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Symmetric majority social choice functions

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Abstract

Under the assumption that individual preferences are linear orders on the set of alternatives, we study the social choice functions which satisfy suitable symmetries and obey the majority principle. In particular, supposing that individuals and alternatives are exogenously partitioned into subcommittees and subclasses, we provide necessary and sufficient conditions for the existence of reversal symmetric majority social choice functions that are anonymous and neutral with respect to the considered partitions. We also determine a general method for constructing and counting all those functions.

Keywords: Social choice function; anonymity; neutrality; reversal symmetry; majority; group theory.

JEL classification: D71.
MSC classification: 20B35.

1 Introduction

Committees are often required to select an alternative among many. Usually, the procedure used to make that selection only depends on committee members' preferences on alternatives. Moreover, among the huge variety of conceivable procedures, those satisfying principles which somehow refer to fairness and equity are typically preferred. Anonymity, neutrality, reversal symmetry and majority are well-known principles of that type: unfortunately, despite of their appeal, they all can be satisfied by a selection procedure only in very special circumstances.

To better explain that fact, consider a committee having $h \geq 2$ members who have to choose one among $n \geq 2$ alternatives. Assume that preferences of committee members are expressed as strict rankings on the set of alternatives, and call preference profile any list of h preferences, each of them associated with one of the individuals in the committee. A function from the set of preference profiles to the set of alternatives is called a social choice function (SCF): it represents a procedure to

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select an alternative whatever individual preferences are. A SCF is said anonymous if the identities of individuals are irrelevant to determine the social outcome, that is, it has the same value over any pair of preference profiles such that we can get one from the other by figuring to permute individual names. A SCF is said neutral if alternatives are equally treated, that is, for every pair of preference profiles such that we can get one from the other by figuring to permute alternative names, the social outcome associated with them coincide up to the considered permutation. A SCF is said reversal symmetric if a complete change in each committee member's mind about her own ranking of alternatives (that is, the best alternative gets the worst, the second best alternative gets the second worst, and so on) implies a change in the social outcome. Finally, a SCF satisfies a majority principle if, for every preference profile, it never selects an alternative which is ranked below an other one in a large enough number of committee member preferences.

Of course, there are several possible qualifications of the majority principle. Here we focus on two versions. Given an integer μ (called majority threshold) not exceeding h but exceeding $\frac{h}{2}$, we say that a SCF satisfies the μ-majority principle if, for every preference profile, there is no alternative which is preferred to the selected one by at least μ individuals. We say instead that a SCF satisfies the minimal majority principle if, for every preference profile and for every majority threshold μ, if there is an alternative which is preferred by at least μ individuals to the social outcome, then for every alternative there is another one which is preferred by at least μ individuals to it. That principle is a version for SCFs of the minimal majority principle introduced by Bubboloni and Gori (2014) for rules (that is, social welfare functions).

Some results about those principles are available in the literature. Moulin (1983, Problem 1, p.25, and Theorem 1, p.23) proves that there exists an anonymous and neutral SCF if and only if

the number of alternatives n cannot be written as sum of divisors greater than 1 of the number h of individuals,

and that there exists an anonymous, neutral and h-majority SCF if and only if

$$\gcd(h, n!) = 1. \tag{1}$$

Observe that the h-majority principle is usually known as unanimity or efficiency principle. Greenberg (1979, Corollary 3) proves[1] that, for every preference profile, there is at least an alternative admitting no alternative which is preferred to it by at least μ individuals if and only if

$$\mu > \frac{n-1}{n}h. \tag{2}$$

That immediately implies that (2) is a necessary and sufficient condition for the existence of a μ-majority SCFs, and that any minimal majority SCF satisfies the μ-majority principle for all μ satisfying (2). Recently, Bubboloni and Gori (2014, Theorem 15) proves that (1) and (2) are necessary and sufficient conditions for the existence of anonymous and neutral μ-majority SCFs.

Of course, condition (1) is a very strong arithmetical condition which is seldom satisfied in concrete situations. When it fails we can only try to design SCFs satisfying weaker versions of the principles of anonymity and neutrality, provided that we are unwilling to renounce to the minimal requirement of h-majority.

Bubboloni and Gori (2015) in the framework of rules, weaken the principle of anonymity assuming that individuals are divided into subcommittees and requiring that, within each subcommittee, individuals equally influence the final collective decision, while different subcommittees may have a different decision power. They also weaken the principle of neutrality, assuming that alternatives are divided into subclasses and requiring that within each subclass alternatives are equally treated,

[1] See Section 2.4 for further details about Greenberg's result.

while different subclasses may have a different treatment. These versions of anonymity and neutrality are actually used in many practical collective decision processes, and can be easily adapted to the case of social choice functions. In fact, given a partition of individuals into subcommittees, we say that a SCF is anonymous with respect to those subcommittees if it has the same value over any pair of preference profiles such that we can get one from the other by figuring to permute the names of individuals belonging to the same subcommittee. Given instead a partition of alternatives into subclasses, we say that a SCF is neutral with respect to those subclasses if, for every pair of preference profiles such that we can get one from the other by figuring to permute the names of alternatives belonging to the same subclass, the social outcome associated with them coincides up to the considered permutation. Of course, requiring that a SCF is anonymous (neutral) is equivalent both to require that it is anonymous (neutral) with respect to the partition whose unique element is the whole set of individuals (alternatives), and to require that it is anonymous (neutral) with respect to any partition of individuals (alternatives).

In the present paper we analyse the SCFs that satisfy anonymity with respect to subcommittees and neutrality with respect to subclasses, obey one between μ-majority and minimal majority principle, and are possibly reversal symmetric. Even though, at the best of our knowledge, conditions assuring the existence of those SCF are not known, it is worth mentioning that, under the assumption that alternatives are two and assuming the possibility of indifference in individual preferences, Campbell and Kelly (2011, 2013) show that the relative majority is implied both by a suitable weak version of anonymity, neutrality and monotonicity, as well as by what they called limited neutrality, anonymity and monotonicity. Moreover, in the general case for the number of alternatives, Kelly (1991) uses the language of permutations groups to make some observations about different levels of anonymity and neutrality a social choice function may have.

The major contribution of our paper can be summarized in the following theorem[2].

Theorem A. *Assume that individuals are partitioned into $r \geq 1$ subcommittees with number of members $b_1, \dots, b_r$, and that alternatives are partitioned into $s \geq 1$ subclasses with number of alternatives $c_1, \dots, c_s$.*

i) There exists a minimal majority SCF *which is anonymous with respect to the considered subcommittees and neutral with respect to the considered subclasses if and only if*

$$\gcd\big(\gcd(b_1, \dots, b_r), \mathrm{lcm}(c_1!, \dots, c_s!)\big) = 1. \tag{3}$$

ii) There exists a reversal symmetric minimal majority SCF *which is anonymous with respect to the considered subcommittees and neutral with respect to the considered subclasses if and only if*

$$\gcd\big(\gcd(b_1, \dots, b_r), \mathrm{lcm}(2, c_1!, \dots, c_s!)\big) = 1, \tag{4}$$

and one of the following conditions hold true:

a) $h \leq 3$;

b) $n \leq 3$;

c) $(h, n) \in \{(4,4), (5,4), (7,4), (5,5)\}$.

iii) Given $\mu \in \mathbb{N} \cap (h/2, h]$, there exists a μ-majority SCF *which is anonymous with respect to the considered subcommittees and neutral with respect to the considered subclasses if and only if* (2) *and* (3) *hold true.*

iv) Given $\mu \in \mathbb{N} \cap (h/2, h]$, there exists a reversal symmetric μ-majority SCF *which is anonymous with respect to the considered subcommittees and neutral with respect to the considered subclasses if and only if* (2) *and* (4) *hold true.*

[2]Theorem A is a rephrase of Theorems 24, 28, 26, and 30

First of all, note that (3) and (4) catch interesting situations. For instance, they are surely both satisfied when there is an individual in the committee having a special role, like a president. Indeed, in that case, we can assume that all the members of the committee but one belongs to the same subcommittee.

Let us observe now that Theorem A i) shows that the consistency of the three principles of anonymity with respect to subcommittees, neutrality with respect to subclasses and minimal majority is equivalent to (3); Theorem A ii) shows instead that the consistency of the three mentioned principles along with the reversal symmetry is equivalent to a set of much stronger conditions, that is, (4) and one among a), b) and c). Indeed, when none of a), b) and c) holds true, minimal majority and reversal symmetry conflict even without any anonymity and neutrality requirement, that is, assuming that every subcommittee and subclass is a singleton. In order to clarify that point consider, for instance, the preference profiles described by the matrices

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 & 4 & 2 \\ 3 & 4 & 2 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 2 & 3 & 1 & 1 & 1 \\ 3 & 4 & 2 & 4 & 2 & 3 \\ 2 & 3 & 4 & 3 & 4 & 2 \\ 1 & 1 & 1 & 2 & 3 & 4 \end{bmatrix}.$$

There, alternatives are denoted by 1, 2, 3 and 4, and each column represents the preferences of an individual in the committee (the higher the alternative is, the better it is). The committee is then supposed to have six members, so that neither a) nor b) nor c) holds true. A simple check shows that any minimal majority SCF should associate with both preference profiles the alternative 1. However, it is immediate to check that the two preference profiles are one obtained one by the other just reversing each individual preference so that a reversal symmetric SCF cannot associated 1 to both of them.

As a consequence, if we strongly desire to have reversal symmetric SCFs, then we should often renounce to the minimal majority principle. Thus, Theorem A iv) clarifies which kind of μ-majority is allowed to require. Note also that (3) and (4) reduce to (1) when we are dealing with anonymity and neutrality, that is, when all individuals are in the same committee and all alternatives in the same subclass. Then, Theorem A iv) implies that (1) and (2) are necessary and sufficient to get an anonymous, neutral and reversal symmetric μ majority SCF, generalizing one of the implications of the previously quoted Theorem 15 in Bubboloni and Gori (2014).

We finally observe that the proof of Theorem A is strongly based on the algebraic approach developed by Bubboloni and Gori (2014, 2015). In the framework of rules, they show how the notion of action of a group on a set can naturally and fruitfully be used to study problems concerning anonymity and neutrality and weaker versions of them, along with reversal symmetry. Here we adapt that algebraic reasoning to treat those same principles in the framework of SCFs. As a novelty, we employ language and methods taken from graph theory. In particular, we associate to each profile and majority threshold a directed graph whose connection and acyclicity properties are crucial for our investigation.

We finally stress that, as in Bubboloni and Gori (2014, 2015), the algebraic machinery provides a method to potentially build and count all the rules described in Theorem A.

2 Definitions and notation

2.1 Permutations and linear orders

Let A, B and C be sets. Given $f : A \to B$ and $g : B \to C$ we denote by gf the right-to-left composition of f and g, that is, the function from A to C defined, for every $a \in A$, as $gf(a) = g(f(a))$.

Let X be a nonempty finite set. We denote by $\mathrm{Sym}(X)$ the group of the bijective functions from X to itself, with product defined, for every $f_1, f_2 \in \mathrm{Sym}(X)$, as $f_1 f_2 \in \mathrm{Sym}(X)$. The neutral element of $\mathrm{Sym}(X)$ is given by the identity function id. $\mathrm{Sym}(X)$ is called the symmetric group[3] on X and its elements permutations on X. Given $\sigma \in \mathrm{Sym}(X)$, we denote its order by $|\sigma|$. For every $k \in \mathbb{N}$, the group $\mathrm{Sym}(\{1, \ldots, k\})$ is simply denoted by S_k.

We denote by $\mathcal{R}(X)$ the set of relations on X. Given $R \in \mathcal{R}(X)$ and $x, y \in X$, we sometimes write $x \geq_R y$ instead of $(x, y) \in R$, as well as $x >_R y$ instead of $(x, y) \in R$ and $(y, x) \notin R$. If $R \in \mathcal{R}(X)$ is antisymmetric, then $x >_R y$ is equivalent to $x \geq_R y$ and $x \neq y$. A relation on X is called a linear order on X if it is complete, transitive and antisymmetric. The set of linear orders on X is denoted by $\mathcal{L}(X)$. If $R_1, R_2 \in \mathcal{L}(X)$, then $R_1 = R_2$ if and only if, for every $x, y \in X$, $x >_{R_1} y$ implies $x >_{R_2} y$.

2.2 Preference relations

From now on, let $n \in \mathbb{N}$ with $n \geq 2$ be fixed, and let $N = \{1, \ldots, n\}$ be the set of alternatives. A *preference relation* on N is an element of $\mathcal{L}(N)$. Throughout the section, let $q \in \mathcal{L}(N)$ be fixed. For every $x, y \in N$, we say that x is preferred to y according to q if $x >_q y$. For every $\psi \in S_n$, we define ψq as the element of $\mathcal{L}(N)$ such that, for every $x, y \in N$, $(x, y) \in \psi q$ if and only if $(\psi^{-1}(x), \psi^{-1}(y)) \in q$. Consider the *order reversing permutation* in S_n, that is, the permutation $\rho_0 \in S_n$ defined, for every $r \in \{1, \ldots, n\}$, as $\rho_0(r) = n - r + 1$. Note that $|\rho_0| = 2$. We define $q\rho_0 \in \mathcal{L}(N)$ as the element in $\mathcal{L}(N)$ such that, for every $x, y \in N$, $(x, y) \in q\rho_0$ if and only if $(y, x) \in q$. We also define $q\, id = q$, where $id \in S_n$. By definition, for every $x, y \in N$ and $\psi \in S_n$, we have that $x >_q y$ if and only if $\psi(x) >_{\psi q} \psi(y)$; $x >_q y$ if and only if $y >_{q\rho_0} x$.

Consider now the set of vectors with n distinct components in N given by

$$\mathcal{V}(N) = \left\{ (x_r)_{r=1}^n \in N^n : x_{r_1} = x_{r_2} \Rightarrow r_1 = r_2 \right\},$$

and think each vector $(x_r)_{r=1}^n \in \mathcal{V}(N)$ as a column vector, that is,

$$(x_r)_{r=1}^n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [x_1, \ldots, x_n]^T.$$

The function $f_1 : \mathcal{V}(N) \to \mathcal{L}(N)$ associating with $(x_r)_{r=1}^n \in \mathcal{V}(N)$ the preference relation

$$\{(x_{r_1}, x_{r_2}) \in N \times N : r_1, r_2 \in \{1, \ldots, n\}, r_1 \leq r_2\},$$

and the function $f_2 : S_n \to \mathcal{L}(N)$ associating with $\sigma \in S_n$ the preference relation

$$\{(\sigma(r_1), \sigma(r_2)) \in N \times N : r_1, r_2 \in \{1, \ldots, n\}, r_1 \leq r_2\}$$

are bijective, so that, in particular, $|S_n| = |\mathcal{V}(N)| = |\mathcal{L}(N)| = n!$. We say that $x \in N$ has rank $r \in \{1, \ldots, n\}$ in q if x is the r-th component of $f_1^{-1}(q)$ or, equivalently, if x is the image of r through $f_2^{-1}(q)$. Note now that, for every $\psi \in S_n$ and $\rho \in \{id, \rho_0\}$, if $f_1^{-1}(q) = [x_1, \ldots, x_n]^T$, then $f_1^{-1}(\psi q) = [\psi(x_1), \ldots, \psi(x_n)]^T$ and $f_1^{-1}(q\rho) = [x_{\rho(1)}, \ldots, x_{\rho(n)}]^T$; if $f_2^{-1}(q) = \sigma$, then $f_2^{-1}(\psi q) = \psi\sigma$ and $f_2^{-1}(q\rho) = \sigma\rho$.

