The comparison test - Not just for nonnegative series

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October 2003

Consider the following question. Is it possible to generalize the comparison test to generic real series? More precisely, is it true that, given $a_n \leq b_n \leq c_n$ for all n, the convergence of $\sum b_n$, follows from the convergence of $\sum a_n$ and $\sum c_n$? At first glance, many of us (certainly the authors) could argue something like "*If it were true then it would be certainly written on some of the books standing on the shelves in my room*". As a matter of fact, all the books on the authors' shelves state the test only for non-negative series. Nonetheless it is true as we show in this note.

1 A generalization of the comparison test

The comparison test is usually stated only for non negative real series both in calculus books (see [1, 4, 5, 8] for some examples) and in more specific-purpose texts (see [2, 3, 6]). There could be many reasons for that, nonetheless its most immediate generalization could have some application in the study of convergence and its proof is so straightforward that, at least, it could be taken into consideration as an exercise in first year calculus courses.

First observe that a series cannot be oscillatory[1] if it is minorized (or, alternatively, majorized) by a convergent series.

Lemma 1 *Let $\sum a_n$ and $\sum b_n$ be two real series such that $a_n \leq b_n$ for all $n \in \mathbb{N}$ and let $\sum a_n$ be convergent. Then $\sum b_n$ is not oscillatory.*

[1] We distinguish among *convergent*, *divergent* and *oscillatory* real series according to the fact that the limit of partial sums *exists and is finite*, *is infinite*, *does not exist*, respectively.

Proof. From $0 \leq b_n - a_n$ we have that $\sum (b_n - a_n)$ is positive and it must converge or diverge. So, the convergence of $\sum a_n$, implies that $\sum b_n$ cannot be oscillatory ■

We can now generalize the comparison test as follows.

Theorem 2 *Let $\sum a_n$, $\sum b_n$, and $\sum c_n$ be three real series such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. We have that*

$$\begin{array}{lllll} (i) & \sum a_n \text{ and } \sum c_n \text{ converge} & \Rightarrow & \sum b_n \text{ converges} & \\ (ii) & \sum a_n \text{ diverges to } +\infty & \Rightarrow & \sum b_n \text{ diverges to } +\infty & . \\ (iii) & \sum c_n \text{ diverges to } -\infty & \Rightarrow & \sum b_n \text{ diverges to } -\infty & \end{array}$$

Proof. Observe that from

$$\sum_{n=1}^{K} a_n \leq \sum_{n=1}^{K} b_n \leq \sum_{n=1}^{K} c_n \tag{1}$$

(ii) and (iii) follow immediatly. In case (i) Lemma 1 applies and so $\sum b_n$ cannot be oscillatory. Finally, (1) implies that $\sum b_n$ cannot diverge ■

Notice that in Theorem 2 $\sum a_n$ and $\sum c_n$ are required to be simply convergent. Clearly, the case of interest is that of conditionally convergent series.[2]

Theorem 2 enables us to prove the following proposition.

Proposition 3 *Let $\sum a_n$ be convergent and $f : \mathbb{R} \to \mathbb{R}$ such that in a neighborhood of zero*

$$f(x) = \alpha x + \beta x^{2k} + o\left(x^{2k}\right), \quad \beta \neq 0, k \in \mathbb{N}.$$

Then $\sum f(a_n)$ converges if and only if $\sum (a_n)^{2k}$ converges.

Proof. By assumption there exists $\varepsilon > 0$ such that for $|x| < \varepsilon$ we have

$$\alpha x + \left(\beta - \frac{|\beta|}{2}\right) x^{2k} \leq f(x) \leq \alpha x + \left(\beta + \frac{|\beta|}{2}\right) x^{2k}$$

Hence, there exists n_ε such that, for each $n > n_\varepsilon$,

$$\alpha a_n + \left(\beta - \frac{|\beta|}{2}\right) (a_n)^{2k} \leq f(a_n) \leq \alpha a_n + \left(\beta + \frac{|\beta|}{2}\right) (a_n)^{2k}$$

Now, an application of Theorem 2 yields the desired result. ■

[2] See [5] p.375.

