

Heterogeneous adaptive expectations and cobweb phenomena.

Domenico Colucci and Vincenzo Valori

DIMAD - Università di Firenze, Via C. Lombroso 6/17, 50134 Firenze, Italy.

August 31, 2009

Abstract

We study a cobweb-type commodity market characterised by a strictly monotone demand and supply, in which n firms operate. The firms are assumed to differ in a key parameter governing price expectations which we suppose to be adaptive. We characterise the unique steady state of the resulting economic dynamics in terms of stability and we study the impact of the number and diversity of firms: to this end we introduce the notions of structural and behavioural degree of instability which prove to be crucial in determining whether stability or instability prevail. We also consider the case of market integration and establish conditions to have stability (or instability) in the aggregated market in terms of the original (structural and behavioural) degrees of instability. We take up the issue of transitional dynamics and speed of convergence when the system is stable and characterise parametric configurations that maximise the speed of convergence. Finally, we assume that the firms - via the parameter which defines their expectations - are sampled independently from a population described by a given probability distribution. In this case the structural degree of instability determines how the number of potentially different firms affects the probability of ending up with a stable outcome.

Journal of Economic Literature Classification Numbers: D83; D84; E17; E32

Key Words: heterogeneous agents, expectations, stability of steady states, market integration, speed of convergence

1 Introduction and related literature

The existence of a certain amount of heterogeneity in economic expectations is uncontroversial. Evidence of heterogeneous expectations in inflation forecasts for example (both by professionals and non-professional forecasters) has been found in Branch [1], Carroll [6] and Mankiw et al. [18]. The extent, the variability and the consequences of such expectations disagreement are an open and interesting research question. The range of applications and models in which such question receives attention includes for example monetary policy theory and design (as in Honkapohja and Mitra [15]), models of exchange rate dynamics (e.g. Manzan and Westerhoff [19]) and asset pricing (for recent examples see Buraschi and Jiltsov [4], Jouini and Napp [17]). Heterogeneous expectations have also been invoked to explain stylized facts such as the volume of trade exchange (see e.g. Frankel and Froot [12]). Further, for some types of agricultural markets in which biological lags naturally suggest the use of cobweb models, heterogeneous expectations have been detected and estimated, for instance by Chavas [7]. A rather comprehensive survey on this thread of literature can be found in Hommes [16].

In essence, in this paper we study conditions under which coordination or disagreement of beliefs among individuals of limited rationality emerge and the eventual impact of a moderate degree of expectations' heterogeneity on dynamic stability. In particular we consider the problem of characterizing dynamic stability of equilibria in a cobweb model in which n firms use adaptive expectations with firm-specific gain parameters. This issue can also be considered a generalization of the problems studied by Nerlove [21] and Carlson [5] a few decades ago. The heterogeneity we take into account is indeed moderate because it is limited to a key parameter governing expectations which are otherwise all drawn from the same expectations mechanism. It turns out that in our model two sources of (potential) instability can be identified: a structural source, linked to the market's fundamentals (such as the shape of demand and supply curves) and a behavioural source, embedded in the average profile of expectations. A necessary and sufficient condition for local stability involving these factors is demonstrated. Such condition implies no particular restriction on individual firms, but only on the entire set of firms as a whole. The structural and behavioural sources of instability also show up as we study the effects of changing the number of market's participants and the various possible outcomes of market integration. For the asymptotically stable configurations, we study the speed of convergence and we characterize the situations in which convergence occurs monotonically and those in which the steady state is approached through oscillations giving rise to the traditional cobweb phenomena. Besides, we explain the factors determining the fastest approach towards the steady state.

Motivated by the difficulty to actually observe expectations, whereas it is easier to measure some structural features of a given market, such as the relevant demand and supply price elasticities, we take the perspective of an observer (e.g. a policy maker) whose information set includes the structural but not the behavioural degree of instability. We therefore consider a setup in which the firms involved in the market are sampled independently from a continuous distribution of such firms. In practice, because what defines each firm is their behavioural parameter, we devise a simple

model of random selection of such parameters from a given distribution. We provide probabilities of convergence when only the structural source of instability is known exactly. A form of polarization of convergence probabilities induced by increasing the number of market's participants is documented. When the number of firms gets large, stability is almost certain for levels of the structural degree of instability up to a certain threshold, while the system is almost certainly unstable past the threshold.

The present work is related to a number of papers, some of which include a similar underlying structure based on a cobweb economy with heterogeneous beliefs of some kind, such as Brock and Hommes [3] (whose basic setup we largely borrow), Chiarella et al. [8], Branch and Evans [2]: our paper is different because it does not address endogenous heterogeneity as in the Brock and Hommes tradition nor do we have proper learning as in the literature described in (and stimulated by) the well known book by Evans and Honkapohja [9]. Our perspective is a slightly different, as it prompts us to understand what is to be expected when changes in the number of (different) firms acting on a market are produced, for instance as the result of market integration or merging into a common, bigger marketplace, considering all other details as given.

Negroni [20] investigates a two-agents problem with adaptive expectations which is akin to ours but for the assumed asymmetry in the roles of the agents which we do not have here. A closely related feature also shows up in the paper by Evans and Guesnerie [10], who name it "structural heterogeneity" and show its potential role of destabilising force when coupled with different beliefs. The kind of spirit animating the present work, namely that of studying the outcomes due to changes in the number of different agents in a dynamic model (or the overall agents' diversity) is shared by Herrendorf et al. [14] and, more recently, by Puu [22].

The paper is organised as follows: Section 2 introduces the model and states a couple of results which are then used throughout the paper. The issue of understanding the specific role of the number of firms in shaping stability is addressed in Section 3. Speed of convergence and the dynamics in the transition to the steady state are analysed in Section 4. Section 5 deals with the probability of convergence when the firms' behavioural characteristics are sampled from a given probability distribution. All the proofs are contained in the Appendix.