Thus, by the functions f_1 and f_2 we are allowed to identify the preference relation q both with the vector $f_1^{-1}(q)$ and with the permutation $f_2^{-1}(q)$, and to naturally interpret the products ψq and $q\rho$ in $\mathcal{V}(N)$ and in S_n. For instance, if $n = 4$ and

$$q = \{(4,2), (2,1), (1,3), (4,1), (4,3), (2,3), (4,4), (2,2), (1,1), (3,3)\} \in \mathcal{L}(\{1,2,3,4\}),$$

[3]The notation and results of group theory about permutation groups and actions, not explicitly discussed in the paper, are standard (see, for instance, Wielandt (1964) and Rose (1978)).

then q is identified with both $f_1^{-1}(q) = [4, 2, 1, 3]^T \in \mathcal{V}(\{1, 2, 3, 4\})$ and $f_2^{-1}(q) = (143) \in S_4$, so that 4 has rank 1, 2 has rank 2, 1 has rank 3, and 3 has rank 4 in q. Moreover, if $\psi = (342) \in S_4$, then we can write

$$\psi q = (342)[4, 2, 1, 3]^T = [2, 3, 1, 4]^T \quad \text{and} \quad q\rho_0 = [4, 2, 1, 3]^T(14)(23) = [3, 1, 2, 4]^T,$$

as well as

$$\psi q = (342)(143) = (123) \quad \text{and} \quad q\rho_0 = (143)(14)(23) = (132).$$

Thus, identifying preference relations with vectors makes computations easy and intuitive. On the other hand, identifying preference relations with permutations allows to transfer the group properties of S_n to the products between preference relations and permutations. In particular, by associativity and cancellation laws, for every $\psi_1, \psi_2 \in S_n$ and $\rho_1, \rho_2 \in \{id, \rho_0\}$, we have that $\psi_1 q = \psi_2 q$ if and only if $\psi_1 = \psi_2$; $q\rho_1 = q\rho_2$ if and only if $\rho_1 = \rho_2$; $(\psi_2\psi_1)q = \psi_2(\psi_1 q)$; $q(\rho_1\rho_2) = (q\rho_1)\rho_2$; $(\psi_1 q)\rho_1 = \psi_1(q\rho_1)$.

Given now $\psi \in S_n$ and $\rho \in \{id, \rho_0\}$, we finally emphasize that the above discussion makes the products ψq and $q\rho$ have interesting interpretations. Indeed, if q represents the preferences of a certain individual, then ψq represents the preferences that the individual would have if, for every $x \in N$, alternative x were called $\psi(x)$; $q\rho$ represents the preferences that the individual would have if, for every $r \in \{1, \dots, n\}$, the alternative whose rank is r is moved to rank $\rho(r)$. As a consequence, even though both ψ and ρ belong to S_n they have different meanings. Indeed, ψ maps alternatives to alternatives, while ρ maps ranks to ranks. Moreover, looking at q as a permutation, we have that q maps ranks to alternatives. In particular, the set $\{1, \dots, n\}$ sometimes refers to the set of alternatives, sometimes to the set of ranks. Although the context always allows to understand the right interpretation, along the paper we denote that set by N in the first case, and by $\{1, \dots, n\}$ in the second one.

2.3 Preference profiles

From now on, let $h \in \mathbb{N}$ with $h \geq 2$ be fixed, and let $H = \{1, \dots, h\}$ be the set of individuals. A *preference profile* is an element of $\mathcal{L}(N)^h$. The set $\mathcal{L}(N)^h$ is denoted by $\mathcal{P}$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preferences of individual i. Any $p \in \mathcal{P}$ can be identified with the matrix whose i-th column is the column vector representing the i-th component of p.

Let us consider the groups $\Omega = \{id, \rho_0\} \leq S_n$ and $G = S_h \times S_n \times \Omega$. For every $(\varphi, \psi, \rho) \in G$ and $p \in \mathcal{P}$, define $p^{(\varphi,\psi,\rho)} \in \mathcal{P}$ as the preference profile such that, for every $i \in H$,

$$\left(p^{(\varphi,\psi,\rho)}\right)_i = \psi p_{\varphi^{-1}(i)}\rho.$$

Since we have given no meaning to $(p_i)^{(\varphi,\psi,\rho)}$ for a single preference relation $p_i \in \mathcal{L}(N)$, we will write the i-th component $p^{(\varphi,\psi,\rho)}$ simply as $p_i^{(\varphi,\psi,\rho)}$, instead of $\left(p^{(\varphi,\psi,\rho)}\right)_i$.

Note that the preference profile $p^{(\varphi,\psi,\rho)}$ is obtained by p according to the following rules: for every $i \in H$, individual i is renamed $\varphi(i)$; for every $x \in N$, alternative x is renamed $\psi(x)$; for every $r \in \{1, \dots, n\}$, alternatives whose rank is r are moved to rank $\rho(r)$. For instance, if $n = 3$, $h = 5$ and

$$p = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad \varphi = (134)(25), \quad \psi = (12), \quad \rho = \rho_0 = (13),$$

then we have

$$p^{(\varphi,id,id)} = \begin{bmatrix} 3 & 2 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 3 \end{bmatrix}, \quad p^{(id,\psi,id)} = \begin{bmatrix} 3 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 2 & 2 \end{bmatrix},$$

$$p^{(id,id,\rho_0)} = \begin{bmatrix} 1 & 3 & 3 & 1 & 1 \\ 2 & 2 & 1 & 2 & 3 \\ 3 & 1 & 2 & 3 & 2 \end{bmatrix}, \quad p^{(\varphi,\psi,\rho_0)} = \begin{bmatrix} 2 & 2 & 2 & 3 & 3 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 1 & 3 & 1 & 2 \end{bmatrix}.$$

As it is easy to verify, if $n = 2$, then $p^{(id,\rho_0,id)} = p^{(id,id,\rho_0)}$ for all $p \in \mathcal{P}$; if $n \geq 3$, then there do not exist $\varphi \in S_h$ and $\psi \in S_n$ such that, for every $p \in \mathcal{P}$, $p^{(\varphi,\psi,id)} = p^{(id,id,\rho_0)}$. Thus, for $n \geq 3$, top-down reversing preference profiles cannot be reduced to a change in individuals and alternatives names.

2.4 Symmetric majority social choice functions

A *social choice function* (SCF) is a function from $\mathcal{P}$ to N. Given a subgroup U of G, we say that a SCF f is *U-symmetric* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in U$,

$$f(p^{(\varphi,\psi,\rho)}) \in \begin{cases} \{\psi f(p)\} & \text{if } \rho = id \\ \\ N \setminus \{\psi f(p)\} & \text{if } \rho = \rho_0 \end{cases} \tag{5}$$

The set of U-symmetric SCFs is denoted by $\mathfrak{F}^U$. Note that if $U' \leq U$, then $\mathfrak{F}^U \subseteq \mathfrak{F}^{U'}$. The concept of symmetry with respect to a subgroup U of G includes some classical requirements for SCFs. For instance, a SCF f is anonymous if and only if $f \in \mathfrak{F}^{S_h \times \{id\} \times \{id\}}$; it is neutral if and only if $f \in \mathfrak{F}^{\{id\} \times S_n \times \{id\}}$; it is reversal symmetric if and only if $f \in \mathfrak{F}^{\{id\} \times \{id\} \times \Omega}$.

Given $\mu \in \mathbb{N} \cap (h/2, h]$, define, for every $p \in \mathcal{P}$, the set

$$D_\mu(p) = \{x \in N : \forall y \in N, |\{i \in H : y >_{p_i} x\}| < \mu\}.$$

Thus, an alternative x belongs to $D_\mu(p)$ if and only if it cannot be found another alternative which is preferred to x by at least μ individuals, according to the preference profile p. Note that the set $D_\mu(p)$ corresponds to the set of μ-majority equilibria associated with p which is first defined in Greenberg (1979) in the more general setting where individual preferences are complete and transitive relations.

For example, consider $h = 9$, $n = 3$ (so that $H = \{1, \dots, 9\}$ and $N = \{1, 2, 3\}$) and the preference profile

$$p = \begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 1 & 3 & 3 & 1 & 1 & 1 & 2 \\ 3 & 3 & 3 & 1 & 1 & 2 & 2 & 2 & 1 \end{bmatrix}. \tag{6}$$

A simple check shows that

$$|\{i \in H : 1 >_{p_i} 2\}| = 5, \qquad |\{i \in H : 1 >_{p_i} 3\}| = 3, \qquad |\{i \in H : 2 >_{p_i} 3\}| = 5,$$
$$|\{i \in H : 2 >_{p_i} 1\}| = 4, \qquad |\{i \in H : 3 >_{p_i} 1\}| = 6, \qquad |\{i \in H : 3 >_{p_i} 2\}| = 4.$$

It is now immediate to compute, for every majority threshold $\mu \in \mathbb{N} \cap (\frac{h}{2}, h] = \{5, 6, 7, 8, 9\}$, the set $D_\nu(p)$. Indeed, $D_5(p) = \varnothing$; $D_6(p) = \{2, 3\}$; $D_7(p) = D_8(p) = D(p) = N$ because, for every $\mu \in \{7, 8, 9\}$ and $x, y \in N$, we have $|\{i \in H : x >_{p_i} y\}| < \mu$.

Observe that

$$\text{if } \ \mu \leq \mu', \text{ then } \ D_\mu(p) \subseteq D_{\mu'}(p) \ \text{ for all } p \in \mathcal{P}. \tag{7}$$

Moreover, as an immediate consequence of Corollary 3 and its proof in Greenberg (1979), we get the following result.

Proposition 1. *Let $\mu \in \mathbb{N} \cap (h/2, h]$. Then the following conditions are equivalent:*

i) for every $p \in \mathcal{P}$, $D_\mu(p) \neq \varnothing$;

ii) $\mu > \frac{n-1}{n}h$.

Given a *majority threshold* $\mu \in \mathbb{N} \cap (h/2, h]$, we say that f is a *μ-majority* SCF if, for every $p \in \mathcal{P}$, $f(p) \in D_\mu(p)$. We denote the set of μ-majority SCFs by $\mathfrak{F}_\mu$. Of course, if $\mu \leq \mu'$, then $\mathfrak{F}_\mu \subseteq \mathfrak{F}_{\mu'}$.

We define the *minimal majority threshold* as $\mu_0 = \left\lceil \frac{h+1}{2} \right\rceil$, that is, the smallest majority threshold that can be taken into consideration. We also define the *Greenberg majority threshold* as

$$\mu_G = \min \left\{ m \in \mathbb{N} \cap (h/2, h] : m > \frac{n-1}{n} h \right\}.$$

Then, Proposition 1 gives

$$\mathfrak{F}_\mu \neq \varnothing \quad \text{if and only if } \mu \geq \mu_G. \tag{8}$$

Moreover, Proposition 1 implies that, for every $p \in \mathcal{P}$, $D_h(p) \neq \varnothing$. Thus, it is well defined the integer

$$\mu(p) = \min\{\mu \in \mathbb{N} \cap (h/2, h] : D_\mu(p) \neq \varnothing\}.$$

For instance, if p is the preference profile defined in (6), we have $\mu(p) = 6$.

Since, by Proposition 1, $D_{\mu_G}(p) \neq \varnothing$ for all $p \in \mathcal{P}$, we immediately get

$$\mu_0 \leq \mu(p) \leq \mu_G. \tag{9}$$

We say that f is a *minimal majority* SCF if, for every $p \in \mathcal{P}$, $f(p) \in D_{\mu(p)}(p)$. The set of minimal majority SCFs is denoted by $\mathfrak{F}_{\min}$ and, by definition, it is always nonempty. It is interesting to note that a minimal majority SCF is, in particular, a μ-majority SCF, for all $\mu \geq \mu_G$, that is, if $\mathfrak{F}_\mu \neq \varnothing$, then $\mathfrak{F}_{\min} \subseteq \mathfrak{F}_\mu$. Indeed, assume that $\mathfrak{F}_\mu \neq \varnothing$ and consider $f \in \mathfrak{F}_{\min}$. Fixed $p \in \mathcal{P}$, by (8) and (9), we have that $\mu \geq \mu(p)$ which, by (7), implies $D_{\mu(p)}(p) \subseteq D_\mu(p)$. Since $f(p) \in D_{\mu(p)}(p)$, we get also $f(p) \in D_\mu(p)$, which says $f \in \mathfrak{F}_\mu$.

For every $U \leq G$ and $\mu \in \mathbb{N} \cap (h/2, h]$, define the set $\mathfrak{F}^U_\mu = \mathfrak{F}^U \cap \mathfrak{F}_\mu$ of the *U-symmetric μ-majority* SCF*s*, and the set $\mathfrak{F}^U_{\min} = \mathfrak{F}^U \cap \mathfrak{F}_{\min}$ of the *U-symmetric minimal majority* SCF*s*. We are going to study under which conditions on U and μ those sets are nonempty.

3 Actions on the set of preference profiles

The next proposition, proved in Bubboloni and Gori (2015), shows that any subgroup U of G naturally acts on the set of preference profiles $\mathcal{P}$. That result is rich of consequences as it allows to exploit many general facts from group theory.

Proposition 2. *Let $U \leq G$. Then:*

i) for every $p \in \mathcal{P}$ and $(\varphi_1, \psi_1, \rho_1), (\varphi_2, \psi_2, \rho_2) \in U$, we have

$$p^{(\varphi_1\varphi_2, \psi_1\psi_2, \rho_1\rho_2)} = \left(p^{(\varphi_2, \psi_2, \rho_2)}\right)^{(\varphi_1, \psi_1, \rho_1)}; \tag{10}$$

ii) the function $\alpha : U \to \mathrm{Sym}(\mathcal{P})$ defined, for every $(\varphi, \psi, \rho) \in U$, as

$$\alpha(\varphi, \psi, \rho) : \mathcal{P} \to \mathcal{P}, \quad p \mapsto p^{(\varphi, \psi, \rho)},$$

is well posed and it is an action of the group U on the set $\mathcal{P}$.

The fact that the function α defined in Proposition 2 is an action, is crucial in our research. To begin with we present a result which is very important from the interpretative point of view.

Proposition 3. *For every $i \in \{1,2\}$, let $U_i \leq G$, where $U_i = Z_i \times R_i$ for some $Z_i \leq S_h \times S_n$ and $R_i \leq \Omega$. Then $\mathfrak{F}^{U_1} \cap \mathfrak{F}^{U_2} = \mathfrak{F}^{\langle U_1, U_2 \rangle}$.*

Proof. Since $\langle U_1, U_2\rangle$ contains both U_1 and U_2, we get immediately $\mathfrak{F}^{\langle U_1,U_2\rangle} \subseteq \mathfrak{F}^{U_1} \cap \mathfrak{F}^{U_2}$. Let us now fix $f \in \mathfrak{F}^{U_1} \cap \mathfrak{F}^{U_2}$ and prove $f \in \mathfrak{F}^{\langle U_1,U_2\rangle}$. Define, for every $k \in \mathbb{N}$, the set $\langle U_1, U_2\rangle_k$ of the elements in $\langle U_1, U_2\rangle$ that can be written as product of k elements of $U_1 \cup U_2$. Then, by definition of generated subgroup, we have $\langle U_1, U_2\rangle = \bigcup_{k\in\mathbb{N}} \langle U_1, U_2\rangle_k$ and to get $f \in \mathfrak{F}^{\langle U_1,U_2\rangle}$ it is enough to show that, for every $k \in \mathbb{N}$, we have that

$$\text{for every } p \in \mathcal{P} \text{ and } g = (\varphi, \psi, \rho) \in \langle U_1, U_2\rangle_k, \text{ (5) holds true.} \tag{11}$$

First of all we show that, for every $k \in \mathbb{N}$, if $(\varphi, \psi, \rho_0) \in \langle U_1, U_2\rangle_k$, then also $(\varphi, \psi, id) \in \langle U_1, U_2\rangle_k$. Being $U_i = Z_i \times R_i$, with $Z_i \leq S_h \times S_h$ and $R_i \in \{\Omega, \{id\}\}$ for all $i \in \{1, 2\}$, that surely holds for $k = 1$. Let now $k \geq 2$ and pick $g = g_1 \cdots g_k = (\varphi, \psi, \rho_0) \in \langle U_1, U_2\rangle_k$, where $g_1, \ldots, g_k \in U_1 \cup U_2$. Then there exists $j \in \{1, \ldots, k\}$ such that the third component of g_j is ρ_0, say $g_j = (\varphi_j, \psi_j, \rho_0)$. Consider $\overline{g}_j = (\varphi_j, \psi_j, id)$ and note that $\overline{g}_j \in U_1 \cup U_2$. Thus also $\overline{g} = g_1 \ldots g_{j-1}\overline{g}_j g_{j+1} \ldots g_k = (\varphi, \psi, id) \in \langle U_1, U_2\rangle_k$.

We now show (11), by induction on k. If $k = 1$, we have $g \in \langle U_1, U_2\rangle_1 = U_1 \cup U_2$ and so (11) is guaranteed by $f \in \mathfrak{F}^{U_1} \cap \mathfrak{F}^{U_2}$. Assume that (11) holds for some $k \in \mathbb{N}$ and show that it holds also for $k + 1$. Let $g \in \langle U_1, U_2\rangle_{k+1}$. Then there exist $g_* = (\varphi_*, \psi_*, \rho_*) \in \langle U_1, U_2\rangle_k$ and $g_1 = (\varphi_1, \psi_1, \rho_1) \in U_1 \cup U_2$ such that $g = g_1 g_* = (\varphi_1\varphi_*, \psi_1\psi_*, \rho_1\rho_*)$.