Remark 1 *Proposition 3 cannot be extended to the case where the expansion ends with an odd power.*[3] *Nevertheless if* $\sum a_n$ *converges and*

$$f(x) = \alpha x + \beta x^{2k+1} + o\left(x^{2k+1}\right), \quad \beta \neq 0, k \in \mathbb{N}$$

then[4]

$$\sum |a_n|^{2k+1} \text{ converges } \Rightarrow \sum f(a_n) \text{ converges.} \tag{2}$$

Finally, notice that a sufficient condition for the convergence of $\sum |a_n|^{2k+1}$ *is the convergence of* $\sum (a_n)^{2i}$ *for some* $i \in \{1, 2, .., k\}$.

Remark 2 *Notice that Remark 1 gives only a sufficient condition for the convergence of* $\sum f(a_n)$. *For this reason, if it is possible, it could be useful to refine the Taylor expansion of* f *to obtain more information about the convergence. Consider the following example:* $\sum \arctan \frac{(-1)^n}{\sqrt[4]{n}}$. *In this case* $\sum \left(\frac{(-1)^n}{\sqrt[4]{n}}\right)^{2k+1}$ *converges for all* k *whereas it converges absolutely if* $k > 1$; *so if we use* $\arctan x = x - \frac{x^3}{3} + o(x^3)$ *nothing can be said about the convergence of the series since* $\sum \left|\frac{(-1)^n}{\sqrt[4]{n}}\right|^3$ *does not converge; on the contrary if we consider the expansion up to the* 5^{th} *order we can conclude that the series is convergent.*[5]

We give now some examples.

Example 4 *Consider the alternating real series* $\sum b_n$ *where the generic term*

$$b_n = \ln\left(1 + \frac{(-1)^n}{n^\gamma}\right)$$

depends on the positive real parameter γ. *First, from the Taylor expansion we know that*

$$\ln(1+x) = x - \frac{x^2}{2} + o\left(x^2\right). \tag{3}$$

[3] More details can be found in Section 2.2.

[4] In fact, we have

$$\alpha x + \beta x^{2k+1} - |x|^{2k+1} \leq f(x) \leq \alpha x + \beta x^{2k+1} + |x|^{2k+1},$$

hence the result.

[5] A less trivial example of such occurrence is provided by example 5.

Second, the series $\sum \frac{(-1)^n}{n^\gamma}$ *converges for all* $\gamma > 0$ *(Leibniz's test) whereas* $\sum \left(\frac{(-1)^n}{n^\gamma}\right)^2 = \sum \frac{1}{n^{2\gamma}}$ *converges if and only if* $\gamma > \frac{1}{2}$*. Hence, applying Proposition 3 we conclude that*

$$\sum b_n \text{ converges if } \gamma > \frac{1}{2}$$
$$\sum b_n \text{ diverges negatively if } 0 < \gamma \leq \frac{1}{2}$$

Notice that Leibniz's test applies to $\sum b_n$ *if and only if* $\gamma \geq 1$ *and* $\sum b_n$ *is absolutely convergent if and only if* $\gamma > 1$.[6]

Example 5 *Consider the real series* $\sum b_n$*, whose generic term, depending on the positive real parameter* γ*, is defined by*

$$b_n = \tan\left(\frac{n(-1)^n + 1}{n^{\gamma+1}}\right)$$

This series is absolutely convergent if and only if $\gamma > 1$*. Furthermore, it is an alternating series but the* Leibniz's test *applies if and only if* $\gamma > 2$.[7]

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[6] Observe that, for $|x|$ sufficiently small, $\frac{|x|}{2} \leq |\ln(1+x)| \leq 2|x|$, hence, for sufficiently large n, $0 < \frac{1}{2n^\gamma} < \left|\ln\left(1 + \frac{(-1)^n}{n^\gamma}\right)\right| < \frac{2}{n^\gamma}$ and so the series is absolutely convergent if and only if $\gamma > 1$. To see that the sequence of absolute values is not decreasing first notice that $\left|\ln\left(1 + \frac{(-1)^n}{n^\gamma}\right)\right| = \ln\left(1 + \frac{2}{2n^\gamma - 1 + (-1)^n}\right)$, then

$$|b_{2k}| < |b_{2k+1}| \Leftrightarrow \ln\left(1 + \frac{1}{(2k)^\gamma}\right) < \ln\left(1 + \frac{1}{(2k+1)^\gamma - 1}\right) \Leftrightarrow (2k+1)^\gamma - 1 < (2k)^\gamma,$$

finally the last inequality is verified if and only if $0 < \gamma < 1$.

