\Delta_i \tilde{N}^{(2)} \geq 1, |\Delta_i M^{(2)}| \leq \sqrt{r_h}(1+p)\} \to 0,$$

as above.

We thus obtained that $\hat{IC} - IC \sim \sqrt{h} + \sum_{i=1}^n \xi_i + \theta_2$. Now we are going to make this more explicit. In (9) we compared $\sqrt{h}$ with $\sum_{i=1}^n \xi_i$. As for θ_2 versus $\sqrt{h}$ we have that:

$$\begin{array}{rl} \theta_2 << \sqrt{h} & \text{if } \alpha_2 < \frac{1}{2u} \\ \theta_2 \sim \sqrt{h} & \text{if } \alpha_2 = \frac{1}{2u} \\ \theta_2 >> \sqrt{h} & \text{if } \alpha_2 > \frac{1}{2u}. \end{array}$$

Comparing now θ_2 with $\sum_{i=1}^n \xi_i$, we reach that

$$\begin{array}{rll} \text{when } \gamma = 1 & \theta_2 >> \sum_{i=1}^n \xi_i & \text{for } \alpha_2 = \alpha_1 = 1\text{: if } u > \frac{1}{4} \\ & & \text{for } (\alpha_1, \alpha_2) \neq (1,1)\text{: } \forall u \in (0, \frac{1}{2}) \\ \text{when } \gamma \in [0,1) & \theta_2 >> \sum_{i=1}^n \xi_i & \text{for } \alpha_1 \leq \alpha_2 u\text{: any } u \in (0, \frac{1}{2}) \\ & & \text{for } \alpha_2 > \alpha_1 > \alpha_2 u\text{: iff } u > \frac{1}{1+\frac{\alpha_2}{\alpha_1}} \\ \text{when } \gamma \in [0,1) & \theta_2 << \sum_{i=1}^n \xi_i & \text{for } \alpha_1 = \alpha_2\text{: any } u \in (0, \frac{1}{2}). \end{array}$$

It follows that

$$\hat{IC} - IC \sim I_{\alpha_2 \geq \frac{1}{2u}} \left(\theta_2 \left[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1),\alpha_1\leq\alpha_2 u\}} + I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u\geq \frac{1}{1+\frac{\alpha_2}{\alpha_1}}\}} \right] \right.$$

$$\left. + \sum_{i=1}^n \xi_i \left[I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u<\frac{\alpha_1}{\alpha_1+\alpha_2}\}} + I_{\{\gamma\in[0,1),\alpha_2=\alpha_1>\alpha_2 u\}} \right] \right)$$

$$I_{\alpha_2<\frac{1}{2u}} \left(\sqrt{h} \left[I_{\{\gamma=1,\alpha_1<\alpha_1^{\star\star}\}} + I_{\{\gamma\in[0,1),\alpha_1<\alpha_1^{\star}\}} \right] + \sum_{i=1}^n \xi_i \left[I_{\gamma=1,\alpha_1\geq\alpha_1^{\star\star}} + I_{\{\gamma\in[0,1),\alpha_1\geq\alpha_1^{\star}\}} \right] \right).$$

However: $1/(2\alpha_2) > 1/4$, so if $\alpha_2 \geq \frac{1}{2u}$ then $u \geq \frac{1}{2\alpha_2}$ so $u > 1/4$; if $\alpha_1 = \alpha_2$ then $\alpha_1 > \alpha_2 u$, since $\alpha_2 > \alpha_2 u$; $\alpha_1 < 1/(2u) \Rightarrow \alpha_1 \leq \alpha_1^{\star\star}$; Thus the above display simplifies as follows:

$$\hat{IC} - IC \sim I_{\alpha_2 \geq \frac{1}{2u}} \left(\theta_2 \left[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1),\alpha_1\leq\alpha_2 u\}} + I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u\geq \frac{1}{1+\frac{\alpha_2}{\alpha_1}}\}} \right] \right.$$

$$\left. + \sum_{i=1}^n \xi_i \left[I_{\{\gamma\in[0,1),\alpha_2>\alpha_1>\alpha_2 u, u<\frac{\alpha_1}{\alpha_1+\alpha_2}\}} + I_{\{\gamma\in[0,1),\alpha_2=\alpha_1\}} \right] \right)$$

$$I_{\alpha_2<\frac{1}{2u}} \left(\sqrt{h} \left[I_{\{\gamma=1\}} + I_{\{\gamma\in[0,1),\alpha_1<\alpha_1^{\star}\}} + \sum_{i=1}^n \xi_i I_{\gamma\in[0,1),\alpha_1\geq\alpha_1^{\star}} \right] \right)$$

and (13) follows.