Assume first that $\rho_1\rho_* = id$. We show that $g = \overline{g}_1\overline{g}_*$, for suitable $\overline{g}_* \in \langle U_1, U_2\rangle_k$ and $\overline{g}_1 \in U_1\cup U_2$ having the third component equal to id. By $\rho_1\rho_* = id$, it follows that ρ_1 and ρ_* are both equal to id or they are both equal to ρ_0. In the first case we take $\overline{g}_* = g_*$, $\overline{g}_1 = g_1$; in the second one we take $\overline{g}_* = (\varphi_*, \psi_*, id)$, $\overline{g}_1 = (\varphi_1, \psi_1, id)$. Then, by (10) and using the inductive hypothesis, we have

$$f(p^g) = f(p^{\overline{g}_1\overline{g}_*}) = f((p^{\overline{g}_*})^{\overline{g}_1}) = \psi_1\psi_* f(p).$$

Next let $\rho_1\rho_* = \rho_0$. Then exactly one among ρ_1, ρ_* is equal to ρ_0 and the other one is equal to id. If $\rho_1 = \rho_0$ and $\rho_* = id$, applying again (10) and the inductive hypothesis, we have

$$f(p^g) = f(p^{g_1 g_*}) = f((p^{g_*})^{g_1}) \neq \psi_1 f(p^{g_*}) = \psi_1\psi_* f(p).$$

If instead $\rho_* = \rho_0$ and $\rho_1 = id$, by (10), we get

$$f(p^g) = f((p^{g_*})^{g_1}) = \psi_1 f(p^{g_*}).$$

On the other hand, the inductive hypothesis gives $f(p^{g_*}) \neq \psi_* f(p)$ and, since ψ_1 is injective, we also get $\psi_1 f(p^{g_*}) \neq \psi_1\psi_* f(p)$, so that $f(p^g) \neq \psi_1\psi_* f(p)$. □

The above proposition has very interesting applications. For instance, it implies that if f is a SCF, then f is anonymous and neutral if and only if $f \in \mathfrak{F}^{S_h\times S_n\times\{id\}}$; f is anonymous and reversal symmetric if and only if $f \in \mathfrak{F}^{S_h\times\{id\}\times\Omega}$; f is neutral and reversal symmetric if and only if $f \in \mathfrak{F}^{\{id\}\times S_n\times\Omega}$; f is anonymous, neutral and reversal symmetric if and only if $f \in \mathfrak{F}^G$.

Thanks to Proposition 2 , we can use in our context notation and results concerning the action of a group on a set. We recall the basic facts that we are going to use. For every $p \in \mathcal{P}$, the set $p^U = \{p^g \in \mathcal{P} : g \in U\}$ is called the U-orbit of p. It is well known that the set $\mathcal{P}^U = \{p^U : p \in \mathcal{P}\}$ of the U-orbits is a partition[4] of $\mathcal{P}$. We use $\mathcal{P}^U$ as set of indexes and denote its elements with j. A vector $(p^j)_{j\in\mathcal{P}^U} \in \times_{j\in\mathcal{P}^U}\mathcal{P}$ is called a system of representatives of the U-orbits if, for every $j \in \mathcal{P}^U$, $p^j \in j$. The set of the systems of representatives of the U-orbits is nonempty and denoted by $\mathfrak{S}(U)$. If $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$, then for each $p \in \mathcal{P}$ there exist $j \in \mathcal{P}^U$ and $(\varphi, \psi, \rho) \in U$ such that $p = p^{j\,(\varphi,\psi,\rho)}$. Moreover if for some $j_1, j_2 \in \mathcal{P}^U$ and some $(\varphi_1, \psi_1, \rho_1), (\varphi_2, \psi_2, \rho_2) \in U$, we have $p^{j_1\,(\varphi_1,\psi_1,\rho_1)} = p^{j_2\,(\varphi_2,\psi_2,\rho_2)}$, then $j_1 = j_2$. For every $p \in \mathcal{P}$, the stabilizer of p in U is the subgroup of U defined by $\mathrm{Stab}_U(p) = \{g \in U : p^g = p\}$.

[4]A partition of a nonempty set X is a set of nonempty pairwise disjoint subsets of X whose union is X.

4 Regular subgroups

Bubboloni and Gori (2015) introduce, in the framework of symmetric rules, the important concept of regular subgroup. We are going to show that regular subgroups have an important role in the setting of symmetric SCFs too.

A subgroup U of G is said to be *regular* if, for every $p \in \mathcal{P}$,

$$\begin{gathered}\text{there exists } \psi_* \in S_n \text{ conjugate to } \rho_0 \text{ such that} \\ \mathrm{Stab}_U(p) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_*\} \times \{\rho_0\}) \,.\end{gathered} \tag{12}$$

Recall that, within our notation, two permutations $\sigma_1, \sigma_2 \in S_n$ are conjugate if there exists $u \in S_n$ such that $\sigma_1 = u\sigma_2 u^{-1}$.

Note that if $p \in \mathcal{P}$ is such that $\mathrm{Stab}_U(p) \not\leq S_h \times S_n \times \{id\}$, then $\psi_* \in S_n$ in (12) is unique. Indeed, assume that $\mathrm{Stab}_U(p) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_*\} \times \{\rho_0\})$ as well as $\mathrm{Stab}_U(p) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_{**}\} \times \{\rho_0\})$, for suitable $\psi_*, \psi_{**} \in S_n$ and pick $(\varphi, \psi, \rho_0) \in \mathrm{Stab}_U(p)$. Then, we have $\psi = \psi_*$ as well as $\psi = \psi_{**}$, so that $\psi_* = \psi_{**}$.

Note also that if U is regular and $W \leq U$, then W is regular too, because $\mathrm{Stab}_W(p) = W \cap \mathrm{Stab}_U(p)$. In particular, G is regular if and only if each subgroup of G is regular.

5 Subcommittees and subclasses

In this section we focus on SCFs that are anonymous with respect to subcommittees, neutral with respect to subclasses and reversal symmetric. To begin with, let us formalize those versions of the principles of anonymity and neutrality in terms of U-symmetry.

Given $B = \{B_j\}_{j=1}^s$ a partition of H, we define

$$V(B) = \{\varphi \in S_h : \varphi(B_j) = B_j \text{ for all } j \in \{1, \dots, s\}\} \,,$$

and given $C = \{C_k\}_{k=1}^t$ a partition of N, we define

$$W(C) = \{\psi \in S_n : \psi(C_k) = C_k \text{ for all } k \in \{1, \dots, t\}\} \,.$$

Note that $V(B)$ is a subgroup of S_h and $W(C)$ is a subgroup of S_n. Moreover, $V(\{H\}) = S_h$ and $W(\{N\}) = S_n$.

A SCF is said to be anonymous with respect to a partition B of H, briefly B-*anonymous*, if it is $V(B) \times \{id\} \times \{id\}$-symmetric. A SCF is said to be neutral with respect to a partition C of N, briefly C-*neutral*, if it is $\{id\} \times W(C) \times \{id\}$-symmetric. Thus, referring to the discussion carried on in the introduction, if B is interpreted as the set of subcommittees in which H is divided, then B-anonymous SCFs are those SCFs which do not distinguish among individuals belonging to the same subcommittee. Analogously, interpreting C as the set of subclasses in which N is divided, we have that C-neutral SCFs are those SCFs equally treating alternatives within each subclass. Note also that, because of Proposition 3, a SCF is B-anonymous and C-neutral if and only if it is $V(B) \times W(C) \times \{id\}$-symmetric. Similarly, a SCF is B-anonymous, C-neutral and reversal symmetric if and only if it is $V(B) \times W(C) \times \Omega$-symmetric.

The following result is proved in Bubboloni and Gori (2015, Theorem 14).

Theorem 4. *Let $B = \{B_j\}_{j=1}^r$ be a partition of H, $C = \{C_k\}_{k=1}^s$ be a partition of N and $R \in \{\{id\}, \Omega\}$. Then $V(B) \times W(C) \times R$ is regular if and only if*

$$\gcd\left(\gcd(|B_j|)_{j=1}^r, \mathrm{lcm}(|R|, |C_{k^*}|!)\right) = 1,$$

where $|C_{k^*}| = \max\{|C_k|\}_{k=1}^s$.

6 Preliminary lemmata

Lemma 5. *Let $\mu \in \mathbb{N} \cap (h/2, h]$, $p \in \mathcal{P}$ and $(\varphi, \psi, id) \in G$. Then $D_\mu(p^{(\varphi,\psi,id)}) = \psi(D_\mu(p))$.*

Proof. For every $a, b \in N$, define the set $H_{a,b}(p) = \{i \in H : a >_{p_i} b\}$. It is easily checked that $H_{a,b}(p^{(\varphi,id,id)}) = \varphi(H_{a,b}(p))$ and $H_{a,b}(p) = H_{\psi(a),\psi(b)}(p^{(id,\psi,id)})$. Thus, using (10), we get $H_{\psi(a),\psi(b)}(p^{(\varphi,\psi,id)}) = \varphi(H_{a,b}(p))$ and, in particular, $|H_{a,b}(p)| = |H_{\psi(a),\psi(b)}(p^{(\varphi,\psi,id)})|$.

Let us prove now that

$$D_\mu(p^{(\varphi,\psi,id))}) = \psi(D_\mu(p)). \tag{13}$$

If $x \in D_\mu(p)$, then, for every $a \in N$, we have $|H_{a,x}(p)| < \mu$. Then also $|H_{\psi(a),\psi(x)}(p^{(\varphi,\psi,id)})| < \mu$ and since $\psi(a)$ describes N when a varies in N, we deduce that $\psi(x) \in D_\mu(p^{(\varphi,\psi,id))})$. That gives $\psi(D_\mu(p)) \subseteq D_\mu(p^{(\varphi,\psi,id))})$. Consider now $y \in D_\mu(p^{(\varphi,\psi,id)})$. By the previous argument we have that $\psi^{-1}(y) \in D_\mu((p^{(\varphi,\psi,id))})^{(\varphi^{-1},\psi^{-1},id))}) = D_\mu(p)$, where the last equality is due again to (10). Then $y \in \psi(D_\mu(p))$, so that $D_\mu(p^{(\varphi,\psi,id))}) \subseteq \psi(D_\mu(p))$, which finally proves (13). □

Corollary 6. *Let $p \in \mathcal{P}$ and $(\varphi, \psi, id) \in G$. Then $\mu(p^{(\varphi,\psi,id)}) = \mu(p)$ and $D_{\mu(p^{(\varphi,\psi,id)})}(p^{(\varphi,\psi,id)}) = \psi(D_{\mu(p)}(p))$.*

Proof. Lemma 5 implies that, for every $\mu \in \mathbb{N} \cap (h/2, h]$, $D_\mu(p) \neq \varnothing$ if and only if $D_\mu(p^{(\varphi,\psi,id))}) \neq \varnothing$, and thus $\mu(p^{(\varphi,\psi,id)}) = \mu(p)$. By applying (13) to $\mu = \mu(p^{(\varphi,\psi,id)}) = \mu(p)$, we then get the desired equality. □

Lemma 7. *Let $p \in \mathcal{P}$ be such that $\mathrm{Stab}_G(p) \not\leq S_h \times S_n \times \{id\}$. Then, for every $(\varphi, \psi, \rho) \in G$, we have $\mu(p^{(\varphi,\psi,\rho)}) = \mu(p)$.*

Proof. Fix $(\varphi, \psi, \rho) \in G$ and prove that $\mu(p^{(\varphi,\psi,\rho)}) = \mu(p)$. If $\rho = id$, we invoke Corollary 6. If instead $\rho = \rho_0$, pick $(\varphi_1, \psi_1, \rho_0) \in \mathrm{Stab}_G(p)$ and use (10) and again Corollary 6 to get

$$\mu(p^{(\varphi,\psi,\rho_0)}) = \mu\left((p^{(\varphi_1,\psi_1,\rho_0)})^{(\varphi,\psi,\rho_0)}\right) = \mu\left(p^{(\varphi\varphi_1,\psi\psi_1,id)}\right) = \mu(p),$$

which completes the proof. □

Note that the assumption $\mathrm{Stab}_G(p) \not\leq S_h \times S_n \times \{id\}$ is crucial for proving Lemma 7. Indeed, consider $h = 3$, $n = 4$ and

$$p = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 4 & 4 & 4 \end{bmatrix}.$$

It is immediate to verify that $\mathrm{Stab}_G(p) \leq S_h \times S_n \times \{id\}$, $\mu(p) = 3$ and $\mu(p^{(id,id,\rho_0)}) = 2$.

7 Existence results: the case $U \leq S_h \times S_n \times \{id\}$

In this section we analyze the sets $\mathfrak{F}^U$, $\mathfrak{F}^U_{\min}$, and $\mathfrak{F}^U_\mu$, where U is a regular subgroup of G included in $S_h \times S_n \times \{id\}$.

7.1 The set $\mathfrak{F}^U$

Proposition 8. *Let $U \leq S_h \times S_n \times \{id\}$ be regular, $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and, for every $j \in \mathcal{P}^U$, let $x_j \in N$. Then there exists a unique $f \in \mathfrak{F}^U$ such that, for every $j \in \mathcal{P}^U$, $f(p^j) = x_j$.*

Proof. Consider $f \in \mathfrak{F}$ defined, for every $p \in \mathcal{P}$, as $f(p) = \psi(x_j)$, where $j \in \mathcal{P}^U$ and $(\varphi, \psi, \rho) \in U$ are such that $p = p^{j\,(\varphi,\psi,\rho)}$. Let us first prove that f is well defined. Consider $j \in \mathcal{P}^U$ and let $(\varphi_1, \psi_1, id), (\varphi_2, \psi_2, id) \in U$ such that $p^{j\,(\varphi_1,\psi_1,id)} = p^{j\,(\varphi_2,\psi_2,id)}$. We need to prove that $f(p^{j\,(\varphi_1,\psi_1,id)}) = f(p^{j\,(\varphi_2,\psi_2,id)})$. Note that, by (10), we have $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1, id) \in \mathrm{Stab}_U(p^j)$ which, since U is regular, gives $\psi_1 = \psi_2$. As a consequence,

$$f(p^{j\,(\varphi_1,\psi_1,id)}) = \psi_1(x_j) = \psi_2(x_j) = f(p^{j\,(\varphi_2,\psi_2,id)}).$$

Since U is a subgroup of G, we have $(id, id, id) \in U$ and thus, for every $j \in \mathcal{P}^U$, we get $f(p^j) = f(p^{j\,(id,id,id)}) = x_j$. Let us now prove that $f \in \mathfrak{F}^U$. Fix $p \in \mathcal{P}$ and $(\varphi, \psi, id) \in U$. Then, there exist $j \in \mathcal{P}^U$ and $(\varphi_1, \psi_1, id) \in U$ such that $p = p^{j\,(\varphi_1,\psi_1,id)}$ and therefore

$$f(p) = f(p^{j\,(\varphi_1,\psi_1,id)}) = \psi_1(x_j).$$

Thus, by (10),

$$f(p^{(\varphi,\psi,id)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,id)}) = \psi\psi_1(x_j) = \psi f(p).$$

We are left with proving that if $f' \in \mathfrak{F}^U$ is such that, for every $j \in \mathcal{P}^U$, $f'(p^j) = x_j$, then $f' = f$, that is, for every $p \in \mathcal{P}$, $f'(p) = f(p)$. Given $p \in \mathcal{P}$, let $p = p^{j\,(\varphi_1,\psi_1,id)}$ for suitable $j \in \mathcal{P}^U$ and $(\varphi_1, \psi_1, id) \in U$. Then

$$f'(p) = f'\left(p^{j\,(\varphi_1,\psi_1,id)}\right) = \psi_1(x_j) = f\left(p^{j\,(\varphi_1,\psi_1,id)}\right) = f(p),$$

and the proof is completed. □

Proposition 9. *Let $U \leq S_h \times S_n \times \{id\}$ be regular. Then $|\mathfrak{F}^U| = n^{|\mathcal{P}^U|}$. In particular, $\mathfrak{F}^U \neq \varnothing$.*

Proof. Fix $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and consider the function $\Phi : \times_{j\in\mathcal{P}^U} N \to \mathfrak{F}^U$ which associates to every $(x_j)_{j\in\mathcal{P}^U} \in \times_{j\in\mathcal{P}^U} N$ the unique $f \in \mathfrak{F}^U$ defined in Proposition 8. Note that Φ is obviously injective. In order to prove that Φ is surjective, consider $f \in \mathfrak{F}^U$, and note that $\Phi\left(\left(f(p^j)\right)_{j\in\mathcal{P}^U}\right) = f$. Then Φ is bijective and the equality $|\mathfrak{F}^U| = n^{|\mathcal{P}^U|}$ follows. □

7.2 The set $\mathfrak{F}^U_{\min}$

Proposition 10. *Let $U \leq S_h \times S_n \times \{id\}$ be regular, $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and, for every $j \in \mathcal{P}^U$, let $x_j \in D_{\mu(p^j)}(p^j)$. Then there exists a unique $f \in \mathfrak{F}^U_{\min}$ such that, for every $j \in \mathcal{P}^U$, $f(p^j) = x_j$.*

Proof. By Proposition 8, there is a unique $f \in \mathfrak{F}^U$ such that, for every $j \in \mathcal{P}^U$, $f(p^j) = x_j$. We complete the proof showing that $f \in \mathfrak{F}_{\min}$, as well. Consider then $p \in \mathcal{P}$ and prove that $f(p) \in D_{\mu(p)}(p)$. Let $p = p^{j\,(\varphi_1,\psi_1,id)}$ for suitable $j \in \mathcal{P}^U$ and $(\varphi_1, \psi_1, id) \in U$. By Corollary 6, we have $\mu(p) = \mu(p^j)$ and $D_{\mu(p)}(p) = \psi_1(D_{\mu(p^j)}(p^j))$. As a consequence

$$f(p) = f\left(p^{j\,(\varphi_1,\psi_1,id)}\right) = \psi_1(x_j) \in \psi_1(D_{\mu(p^j)}(p^j)) = D_{\mu(p)}(p),$$

and the proof is completed. □

Proposition 11. *Let $U \leq S_h \times S_n \times \{id\}$ be regular and $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$. Then*

$$|\mathfrak{F}^U_{\min}| = \prod_{j\in\mathcal{P}^U} \left|D_{\mu(p^j)}(p^j)\right|. \tag{14}$$

In particular, $\mathfrak{F}^U_{\min} \neq \varnothing$.