[7] Observe that $|x| < |\tan x| < 2|x|$, hence $0 < \frac{1}{n^\gamma} + \frac{(-1)^n}{n^{\gamma+1}} < \left|\tan\left(\frac{n(-1)^n+1}{n^{\gamma+1}}\right)\right| < \frac{2}{n^\gamma} + \frac{2(-1)^n}{n^{\gamma+1}}$ and so the series is absolutely convergent if and only if $\gamma > 1$. Moreover, its terms are not decreasing in absolute value. Indeed, the equivalence

$$\left|\frac{(2k-1)(-1)^{2k-1}+1}{(2k-1)^{\gamma+1}}\right| < \left|\frac{(2k)(-1)^{2k}+1}{(2k)^{\gamma+1}}\right| \Leftrightarrow \frac{2k-2}{2k+1} < \left(\frac{2k-1}{2k}\right)^{\gamma+1},$$

the monotonicity of $\tan x$ and the fact that $|\tan x| = \tan|x|$ (if $|x| \leq \frac{\pi}{2}$) imply that

$$|b_{2k-1}| < |b_{2k}|$$

if and only if $\gamma \leq 2$.

Now, from the Taylor expansion we have

$$\tan x = x + \frac{x^3}{3} + o\left(x^3\right) \tag{4}$$

Moreover, the series $\sum \frac{n(-1)^n+1}{n^{\gamma+1}}$ *converges for all* $\gamma > 0$ *whereas* $\sum \left|\frac{n(-1)^n+1}{n^{\gamma+1}}\right|^3 = \sum \left(\frac{1}{n^\gamma} + \frac{(-1)^n}{n^{\gamma+1}}\right)^3$ *converges if and only if* $\gamma > \frac{1}{3}$. *Hence, by Remark 1 we conclude that* $\sum b_n$ *converges if* $\gamma > \frac{1}{3}$. *It is worth noting that, on the basis of Remark 1, nothing can be said about the behaviour of this series when* $0 < \gamma \leq \frac{1}{3}$. *Nevertheless this series converges for all* $\gamma > 0$ *as can be seen along the lines of what suggested in Remark 2.*[8]

The following example concerns a series with no regularity in sign.

Example 6 *Consider the real series* $\sum b_n$, *whose generic term, depending on the parameters* $\alpha, \gamma \in \mathbb{R}$, $\gamma > 0$, *is defined by*

$$b_n = \exp\left(\frac{\sin \alpha n}{n^\gamma}\right) - 1.$$

If α *is multiple of* π *then the series is null. For* $\alpha \neq k\pi$ *consider the Taylor expansion*

$$e^x - 1 = x + \frac{x^2}{2} + o\left(x^2\right). \tag{5}$$

The series $\sum \frac{\sin \alpha n}{n^\gamma}$ *converges for all* $\alpha, \gamma \in \mathbb{R}$, $\gamma > 0$ *(Dirichlet's test) whereas* $\sum \frac{\sin^2 \alpha n}{n^{2\gamma}} = \sum \frac{1-\cos 2\alpha n}{2n^{2\gamma}}$ *converges if and only if* $\gamma > \frac{1}{2}$. *Hence, applying Proposition 3 we have that*

$$\sum b_n \text{ converges if } \gamma > \frac{1}{2}$$
$$\sum b_n \text{ diverges positively if } 0 < \gamma \leq \frac{1}{2}$$

Finally observe that $\sum b_n$ *is absolutely convergent if and only if* $\gamma > 1$.[9]

[8] The proof for the case $0 < \gamma \leq \frac{1}{3}$ is left to the reader (See the Appendix for our suggested solution).

[9] The proof is left to the reader (See the Appendix for our suggested solution).

We conclude the section with an exercise to be solved along the lines of the previous examples.

Exercise 1.1 *Discuss the series* $\sum a_n$ *where*

$$a_n = \tanh\left(\frac{\cos \alpha n}{n^\gamma}\right)$$

and $\alpha, \gamma \in \mathbb{R}$*,* $\gamma > 0$*.*

2 Further discussion

In this Section we provide counterexamples to some desirable extension of the previous results.