2 The Model

Consider a cobweb-type commodity market in which each of n firms needs to allow for a production lag and so choose optimal supplied quantities conditioned on the forecasted future price p_i^e . The optimal supply is proportional to firm's size, $\psi_i > 0$, hence

$$S_i(p_i^e) = \psi_i s(p_i^e)$$

where $s(p_i^e)$ depends on the available technology. Demand is a function of the current price $D(p)$. We assume a strictly increasing supply and a strictly decreasing demand, which are smooth and intersecting at a point p^* . By defining $\Psi = \sum_i \psi_i$ as the aggregate production scale factor, $S(\cdot) = \Psi s(\cdot)$ and $\phi_i = \frac{\psi_i}{\Psi}$ as the firm i market

share (with $\sum_{i=1}^n \phi_i = 1$) the aggregate supply becomes

$$\hat{S}(p_{t,1}^e, \dots, p_{t,n}^e) = \Psi \sum_{i=1}^n \phi_i s(p_{t,i}^e) = \sum_{i=1}^n \phi_i S(p_{t,i}^e).$$

Market clearing requires that $D(p_t) = \sum_{i=1}^n \phi_i S(p_{t,i}^e)$ which, because demand is strictly increasing, can be written explicitly as

$$p_t = D^{-1} \left(\sum_{i=1}^n \phi_i S(p_{t,i}^e) \right) \equiv F(p_{t,1}^e, \dots, p_{t,n}^e) \quad (1)$$

Up to this point, the model is rather standard: it differs for example from Brock and Hommes [3] only in the fact that firms dimensions, ψ_i , are kept separated from market shares, ϕ_i . While this is costless in terms of the required algebra - we only need to add the constant Ψ to an otherwise standard model - it will help us in a later discussion about the effects of market size variations on the stability properties of equilibria.

In a different but equally common formulation of this model, aggregation uses a weighted average of expectations as the argument of the actual law of motion (see Branch and Evans [2]): we compare results obtained under such different modelling strategy in Section 2.3.

We close the model assuming that expectations are adaptive with gains that differ across different firms $p_{t+1,i}^e = p_{t,i}^e + \alpha_i (p_t - p_{t,i}^e)$, $i = 1, \dots, n$. Summing up, the evolution of the system can be described by the following system of difference equations

$$\begin{cases} p_{t+1,1}^e = p_{t,1}^e + \alpha_1 (F(p_{t,1}^e, \dots, p_{t,n}^e) - p_{t,1}^e) \\ \vdots \\ p_{t+1,n}^e = p_{t,n}^e + \alpha_n (F(p_{t,1}^e, \dots, p_{t,n}^e) - p_{t,n}^e) \end{cases} \quad (2)$$

The above assumptions on the monotonicity of supply and demand guarantee that there will be a unique steady state for the system (2), corresponding to the supply-demand equilibrium price p^* . Notice that, in spite of the simplicity of adaptive expectations, the number of different firms determines the dimension of the dynamical system (2): this is a distinguishing feature of the model. Assuming for example that the n firms use $AR(p)$ forecasting models (with lags up to a given p) would make the dimension of the system independent of n : so a large number of firms would not, as it does here, complicate the tractability of the model.

2.1 Special case: one representative firm

It is useful to see what happens if there is only one firm. In this case the price equation (1) reduces to $p_t = D^{-1}(\Psi s(p_t^e)) = D^{-1}(S(p_t^e))$ so the system evolves according to

$$p_{t+1}^e = p_t^e + \alpha (D^{-1}(S(p_t^e)) - p_t^e)$$

and the stability condition is $-1 < 1 - \alpha + \alpha \frac{S'(p^*)}{D'(p^*)} < 1$ which, defining $\delta = -\frac{S'(p^*)}{D'(p^*)}$, we can write as $-1 < 1 - \alpha - \alpha\delta < 1$. Using the definition $\beta = \frac{\alpha}{2-\alpha}$ and the fact $\delta > 0$, stability requires that

$$\delta\beta < 1 \quad (3)$$

As it turns out, the two parameters¹ δ, β play a key role throughout the paper. We label δ the *structural degree of instability*. Notice that as δ approaches 1 condition (3) is automatically satisfied for any choice of $\alpha \in (0, 1)$, and if $\delta \leq 1$ stability is always warranted under adaptive expectations. Therefore we assume $\delta > 1$. The parameter β will be called the *behavioural degree of instability*.

2.2 General case: n firms

We now turn to the issue of how stability for the model in its general form with n firms relates to (behavioural) characteristics of the individual firms and to the market's exogenous structure (as given by the demand and supply functions). To this end notice preliminarily that

$$\left. \frac{\partial F(p_{t,1}^e, \dots, p_{t,n}^e)}{\partial p_{t,i}^e} \right|_{p_{t,1}^e = \dots = p_{t,n}^e = p^*} = \phi_i \frac{S'(p^*)}{D'(p^*)} = -\phi_i \delta$$

so the Jacobian matrix of the system (2) evaluated at p^* is:

$$J_n = \begin{pmatrix} 1 - \alpha_1(\phi_1 \delta + 1) & -\alpha_1 \phi_2 \delta & \dots & -\alpha_1 \phi_n \delta \\ -\alpha_2 \phi_1 \delta & 1 - \alpha_2(\phi_2 \delta + 1) & \dots & -\alpha_2 \phi_n \delta \\ \dots & \dots & \ddots & \dots \\ -\alpha_n \phi_1 \delta & -\alpha_n \phi_2 \delta & \dots & 1 - \alpha_n(\phi_n \delta + 1) \end{pmatrix} \quad (4)$$