□

6 Notations list

$\Delta H_t^{(m)} = H_t - H_{t-}$, $\Delta_i H^{(m)} \doteq H_{t_i}^{(m)} - H_{t_{i-1}}^{(m)}$, for any process H
$X^{(m)} = D^{(m)} + Z^{(m)} = Y^{(m)} + M^{(m)}$:
$D_t^{(m)} = \int_0^t a_s^{(m)} ds + \int_0^t \sigma_s^{(m)} dW_s^{(m)}$
$Z^{(m)} = J^{(m)} + M^{(m)}$
$J^{(m)} = \int_0^{\cdot} \int_{\{|\gamma(s,\omega,x)|>1\}} \gamma(s,\omega,x)\mu^{(m)}(\omega,dx,ds)$
$M^{(m)} = \int_0^{\cdot} \int_{\{|\gamma(s,\omega,x)|\leq 1\}} \gamma(s,\omega,x)\tilde{\mu}^{(m)}(\omega,dx,ds)$
$L^{(m)}$: stable Lévy process with characteristic triplet $(z^{(m)}, 0, \nu^{(m)}(dx))$:
$M_t^{(m)} = L_t^{(m)} - z^{(m)} t - \sum_{s\leq t} \Delta L_s^{(m)} I_{\{|\Delta L_s^{(m)}|>1\}}$
$N_t^{(m)} = \sum_{s\leq t} I_{\{\Delta J_s^{(m)}\neq 0\}} = \sum_{s\leq t} I_{\{|\Delta X_s^{(m)}|>1\}}$
$\tilde{N}_t^{(m)} = \sum_{s\leq t} I_{\{|\Delta X_s^{(m)}|>\sqrt{r_h}\}}$
$\tilde{V}_t^{(m)} = \sum_{s\leq t} I_{\{|\Delta M_s^{(m)}|>\sqrt{r_h}\}}$
$M'^{(m)}, H_1', \tilde{L}'$: for any process H, $H_t' \doteq H_t - \sum_{s\leq t} \Delta H_s I_{\{|\Delta H_s|>\varepsilon\}}$
$\tilde{L}$ is a symmetric stable process: $\tilde{L} = H_1 - H_2$ with $H_{1t} = L_t - zt$
$\tilde{\tilde{N}}_t = \sum_{s\leq t} \Delta I_{\{|\Delta\tilde{L}_s|>\varepsilon\}}$
$\xi_j = \xi_j^{\varepsilon} \doteq \Delta_j M'^{(1)} \Delta_j M'^{(2)}$.
α_m: for $Z^{(m)}$ Lévy, $m = 1, 2$, α_m is its Blumenthal Getoor index (see [6]).
$\Delta_i H_{\star} := \Delta_i H \; I_{\{(\Delta_i H)^2\leq r_h\}}$, for any processes H (e.g. $H = X^{(m)}$ or $M^{(m)}$, etc.)
$IC \doteq \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt$
$\hat{IC} = \sum_{j=1}^n \Delta_j X^{(1)} I_{\{(\Delta_j X^{(1)})^2\leq r(h)\}} \Delta_j X^{(2)} I_{\{(\Delta_j X^{(2)})^2\leq r(h)\}} = \tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T$
$\tilde{v}_{r,l}^{(n)}(X^{(1)}, X^{(2)})_T = h^{1-\frac{r+l}{2}} \sum_{j=1}^n (\Delta_j X^{(1)})^r I_{\{(\Delta_j X^{(1)})^2\leq r_h\}} (\Delta_j X^{(2)})^l I_{\{(\Delta_j X^{(2)})^2\leq r_h\}}$,