2.1 Limit comparison test

The following question arises naturally: Is it possible to generalize the limit comparison test along the lines of the previous section; that is, to extend the statement by relaxing the restriction on the sign of the series? In other words, we wonder if $\lim_{n\to+\infty} \frac{a_n}{b_n} = L \neq 0$ implies that $\sum a_n$ and $\sum b_n$ behave the same no matter the sign of a_n and b_n. Unfortunately this is not true as the following simple example shows.[10]

Example 7 *Let* $a_n = \frac{(-1)^n}{n}$ *and* $b_n = \frac{(-1)^n}{n} + \frac{1}{n \ln n}$*. In this case* $\sum a_n$ *converges and* $\sum b_n$ *diverges while* $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$*.*

Working with the sequences $a_n = \frac{(-1)^n}{n} + \frac{\operatorname{sgn}(f(n))}{n \ln n}$ *and* $b_n = \frac{(-1)^n}{n} + \frac{\operatorname{sgn}(g(n))}{n \ln n}$ *where the functions* f *and* g *are properly defined it is easy to build examples of every other possible combination.*

We observe that Examples 4 and 6 provide other two cases in which $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$ but $\sum a_n$ and $\sum b_n$ have different behaviour. So a generalization of the limit comparison test in the sense of relaxing the sign requirements is not possible.

[10] Notice that condition $\lim \frac{a_n}{b_n} = L$ is a strong requirement as it calls for the product $a_n b_n$ to have the same sign from a certain stage on. Alternatively we could consider condition $\lim \left|\frac{a_n}{b_n}\right| = L \neq 0$ or the combination of $\max\lim \frac{a_n}{b_n} = L \neq 0$ and $\min\lim \frac{a_n}{b_n} = l \neq 0$. Obviously, also this conditions cannot work as they are weaker than that which we consider.

Possibly, an estension could be proved under stronger conditions. Example 5 could suggest the following version: if f is odd, a_n is alternating and $f(x) \sim x$ then $\sum f(a_n)$ and $\sum a_n$ have the same behaviour. Also this statement is not true as we now show.

Example 8 *Let a_n be defined as follows*

$$a_{4n-3} = \frac{1}{\sqrt[3]{n}}; \quad a_{4n-2} = -\frac{1}{2\sqrt[3]{n}}; \quad a_{4n-1} = \frac{1}{4\sqrt[3]{n}}; \quad a_{4n} = -\frac{3}{4\sqrt[3]{n}}$$

The series $\sum a_n$ converges by the Dirichlet test (see [7], Theorem 3 p. 137). Nevertheless the series $\sum (a_n + a_n^3)$ does not converge. To see this observe that the subsequence of the partial sums

$$S_{4k} = \sum_{n=1}^{4k} a_n^3 = \frac{15}{32} \sum_{n=1}^{k} \frac{1}{n}$$

is such that $\lim_{k\to\infty} S_{4k} = +\infty$.

So a generalization of the limit comparison test is not straightforward.

2.2 Other counterexamples

We begin this section by showing that the result of Proposition 3 cannot be extended to odd powers much more than it is done in Remark 1. That is, if $\sum a_n$ is convergent but the function $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(x) = \alpha x + \beta x^{2k+1} + o\left(x^{2k+1}\right), \quad \beta \neq 0, k \in \mathbb{N},$$

then

$$\sum (a_n)^{2k+1} \text{ converges } \quad \begin{matrix} \not\Rightarrow \\ \not\Leftarrow \end{matrix} \quad \sum f(a_n) \text{ converges.}$$

Example 9 **($\not\Rightarrow$)** *Consider the real series $\sum a_n$, where*

$$a_n = \frac{(-1)^n}{\sqrt[4]{n}}$$

and the function $f(x) = x + x^3 + x^4$.

Example 10 ($\not\Leftarrow$) *Consider the real series* $\sum a_n$*, where*

$$a_{2k} = \frac{(-1)^k}{\sqrt[4]{2k}} + \frac{1}{3\sqrt{2k}} \quad \textit{and} \quad a_{2k+1} = -\frac{1}{3\sqrt{2k}}$$

and the function $f(x) = x + x^3 - x^4$*. In this case* $\sum a_n$ *and* $\sum f(a_n)$ *converge, but* $\sum a_n^3$ *and* $\sum a_n^4$ *do not.*[11]

The following example shows that also the converse of (2) is not true, that is if $\sum a_n$ converges and $f(x) = \alpha x + \beta x^{2k+1} + o\left(x^{2k+1}\right)$ then

$$\sum |a_n|^{2k+1} \text{ converges } \not\Rightarrow \sum f(a_n) \text{ converges.}$$