It seems fairly intuitive that aggregation should preserve stability if the conditions for individual stability derived above in (3) are met for each firm. Indeed this needs not be the case in general, as Franke and Nesemann [11] have shown in a specific case in which two "unstable" learning rules offset each other bringing about a stable outcome. In our context, given the limited degree of freedom in firms behaviour, such phenomenon is not possible and stability (or instability) at the individual level suffice for stability (instability) with many firms. An intuition of this can be given as follows: suppose n firms for which individual stability conditions (3) are met, are aggregated. Let λ be a real eigenvalue² of J_n , with an associated eigenvector $v = (v_1, \dots, v_n)^T$. Then, either $\sum_{j=1}^n \phi_j v_j = 0$ or we can assume, without loss of generality, $\sum_{j=1}^n \phi_j v_j = 1$. In the first case, letting $v_i \neq 0$, we have $\lambda v_i = (1 - \alpha_i) v_i$, so $\lambda = 1 - \alpha_i \in (0, 1)$. Otherwise, there is i such that $v_i \geq \sum_{j=1}^n \phi_j v_j = 1$ so

$$\lambda v_i = (1 - \alpha_i) v_i - \alpha_i \delta \sum_{j=1}^n \phi_j v_j \implies \lambda = 1 - \alpha_i - \alpha_i \delta \frac{1}{v_i}$$

and therefore, using $v_i \geq 1$ and the assumption that conditions for individual stability, $-1 < 1 - \alpha_i - \alpha_i \delta < 1$, are met for every i , we can conclude that the n -dimensional system is stable.

¹Notice that in turn δ depends on a third parameter, namely Ψ , the overall dimension of the economy. This is relevant only when market integration is discussed.

²The Jacobian matrix J_n has real eigenvalues only, as we show in Lemma 1.

It can also be shown, with a little more effort, that if individual instability holds for each firm then instability follows. Notice further that the market can be stable even though stability is not the case for all individual firms: consider, for example, the case of a market with two firms of equal size having $\delta = 2$, $\beta_1 = \frac{1}{4}$ and $\beta_2 = \frac{2}{3}$. As $\delta\beta_1 < 1 < \delta\beta_2$, condition (3) entails that, in isolation, the first firm implies stability whereas the second implies instability, but the aggregated market is stable³. So it is interesting to establish conditions by which stability is produced when two (not both necessarily stable) markets are merged. A more general result encompassing these cases is the focus of this Section.

We shall require the following preliminary result.

Lemma 1 *Consider a matrix*

$$M = \begin{pmatrix} a_1c_1 + b_1 & a_1c_2 & \cdots & a_1c_n \\ a_2c_1 & a_2c_2 + b_2 & \cdots & a_2c_n \\ \vdots & \vdots & \ddots & \vdots \\ a_nc_1 & a_nc_2 & \cdots & a_nc_n + b_n \end{pmatrix} \quad (5)$$

with $a_i < 0, b_i, c_i > 0$ for all i . Then

i) $\det M = \prod_{i=1}^n b_i + \sum_{i=1}^n c_i a_i \prod_{j \neq i} b_j$

ii) M has real eigenvalues

iii) $(n-1)$ eigenvalues of M belong to the interval $[\min_i \{b_i\}, \max_i \{b_i\}]$ and for the smallest eigenvalue of M , $\lambda_{\min}$, it is $\lambda_{\min} < \min_i \{b_i\}$

iv) $\lambda_{\min}$ is greater than -1 if and only if the characteristic polynomial, $P(\lambda)$, is positive at $\lambda = -1$.

Notice that the Jacobian (4) is a particular specification of matrix (5) with $a_i = -\alpha_i, b_i = 1 - \alpha_i, c_i = \phi_i \delta$. Therefore, as a consequence of part ii) and iii) of the above Lemma applied to (4), the steady state of system (2) can loose (acquire) stability only through a Period-doubling bifurcation. When the eigenvalues of (4) are all non-negative the local convergence of expectations, quantities and price to their steady state value is monotone: in that case, a perturbation of the model's parameters does not result in a qualitative change of the dynamics around the steady state. Section 4 returns to this topic at some length.

Let us now turn to the stability properties of the steady state of the market dynamics.

We define the *market degree of behavioural instability* for the n heterogeneous firms case as $\bar{\beta}_n = \sum_{i=1}^n \phi_i \beta_i = \sum_{i=1}^n \phi_i \frac{\alpha_i}{2 - \alpha_i}$. Perhaps surprisingly, stability can be characterised in terms of $\bar{\beta}_n$ and δ in the same way as in the homogeneous case.

Proposition 2 *The steady state of the system (2) is locally stable and hyperbolic (i.e. with eigenvalues strictly inside the unit circle) if and only if $\delta \bar{\beta}_n < 1$.*

The result says that, in order to have stability, the multiplicative combination of structural and behavioural instability must not exceed one. So this establishes a threshold for the aggregate sources of instability in the market, marking the frontier between the stable and the unstable regimes.

³Direct calculation or Proposition 2 below show that such is the case.

2.3 Heterogenous versus Average Representative Agent's markets

At this point the reader might raise the following doubt: if the heterogeneity in the market were incorporated within a single representative firm, would there be any real loss with respect to our more complex model? It turns out that the answer is "yes". We indeed compare the conditions for stability in the homogeneous market with a single firm (that we call *average representative firm*) which is representative in the sense that its adaptive gain parameter is equal to the weighted average of parameters of n given firms, to those for the heterogeneous market with those n firms actually playing directly.

Proposition 3 *Consider a market with n firms defined by gains $\alpha_1, \dots, \alpha_n$ and weights $\phi_1, \dots, \phi_n$ and a market with an average single firm with gain $\alpha = \sum_{i=1}^n \phi_i \alpha_i$. Conditions for stability in the heterogeneous market are sufficient but not necessary for the average homogeneous market.*

The following numerical example shows that indeed one can have a stable average representative firm such that if each firm reproduced by the average were to act directly the outcome would be unstable. Consider $n = 2$, $\phi_{1,2} = 1/2$ and $\delta = 2$. If $\alpha_1 = 1/3$ and $\alpha_2 = 9/10$ we have

$$\begin{aligned}\bar{\beta}_{av} &= \frac{\frac{1}{2} \left(\frac{1}{3} + \frac{9}{10} \right)}{2 - \frac{1}{2} \left(\frac{1}{3} + \frac{9}{10} \right)} = \frac{37}{83} < \frac{1}{2} \Rightarrow \bar{\beta}_{av} \delta < 1 \\ \bar{\beta}_{het} &= \frac{1}{2} \left(\frac{\frac{1}{3}}{2 - \frac{1}{3}} \right) + \frac{1}{2} \left(\frac{\frac{9}{10}}{2 - \frac{9}{10}} \right) = \frac{28}{55} > \frac{1}{2} \Rightarrow \bar{\beta}_{het} \delta > 1\end{aligned}$$

Proposition 3 argues in favour of the idea that heterogeneity matters, from the dynamic stability/instability viewpoint, in that it cannot be safely sterilized by using an average representation instead of the whole heterogeneous picture. In a sense this result also appears to indicate that, as opposed to the average representative firm, heterogeneity implies (or has a potential for) destabilization. More in general, while it is crucial to fix ideas precisely as to what (de)stabilizing heterogeneity means, much depends on the level of structural degree of instability, δ : Section 5 is specifically devoted to this issue.