Proof. Consider the bijective function Φ defined in the proof of Proposition 9. We prove (14) showing that $\Phi\left(\times_{j\in\mathcal{P}^U} D_{\mu(p^j)}(p^j)\right) = \mathfrak{F}^U_{\min}$. By Proposition 10, we know that $\Phi\left(\times_{j\in\mathcal{P}^U} D_{\mu(p^j)}(p^j)\right) \subseteq \mathfrak{F}^U_{\min}$. In order to prove the other inclusion, simply note that, given $f \in \mathfrak{F}^U_{\min}$, for every $j \in \mathcal{P}^U$, $f(p^j) \in D_{\mu(p^j)}(p^j)$ and $\Phi\left(\left(f(p^j)\right)_{j\in\mathcal{P}^U}\right) = f$. Thus $|\mathfrak{F}^U_{\min}| = \prod_{j\in\mathcal{P}^U}\left|D_{\mu(p^j)}(p^j)\right|$. Since, for every $j \in \mathcal{P}^U$, we have that $D_{\mu(p^j)}(p^j) \neq \varnothing$ it follows that $\mathfrak{F}^U_{\min} \neq \varnothing$. □

7.3 The set $\mathfrak{F}^U_\mu$

Propositions 12 and 13 below can be proved simply mimicking the proof of Propositions 10 and 11 and using Lemma 5.

Proposition 12. *Let $U \leq S_h \times S_n \times \{id\}$ be regular and $\mu \in \mathbb{N} \cap (h/2, h]$. Let $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and, for every $j \in \mathcal{P}^U$, let $x_j \in D_\mu(p^j)$. Then there exists a unique $f \in \mathfrak{F}^U_\mu$ such that, for every $j \in \mathcal{P}^U$, $f(p^j) = x_j$.*

Proposition 13. *Let $U \leq S_h \times S_n \times \{id\}$ be regular, $\mu \in \mathbb{N} \cap (h/2, h]$ and $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$. Then*

$$|\mathfrak{F}^U_\mu| = \prod_{j\in\mathcal{P}^U} \left|D_\mu(p^j)\right|.$$

Theorem 14. *Let $U \leq S_h \times S_n \times \{id\}$ be regular and $\mu \in \mathbb{N} \cap (h/2, h]$. Then $\mathfrak{F}^U_\mu \neq \varnothing$ if and only if $\mu \geq \mu_G$.*

Proof. Assume first $\mu \geq \mu_G$. Then for every $p \in \mathcal{P}$, by Proposition 1, we get $D_\mu(p) \neq \varnothing$. That implies $\mathfrak{F}^U_\mu \neq \varnothing$, using Proposition 13. Assume now that $\mathfrak{F}^U_\mu \neq \varnothing$. Then, in particular, $\mathfrak{F}_\mu \neq \varnothing$ and so $\mu \geq \mu_G$ by (8). □

8 Existence results: the case $U \not\leq S_h \times S_n \times \{id\}$

In this section, we analyze the sets $\mathfrak{F}^U$, $\mathfrak{F}^U_{\min}$ and $\mathfrak{F}^U_\mu$, where U is a regular subgroup of G not included in $S_h \times S_n \times \{id\}$. Note that the most easy example of such a kind of regular subgroup is $U = \{id\} \times \{id\} \times \Omega$. This is a consequence of the fact that there exists no $p \in \mathcal{P}$ such that $p^{(id,id,\rho_0)} = p$ and thus, for all $p \in \mathcal{P}$, we have $\mathrm{Stab}_U(p) = \{id\} \times \{id\} \times \{id\}$.

To begin with, observe that, for every $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in U$, we have

$$\mathrm{Stab}_U(p^{(\varphi,\psi,\rho)}) = (\varphi, \psi, \rho)\mathrm{Stab}_U(p)(\varphi, \psi, \rho)^{-1}.$$

This implies that if V is a normal subgroup of U and $p \in \mathcal{P}$, then $\mathrm{Stab}_U(p) \leq V$ if and only if we have $\mathrm{Stab}_U(p^{(\varphi,\psi,\rho)}) \leq V$ for all $(\varphi, \psi, \rho) \in U$. Being $S_h \times \{id\} \times \{id\}$ normal in G, by an elementary group theory result, we have that $U \cap (S_h \times \{id\} \times \{id\})$ is normal in U. Thus the above argument guarantees that, for every $p \in \mathcal{P}$, exactly one the two following conditions holds true:

- for every $p' \in p^U$, $\mathrm{Stab}_U(p') \leq S_h \times \{id\} \times \{id\}$;
- for every $p' \in p^U$, $\mathrm{Stab}_U(p') \not\leq S_h \times \{id\} \times \{id\}$.

Given now U a regular subgroup of G, define

$$\mathcal{P}_1^U = \left\{ j \in \mathcal{P}^U : \forall p \in j, \, \mathrm{Stab}_U(p) \leq S_h \times \{id\} \times \{id\} \right\},$$

$$\mathcal{P}_2^U = \left\{ j \in \mathcal{P}^U : \forall p \in j, \, \mathrm{Stab}_U(p) \not\leq S_h \times \{id\} \times \{id\} \right\}.$$

Of course, $\mathcal{P}_1^U \cup \mathcal{P}_2^U = \mathcal{P}^U$ and $\mathcal{P}_1^U \cap \mathcal{P}_2^U = \varnothing$. In particular $\mathcal{P}_1^U, \mathcal{P}_2^U$ cannot be both empty. Moreover we can write

$$(p^j)_{j \in \mathcal{P}^U} = ((p^j)_{j \in \mathcal{P}_1^U}, (p^j)_{j \in \mathcal{P}_2^U}).$$

8.1 The set $\mathfrak{F}^U$

Proposition 15. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$. Consider $(p^j)_{j \in \mathcal{P}^U} \in \mathfrak{S}(U)$ and fix $(\varphi_*, \psi_*, \rho_0) \in U$. For every $j \in \mathcal{P}_1^U$, let*

$$(y_j, z_j) \in \{(y, z) \in N \times N : z \neq \psi_*(y)\},$$

and, for every $j \in \mathcal{P}_2^U$, let

$$x_j \in \{x \in N : x \neq \psi_j(x)\},$$

where ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then there exists a unique $f \in \mathfrak{F}^U$ such that $f(p^j) = y_j$ and $f(p^{j\,(\varphi_*, \psi_*, \rho_0)}) = z_j$ for all $j \in \mathcal{P}_1^U$, and $f(p^j) = x_j$ for all $j \in \mathcal{P}_2^U$.

Proof. Given $j \in \mathcal{P}_2^U$, consider the set $K^U(p^j) = \left\{ \sigma \in S_n : \psi_j = \sigma \rho_0 \sigma^{-1} \right\}$. Since U is regular, $K^U(p^j)$ is nonempty so that we can choose an element σ_j in $K^U(p^j)$. Note that, for every $j \in \mathcal{P}_2^U$ and $(\varphi, \psi, \rho) \in \mathrm{Stab}_U(p^j)$, we then have $\psi = \sigma_j \rho \sigma_j^{-1}$. Let us consider then $f \in \mathfrak{F}$ defined as follows. For $p \in \mathcal{P}$, find the unique $j \in \mathcal{P}^U$ such that $p \in j$. Then consider $(\varphi, \psi, \rho) \in U$ such that $p = p^{j\,(\varphi, \psi, \rho)}$ and let

$$f(p) = \begin{cases} \psi(y_j) & \text{if } j \in \mathcal{P}_1^U \text{ and } \rho = id \\ \psi \psi_*^{-1}(z_j) & \text{if } j \in \mathcal{P}_1^U \text{ and } \rho = \rho_0 \\ \psi \sigma_j \rho \sigma_j^{-1}(x_j) & \text{if } j \in \mathcal{P}_2^U \end{cases}$$

We are going to prove that f satisfies all the desired properties.

Let us prove at first that f is well defined. Consider $j \in \mathcal{P}^U$ and let $(\varphi_1, \psi_1, \rho_1), (\varphi_2, \psi_2, \rho_2) \in U$ such that $p^{j\,(\varphi_1, \psi_1, \rho_1)} = p^{j\,(\varphi_2, \psi_2, \rho_2)}$. We prove that $f(p^{j\,(\varphi_1, \psi_1, \rho_1)}) = f(p^{j\,(\varphi_2, \psi_2, \rho_2)})$. To start with, note that (10) implies $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1, \rho_2^{-1}\rho_1) \in \mathrm{Stab}_U(p^j)$.

- If $j \in \mathcal{P}_1^U$, then $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1, \rho_2^{-1}\rho_1) \in \mathrm{Stab}_U(p^j)$ implies $\rho_2 = \rho_1$ and $\psi_1 = \psi_2$. As a consequence, if $\rho_1 = \rho_2 = id$, then $f(p^{j\,(\varphi_1, \psi_1, \rho_1)}) = \psi_1(y_j) = \psi_2(y_j) = f(p^{j\,(\varphi_2, \psi_2, \rho_2)})$, while if $\rho_1 = \rho_2 = \rho_0$, then $f(p^{j\,(\varphi_1, \psi_1, \rho_1)}) = (\psi_1\psi_*^{-1})(z_j) = (\psi_2\psi_*^{-1})(z_j) = f(p^{j\,(\varphi_2, \psi_2, \rho_2)})$.
- If $j \in \mathcal{P}_2^U$, then $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1, \rho_2^{-1}\rho_1) \in \mathrm{Stab}_U(p^j)$ implies $\psi_2^{-1}\psi_1 = \sigma_j \rho_2^{-1}\rho_1 \sigma_j^{-1}$, that is, $\psi_1 \sigma_j \rho_1 \sigma_j^{-1} = \psi_2 \sigma_j \rho_2 \sigma_j^{-1}$, as for each $\rho \in \Omega$ we have $\rho = \rho^{-1}$. We then get

$$f(p^{j\,(\varphi_1, \psi_1, \rho_1)}) = \psi_1 \sigma_j \rho_1 \sigma_j^{-1}(x_j) = \psi_2 \sigma_j \rho_2 \sigma_j^{-1}(x_j) = f(p^{j\,(\varphi_2, \psi_2, \rho_2)}).$$

Let us prove now that $f \in \mathfrak{F}^U$. Fix $p \in \mathcal{P}$ and $(\varphi, \psi, \rho) \in U$. Of course, there exist $j \in \mathcal{P}^U$ and $(\varphi_1, \psi_1, \rho_1) \in U$ such that $p = p^{j\,(\varphi_1, \psi_1, \rho_1)}$.

- Assume $j \in \mathcal{P}_1^U$ and $\rho_1 = id$, so that $f(p) = f(p^{j\,(\varphi_1,\psi_1,\rho_1)}) = \psi_1(y_j)$. If $\rho = id$, then

$$f(p^{(\varphi,\psi,\rho)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,id)}) = \psi\psi_1(y_j) = \psi f(p).$$

  while if $\rho = \rho_0$, then

$$f(p^{(\varphi,\psi,\rho)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,\rho_0)}) = \psi\psi_1\psi_*^{-1}(z_j) \neq \psi\psi_1(y_j) = \psi f(p),$$

  since $z_j \neq \psi_*(y_j)$.

- Assume $j \in \mathcal{P}_1^U$ and $\rho_1 = \rho_0$ so that $f(p) = f(p^{j\,(\varphi_1,\psi_1,\rho_0)}) = \psi_1\psi_*^{-1}(z_j)$. If $\rho = id$, then

$$f(p^{(\varphi,\psi,\rho)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,\rho_0)}) = \psi\psi_1\psi_*^{-1}(z_j) = \psi f(p),$$

  while if $\rho = \rho_0$, then

$$f(p^{(\varphi,\psi,\rho)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,id)}) = \psi\psi_1(y_j) \neq \psi\psi_1\psi_*^{-1}(z_j) = \psi f(p),$$

  since $z_j \neq \psi_*(y_j)$.

- Assume $j \in \mathcal{P}_2^U$. Then

$$f(p) = f(p^{j\,(\varphi_1,\psi_1,\rho_1)}) = \psi_1\sigma_j\rho_1\sigma_j^{-1}(x_j),$$

  and

$$f(p^{(\varphi,\psi,\rho)}) = f(p^{j\,(\varphi\varphi_1,\psi\psi_1,\rho\rho_1)}) = \psi\psi_1\sigma_j\rho\rho_1\sigma_j^{-1}(x_j).$$

  As a consequence, if $\rho = id$, we get $f(p^{(\varphi,\psi,\rho)}) = \psi f(p)$. If instead $\rho = \rho_0$, we have that $f(p^{(\varphi,\psi,\rho)}) \neq \psi f(p)$ if and only if $\psi\psi_1\sigma_j\rho_0\rho_1\sigma_j^{-1}(x_j) \neq \psi\psi_1\sigma_j\rho_1\sigma_j^{-1}(x_j)$ if and only if $\sigma_j\rho_0\sigma_j^{-1}(x_j) \neq x_j$. However, the last relation holds true since $\sigma_j\rho_0\sigma_j^{-1} = \psi_j$ and $x_j \in \{x \in N : x \neq \psi_j(x)\}$.

Note also that the definition of f immediately implies that $f(p^j) = y_j$ and $f(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = z_j$ for all $j \in \mathcal{P}_1^U$, and $f(p^j) = x_j$ for all $j \in \mathcal{P}_2^U$.