Example 11 *Consider the function* $f(x) = \sin x = x + \frac{x^3}{6} + o(x^3)$ *and the series* $\sum \frac{(-1)^n}{\ln n}$*. Then* $\sum \sin\left(\frac{(-1)^n}{\ln n}\right)$ *converges (Leibniz's test) but* $\sum \left|\frac{(-1)^n}{\ln n}\right|^3 = \sum \frac{1}{\ln^3 n}$ *does not.*

We conclude observing that the main restriction to the applicability of the comparison test as suggested by Proposition 3 stems from the constraints given in the case of a Taylor expansion which ends with odd powers. Example 5 shows a case where the procedure suggested in Remarks 1 and 2 applies. Nonetheless refining the Taylor expansion when the function f is odd leaves us with a number of series ($\sum a_n, \sum a_n^3, \ldots, \sum a_n^{2k+1}$) to be studied separately. It would be useful to have the opportunity of deriving some conclusion on $\sum a_n^{2k+1}$ from the behaviour of $\sum a_n^{2h+1}$ for some couple of $k, h \in \mathbb{N}$. Unfortunately this is not possible; in particular neither the convergence nor the divergence of one can be deducted from the convergence or the divergence of the other. Indeed

$$\sum a_n^{2h+1} \begin{array}{c}\text{converges}\\ \text{diverges}\end{array} \quad \not\Rightarrow \quad \sum a_n^{2k+1} \begin{array}{c}\text{converges}\\ \text{diverges}\end{array}$$

[11] Indeed $\sum a_n = \sum \frac{(-1)^k}{\sqrt[4]{2k}}$ converges, also $\sum a_n^3 = \sum \left(\frac{(-1)^{3k}}{(2k)^{\frac{3}{4}}} + \frac{1}{2k} + \frac{(-1)^k}{3(2k)^{\frac{5}{4}}}\right)$ and $\sum a_n^4 = \sum \left(\frac{1}{2k} + 4\frac{(-1)^{3k}}{3(2k)^{\frac{5}{4}}} + 2\frac{1}{3(2k)^{\frac{3}{2}}} + 4\frac{(-1)^k}{27(2k)^{\frac{7}{4}}}\right)$ so $\sum a_n^3$ and $\sum a_n^4$ do not converge; finally $\sum f(a_n) = \sum \left(a_n + a_n^3 - a_n^4\right) = \sum \left(\frac{(-1)^k}{(2k)^{\frac{1}{4}}} + \frac{(-1)^{3k}}{(2k)^{\frac{3}{4}}} + 5\frac{(-1)^k}{3(2k)^{\frac{5}{4}}} + 2\frac{1}{3(2k)^{\frac{3}{2}}} + 4\frac{(-1)^k}{27(2k)^{\frac{7}{4}}}\right)$ which converges.

Example 12 *Given the series* $\sum a_n$ *where*

$$a_{3n-2} = \frac{1}{\sqrt[2k+1]{n}}; \quad a_{3n-1} = -\frac{1}{\sqrt[2h+1]{2}\,\sqrt[2k+1]{n}}; \quad a_{3n} = -\frac{1}{\sqrt[2h+1]{2}\,\sqrt[2k+1]{n}}$$

we have that $\sum a_n^{2h+1}$ *converges to zero but* $\sum a_n^{2k+1}$ *diverges.*

Also if $\sum a_n^{2h+1}$ is oscillatory nothing can be said in general about $\sum a_n^{2k+1}$.

Example 13 *Consider the series* $\sum a_n^{2h+1}$ *with*

$$a_n = \frac{(-1)^i}{n} \text{ where } i \in \mathbb{N} \text{ is such that } 2^{i-1} < n \leq 2^i.$$

As $\left|\frac{(-1)^n}{2^{n-1}+1} + \frac{(-1)^n}{2^{n-1}+2} + \ldots + \frac{(-1)^n}{2^n}\right| > \frac{1}{2}$ *for all* $n \in \mathbb{N}$ *we have that* $\sum a_n^{2h+1}$ *is oscillatory, whereas* $\sum a_n^{2k+1}$ *is convergent for* $k > h$ *as* $\sum \frac{-1}{n^{\frac{2k+1}{2h+1}}} < \sum a_n^{2k+1} < \sum \frac{1}{n^{\frac{2k+1}{2h+1}}}$ *and is oscillatory for* $k < h$.