3 Some comparative statics and market integration

This section deals with the general issue of assessing the effect of changes in the number of firms (and hence of the amount of behavioural heterogeneity) on stability, first in the context of a comparative statics exercise then as a by-product of a process of market integration (whereby two separate markets are merged).

To begin with, suppose there is a change in the number of firms, n , while the aggregate supply and the equilibrium price are unchanged. This is a kind of thought experiment in which two economies, A and B, have the same aggregate demand

and supply schedules, the only difference being in the number of firms behind the aggregation on the supply side. The purpose is to isolate the effect of changing the assortment of different beliefs held by the firms. It is indeed a thought experiment in the sense that it is difficult to imagine real situations in which a change in n leaves the aggregate supply and therefore the equilibrium price unchanged. A more realistic situation arises for example with market integration and we shall move to this problem later in the section.

We assume that economy A is populated by a given set of firms whereas in economy B an extra group of firms, J, join in the supply side. Because the aggregate supply has to remain the same we shall make the following assumptions: *i*) the overall weight of the entering firms is $1 - \rho$ compared to that of the original firms which is ρ ; *ii*) the relative dimension of each firm in economy B, $\bar{\phi}_i$, is obtained by rescaling its original dimension, ϕ_i , with the appropriate overall weight (ρ or $1 - \rho$).

Letting $\bar{\beta}_A$, $\bar{\beta}_J$ and $\bar{\beta}_B$ the aggregate degree of behavioural instability in economy A, of the joining firms, and in economy B, we have

$$\begin{aligned}\bar{\beta}_B &= \sum_{i \in B} \bar{\phi}_i \frac{\alpha_i}{2 - \alpha_i} = \sum_{i \in A} \rho \phi_i \frac{\alpha_i}{2 - \alpha_i} + \sum_{i \in J} (1 - \rho) \phi_i \frac{\alpha_i}{2 - \alpha_i} \\ &= \rho \bar{\beta}_A + (1 - \rho) \bar{\beta}_J\end{aligned}$$

In light of the above assumptions, the degree of structural instability, δ , is the same in both economies.

The following result is now easy to prove. Due to Proposition 2 stability (or instability) in economy B depends on whether $\delta \bar{\beta}_B$ is smaller (or larger) than 1. Trivially $\delta \bar{\beta}_B = \delta (\rho \bar{\beta}_A + (1 - \rho) \bar{\beta}_J) \leq 1$ when $\delta \bar{\beta}_A, \delta \bar{\beta}_J \leq 1$; therefore if economy A is stable so is economy B if $\delta \bar{\beta}_J < 1$. If instead $\delta \bar{\beta}_J > 1$ then economy B is stable if and only if $\rho > \frac{\delta \bar{\beta}_J - 1}{\delta \bar{\beta}_J - \delta \bar{\beta}_A}$. The same goes for instability. So, in this exercise, stability (or instability) persists when a larger span of jointly stable (or unstable) firms is allowed for.

Also, when the firms in group J would by themselves imply instability, the outcome depends crucially on their relative weight $1 - \rho$: such weight needs to be under (above) a threshold which is a function of the structural and behavioural instability, $\delta, \bar{\beta}_A$ and $\bar{\beta}_J$, in order for the outcome to be stable (unstable). Again this is a rather intuitive result. We can also derive a measure of (in)stability robustness to entry for a given market. In other words we can answer the following question: what is the minimum relative weight of joining firms that can destabilise the market? Intuitively, the worst that can happen is the entry of a group of firms of weight $1 - \rho$ having a behavioural degree of instability of 1 (i.e. with $\alpha_i = 1$ for all i). In that case $\rho < \frac{1 - \delta \bar{\beta}_A}{\delta \bar{\beta}_J - \delta \bar{\beta}_A}$ would make the system unstable. Remark that this argument relies on the idea that choosing $\alpha_i = 1$ for all entrant firms is the "worst that can happen". This is indeed the case because the largest eigenvalue is always smaller than 1 while the smallest one is strictly decreasing in α_i as we show in the following Lemma.

Lemma 4 *If $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of the Jacobian (4), then $\lambda_{\min}$ strictly decreases with α_i , for all i , while $\lambda_{\max}$ weakly decreases with α_i , for all i and $\lambda_{\max} \leq 1 - \alpha_{\min}$.*

Finally we can also determine a kind of "central value" for the degree of structural instability, δ . To do so, we compute the stability threshold ratio of firms in the polar case of $\bar{\beta}_A = 0$ and $\bar{\beta}_B = 1$: it is easy to see that in this case it equals $1/\delta$, therefore when $\delta = 2$ there is a threshold of 50% separating the stable and the unstable regimes when firms are only either static or myopic (i.e. when the gain α_i is either 0 or 1). Remarkably, the value of $\delta = 2$ can be described as a central value also for other reasons, as we show in Section 5.

To make these points more persuasive, though, it is worth developing more carefully on the issue of market integration.