$\tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \doteq h^{-1}\sum_{j=1}^{n-1}\prod_{i=0}^{1}\Delta_{j+i}X^{(1)}I_{\{(\Delta_{j+i}X^{(1)})^2\le r_h\}}\prod_{i=0}^{1}\Delta_{j+i}X^{(2)}I_{\{(\Delta_{j+i}X^{(2)})^2\le r_h\}}$,
$\nu^{(m)}(dx_m) = c_m x_m^{-1-\alpha_m} I_{\{x_m>0\}}dx_m$
ν_γ Lévy measure of the Lévy process $(L^{(1)}, L^{(2)})$
$\nu_\varepsilon(dx_1, dx_2) = I_{\{0\le x_1, x_2\le\varepsilon\}}\nu_\gamma(dx_1, dx_2)$,
$U_m(x_m) := \nu^{(m)}\big([x_m, +\infty[\big) = c_m \frac{x_m^{-\alpha_m}}{\alpha_m}, \quad x_m > 0$
$U(x_1, x_2) = \nu_\gamma([x_1, +\infty) \times [x_2, +\infty)) = C_\gamma(U_1(x_1), U_2(x_2))$
$C_\gamma(u, v) = \gamma C_\perp(u, v) + (1-\gamma)C_\parallel(u, v) : C_\perp(u, v) = uI_{\{v=\infty\}} + vI_{\{u=\infty\}}$, $C_\parallel(u, v) = u \wedge v$
c, K are mute names for constants
$C(k, m) \doteq c_2\left(\frac{\alpha_2 c_1}{\alpha_1 c_2}\right)^{\frac{k}{\alpha_1}} \frac{1}{m+\frac{\alpha_2}{\alpha_1}k-\alpha_2} > 0$;
$C_m(k) = \frac{c_m}{k-\alpha_m}$
$\mathcal{A} = \gamma C_1(2)I_{\{\alpha_1\le\alpha_2\}} + (1-\gamma)C(2, 0)I_{\{\alpha_1=\alpha_2\}}$
$A_m^\varepsilon \doteq \int_{\varepsilon\le x_m\le 1} x_m\nu^{(m)}(dx_m)$, $c_{A_m} \doteq \frac{c_m}{1-\alpha_m}I_{\alpha_m\neq 1} + c_m I_{\alpha_m=1}$.
$X_m^\varepsilon \doteq \int_0^h\int_{|x|\le\varepsilon} x\tilde{\mu}^{(m)}(dx, dt)$
$\varepsilon = h^u = \sqrt{r_h},\ u \in (0, \frac{1}{2})$
$x_\star = \frac{1+2u-\sqrt{-4(2\alpha_2-1)u^2+4u+1}}{2u} \in (0, \alpha_2)$.
$\alpha_1^\star \doteq \frac{\alpha_2 u}{\alpha_2 u - u + 1/2} \in (2u, 1)$
$\alpha_1^{\star\star} \doteq \frac{1+2u(2-\alpha_2)}{2u} > \frac{1}{2u} > 1$,
$\theta_m = hr_h^{-\frac{\alpha_m}{2}}$
$\tilde{\theta}_m = h\varepsilon^{\alpha_m}$
$\mathcal{B}_i = \{|\Delta_i X^{(1)} \le \sqrt{r_h}|, |\Delta_i X^{(2)} \le \sqrt{r_h}|\}^c \cap \{|\Delta_i Y^{(1)}| \le 2\sqrt{r_h}, |\Delta_i Y^{(2)}| \le 2\sqrt{r_h}\}$
$||U||_p = E^{\frac{1}{p}}[|U|^p]$, for any r.v. U
$\mathcal{N}$ is a standard Gaussian rv
$\sigma_j = \sigma_{t_j}$
$f(h) \sim g(h)$: for two deterministic functions f, g, $f(h) \sim g(h)$ means that as $h \to 0$ we have both
$f(h) = O(g(h))$ and $g(h) = O(f(h))$
$f(h) << g(h)$: means that $f(h) = o(g(h))$
O_P: given two random sequences U_n and V_n, $U_n = O_P(V_n)$ if there exists $\bar{n}$: for all $n \ge \bar{n}$ we have that for any $\epsilon > 0$, there exists a constant $\eta > 0$ such that $\mathcal{P}(|U_n| > \eta|V_n|) < \epsilon$
$U_n \sim V_n$: for two random sequences U_n and V_n, $U_n \sim V_n$ when as $n \to \infty$ we have both $U_n = O_P(V_n)$ and $V_n = O_P(U_n)$
$U_n \approx aV_n$: for two random sequences U_n and V_n with $V_n \neq 0, \forall n$, and a a constant, let us denote $U_n \approx aV_n$ when as $n \to \infty$ we have $U_n/V_n \to a$ in probability.
$\stackrel{ucp}{\to}$ denotes convergence in probability uniformly on $[0, T]$
$\stackrel{P}{\to}$ denotes convergence in probability
$\stackrel{st}{\to}$ denotes stable convergence in law
$\stackrel{d}{\to}$ denotes convergence in law
SM 0 semimartingale, BSM = Brownian semimartingale
wlg= without loss of generality
rhs= right hand side, lhs= left hand side
wrt = with respect to
iff = if and only if
rv = random variable