In order to prove uniqueness, let $f' \in \mathfrak{F}^U$ such that $f'(p^j) = y_j$ and $f'(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = z_j$ for all $j \in \mathcal{P}_1^U$, and $f'(p^j) = x_j$ for all $j \in \mathcal{P}_2^U$, and prove that, for every $p \in \mathcal{P}$, $f'(p) = f(p)$. Given $p \in \mathcal{P}$, let $j \in \mathcal{P}^U$ and $(\varphi_1,\psi_1,\rho_1) \in U$ such that $p = p^{j\,(\varphi_1,\psi_1,\rho_1)}$. Thus, we have that:

- if $j \in \mathcal{P}_1^U$ and $\rho_1 = id$, then $f'(p) = f'(p^{j\,(\varphi_1,\psi_1,id)}) = \psi_1(y_j) = f(p^{j\,(\varphi_1,\psi_1,id)}) = f(p)$,

- if $j \in \mathcal{P}_1^U$ and $\rho_1 = \rho_0$, then by (10), and since f and f' are both U-symmetric SCF, we have

$$f'(p) = f'(p^{j\,(\varphi_1,\psi_1,\rho_0)}) = f'\left(\left(p^{j\,(\varphi_*,\psi_*,\rho_0)}\right)^{(\varphi_1\varphi_*^{-1},\psi_1\psi_*^{-1},id)}\right) = \psi_1\psi_*^{-1}f'(p^{j\,(\varphi_*,\psi_*,\rho_0)})$$

$$= \psi_1\psi_*^{-1}f(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = f\left(\left(p^{j\,(\varphi_*,\psi_*,\rho_0)}\right)^{(\varphi_1\varphi_*^{-1},\psi_1\psi_*^{-1},id)}\right) = f(p^{j\,(\varphi_1,\psi_1,\rho_0)}) = f(p).$$

- if $j \in \mathcal{P}_2^U$, then $f'(p) = f'(p^{j\,(\varphi_1,\psi_1,\rho_1)}) = \psi_1\sigma_j\rho_1\sigma_j^{-1}(x_j) = f(p^{j\,(\varphi_1,\psi_1,\rho_0)}) = f(p)$,

so that the proof is completed. $\square$

Proposition 16. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$. Let $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and $(\varphi_*, \psi_*, \rho_0) \in U$ be fixed. If $\mathcal{P}_1^U \neq \varnothing$ define the set*

$$A^1 = \times_{j\in\mathcal{P}_1^U}\{(y,z) \in N \times N : z \neq \psi^*(y)\},$$

and, if $\mathcal{P}_2^U \neq \varnothing$ define the set

$$A^2 = \times_{j\in\mathcal{P}_2^U}\{x \in N : \psi_j(x) \neq x\},$$

where, for every $j \in \mathcal{P}_2^U$, ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then

$$|\mathfrak{F}^U| = \begin{cases} |A^1| & \text{if } \mathcal{P}_2^U = \varnothing \\ |A^2| & \text{if } \mathcal{P}_1^U = \varnothing \\ |A^1| \cdot |A^2| & \text{if } \mathcal{P}_1^U \neq \varnothing \text{ and } \mathcal{P}_2^U \neq \varnothing \end{cases}$$

In particular, $\mathfrak{F}^U \neq \varnothing$.

Proof. Assume first that $\mathcal{P}_1^U$ and $\mathcal{P}_2^U$ are both nonempty. Consider the function $\Psi : A^1 \times A^2 \to \mathfrak{F}^U$ which associates to every $((y_j,z_j)_{j\in\mathcal{P}_1^U}, (x_j)_{j\in\mathcal{P}_2^U}) \in A^1 \times A^2$, the unique $f \in \mathfrak{F}^U$ defined in Proposition 15. Note that the function Ψ is obviously injective. Consider now $f \in \mathfrak{F}^U$, and note that,

$$\left((f(p^j), f(p^{j(\varphi_*,\psi_*,\rho_0)}))_{j\in\mathcal{P}_1^U}, (f(p_j))_{j\in\mathcal{P}_2^U}\right) \in A^1 \times A^2,$$

and

$$\Psi\left((f(p^j), f(p^{j(\varphi_*,\psi_*,\rho_0)}))_{j\in\mathcal{P}_1^U}, (f(p_j))_{j\in\mathcal{P}_2^U}\right) = f.$$

That proves that Ψ is surjective. Then Ψ is bijective and the equality $|\mathfrak{F}^U| = |A^1||A^2|$ follows. The case $\mathcal{P}_1^U = \varnothing$ and the case $\mathcal{P}_2^U = \varnothing$ are similar and then omitted.

In order to show that $\mathfrak{F}^U \neq \varnothing$, simply note that $|A^1| = |\mathcal{P}_1^U||\{(y,z) \in N \times N : z \neq \psi_*(y)\}| = |\mathcal{P}_1^U|n(n-1)$ and, since ψ_j has at most one fixed point, $|\{x \in N : x \neq \psi_j(x)\}| \geq n-1$ for all $j \in \mathcal{P}_2^U$, so that $|A^2| \geq |\mathcal{P}_2^U|(n-1)$. □

8.2 The set $\mathfrak{F}^U_{\min}$

Proposition 17. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$. Consider $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and fix $(\varphi_*, \psi_*, \rho_0) \in U$. For every $j \in \mathcal{P}_1^U$, let*

$$(y_j, z_j) \in \left\{(y,z) \in D_{\mu(p^j)}(p^j) \times D_{\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)})}(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\},$$

and, for every $j \in \mathcal{P}_2^U$, let

$$x_j \in \left\{x \in D_{\mu(p^j)}(p^j) : \psi_j(x) \neq x\right\},$$

where ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then there exists a unique $f \in \mathfrak{F}^U_{\min}$ such that $f(p^j) = y_j$ and $f(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = z_j$ for all $j \in \mathcal{P}_1^U$, and $f(p^j) = x_j$ for allj $\in \mathcal{P}_2^U$.

Proof. Let us consider $f \in \mathfrak{F}^U$ defined in Proposition 15. We complete the proof showing that $f \in \mathfrak{F}_{\min}$, as well. Consider then $p \in \mathcal{P}$ and prove that $f(p) \in D_{\mu(p)}(p)$. Let $j \in \mathcal{P}^U$ and $(\varphi_1, \psi_1, \rho_1) \in U$ such that $p = p^{j\,(\varphi_1,\psi_1,\rho_1)}$. We analyse three different cases: in each of them Corollary 6 proves crucial. Assume first that $j \in \mathcal{P}_1^U$ and $\rho_1 = id$. Then

$$f(p) = f(p^{j(\varphi_1,\psi_1,id)}) = \psi_1 f(p^j) = \psi_1(y_j) \in \psi_1(D_{\mu(p^j)}(p^j)) = D_{\mu(p)}(p)$$

Assume now that $j \in \mathcal{P}_1^U$ and $\rho_1 = \rho_0$. Then, by (10),

$$\begin{aligned} f(p) &= f(p^{j(\varphi_1,\psi_1,\rho_0)}) = f((p^{j(\varphi_*,\psi_*,\rho_0)})^{(\varphi_1\varphi_*^{-1},\psi_1\psi_*^{-1},id)}) \\ &= \psi_1\psi_*^{-1}(f(p^{j(\varphi_*,\psi_*,\rho_0)})) = \psi_1\psi_*^{-1}(z_j) \in \psi_1\psi_*^{-1}\left(D_{\mu(p^{j\,(\varphi^*,\psi^*,\rho_0)})}(p^{j\,(\varphi^*,\psi^*,\rho_0)})\right) \\ &= D_{\mu((p^{j(\varphi_*,\psi_*,\rho_0)})^{(\varphi_1\varphi_*^{-1},\psi_1\psi_*^{-1},id)})}((p^{j(\varphi_*,\psi_*,\rho_0)})^{(\varphi_1\varphi_*^{-1},\psi_1\psi_*^{-1},id)}) = D_{\mu(p)}(p) \end{aligned}$$

Assume finally that $j \in \mathcal{P}_2^U$ and consider $(\varphi_2, \psi_2, \rho_0) \in U$ such that $p^{j(\varphi_2,\psi_2,\rho_0)} = p^j$. Then

$$p = p^{j(\varphi_1,\psi_1,\rho_0)} = (p^{j(\varphi_2,\psi_2,\rho_0)})^{(\varphi_1\varphi_2^{-1},\psi_1\psi_2^{-1},id)} = p^{j(\varphi_1\varphi_2^{-1},\psi_1\psi_2^{-1},id)},$$

and

$$\begin{aligned} f(p) &= f(p^{j(\varphi_1\varphi_2^{-1},\psi_1\psi_2^{-1},id)}) = \psi_1\psi_2^{-1}(f(p^j)) = \psi_1\psi_2^{-1}(x_j) \in \psi_1\psi_2^{-1}\left(D_{\mu(p^j)}(p^j)\right) \\ &= D_{\mu(p^{j(\varphi_1\varphi_2^{-1},\psi_1\psi_2^{-1},id)})}(p^{j(\varphi_1\varphi_2^{-1},\psi_1\psi_2^{-1},id)}) = D_{\mu(p)}(p). \end{aligned}$$

□

Proposition 18. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$. Consider $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and fix $(\varphi_*, \psi_*, \rho_0) \in U$. If $\mathcal{P}_1^U \neq \varnothing$ define the set*

$$A^1_{\min} = \times_{j\in\mathcal{P}_1^U}\left\{(y,z) \in D_{\mu(p^j)}(p^j) \times D_{\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)})}(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\},$$

and, if $\mathcal{P}_2^U \neq \varnothing$ define the set

$$A^2_{\min} = \times_{j\in\mathcal{P}_2^U}\left\{x \in D_{\mu(p^j)}(p^j) : \psi_j(x) \neq x\right\},$$

where, for every $j \in \mathcal{P}_2^U$, ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then

$$|\mathfrak{F}^U_{\min}| = \begin{cases} |A^1_{\min}| & \textit{if } \mathcal{P}_2^U = \varnothing \\ |A^2_{\min}| & \textit{if } \mathcal{P}_1^U = \varnothing \\ |A^1_{\min}| \cdot |A^2_{\min}| & \textit{if } \mathcal{P}_1^U \neq \varnothing \textit{ and } \mathcal{P}_2^U \neq \varnothing \end{cases}$$

Proof. Assume first that $\mathcal{P}_1^U$ and $\mathcal{P}_2^U$ are both nonempty. Consider the bijective function Ψ defined in the proof of Proposition 16. We complete the proof showing that $\Psi\left(A^1_{\min} \times A^2_{\min}\right) = \mathfrak{F}^U_{\min}$.

By Proposition 17, we know that $\Psi\left(A^1_{\min} \times A^2_{\min}\right) \subseteq \mathfrak{F}^U_{\min}$. In order to prove the opposite inequality, simply note that, given $f \in \mathfrak{F}^U_{\min}$, we have

$$\left((f(p^j), f(p^{j(\varphi_*,\psi_*,\rho_0)}))_{j\in\mathcal{P}_1^U}, (f(p_j))_{j\in\mathcal{P}_2^U}\right) \in A^1_{\min} \times A^2_{\min},$$

and

$$\Psi\left((f(p^j), f(p^{j(\varphi_*,\psi_*,\rho_0)}))_{j\in\mathcal{P}_1^U}, (f(p_j))_{j\in\mathcal{P}_2^U}\right) = f.$$

The case $\mathcal{P}_1^U = \varnothing$ and the case $\mathcal{P}_2^U = \varnothing$ are similar and then omitted. □

Proposition 18 does not imply the nonemptyness of $\mathfrak{F}^U_{\min}$. The following crucial result, whose technical proof can be found in Section 10, gives necessary and sufficient conditions for being $\mathfrak{F}^U_{\min} \neq \varnothing$. In what follows, it proves fundamental the set

$$\begin{aligned} T = \quad & \{(h,n) \in \mathbb{N}^2 : 2 \leq h \leq 3,\, n \geq 2\} \cup \\ & \{(h,n) \in \mathbb{N}^2 : h \geq 2,\, 2 \leq n \leq 3\} \cup \\ & \{(4,4),(5,4),(7,4),(5,5)\}. \end{aligned}$$

Theorem 19. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$. Then $\mathfrak{F}^U_{\min} \neq \varnothing$ if and only if $(h,n) \in T$.*

Note that the above conditions are of purely arithmetic flavour. In particular, they do not depend on U. Thus, having in mind also Proposition 11, we immediately get the following result.

Corollary 20. *The following facts are equivalent:*

i) $\mathfrak{F}^{\{id\}\times\{id\}\times\Omega}_{\min} \neq \varnothing$*;*

ii) $\mathfrak{F}^U_{\min} \neq \varnothing$ *for all* $U \leq G$ *regular;*

iii) there exists $U \leq G$ *regular with* $U \not\leq S_h \times S_n \times \{id\}$ *such that* $\mathfrak{F}^U_{\min} \neq \varnothing$*.*

8.3 The set $\mathfrak{F}^U_\mu$

Propositions 21 and 22 below can be proved simply mimicking the proof of Propositions 17 and 18 and using Lemma 5.

Proposition 21. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$, and $\mu \in \mathbb{N} \cap (h/2, h]$. Consider $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and fix $(\varphi_*, \psi_*, \rho_0) \in U$. For every $j \in \mathcal{P}^U_1$, let*

$$(y_j, z_j) \in \left\{(y,z) \in D_\mu(p^j) \times D_\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\},$$

and, for every $j \in \mathcal{P}^U_2$, let

$$x_j \in \left\{x \in D_\mu(p^j) : \psi_j(x) \neq x\right\},$$

where ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then there exists a unique $f \in \mathfrak{F}^U_\mu$ such that $f(p^j) = y_j$ and $f(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = z_j$ for all $j \in \mathcal{P}^U_1$, and $f(p^j) = x_j$ for all $j \in \mathcal{P}^U_2$.

Proposition 22. *Let $U \leq G$ be regular and such that $U \not\leq S_h \times S_n \times \{id\}$, and $\mu \in \mathbb{N} \cap (h/2, h]$. Consider $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ and fix $(\varphi_*, \psi_*, \rho_0) \in U$. If $\mathcal{P}^U_1 \neq \varnothing$ define the set*

$$A^1_\mu = \times_{j\in\mathcal{P}^U_1} \left\{(y,z) \in D_\mu(p^j) \times D_\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\},$$

and, if $\mathcal{P}^U_2 \neq \varnothing$ define the set

$$A^2_\mu = \times_{j\in\mathcal{P}^U_2} \left\{x \in D_\mu(p^j) : \psi_j(x) \neq x\right\},$$

where, for every $j \in \mathcal{P}^U_2$, ψ_j is the unique element in S_n such that

$$\mathrm{Stab}_U(p^j) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_j\} \times \{\rho_0\}).$$

Then

$$|\mathfrak{F}^U_\mu| = \begin{cases} |A^1_\mu| & \text{if } \mathcal{P}^U_2 = \varnothing \\ |A^2_\mu| & \text{if } \mathcal{P}^U_1 = \varnothing \\ |A^1_\mu| \cdot |A^2_\mu| & \text{if } \mathcal{P}^U_1 \neq \varnothing \text{ and } \mathcal{P}^U_2 \neq \varnothing \end{cases}$$

Proposition 22 does not imply the nonemptyness of $\mathfrak{F}_\mu^U$. The following result provides necessary and sufficient conditions for being $\mathfrak{F}_\mu^U \neq \varnothing$. Its proof can be found in Section 10.

Theorem 23. *Let $U \le G$ be regular, such that $U \not\le S_h \times S_n \times \{id\}$, and $\mu \in \mathbb{N} \cap (h/2, h]$. Then $\mathfrak{F}_\mu^U \neq \varnothing$ if and only if $\mu \ge \mu_G$.*

9 Existence results: subcommittees and subclasses

When we are dealing with symmetries associated with subcommittees and subclasses, the following hold true.

Theorem 24. *Let $B = \{B_j\}_{j=1}^s$ be a partition of H, and $C = \{C_k\}_{k=1}^s$ be a partition of N with $|C_{k^*}| = \max\{|C_k|\}_{k=1}^s$. Then the two following conditions are equivalent:*

i) $\mathfrak{F}_{\min}^{V(B)\times W(C)\times\{id\}} \neq \varnothing$;

ii) $\gcd\left(\gcd(|B_j|)_{j=1}^r, |C_{k^*}|!\right) = 1$.

Proof. Let us prove first that *ii)* implies *i)*. Indeed, by Theorem 4, *ii)* implies that $V(B) \times W(C) \times \{id\}$ is regular and then we can apply Proposition 11.