3.1 Market integration

Suppose two markets that were previously independent are aggregated. Assume that demand and supply are both strictly monotone in both markets so that the equilibrium price in the integrated market will be intermediate between the two original equilibria. The two markets are characterised by the vectors of parameters $(\bar{\beta}_A, \delta_A, \Psi_A)$, $(\bar{\beta}_B, \delta_B, \Psi_B)$. Let $p_A^* < p_B^*$ be the market equilibria defined by supply and demand. The aggregated market will have $(\bar{\beta}, \delta, p^*, \Psi)$ where $\Psi = \Psi_A + \Psi_B$, $\bar{\beta} = \frac{\Psi_A}{\Psi}\bar{\beta}_A + \frac{\Psi_B}{\Psi}\bar{\beta}_B$, and $p_A^* < p^* < p_B^*$. In general allowing for different demand and supply functions in the two original markets implies $\delta = -\frac{\Psi_A s'_A(p^*) + \Psi_B s'_B(p^*)}{D'_A(p^*) + D'_B(p^*)}$ which does not trivially compare with δ_A and δ_B . However, assuming linear demand functions $D_A(\cdot)$ and $D_B(\cdot)$ with slopes D'_A and D'_B and a common linear $s(\cdot)$ with slope s' we have $\delta_A = -\frac{\Psi_A s'}{D'_A}$, $\delta_B = -\frac{\Psi_B s'}{D'_B}$ and

$$\delta = -\frac{\Psi_A s' + \Psi_B s'}{D'_A + D'_B} = \frac{1}{\frac{\Psi_A}{\Psi} \frac{1}{\delta_A} + \frac{\Psi_B}{\Psi} \frac{1}{\delta_B}}.$$

Hence

$$\bar{\beta}\delta = \left(\frac{\Psi_A}{\Psi}\bar{\beta}_A + \frac{\Psi_B}{\Psi}\bar{\beta}_B \right) \frac{1}{\frac{\Psi_A}{\Psi} \frac{1}{\delta_A} + \frac{\Psi_B}{\Psi} \frac{1}{\delta_B}} \geq \frac{\frac{\Psi_A}{\Psi} \frac{1}{\delta_A} + \frac{\Psi_B}{\Psi} \frac{1}{\delta_B}}{\frac{\Psi_A}{\Psi} \frac{1}{\delta_A} + \frac{\Psi_B}{\Psi} \frac{1}{\delta_B}} = 1$$

if $\bar{\beta}_A \delta_A \geq 1$, $\bar{\beta}_B \delta_B \geq 1$ ⁴. Therefore, under such strong assumptions things remain very similar to our comparative statics scenario above: stability (resp. instability) of the integrated market immediately follows from stability (resp. instability) of the original ones. It is nevertheless worth remarking that this needs not be the case if less is assumed about the demand and the supply functions. Here is an example in which stability in the original markets is not robust to market integration (see Figure 1):

$S(p) = \tanh(p - 1) + 1$	$D_A(p) = \frac{9-5\ln 2}{10} - \frac{1}{2}p$	$D_B(p) = \frac{21+5\ln 2}{10} - \frac{1}{2}p$
$p_A^* = 1 - \ln 2$	$p_B^* = 1 + \ln 2$	$p^* = 1$
$\delta_A = \delta_B = 2S'(1 - \ln 2) = \frac{32}{25}$	$\delta = \frac{2S'(1)}{\frac{1}{2} + \frac{1}{2}} = 2$	$\bar{\beta}_A = \bar{\beta}_B = \frac{5}{8}$

⁴Observe that this reflects a well known equality involving arithmetic and harmonic means. Namely, that the harmonic mean of given non-zero numbers $a_1, \dots, a_n$ times the arithmetic mean of $a_1^{-1}, \dots, a_n^{-1}$ equals 1. See e.g. Hardy et al. [13], p. 14.

$$\Rightarrow \delta_A \bar{\beta}_A = \delta_B \bar{\beta}_B = \frac{4}{5} < 1 < \delta \bar{\beta} = \frac{5}{4}.$$

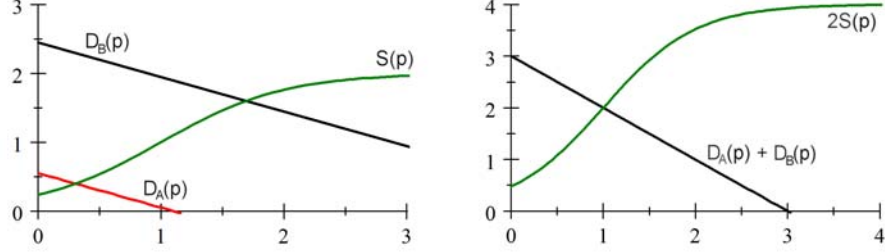


Figure 1: Stability for markets considered separately is lost under aggregation.

Analogous examples where unstable markets integrate into a stable larger market can also be given. One may wonder whether the result in our example is driven by the change of concavity in the supply. To clarify this point we focus on the case where demand and supply in the two original markets, A and B, differ only by a scale factor. So we have

$$\begin{aligned} S_A(\cdot) &= \Psi_A s(\cdot), & D_A(\cdot) &= \Theta_A d(\cdot) \\ S_B(\cdot) &= \Psi_B s(\cdot), & D_B(\cdot) &= \Theta_B d(\cdot) \end{aligned}$$

with $s'(\cdot) > 0$ and $d'(\cdot) < 0$. As before

$$\delta_A = -\frac{\Psi_A s'(p_A^*)}{\Theta_A d'(p_A^*)}, \quad \delta_B = -\frac{\Psi_B s'(p_B^*)}{\Theta_B d'(p_B^*)}$$

Proposition 5 Assume that $\delta_A \bar{\beta}_A < 1$, $\delta_B \bar{\beta}_B < 1$ and $\frac{\Psi_A}{\Theta_A} > \frac{\Psi_B}{\Theta_B}$. If either

$$s'(p) d''(p) - s''(p) d'(p) \geq 0 \quad \forall p \in [p_A^*, p_B^*] \quad \wedge \quad \frac{\Psi_A/\Theta_A}{\Psi_B/\Theta_B} \delta_B \bar{\beta}_A < 1 \quad (6)$$

or, alternatively

$$s'(p) d''(p) - s''(p) d'(p) \leq 0 \quad \forall p \in [p_A^*, p_B^*] \quad \wedge \quad \frac{\Psi_B/\Theta_B}{\Psi_A/\Theta_A} \delta_A \bar{\beta}_B < 1 \quad (7)$$

then $\delta \bar{\beta} < 1$ and hence stability carries through to the integrated market.