References

[1] Aït-Sahalia, Y. and Jacod, J. (2011). Testing whether jumps have finite or infinite activity. Annals of Statistics, 39, 1689-1719.

[2] Y. Aït-Sahalia, J. Jacod, Estimating the Degree of Activity of Jumps in High Frequency Data, Annals of Statistics 37 (2009) 2202–2244.

[3] Andersen, T.G., Bollerslev, T., Diebold, F.X. (2007): *Roughing it up: Including jump components in the measurement, modeling and forecasting of return volatility.* The Review of Economics and Statistics, MIT Press, vol. 89(4), pages 701-720, 04.

[4] Barndorff-Nielsen, O.E., Shephard, N. (2004): Econometric analysis of realized covariation: high frequency based covariance, regression and correlation in financial economics. *Econometrica*, (2004), *72*, 885-925.

[5] R. Cont, C. Mancini (2011) Nonparametric tests for pathwise properties of semimartingales, Bernoulli, 17(2), 781-813

[6] Cont, R., Tankov, P. (2004): *Financial Modelling with jump processes.* Chapman and Hall-CRC.

[7] Gobbi, F., Mancini, C. (2013) Diffusion covariation and co-jumps in bidimensional asset price processes with stochastic volatility and infinite activity Lévy jumps, http://arxiv.org/pdf/0705.1268.pdf

[8] Jacod, J. (2012). Statistics and high-frequency data. In: "Statistical Methods for Stochastic Differential Equations" Eds: Kessler, Lindner and Sorensen

[9] Jacod, J. (2008) Asymptotic properties of realized power variations and related functionals of semimartingales, *Stoch. Proc. Appl.*, 118, pp.517-559

[10] Jacod, J. & Shiryaev, A.N. (2003) *Limit theorems for stochastic processes*, 2nd edition, Springer-Verlag, Berlin

[11] Kallsen, J., Tankov, P. (2006) Characterization of dependence of multidimensional Lévy processes using Lévy copulas, *Journal of Multivariate Analysis*, Vol. 97, 1551-1572

[12] Karatzas, I. & Shreve, S.E. (1999). *Brownian motion and stochastic calculus*, Springer.

[13] Mancini, C. (2009) Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps, Scandinavian Journal of Statistics, 36, 270-296

[14] Mancini, C. (2011) The speed of convergence of the threshold estimator of integrated variance. *Stochastic processes and their applications*, 121(4), 845-855

[15] Mancini, C., Calvori, F. (2012): Jumps, Chapter 17 of the Wiley *Handbook in Financial Engineering and Econometrics: Volatility Models and Their Applications*, Editors: Luc Bauwens, Christian Hafner and Sébastien Laurent, April

[16] C. Mancini, F. Gobbi (2012) Identifying the Brownian covariation from the co-jumps given discrete observations, Econometric Theory, 28 (2), pp. 249-273, April

[17] Tankov, P. (2003) Dependence structure of Lévy processes with applications in risk management, *Internal report* 502 of CMAP, Ecole Polytechnique