Let us prove now that *i)* implies *ii)*. Assume there exists $f \in \mathfrak{F}_{\min}^{V(B)\times W(C)\times\{id\}}$ and assume by contradiction that *ii)* does not hold true. Then there exists a prime π which divides $\gcd\left(\gcd(|B_j|)_{j=1}^r, |C_{k^*}|!\right)$. Then, for every $j \in \{1,\dots,r\}$, $\pi \mid |B_j|$ and $\pi \le |C_{k^*}|$. Let us consider then distinct alternatives $\widehat{x}_1,\dots,\widehat{x}_\pi \in C_{k^*}$ and denote by $y_1,\dots,y_{n-\pi}$ the remaining alternatives. For every $j \in \{1,\dots,r\}$, let $b_j = |B_j|$ and $i_1^j,\dots,i_{b_j}^j$ be the list of all the elements in B_j. Let $\widehat{\psi} = (\widehat{x}_1 \dots \widehat{x}_\pi) \in S_n$ and note that, in fact, $\widehat{\psi} \in W(C)$. Let us consider also

$$\widehat{\varphi} = (i_1^1,\dots,i_{b_1}^1)(i_1^2,\dots,i_{b_2}^2)\dots(i_1^r,\dots,i_{b_r}^r) \in S_h,$$

and note that $\widehat{\varphi} \in V(B)$. Consider then the preference

$$p_0 = [\widehat{x}_1,\dots,\widehat{x}_\pi, y_1,\dots,y_{n-\pi}]^T \in \mathcal{L}(N),$$

and the preference profile p such that, for every $j \in \{1,\dots,r\}$ and $s \in \{1,\dots,b_j\}$, $p_{i_s^j} = \widehat{\psi}^{s-1} p_0$. A simple check shows that $f(p) \in D_{\mu(p)}(p) \subseteq \{\widehat{x}_1,\dots,\widehat{x}_\pi\}$ and that that $p^{(\widehat{\varphi},\widehat{\psi},id)} = p$. Then

$$f(p) = f\left(p^{(\widehat{\varphi},\widehat{\psi},id)}\right) = \widehat{\psi} f(p).$$

Then $f(p)$ is a fixed point of $\widehat{\psi}$ and that is a contradiction as $\widehat{\psi}$ has no fixed point in the set $\{\widehat{x}_1,\dots,\widehat{x}_\pi\}$. ☐

Corollary 25. $\mathfrak{F}_{\min}^{S_h\times S_n\times\{id\}} \neq \varnothing$ *if and only if* $\gcd(h, n!)$=*1.*

Proof. Note that $S_h = V(\{H\})$ and $S_n = W(\{N\})$, and apply Theorem 24. ☐

Theorem 26. *Let $B = \{B_j\}_{j=1}^s$ be a partition of H, $C = \{C_k\}_{k=1}^s$ be a partition of N with $|C_{k^*}| = \max\{|C_k|\}_{k=1}^s$, and $\mu \in \mathbb{N} \cap (h/2, h]$. Then the two following conditions are equivalent:*

i) $\mathfrak{F}_\mu^{V(B)\times W(C)\times\{id\}} \neq \varnothing$;

ii) $\gcd\left(\gcd(|B_j|)_{j=1}^r, |C_{k^*}|!\right) = 1$ *and* $\mu \ge \mu_G$.

Proof. Let us prove first that *ii*) implies *i*). Indeed, by Theorem 4, *ii*) implies that $V(B)\times W(C)\times \{id\}$ is regular and then we can apply Theorem 14. The proof that *i*) implies *ii*) can be made following the same argument as in the proof of Theorem 24 and recalling (8). □

Corollary 27. *Let* $\mu\in\mathbb{N}\cap(h/2,h]$. $\mathfrak{F}_{\mu}^{S_h\times S_n\times\{id\}}\neq\varnothing$ *if and only if* $\gcd(h,n!)=1$ *and* $\mu\geq\mu_G$.

Proof. Note that $S_h=V(\{H\})$ and $S_n=W(\{N\})$, and apply Theorem 26. □

Theorem 28. *Let* $B=\{B_j\}_{j=1}^s$ *be a partition of* H*, and* $C=\{C_k\}_{k=1}^s$ *be a partition of* N *with* $|C_{k^*}|=\max\{|C_k|\}_{k=1}^s$*. Then the two following conditions are equivalent:*

i) $\mathfrak{F}_{\min}^{V(B)\times W(C)\times\Omega}\neq\varnothing$;

ii) $\gcd\left(\gcd(|B_j|)_{j=1}^r,\, \mathrm{lcm}(2,|C_{k^*}|!)\right)=1$ *and* $(h,n)\in T$.

Proof. Let us prove first that *ii*) implies *i*). Indeed, by Theorem 4, *ii*) implies that $V(B)\times W(C)\times\Omega$ is regular and then we can apply Theorem 19.

Let us prove now that *i*) implies *ii*). Assume there exists $f\in\mathfrak{F}_{\min}^{V(B)\times W(C)\times\Omega}$ and assume by contradiction that *ii*) does not hold true. Then we have to discuss two cases.

Assume first that there exists a prime π which divides $\gcd\left(\gcd(|B_j|)_{j=1}^r,\, \mathrm{lcm}(2,|C_{k^*}|!)\right)$. If $\pi\geq 3$, then we have that, for every $j\in\{1,\dots,r\}$, $\pi\mid|B_j|$ and $\pi\leq|C_{k^*}|$: thus, we can work exactly as in the proof of Theorem 24 and find the contradiction. If instead $\pi=2$, then we need a different argument. For every $j\in\{1,\dots,r\}$, let $b_j=|B_j|$ and $i_1^j,\dots,i_{b_j}^j$ be the list of all the elements in B_j. Let us consider then

$$\widehat{\varphi}=(i_1^1,\dots,i_{b_1}^1)(i_1^2,\dots,i_{b_2}^2)\dots(i_1^r,\dots,i_{b_r}^r)\in S_h,$$

and note that in fact $\widehat{\varphi}\in V(B)$. Fix $p_0\in\mathcal{L}(N)$ and consider the preference profile p such that, for every $j\in\{1,\dots,r\}$ and $s\in\{1,\dots,b_j\}$, $p_{i_s^j}=p_0\rho_0^{s-1}$. Of course, we have that $id\in W(C)$, and a simple check shows that $p^{(\widehat{\varphi},id,\rho_0)}=p$. Then

$$f(p)=f\left(p^{(\widehat{\varphi},id,\rho_0)}\right)\neq f(p),$$

and the contradiction is found again.

Assume now that $\gcd\left(\gcd(|B_j|)_{j=1}^r,\, \mathrm{lcm}(2,|C_{k^*}|!)\right)=1$ but $(h,n)\notin T$. By Theorem 4, we have that $V(B)\times W(C)\times\Omega$ is regular and, by Theorem 19, we get $(h,n)\in T$, a contradiction. □

Corollary 29. $\mathfrak{F}_{\min}^{G}\neq\varnothing$ *if and only if one of the three following conditions hold true:*

i) $n=2$ *and* h *is odd;*

ii) $n=3$ *and* h *is odd and not divisible by* 3*;*

iii) $n=4$ *and* $h\in\{5,7\}$.

Proof. From $G=V(\{H\})\times W(\{N\})\times\Omega$ and Theorem 28, we get that $\mathfrak{F}_{\min}^{G}\neq\varnothing$ is equivalent to $\gcd(h,n!)=1$ and $(h,n)\in T$. It then immediate to show that the pair (h,n) satisfies $\gcd(h,n!)=1$ and $(h,n)\in T$ if and only if one among *i*), *ii*) and *iii*) holds true. □

Theorem 30. *Let* $B=\{B_j\}_{j=1}^s$ *be a partition of* H*,* $C=\{C_k\}_{k=1}^s$ *be a partition of* N *with* $|C_{k^*}|=\max\{|C_k|\}_{k=1}^s$*, and* $\mu\in\mathbb{N}\cap(h/2,h]$*. Then the two following conditions are equivalent:*

i) $\mathfrak{F}_{\mu}^{V(B)\times W(C)\times\Omega}\neq\varnothing$;

ii) $\gcd\left(\gcd(|B_j|)_{j=1}^r,\, \mathrm{lcm}(2,|C_{k^*}|!)\right)=1$ *and* $\mu\geq\mu_G$.

Proof. Let us prove first that *ii*) implies *i*). Indeed, by Theorem 4, *ii*) implies that $V(B)\times W(C)\times\Omega$ is regular and then we can apply Theorem 23. The proof that *i*) implies *ii*) can be made following the same argument as in the proof of Theorem 28 and recalling (8). □

10 Proof of Theorems 19 and 23

10.1 Preliminaries for Theorem 19

Lemma 31. *Let $U \le G$ be regular and such that $U \not\le S_h \times S_n \times \{id\}$. Let $p \in \mathcal{P}$ and $(\varphi_*, \psi_*, \rho_0) \in U$ such that, for every $x \in N$, we do not have $D_{\mu(p)}(p) = \{x\}$ and $D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)}) = \{\psi_*(x)\}$. Then*

$$\left\{(y,z) \in D_{\mu(p)}(p) \times D_{\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)})}(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\} \neq \varnothing,$$

and if $\mathrm{Stab}_U(p) \not\le S_h \times \{id\} \times \{id\}$, *then*

$$\left\{x \in D_{\mu(p)}(p) : \psi_p(x) \neq x\right\} \neq \varnothing,$$

where ψ_p is the unique element in S_n such that

$$\mathrm{Stab}_U(p) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_p\} \times \{\rho_0\}).$$

Proof. The first part of the statement is an immediate consequence of the assumptions. Assume now that there exists $(\varphi_1, \psi_1, \rho_0) \in \mathrm{Stab}_U(p)$ and assume by contradiction that

$$\left\{x \in D_{\mu(p)}(p) : \psi_p(x) \neq x\right\} = \varnothing.$$

Then, for every $x \in D_{\mu(p)}(p)$, we have that $\psi_p(x) = x$. If n is even, then ψ_p has no fixed point and we get a contradiction. If instead n is odd, we have that ψ_p has a unique fixed point x_0 and then $D_{\mu(p)}(p) = \{x_0\}$. Now, by the regularity of U, we get $\psi_1 = \psi_p$ and thus $\psi_1(x_0) = x_0$. By Lemma 7 and Corollary 6, we then have that

$$D_{\mu(p^{\varphi_*,\psi_*,\rho_0})}(p^{\varphi_*,\psi_*,\rho_0}) = D_{\mu(p)}(p^{\varphi_*,\psi_*,\rho_0}) = D_{\mu(p)}\left(\left(p^{(\varphi_1,\psi_1,\rho_0)}\right)^{(\varphi_*\varphi_1^{-1},\psi_*\psi_1^{-1},id)}\right)$$

$$= D_{\mu(p)}\left(p^{(\varphi_*\varphi_1^{-1},\psi_*\psi_1^{-1},id)}\right) = \psi_*\psi_1^{-1}\left(D_{\mu(p)}(p)\right) = \psi_*\psi_1\left(\{x_0\}\right) = \{\psi^*(x_0)\},$$

so that a contradiction is found. □

Lemma 32. *Let $p \in \mathcal{P}$ and $(\varphi_*, \psi_*, \rho_0) \in U$. Then*

$$D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)}) = \psi_*\left(D_{\mu(p^{(id,id,\rho_0)})}(p^{(id,id,\rho_0)})\right).$$

Proof. Indeed, by (10) and Corollary 6, we have

$$D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)}) = D_{\mu\left((p^{(id,id,\rho_0)})^{(\varphi_*,\psi_*,id)}\right)}\left(\left(p^{(id,id,\rho_0)}\right)^{(\varphi_*,\psi_*,id)}\right)$$

$$= \psi_*\left(D_{\mu(p^{(id,id,\rho_0)})}\left(p^{(id,id,\rho_0)}\right)\right).$$

□

Lemma 33. *Let $U \le G$ be regular and such that $U \not\le S_h \times S_n \times \{id\}$. Then the following conditions are equivalent:*

i) $\mathfrak{F}^U_{\min} \neq \varnothing$;

ii) *there exists $(\varphi_*, \psi_*, \rho_0) \in U$ such that, for every $p \in \mathcal{P}$ and $x \in N$, we do not have $D_{\mu(p)}(p) = \{x\}$ and $D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)}) = \{\psi_*(x)\}$;*

iii) there exist $(p^j)_{j\in\mathcal{P}^U} \in \mathfrak{S}(U)$ *and* $(\varphi_*, \psi_*, \rho_0) \in U$ *such that, for every* $j \in \mathcal{P}^U$ *and* $x \in N$, *we do not have* $D_{\mu(p^j)}(p^j) = \{x\}$ *and* $D_{\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)})}(p^{j\,(\varphi_*,\psi_*,\rho_0)}) = \{\psi_*(x)\}$;

iv) for every $p \in \mathcal{P}$ *and* $x \in N$, *we do not have* $D_{\mu(p)}(p) = D_{\mu(p^{(id,id,\rho_0)})}(p^{(id,id,\rho_0)}) = \{x\}$.

Proof. Since, by Lemma 32, the equivalence of *ii)* and *iv)* is immediate, it is enough to show that *i), ii), iii)* are equivalent.

i) $\Rightarrow$ *ii)* Let $f \in \mathfrak{F}^U_{\min}$ and assume, by contradiction, that for every $(\varphi_*, \psi_*, \rho_0) \in U$ there exists $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = \{x\}$ and $D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)}) = \{\psi_*(x)\}$. Then, since $f(p) \in D_{\mu(p)}(p)$ and $f(p^{(\varphi_*,\psi_*,\rho_0)}) \in D_{\mu(p^{(\varphi_*,\psi_*,\rho_0)})}(p^{(\varphi_*,\psi_*,\rho_0)})$, we have $f(p) = x$ as well as $f(p^{(\varphi_*,\psi_*,\rho_0)}) = \psi_*(x)$. On the other hand, the U-symmetry requires $f(p^{(\varphi_*,\psi_*,\rho_0)}) \neq \psi_*(f(p)) = \psi_*(x)$, a contradiction.

ii) $\Rightarrow$ *iii)* This implication is clear, because it is enough to choose any representative system.

iii) $\Rightarrow$ *i)* It follows immediately from Lemma 31 and Proposition 18. □

The following lemma immediately implies Theorem 19 via Lemma 33. Its proof is difficult and can be found in Section 10.3.3.

Lemma 34. *The following conditions are equivalent:*

i) for every $p \in \mathcal{P}$ *and* $x \in N$, *we do not have* $D_{\mu(p)}(p) = D_{\mu(p^{(id,id,\rho_0)})}(p^{(id,id,\rho_0)}) = \{x\}$;

ii) $(h, n) \in T$.

10.2 Preliminaries for Theorem 23

In order to prove Theorem 23, note that if $U \leq G$ and $\mu \in \mathbb{N} \cap (h/2, h]$ are such that $\mathfrak{F}_\mu \neq \varnothing$, then, by (8), $\mu \geq \mu_G$. Then we are left with proving that $\mu \geq \mu_G$ implies $\mathfrak{F}^U_\mu \neq \varnothing$. We need several lemmata to get that result. The proofs of Lemmata 35, 36 and 37 below are similar to the ones of Lemmata 31, 32 and 33, and then omitted.

Lemma 35. *Let* $U \leq G$ *be regular and such that* $U \not\leq S_h \times S_n \times \{id\}$, *and* $\mu \in \mathbb{N} \cap (h/2, h]$ *with* $\mu \geq \mu_G$. *Let* $p \in \mathcal{P}$ *and* $(\varphi_*, \psi_*, \rho_0) \in U$ *such that, for every* $x \in N$, *we do not have* $D_\mu(p) = \{x\}$ *and* $D_\mu(p^{(\varphi_*,\psi_*,\rho_0)}) = \{\psi_*(x)\}$. *Then*

$$\left\{(y, z) \in D_\mu(p) \times D_\mu(p^{j\,(\varphi_*,\psi_*,\rho_0)}) : z \neq \psi_*(y)\right\} \neq \varnothing,$$

and if $\mathrm{Stab}_U(p) \not\leq S_h \times \{id\} \times \{id\}$, *then*

$$\{x \in D_\mu(p) : \psi_p(x) \neq x\} \neq \varnothing,$$

where ψ_p *is the unique element in* S_n *such that*

$$\mathrm{Stab}_U(p) \subseteq (S_h \times \{id\} \times \{id\}) \cup (S_h \times \{\psi_p\} \times \{\rho_0\}).$$

Lemma 36. *Let* $p \in \mathcal{P}$, $(\varphi_*, \psi_*, \rho_0) \in U$, *and* $\mu \in \mathbb{N} \cap (h/2, h]$ *with* $\mu \geq \mu_G$. *Then*

$$D_\mu(p^{(\varphi_*,\psi_*,\rho_0)}) = \psi_*\left(D_\mu(p^{(id,id,\rho_0)})\right).$$

Lemma 37. *Let* $U \leq G$ *be regular and such that* $U \not\leq S_h \times S_n \times \{id\}$, *and* $\mu \in \mathbb{N} \cap (h/2, h]$ *with* $\mu \geq \mu_G$. *Then the following conditions are equivalent:*

i) $\mathfrak{F}^U_\mu \neq \varnothing$;

ii) there exists $(\varphi_*, \psi_*, \rho_0) \in U$ such that, for every $p \in \mathcal{P}$ and $x \in N$, we do not have $D_\mu(p) = \{x\}$ and $D_\mu(p^{(\varphi_*, \psi_*, \rho_0)}) = \{\psi_*(x)\}$;

iii) there exist $(p^j)_{j \in \mathcal{P}^U} \in \mathfrak{S}(U)$ and $(\varphi_*, \psi_*, \rho_0) \in U$ such that, for every $j \in \mathcal{P}^U$ and $x \in N$, we do not have $D_\mu(p^j) = \{x\}$ and $D_\mu(p^{j\,(\varphi_*, \psi_*, \rho_0)}) = \{\psi_*(x)\}$;

iv) for every $p \in \mathcal{P}$ and $x \in N$, we do not have $D_\mu(p) = D_\mu(p^{(id, id, \rho_0)}) = \{x\}$.