The above proposition shows some of the possible extra requirements that guarantee that stability be robust to market integration. A case in which things are easy is when $d(p) = p^{-k}$, $s(p) = p^h$, $k, h > 0$ because δ is a constant equal to $\frac{h}{k}$.

Observe that the assumption $s'(p)d''(p) - s''(p)d'(p) \stackrel{(\leq)}{\geq} 0$, which ensures that the price equilibrium map is monotone, is the same as $d''(p)/d'(p) \stackrel{(\geq)}{\leq} s''(p)/s'(p)$ which in turn means that the elasticity of $d'(p)$ has to be smaller (larger) than the elasticity of $s'(p)$. Further, notice the role played by each market's specific parameters in the technical condition $\frac{\Psi_A/\Theta_A}{\Psi_B/\Theta_B}\delta_B\bar{\beta}_A < 1$ $\left(\frac{\Psi_B/\Theta_B}{\Psi_A/\Theta_A}\delta_A\bar{\beta}_B < 1\right)$, which imposes a cross-market constraint on the parameters compatible with persistence of stability under aggregation.

4 Speed of convergence and cobweb phenomena

When the steady state is locally stable it is interesting to look for more insights about the path of convergence to the equilibrium. The persistent fluctuating pattern of prices in specific agricultural markets, originally attracted the attention of the economics profession in the 1930s and propelled the development of the cobweb literature. Such ongoing phenomena of recurring price oscillations, which fully retain their interest, prompt us to identify conditions under which our model implies oscillatory dynamics, in particular along converging paths.

First observe that the model allows both for monotone and for non-monotone convergence, depending on the parameters values.

Proposition 6 *The system (2) shows monotonic local convergence to the steady state if and only if $\sum_{i=1}^n \phi_i \frac{\alpha_i}{(1-\alpha_i)} < \frac{1}{\delta}$*

Notice that the left hand side of the above inequality tends to 0 with the α_i , so monotone convergence is always possible independently of the market's structural degree of instability level. Furthermore, as the greatest eigenvalue cannot exceed 1, the robustness of the market stability to parameters perturbations is stronger when convergence is monotone and also, due to Proposition (4), it increases when the α_i decrease. On the contrary, the speed of convergence⁵ to the steady state is higher when convergence is non-monotone as stated in the following Proposition:

Proposition 7 *For the system (2), with largest and smallest eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$ of the Jacobian (4), the speed of convergence to the steady state is maximised only if $\lambda_{\max} = -\lambda_{\min}$.*

Notice that the above symmetry condition on the largest and smallest eigenvalues is necessary but it is not sufficient.

In order to fully characterise parametric configurations that maximise the speed of convergence a few more steps are required. First remark that, if both gain parameters and firms' weights are variable, then the problem of maximising the speed of convergence is unbounded. Indeed, to see this consider the following case

$$\begin{aligned}\phi_1 &= 1 - \varepsilon, \quad \alpha_1 = \frac{1}{1 + \delta} \\ \phi_2 &= \dots = \phi_n = \frac{\varepsilon}{n - 1}, \quad \alpha_2 = \dots = \alpha_n = 1 - \varepsilon\end{aligned}$$

⁵We define *Speed of Convergence* to the steady state the quantity $\sigma = -\ln(\rho(J_n))$, where $\rho(J_n)$ is the spectral radius of the Jacobian matrix (4).

implying that, as $\varepsilon \rightarrow 0$ the Jacobian matrix approaches the following

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\delta & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ -\delta & 0 & \cdots & 0 \end{pmatrix}$$

for which the speed of convergence is unbounded.

Suppose instead that market shares are equal to $1/n$. In this case we prove that the configuration of gain parameters that maximises the speed of convergence is homogeneous.

Proposition 8 *If $\phi_1 = \cdots = \phi_n = 1/n$ then the maximum speed of convergence to the steady state is $\ln\left(\frac{\delta+2}{\delta}\right)$ and it is attained if and only if $\alpha_1 = \cdots = \alpha_n = \frac{2}{\delta+2}$.*

5 Implications of firms' number and composition on the probability of stability

In the kind of markets we have in mind, heterogeneity has to do with the number of different types of firms operating in the market; in turn different firms are characterised by a different behavioural degree of instability, β_i . Therefore the number of firms in the market is a rough measure of heterogeneity. If the level of heterogeneity and/or the firms' composition in the market changes as a result of policies or exogenous structural breaks, what kind of consequences are to be expected on the system's stability? Knowing little (or nothing) about the nature of the process, we can make inferences on the behavioural characteristic of the firms entering the market under reasonable assumptions on the distribution of characters in the whole population.

Consider, for example, a market with a representative firm. Suppose that its behavioural parameter β is unknown and that it can be considered as the realization of a random variable uniformly distributed on the unit interval, $U(0, 1)$. Stability is warranted in this case if $\delta\beta < 1$ (see (3)), so the probability of such event, for a given structural degree of instability $\delta > 1$, will be $\int_0^{1/\delta} dx = \frac{1}{\delta}$. One may wonder how this probability will be affected if $n > 1$ or more in general if n varies. We shall define a *stable sample* of behavioural parameters as one entailing the corresponding system (2) has a locally stable steady state (p^*), which means, thanks to the characterisation provided by Proposition 2, that $\delta\bar{\beta}_n < 1$. We look for the probability of drawing a stable sample as a function of δ , for a given n . Assuming that the β_i are drawn independently from $U(0, 1)$ the expected value of $\bar{\beta}_n$ is $1/2$. This means the value $\delta = 2$ makes the expected value of $\delta\bar{\beta}_n$ equal 1. But because the distribution of $\bar{\beta}_n$ is symmetric the probability of a stable sample when $\delta = 2$ is exactly $1/2$. Notice that in this case using the known form of the density for $\bar{\beta}_n$, $f_n(x)$, we can write down the explicit probability functions for any n , mapping values of δ with the probability of a stable sample:

$$P_n(\delta) = \int_0^{1/\delta} f_n(x) dx \quad (8)$$

so we can write the probability functions explicitly⁶. These functions are obviously decreasing in δ . Figure 2 depicts such functions for various n . Notice that heterogeneity appears to have a stabilising/destabilising impact depending on whether δ is less/more than a critical value (2 in this case). Some sort of *polarization effect* seems to be at work. Indeed we show that both these observations generalise easily beyond this example based on the uniform distribution. First, because of the stability condition $\delta\bar{\beta}_n < 1$, the probability of a stable n -sample is the probability of having $\bar{\beta}_n < 1/\delta$, which decreases with δ irrespective of the population distribution. Second, we show that polarization is robust, using an argument based on what happens taking the limit for $n \rightarrow \infty$ in a fairly general setting.

Proposition 9 *Let $f_n(x)$ the density for $\bar{\beta}_n$ as the result of sampling the β_i from some distribution over the unit interval with $E(\bar{\beta}_n) = \bar{\beta}$, and $P_n(\delta)$ the probability of a stable sample. As $n \rightarrow \infty$, $P_n(\delta)$ converges pointwise to*

$$P_\infty(\delta) = \begin{cases} 1 & \text{if } 1 < \delta < 1/\bar{\beta} \\ 0 & \text{if } \delta \geq 1/\bar{\beta} \end{cases}$$

The above proposition entails that increasing n has the effect of making stability or instability (depending on δ) more and more likely. Figure 2 witnesses this fact quite clearly.

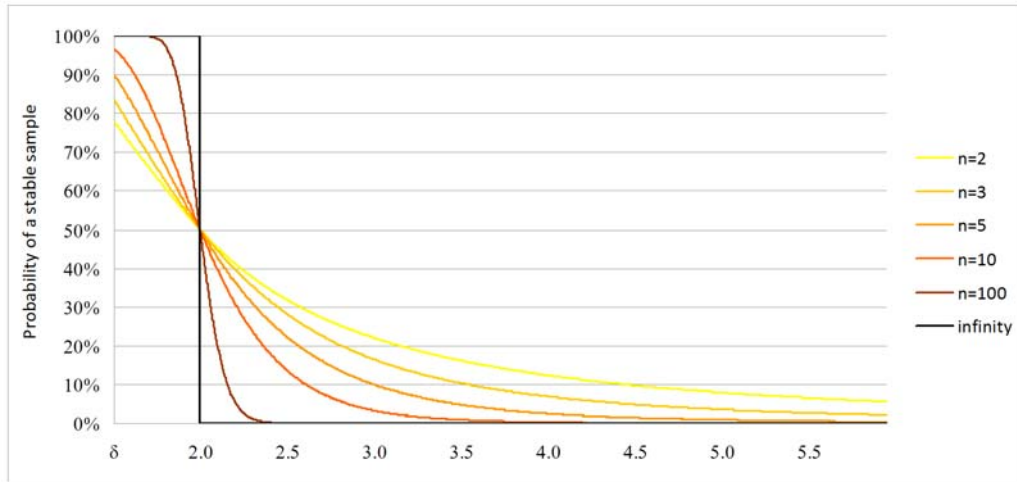


Figure 2: Probability of a stable sample of betas.

⁶The density takes the form $\frac{\sum_{k=0}^n (-1)^k \binom{n}{k} (x-k)^{n-1} \text{sign}(x-k)}{2(n-1)!}$. So, for example, for $n = 2, 3, 4$ we have $P_2(\delta) = \begin{cases} \frac{-2+4\delta-\delta^2}{\delta^2} & \text{if } 1 < \delta \leq 2 \\ 0 & \text{if } \delta > 2 \end{cases}$, $P_3(\delta) = \begin{cases} \frac{9-27\delta+27\delta^2-7\delta^3}{2\delta^3} & \text{if } 1 < \delta \leq 3/2 \\ \frac{-18+27\delta-9\delta^2+\delta^3}{2\delta^3} & \text{if } 3/2 < \delta \leq 3 \\ 0 & \text{if } \delta > 3 \end{cases}$, $P_4(\delta) = \begin{cases} \frac{-32+128\delta-192\delta^2+128\delta^3-29\delta^4}{3\delta^4} & \text{if } 1 < \delta \leq 4/3 \\ \frac{192-512\delta+480\delta^2-176\delta^3+23\delta^4}{6\delta^4} & \text{if } 4/3 < \delta \leq 2 \\ \frac{-192+256\delta-96\delta^2+16\delta^3-\delta^4}{6\delta^4} & \text{if } 2 < \delta \leq 4 \\ 0 & \text{if } \delta > 4 \end{cases}$

Another example, useful to illustrate the polarization effect, is as follows: imagine to draw a sample of n values for the α_i . The expected value of $\bar{\beta}_n$ in this case is $E\left(\sum_{i=1}^n \phi_i \frac{\alpha_i}{2-\alpha_i}\right) = \sum_{i=1}^n \phi_i E\left(\frac{\alpha_i}{2-\alpha_i}\right) = \int_0^1 \frac{\alpha_i}{2-\alpha_i} d\alpha_i = \ln 4 - 1 \simeq 0.39$ which shows that sampling these behavioural parameters instead of the instability degrees returns a distribution for $\bar{\beta}_n$ more geared towards low values. In principle it would be possible to work out the distribution for $\bar{\beta}_n$, just as above: since it does not add much insight (while the algebra is more tedious), we just provide a (numerically obtained) picture similar to the first example, in Figure 3.