3 Appendix

Solution to footnote 8.

Let $a_n = \frac{(-1)^n}{n^\gamma} + \frac{1}{n^{\gamma+1}}$. We must show that $\sum b_n = \sum \tan(a_n)$ converges for $0 < \gamma \leq \frac{1}{3}$.

Observe that

$$a_n^k = \left(\frac{(-1)^n}{n^\gamma} + \frac{1}{n^{\gamma+1}}\right)^k = \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{n(k-i)}}{n^{\gamma(k-i)}} \frac{1}{n^{i(\gamma+1)}} = \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{n(k-i)}}{n^{\gamma k+i}}.$$

For all $i > 0$ the series $\sum_n \frac{(-1)^{n(k-i)}}{n^{\gamma k+i}}$ converges absolutely. For $i = 0$ we have two cases; if k is odd then $\sum_n \frac{(-1)^{nk}}{n^{\gamma k}}$ converges by the Leibniz's test whereas if k is even then $\sum_n \frac{(-1)^{nk}}{n^{\gamma k}}$ converges if and only if $\gamma > \frac{1}{k}$; in conclusion

1. if k is odd $\sum_n a_n^k$ converges for all $\gamma > 0$
2. if k is even $\sum_n a_n^k$ converges for all $\gamma > \frac{1}{k}$ (6)

Given $\gamma \in \left(0, \frac{1}{3}\right]$ there is $h \in \mathbb{N}$ such that $\gamma > \frac{1}{2h}$. Now consider the Taylor expansion of $\tan x$ up to order $2h - 1$

$$\tan x = \sum_{k=1}^{h} \frac{x^{2k-1}}{2k-1} + o\left(x^{2h}\right)$$

so

$$\sum_{k=1}^{h} \frac{x^{2k-1}}{2k-1} - x^{2h} < \tan x < \sum_{k=1}^{h} \frac{x^{2k-1}}{2k-1} + x^{2h}$$

and substituting

$$a_n + \frac{a_n^3}{3} + \ldots + \frac{a_n^{2h-1}}{2h-1} - a_n^{2h} < \tan a_n < a_n + \frac{a_n^3}{3} + \ldots + \frac{a_n^{2h-1}}{2h-1} + a_n^{2h}.$$

The result follows from (6) and Theorem 2.

Solution to footnote 9.

Without loss of generality we consider $\alpha \in \left(0, \frac{\pi}{2}\right)$. We want to prove that $\sum \frac{|\sin \alpha n|}{n^\gamma}$ diverges for $0 < \gamma \leq \frac{1}{2}$. Observe that there is a monotone increasing sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\begin{aligned}
\frac{1}{4}\pi &\leq \alpha n_0 \leq \frac{3}{4}\pi \\
\left(1 + \frac{1}{4}\right)\pi &\leq \alpha n_1 \leq \left(1 + \frac{3}{4}\right)\pi \\
&\vdots \\
\left(k + \frac{1}{4}\right)\pi &\leq \alpha n_k \leq \left(k + \frac{3}{4}\right)\pi \\
&\vdots
\end{aligned}$$

from which we obtain

$$\left(\frac{\alpha}{\pi\left(k + \frac{3}{4}\right)}\right)^\gamma \leq \frac{1}{n_k^\gamma} \leq \left(\frac{\alpha}{\pi\left(k + \frac{1}{4}\right)}\right)^\gamma$$

hence

$$\frac{|\sin \alpha n_k|\, \alpha^\gamma}{\pi^\gamma \left(k + \frac{3}{4}\right)^\gamma} \leq \frac{|\sin \alpha n_k|}{n_k^\gamma} \leq \frac{|\sin \alpha n_k|\, \alpha^\gamma}{\pi^\gamma \left(k + \frac{1}{4}\right)^\gamma}$$

and

$$\frac{\alpha^\gamma}{\sqrt{2}\pi^\gamma \left(k + \frac{3}{4}\right)^\gamma} \leq \frac{|\sin \alpha n_k|}{n_k^\gamma} \leq \frac{\alpha^\gamma}{\pi^\gamma \left(k + \frac{1}{4}\right)^\gamma}$$

Therefore $\sum_k \frac{|\sin \alpha n_k|}{n_k^\gamma}$ diverges and so does the series $\sum_n \frac{|\sin \alpha n|}{n^\gamma}$.

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