The following lemma immediately allows to complete the proof of Theorem 23 via Lemma 37. Its proof can be found in Section 10.3.2.

Lemma 38. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ with $\mu \geq \mu_G$. Then, for every $p \in \mathcal{P}$ and $x \in N$, we do not have $D_\mu(p) = D_\mu(p^{(id, id, \rho_0)}) = \{x\}$.*

10.3 The proofs of Lemmata 34 and 38

10.3.1 Graphs

In this section, we recall some basic facts and notation from graph theory, which we are going to use in the sequel[5]. A (directed) graph is a pair (V, A), where V is a nonempty set called vertex set and A is a subset of $\{(x, y) \in V^2 : x \neq y\}$ called arc set. Note that if $\Gamma = (V, A)$ is a graph and $|V| = 1$, then $A = \varnothing$. Given two graphs $\Gamma_1 = (V_1, A_1)$ and $\Gamma_2 = (V_2, A_2)$, we say that Γ_2 is a subgraph of Γ_1 if $V_2 \subseteq V_1$ and $A_2 \subseteq A_1$. If Γ_2 is a subgraph of Γ_1, we write $\Gamma_2 \leq \Gamma_1$.

Consider now a graph $\Gamma = (V, A)$. We say that $x \in V$ is *maximal* [*minimal*] for Γ if there exists no $y \in V$ such that $(y, x) \in A$ [$(x, y) \in A$]. We denote by $\max(\Gamma)$ [$\min(\Gamma)$] the set of maximal [minimal] vertices for Γ. Note that those sets may be empty. We say that $x \in V$ is a *maximum* [*minimum*] of Γ if, for every $y \in V \setminus \{x\}$, we have that $(x, y) \in A$ [$(y, x) \in A$][6]. We say that $x \in N$ is *isolated* in Γ if, for every $y \in V \setminus \{x\}$, we have that $(x, y), (y, x) \notin A$. We denote with $\mathcal{I}(\Gamma)$ the set of the isolated vertices of Γ. It is useful to note that

$$\max(\Gamma) \cap \min(\Gamma) = \mathcal{I}(\Gamma). \tag{15}$$

Γ is said to be *connected* if, for every $x, y \in V$ with $x \neq y$, there exist $k \geq 2$ and an ordered sequence $x_1, \ldots, x_k$ of distinct elements of V such that $x_1 = x$, $x_k = y$, and, for every $j \in \{1, \ldots, k-1\}$, $(x_j, x_{j+1}) \in A$ or $(x_{j+1}, x_j) \in A$. Note that if Γ has a maximum [minimum], then Γ is connected. It is well known that there exist, uniquely determined, $c \in \mathbb{N}$ and $\Gamma_1 = (V_1, A_1), \ldots, \Gamma_c = (V_c, A_c)$ connected subgraphs of Γ such that $\cup_{i=1}^c V_i = V$, $\cup_{i=1}^c A_i = A$, and for every $i, j \in \{1, \ldots, c\}$ with $i \neq j$, $V_i \cap V_j = A_i \cap A_j = \varnothing$. Those subgraphs $\Gamma_1, \ldots, \Gamma_c$ are called the *connected components* of Γ. They are maximal among the connected subgraphs of Γ, that is, if $\Gamma' \leq \Gamma$ is connected and $\Gamma' \geq \Gamma_i$ for some $i \in \{1, \ldots, c\}$, then $\Gamma' = \Gamma_i$. In particular, for every $i \in \{1, \ldots, c\}$, $x \in V_i$ and $y \in V \setminus V_i$ imply $(x, y), (y, x) \notin A$; $x, y \in V_i$ and $(x, y) \in A$ imply $(x, y) \in A_i$. Note that $x \in N$ is isolated in Γ if and only if the connected component of Γ containing x is $(\{x\}, \varnothing)$. Given $l \geq 2$, Γ is said to be a *l-cycle* if $|V| = l$ and there exists an ordered sequence $x_1, \ldots, x_l$ of the elements of V such that, once defined $x_{l+1} = x_1$, we have that $A = \{(x_j, x_{j+1}) \ : 1 \leq j \leq l\}$. Γ is said to be a *cycle* if it is a l-cycle for some $l \geq 2$. Fixed $l \geq 2$, Γ is said to be *l-cyclic* if there exists a l-cycle $\Gamma_1 \leq \Gamma$, *l-acyclic* otherwise. Γ is said to be *acyclic* if it is l-acyclic for all $l \geq 2$. Note that if $|V| = 1$, then Γ is acyclic. The following lemma states some interesting properties of 2-acyclic graphs which we are going to use later.

[5] All unexplained notation is standard. See, for instance, Diestel (2010).

[6] Note that if x is a maximum [minimum] of Γ it is not necessarily maximal [minimal] for Γ. In fact, given $\Gamma = (\{1, 2\}, \{(1, 2), (2, 1)\})$, we have that 1 and 2 are both a maximum [minimum] but none of them is maximal [minimal].

Lemma 39. *Let $\Gamma = (V, A)$ be a 2-acyclic graph. Then Γ has at most one maximum. Moreover, if $x \in V$ is a maximum of Γ, then $\max(\Gamma) = \{x\}$.*

Proof. Assume by contradiction that there exist $x, y \in V$ with $x \neq y$ such that they are both maxima of Γ. Then $(x, y) \in A$ and $(y, x) \in A$, so that $\Gamma_1 = (\{x, y\}, \{(x, y), (y, x)\}) \leq \Gamma$. Since Γ_1 is a 2-cycle, that contradicts the fact that Γ is 2-acyclic.

Assume now that $x \in V$ is the unique maximum of Γ. If by contradiction $x \notin \max(\Gamma)$, then there is $y \in V$ such that $(y, x) \in A$. Since we know that $(x, y) \in V$, the 2-cycle $\Gamma_1 = (\{x, y\}, \{(x, y), (y, x)\})$ is a subgraph of Γ and the contradiction is found. We complete the proof simply noticing that, being x a maximum of Γ, for every $y \in V \setminus \{x\}$, we have that $(x, y) \in A$ so that $y \notin \max(\Gamma)$. □

10.3.2 Majority graphs and proof of Lemma 38

In the rest of the paper, given $p \in \mathcal{P}$, we use the writing p^{ρ_0} instead of $p^{(id,id,\rho_0)}$. Of course, according to the new notation, by (10), we have $(p^{\rho_0})^{\rho_0} = p$ for all $p \in \mathcal{P}$.

Given $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$, we consider the relation on N given by

$$\Sigma_\mu(p) = \Big\{(x, y) \in N \times N : |\{i \in H : x >_{p_i} y\}| \geq \mu\Big\}.$$

Thus, $(x, y) \in \Sigma_\mu(p)$ means that $x \neq y$ and at least μ individuals prefer x to y with respect to the preference profile p. We usually write $x >^p_\mu y$ instead of $(x, y) \in \Sigma_\mu(p)$. In a natural way, we associate with $\Sigma_\mu(p)$ the graph $\Gamma_\mu(p) = (N, \Sigma_\mu(p))$, called the *$\mu$-majority graph* of p. Obviously, the properties of the graph $\Gamma_\mu(p)$ are nothing more than the properties of the relation $\Sigma_\mu(p)$ translated into the graph theory language. To start with, note that $\mu \in \mathbb{N} \cap (h/2, h]$ implies that $\Gamma_\mu(p)$ is 2-acyclic. Thus, as an immediate application of Lemma 39, we get the following.

Lemma 40. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then $\Gamma_\mu(p)$ has at most one maximum. Moreover, if $\Gamma_\mu(p)$ has a maximum $x \in N$, then $D_\mu(p) = \{x\}$.*

Lemma 41. *Let $\mu, \mu' \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\mu' \leq \mu$, then $\Gamma_\mu(p) \leq \Gamma_{\mu'}(p)$. In particular, $\Sigma_\mu(p) \subseteq \Sigma_{\mu_0}(p)$.*

Proof. Simply note that the graphs $\Gamma_\mu(p)$ and $\Gamma_{\mu'}(p)$ have the same vertex set and that $\mu' \leq \mu$ implies $\Sigma_\mu(p) \subseteq \Sigma_{\mu'}(p)$. □

Lemma 42. *Let $\mu \in \mathbb{N} \cap (h/2, h]$. Then $\Gamma_\mu(p)$ is acyclic for all $p \in \mathcal{P}$ if and only if $\mu \geq \mu_G$. In particular $\Gamma_h(p)$ is acyclic for all $p \in \mathcal{P}$.*

Proof. It is an immediate consequence of Proposition 6 and 7 in Bubboloni and Gori (2014). □

It will be immediately apparent how changing the language could help to improve our comprehension of the set $D_\mu(p)$. Let us present some useful lemmata.

Lemma 43. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then $D_\mu(p) = \max(\Gamma_\mu(p)) = \min(\Gamma_\mu(p^{\rho_0}))$ and $D_\mu(p) \cap D_\mu(p^{\rho_0}) = \mathcal{I}(\Gamma_\mu(p)) = \mathcal{I}(\Gamma_\mu(p^{\rho_0}))$.*

Proof. The first equality immediately follows by the definitions of $D_\mu(p)$ and p^{ρ_0}. Then we also get $D_\mu(p) = \max(\Gamma_\mu(p))$ and $D_\mu(p^{\rho_0}) = \max(\Gamma_\mu(p^{\rho_0})) = \min(\Gamma_\mu(p))$. Thus,(15), completes the proof. □

Lemma 44. *If h is odd, then, for every $p \in \mathcal{P}$, $\mathcal{I}(\Gamma_{\mu_0}(p)) = \varnothing$. Moreover, if $D_{\mu_0}(p) \neq \varnothing$ then $\Gamma_{\mu_0}(p)$ admits maximum $x \in N$ and $D_{\mu_0}(p) = \{x\}$.*

Proof. Let us fix $p \in \mathcal{P}$. We prove that $\mathcal{I}(\Gamma_{\mu_0}(p)) = \varnothing$ showing that the relation $\Sigma_{\mu_0}(p)$ is complete. To begin with note that, being h odd, we have $\mu_0 = \frac{h+1}{2}$. Assume now by contradiction that there exist $x, y \in N$ with $x \not>^p_{\mu_0} y$ and $y \not>^p_{\mu_0} x$. Then

$$h = |\{i \in H : x >_{p_i} y\}| + |\{i \in H : y >_{p_i} x\}| \leq \mu_0 - 1 + \mu_0 - 1 = 2\left(\frac{h+1}{2}\right) - 2 = h - 1,$$

that is, a contradiction. In order to prove the second part, assume that $D_{\mu_0}(p) \neq \varnothing$ and pick $x \in D_{\mu_0}(p)$. Since the relation $\Sigma_{\mu_0}(p)$ is complete, then we have $x >^p_{\mu_0} y$ for all $y \in N \setminus \{x\}$, that is, x is a maximum in $\Gamma_{\mu_0}(p)$. Then, by Lemma 40, $D_{\mu(p)}(p) = \{x\}$. ☐

Let us denote by $\mathcal{C}(\Gamma_\mu(p))$ the set of the connected components of $\Gamma_\mu(p)$ and define $\mathcal{A}(\Gamma_\mu(p)) = \{\Gamma \in \mathcal{C}(\Gamma_\mu(p)) : \Gamma \text{ is acyclic}\}$. We are ready for a crucial lemma giving a lower bound for $|D_\mu(p)|$.

Lemma 45. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. Then*

$$D_\mu(p) = \bigcup_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \bigcup_{\Gamma \in \mathcal{A}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \mathcal{I}(\Gamma_\mu(p)),$$

and

$$|D_\mu(p)| = \sum_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} |\max(\Gamma)| \geq |\mathcal{A}(\Gamma_\mu(p))| \geq |\mathcal{I}(\Gamma_\mu(p))|.$$

Proof. Let $\Gamma = (V, A) \in \mathcal{C}(\Gamma_\mu(p))$. Then $\Gamma \leq \Gamma_\mu(p)$, so that $V \subseteq N$ and $A \subseteq \Sigma_\mu(p)$. Since Γ is a connected component of $\Gamma_\mu(p)$, we have that, for every $x \in V$ and $y \in N \setminus V$, $y \not>^p_\mu x$. This immediately gives that each $x \in \max(\Gamma)$ belongs to $D_\mu(p)$, so that $D_\mu(p) \supseteq \bigcup_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} \max(\Gamma)$. The other inclusion is trivial and thus $D_\mu(p) = \bigcup_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} \max(\Gamma)$. Since $\mathcal{A}(\Gamma_\mu(p)) \subseteq \mathcal{C}(\Gamma_\mu(p))$ and, for every $x \in \mathcal{I}(\Gamma_\mu(p))$, $(\{x\}, \varnothing) \in \mathcal{A}(\Gamma_\mu(p))$, we also get

$$\bigcup_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \bigcup_{\Gamma \in \mathcal{A}(\Gamma_\mu(p))} \max(\Gamma) \supseteq \mathcal{I}(\Gamma_\mu(p)).$$

In particular, since there is no overlap between vertices of different connected components, we deduce $|D_\mu(p)| = \sum_{\Gamma \in \mathcal{C}(\Gamma_\mu(p))} |\max(\Gamma)|$. We complete the proof showing that for every $\Gamma \in \mathcal{A}(\Gamma_\mu(p))$, we have $\max(\Gamma) \neq \varnothing$. Pick $x_1 \in V$. If $y \not>^p_\mu x_1$ for all $y \in V$, then we have $x_1 \in \max(\Gamma)$ and we have finished. Assume instead there exists $x_2 \in V$ with $x_2 >^p_\mu x_1$. Obviously, we have $x_2 \neq x_1$. Then, repeat the argument for x_2. Since the set N is finite and Γ contains no cycle, in a finite number $k \leq n$ of steps, we obtain an element $x_k \in \max(\Gamma)$. ☐

Corollary 46. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ is acyclic and $D_\mu(p)$ is a singleton, then $\Gamma_\mu(p)$ is connected.*

Proof. Since $\Gamma_\mu(p)$ is acyclic, we have that $\mathcal{C}(\Gamma_\mu(p)) = \mathcal{A}(\Gamma_\mu(p))$. Then, using Lemma 45, we get $1 = |D_\mu(p)| \geq |\mathcal{C}(\Gamma_\mu(p))| \geq 1$. That implies $|\mathcal{C}(\Gamma_\mu(p))| = 1$, that is, $\Gamma_\mu(p)$ is connected. ☐

We are ready for a crucial result.

Corollary 47. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ is acyclic, then, for every $x \in N$, we do not have $D_\mu(p) = D_\mu(p^{\rho_0}) = \{x\}$.*

Proof. Assume by contradiction that $D_{\mu_0}(p) = D_{\mu_0}(p^{\rho_0}) = \{x\}$, for some $x \in N$. Then, by Lemma 43, we have that x is isolated in $\Gamma_\mu(p)$. On the other hand, by Corollary 46, $\Gamma_\mu(p)$ is connected so that its only vertex is x, against $n \geq 2$. ☐

Proof of Lemma 38. Let $\mu \geq \mu_G$. Then by Lemma 42, we have that $\Gamma_\mu(p)$ is acyclic for all $p \in \mathcal{P}$, so that Corollary 47 applies. ☐

10.3.3 Majority graphs and proof of Lemma 34

We now focus on the minimal majority threshold. Note that, for every $p \in \mathcal{P}$, $\max(\Gamma_{\mu(p)}(p)) = D_{\mu(p)}(p) \neq \varnothing$.

Lemma 48. *Let $p \in \mathcal{P}$. If $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic, then there exists no $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$.*

Proof. Let both $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ be acyclic and assume, by contradiction, that there exists $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$. Being $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ acyclic, by Corollary 46, we have that they are connected. Without loss of generality, assume now that $\mu(p^{\rho_0}) \geq \mu(p)$. We show that there exists $y \in N$ with $x >^{p^{\rho_0}}_{\mu(p^{\rho_0})} y$. Assume the contrary and note that, due to $n \geq 2$, the connection of $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ implies the existence of $z \in N$ such that $z >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$, against $x \in D_{\mu(p^{\rho_0})}(p^{\rho_0})$. Then, it follows that $y >^{p}_{\mu(p)} x$, against $x \in D_{\mu(p)}(p)$. ☐

Corollary 49. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$. If $\Gamma_\mu(p)$ admits at least an acyclic connected component, then $\mu(p) \leq \mu$.*

Proof. By Lemma 45, we have $|D_\mu(p)| \geq 1$, so that $D_\mu(p) \neq \varnothing$. ☐

Lemma 50. *Let h be odd and $p \in \mathcal{P}$. If $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0})$, then $\mu(p^{\rho_0}) > \mu_0$.*

Proof. Let $p \in \mathcal{P}$ and assume that $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0})$. Then $D_{\mu_0}(p) \neq \varnothing$ and, using Lemma 44, $\Gamma_{\mu(p)}(p)$ has a maximum $x \in N$ and $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$. Assume by contradiction that $\mu(p^{\rho_0}) = \mu_0$. By Lemma 43, we get that x is isolated in $\Gamma_{\mu(p)}(p)$, against the fact that x is the maximum of $\Gamma_{\mu(p)}(p)$ and $n \geq 2$. ☐

Corollary 51. *Let $p \in \mathcal{P}$ such that $\mu(p) = \mu_0$. If $\Gamma_{\mu(p)}(p)$ is acyclic, then there exists no $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$.*

Proof. The acyclicity of $\Gamma_{\mu(p)}(p)$ implies that of $\Gamma_{\mu(p)}(p^{\rho_0})$, so that, by Corollary 49 we have $\mu_0 \leq \mu(p^{\rho_0}) \leq \mu(p) = \mu_0$. It follows that $\mu(p^{\rho_0}) = \mu(p)0\mu_0$ and Corollary 47 applies. ☐

Due to the previous results, it is important to understand which conditions guarantee the acyclicity of $\Gamma_{\mu(p)}(p)$. We have observed that, for every $\mu \in \mathbb{N} \cap (h/2, h]$, $\Gamma_\mu(p)$ is 2-acyclic. Anyway, it can admit l-cycles for some $l \geq 3$. We explore this possibility through Propositions 6 and 7 in Bubboloni and Gori (2014).

Proposition 52. *Let $\mu \in \mathbb{N} \cap (h/2, h]$ and $l \in \mathbb{N} \cap [2, n]$. Then there exists $p \in \mathcal{P}$ such that $\Gamma_\mu(p)$ is l-cyclic if and only if $\mu \leq \frac{l-1}{l} h$.*

Proof. Consider $\mu > \frac{l-1}{l} h$ and assume by contradiction that there exists $p \in \mathcal{P}$ and an l-cycle $\Gamma \leq \Gamma_\mu(p)$ with vertex set V. Then $V \subseteq N$ and $|V| = l \leq n$. Consider the preference profile p' on the set of l alternatives V obtained from p eliminating (if any) those entries in $N \setminus V$. By Proposition 6 in Bubboloni and Gori (2014), we have that $\Gamma_\mu(p')$ is acyclic, against the fact that $\Gamma \leq \Gamma_\mu(p')$.

Let now $\mu \leq \frac{l-1}{l} h$ and let $V \subseteq N$ with $|V| = l$. By Proposition 7 in Bubboloni and Gori (2014), there exists a preference profile p' on the set of alternatives V such that $\Gamma_\mu(p')$ contains a l-cycle Γ whose set of vertices is V. Consider a preference profile p on the set of alternatives N, in which every individual $i \in H$ ranks in the first l positions the alternatives in V as p'_i and those in $N \setminus V$ as she likes. Then $\Gamma \leq \Gamma_\mu(p)$. ☐

Let us define now

$$\mu_a = \min\left\{ m \in \mathbb{N} \cap (h/2, h] : m > \frac{n-2}{n-1} h \right\},$$

and note that μ_a is well posed because $h > \frac{n-2}{n-1} h$. Moreover, we have that $\mu_0 \leq \mu_a \leq \mu_G$ and, when $n \in \{2, 3\}$, $\mu_a = \mu_0$. We call μ_a the *acyclicity threshold*, due to the following result.

Corollary 53. *Let $p \in \mathcal{P}$. If $\mu(p) \geq \mu_a$, then $\Gamma_{\mu(p)}(p)$ is acyclic. In particular, for every $n \in \{2, 3\}$, $\Gamma_{\mu(p)}(p)$ is acyclic.*

Proof. Consider $\Gamma_{\mu(p)}(p)$. It admits no n-cycle, because having such a cycle obviously implies the contradiction $D_{\mu(p)}(p) = \varnothing$. On the other hand, by Proposition 52, it does not have l-cycles for all $l \in \{2, \ldots, n-1\}$ because $\mu(p) > \frac{n-2}{n-1}h \geq \frac{l-1}{l}h$. Finally note that, if $n \in \{2, 3\}$, then $\mu(p) \geq \mu_0 = \mu_a$. $\square$

Proposition 54. *If $n \geq 4$ and h is odd and such that $h \geq \frac{3(n-1)}{n-3}$, then there exists $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$.*

Proof. First of all, note that $\mu_0 = \frac{h+1}{2}$. Define then $\mu = \frac{h+3}{2} = \mu_0 + 1$ and $V = N \setminus \{n\}$. The assumption $h \geq \frac{3(n-1)}{n-3}$ is equivalent to $\mu \leq \frac{(n-1)-1}{n-1}h$ and thus, by Proposition 52, there exists p', a preference profile on the set of alternatives V, such that $\Gamma_\mu(p')$ has a $(n-1)$-cycle Γ. We define now the preference profile $p \in \mathcal{P}$ defining[7], for every $i \in H$, the preference p_i. If $i \leq \mu_0$, then let $p_i(1) = n$ and $p_i(j) = p'_i(j-1)$ for all $j \in \{2, \ldots, n\}$; if $\mu_0 < i \leq h$, then let $p_i(j) = p'_i(j)$ for all $j \in \{1, \ldots, n-1\}$ and $p_i(n) = n$. Note that in p, the alternative n is ranked first μ_0 times and last $h - \mu_0$ times. Thus, n is a maximum in $\Gamma_{\mu_0}(p)$. Moreover $\Gamma \leq \Gamma_\mu(p)$ so that also $\Gamma_\mu(p^{\rho_0})$ contains a $(n-1)$-cycle Γ_0, with inverted orientation, whose vertex set is V. By Lemma 40 and by (9), it follows that $\mu(p) = \mu_0$ and $D_{\mu(p)}(p) = \{n\}$. On the other hand, $D_{\mu_0}(p^{\rho_0}) = \varnothing$. Indeed, n is not maximal in $\Gamma_{\mu_0}(p^{\rho_0})$, being beaten μ_0 times by any other alternative, and each alternative in V is not maximal in $\Gamma_{\mu_0}(p^{\rho_0})$ because, due to the presence of the cycle Γ_0, it is beaten μ times by a suitable alternative in V. Anyway $D_\mu(p^{\rho_0}) = \{n\}$, because n is isolated and thus maximal in $\Gamma_\mu(p^{\rho_0})$, by (15). Then $\mu(p^{\rho_0}) = \mu$ and $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{n\}$. $\square$

Proposition 55. *If $n \geq 4$ and h is even and such that $h \geq \frac{2(n-1)}{n-3}$, then there exists $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$.*

Proof. First of all, note that $\mu_0 = \frac{h+2}{2}$ and define $V = N \setminus \{n\}$. The assumption $h \geq \frac{2(n-1)}{n-3}$ is equivalent to $\mu_0 \leq \frac{(n-1)-1}{n-1}h$ and thus, by Proposition 52, there exist a preference profile p' on the set of alternatives V, such that $\Gamma_{\mu_0}(p')$ has a $(n-1)$-cycle Γ. We define now the preference profile $p \in \mathcal{P}$, defining, for every $i \in H$, the preference p_i. If $i \leq \frac{h}{2}$, then let $p_i(1) = n$ and $p_i(j) = p'_i(j-1)$ for all $j \in \{2, \ldots, n\}$; if $\frac{h}{2} < i \leq h$, then let $p_i(j) = p'_i(j)$ for all $j \in \{1, \ldots, n-1\}$ and $p_i(n) = n$. Note that in p, the alternative n is ranked first $\frac{h}{2}$ times and last $\frac{h}{2}$ times. Thus, by (15), n is isolated and maximal both in $\Gamma_{\mu_0}(p)$ and in $\Gamma_{\mu_0}(p^{\rho_0})$. Moreover, no further alternative is maximal in $\Gamma_{\mu_0}(p)$ because each element in V is involved in the cycle $\Gamma \leq \Gamma_{\mu_0}(p)$. Since each cycle in $\Gamma_{\mu_0}(p)$ determines a cycle with inverted orientation in $\Gamma_{\mu_0}(p^{\rho_0})$, the same consideration holds for $\Gamma_{\mu_0}(p^{\rho_0})$, as well. Then, we can conclude that $\mu(p) = \mu(p^{\rho_0}) = \mu_0$ and $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{n\}$. $\square$

Corollary 56. *If $(h, n) \in \mathbb{N}^2 \setminus T$, then there exists $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$.*

Proof. We get the proof showing that the assumptions of Propositions 54 or 55 hold. First of all, note that $(h, n) \in \mathbb{N}^2 \setminus T$ implies $h \geq 4$ and $n \geq 4$. If $n = 4$, then either h is even with $h \geq 6$ and so satisfies $h \geq \frac{2(n-1)}{n-3}$, or h is odd with $h \geq 9$ and so satisfies $h \geq \frac{3(n-1)}{n-3}$. If $n = 5$, then the same argument applies. If $n \geq 6$, then we have $\frac{2(n-1)}{n-3} \leq 4 \leq h$ for every h even as well as $\frac{3(n-1)}{n-3} \leq 5 \leq h$ for every h odd. $\square$

[7] Recall that, as discussed in Section 2.2, preferences can be thought as functions from the set of rankings to the set of alternatives.

Proof of Lemma 34. The fact that i) implies ii) is exactly Corollary 56. Thus, we are left with proving that ii) implies i). Consider then $(h,n) \in T$ and assume by contradiction that there exist $p \in \mathcal{P}$ and $x \in N$ such that $D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{x\}$. By Corollary 6, we can assume that $x = n$ so that

$$D_{\mu(p)}(p) = D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{n\}. \tag{16}$$

There are several cases to study.

If $n \in \{2,3\}$, then, by Corollary 53, we have that $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic so that Lemma 48 applies contradicting (16).

If $h = 2$, then $\mu_0 = 2$ and $\mu(p) = \mu(p^{\rho_0}) = 2$. By Lemma 43, n is then the only isolated vertex in $\Gamma_{\mu(p)}(p)$. Let $x_1 = p_1(1)$ and $x_2 = p_2(1)$ be the alternatives which are ranked first by the two individuals in $H = \{1,2\}$. If $x_1 \neq x_2$, then x_1, x_2 are both isolated in $\Gamma_{\mu(p)}(p)$ and we get the contradiction. On the other hand, if $x_1 = x_2$, then x_1 is a maximum in $\Gamma_{\mu(p)}(p)$ and so it is not isolated. It follows that $x_1 \neq n$ while, by Lemma 40, $D_{\mu(p)}(p) = \{x_1\}$ and the contradiction is found again.

If $h = 3$, then $\mu_0 = 2$ and $\mu(p), \mu(p^{\rho_0}) \in \{2,3\}$. We examine separately the two possibilities. If $\mu(p) = 2$, then, by Lemma 50, we have that $\mu(p^{\rho_0}) = 3$ so that $D_{\mu(p^{\rho_0})}(p^{\rho_0}) = \{n\}$ and $D_{\mu(p)}(p^{\rho_0}) = \varnothing$. Let $V = N \setminus \{n\}$. Since n is the only maximal element in $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$, for every $x \in V$, there exists $y \in N$ with $y >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$. Note that if y were equal to n, then from $n >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$ we would get $x >^{p}_{\mu(p^{\rho_0})} n$ against the maximality of n in $\Gamma_{\mu(p)}(p)$. Thus, there exists a cycle in $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ involving some vertices of V. That leads to a contradiction since, by Lemma 42, $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ is acyclic. If $\mu(p^{\rho_0}) = 2$ the previous argument applies to p^{ρ_0}. If $\mu(p) = \mu(p^{\rho_0}) = 3$, then we reach a contradiction applying Lemma 42 and Corollary 47.

If $(h,n) = (4,4)$, then $\mu_0 = \mu_a = 3$ and $\mu(p),\ \mu(p^{\rho_0}) \in \{3,4\}$. Thus, by Corollary 53, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic, so that Lemma 48 applies contradicting (16).

If $(h,n) = (5,4)$, then $\mu_0 = 3$, $\mu_a = \mu_G = 4$, and $\mu(p),\ \mu(p^{\rho_0}) \in \{3,4\}$. If $\mu(p) = \mu(p^{\rho_0}) = 4$, then by Corollary 53, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic and we contradict (16), using Lemma 48. Therefore, by Lemma 50, we can reduce to the case $\mu(p) = 3 = \mu_0$ and $\mu(p^{\rho_0}) = 4$. By Corollary 53 we have that $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ is acyclic and then, by Corollary 46, connected. Assume there exists $x \in V = \{1,2,3\}$ such that $4 >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$. Then $x >^{p}_{\mu(p^{\rho_0})} 4$, against $4 \in D_{\mu(p)}(p)$. So, we have $4 \not>^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$, for all $x \in V$. On the other hand, from $4 \in D_{\mu(p^{\rho_0})}(p^{\rho_0})$, we deduce that $x \not>^{p^{\rho_0}}_{\mu(p^{\rho_0})} 4$. Thus, 4 is isolated in $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$, against the connection of $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$.

If $(h,n) = (7,4)$, then $\mu_0 = 4$, $\mu_a = 5$, $\mu_G = 6$ and $\mu(p),\ \mu(p^{\rho_0}) \in \{4,5,6\}$. If $\mu(p),\ \mu(p^{\rho_0}) \in \{5,6\}$, then by Corollary 53, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic and we contradict (16), using Lemma 48. Therefore, we can reduce to the case $\mu(p) = 4$ and, by Lemma 50, $\mu(p^{\rho_0}) \in \{5,6\}$. By Corollary 53, $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ is acyclic and then, by Corollary 46, connected. Assume there exists $x \in V = \{1,2,3\}$ such that $4 >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$. Then $x >^{p}_{\mu(p^{\rho_0})} 4$, against $4 \in D_{\mu(p)}(p)$. So, we have $4 \not>^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$ for all $x \in V$. On the other hand, from $4 \in D_{\mu(p^{\rho_0})}(p^{\rho_0})$ we deduce that $x \not>^{p^{\rho_0}}_{\mu(p^{\rho_0})} 4$ for all $x \in V$. Thus, 4 is isolated in $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$, against the connection of $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$.

If $(h,n) = (5,5)$, then $\mu_0 = 3$, $\mu_a = 4$, $\mu_G = 5$ and $\mu(p),\ \mu(p^{\rho_0}) \in \{3,4,5\}$. If $\mu(p),\ \mu(p^{\rho_0}) \in \{4,5\}$, then by Corollary 53, $\Gamma_{\mu(p)}(p)$ and $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ are both acyclic and we contradict (16), using Lemma 48. Therefore, we can reduce to the case $\mu(p) = 3$ and, by Lemma 50, $\mu(p^{\rho_0}) \geq 4$. By Corollary 53, $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$ is acyclic and then, by Corollary 46, connected. Assume there exists $x \in V = \{1,2,3,4\}$ such that $5 >^{p^{\rho_0}}_{\mu(p^{\rho_0})} x$. Then $x >^{p}_{\mu(p^{\rho_0})} 5$, against $5 \in D_{\mu(p)}(p)$. On the other hand, from $5 \in D_{\mu(p^{\rho_0})}(p^{\rho_0})$ we deduce that $x \not>^{p^{\rho_0}}_{\mu(p^{\rho_0})} 5$ for all $x \in V$. Thus, 5 is isolated in $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$, against the connection of $\Gamma_{\mu(p^{\rho_0})}(p^{\rho_0})$. $\square$

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