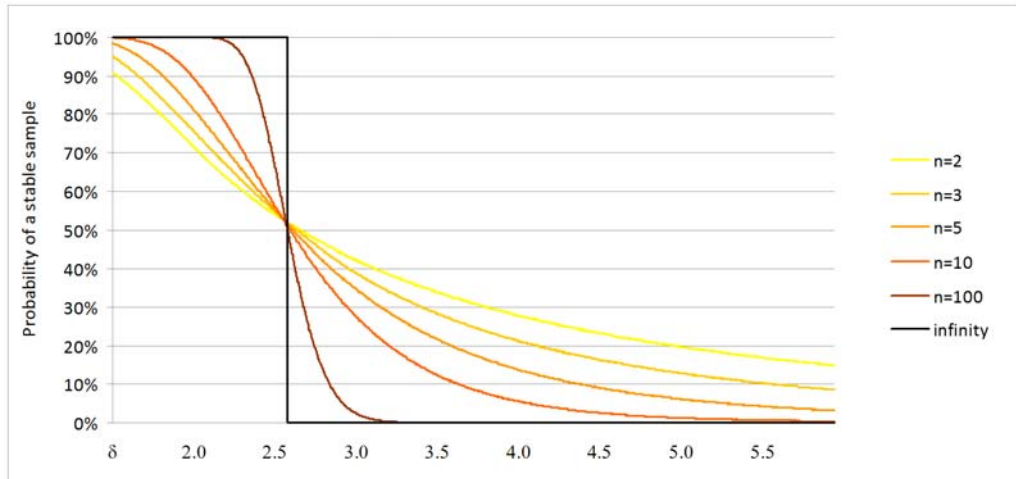


Figure 3: Probability of a stable sample of alphas.

6 Conclusions

We have analysed the dynamic consequences of expectations heterogeneity in a fairly general cobweb model with n firms, each resorting to adaptive expectations with a specific gain parameter. The concepts of structural and behavioural degree of instability were introduced to distinguish the different possible sources of failures to converge to the unique steady state in the model. In particular the behavioural degree of instability depends exclusively on the sensitivity of firms' expectations. Stability is shown to obtain if and only if the product of the two sources of instability is less than one. Within the model, we have clarified how marketwise outcomes are grounded in individual firms' characteristics and how a representative agent assumption can inaccurately predict a stable outcome when the whole heterogeneous picture implies otherwise. Conditions that make stability robust to market aggregation are provided and the speed of convergence to the steady state for stable configurations has been investigated. Finally we have studied a simple model of random selection of firms that takes into account the difficulty of observing individual expectations reliably and directly; our setup allows us to calculate the probability of a stable outcome, given the number of firms and the structural degree of instability. A form of polarization is documented, by which when the number of firms is large, stability most likely

obtains for levels of the structural degree of instability up to a certain threshold, while instability is almost certain past the threshold.

7 Appendix

Proof of Lemma 1.

i) Consider first the simpler case in which $c_1 = \dots = c_n = 1$. Observe that

$$\det M = \det N$$

$$N = \begin{pmatrix} a_1 + b_1 & -b_1 & \dots & -b_1 \\ a_2 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \dots & b_n \end{pmatrix}$$

where N is obtained from M subtracting its first column from the remaining columns. The equality of the two determinants stems from multilinearity in columns. Developing the determinant of N along the first column clearly shows that it is linear in each a_i . As a consequence, expressing $\det M$ as sum of products along permutations of the column indices, the only terms that do not cancel out are contained in the product of terms along the diagonal (any other permutation contains products of the type $a_i a_j$, $i \neq j$). Eliminating terms that involve such products between different a_i 's from the product along the diagonal gives the required result. Now let

$$A = \begin{pmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_2 & \dots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \dots & a_n \end{pmatrix}, B = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{pmatrix}, C = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{pmatrix}$$

For values of the $c_i \neq 1$ the same results applies remarking that

$$M = AC + B = (A + BC^{-1})C$$

therefore $\det M = \left(\prod_{i=1}^n \frac{b_i}{c_i} + \sum_{i=1}^n a_i \prod_{j \neq i} \frac{b_j}{c_j} \right) \prod_{i=1}^n c_i = \prod_{i=1}^n b_i + \sum_{i=1}^n c_i a_i \prod_{j \neq i} b_j$ as stated.

ii) Using i) the characteristic polynomial of M writes as

$$P(\lambda) = \prod_{i=1}^n (b_i - \lambda) + \sum_{i=1}^n c_i a_i \prod_{j \neq i} (b_j - \lambda)$$

Suppose that $b_1 > b_2 > \dots > b_n$. Then

$$\begin{aligned} n \text{ even} &\Rightarrow P(b_1) > 0, P(b_2) < 0, \dots, P(b_n) < 0 \\ n \text{ odd} &\Rightarrow P(b_1) < 0, P(b_2) > 0, \dots, P(b_n) < 0 \end{aligned} \tag{9}$$

Therefore $P(\lambda)$ has $n-1$ real roots and hence n real roots. Suppose more in general that the set of b_i s is as follows:

$$\left\{ \underbrace{b_1, \dots, b_1}_{n_1 \text{ times}}, \underbrace{b_2, \dots, b_2}_{n_2 \text{ times}}, \dots, \underbrace{b_m, \dots, b_m}_{n_m \text{ times}} \right\}$$

with $\sum_{j=1}^m n_j = n$. Let also $k_1 = \{1, \dots, n_1\}$, $k_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, $\dots$, $k_m = \{\sum_{j=1}^{m-1} n_j + 1, \dots, n\}$ and $\bar{a}_i = \sum_{j \in k_i} c_j a_j$. Then

$$\begin{aligned} P(\lambda) &= \prod_{i=1}^m (b_i - \lambda)^{n_i} + \sum_{i=1}^n \bar{a}_i (b_i - \lambda)^{n_i-1} \prod_{j \neq i} (b_j - \lambda)^{n_j} \\ &= \prod_{i=1}^m (b_i - \lambda)^{n_i-1} \left(\prod_{i=1}^m b_i + \sum_{i=1}^n \bar{a}_i \prod_{j \neq i} (b_j - \lambda) \right) \\ &\equiv P_1(\lambda) P_2(\lambda) \end{aligned} \quad (10)$$

where, counting multiplicity, $P_1(\lambda)$ has $n - m$ real roots and $P_2(\lambda)$ has m real roots (this stems from what we showed for the case of distinct b_i s).

iii) and iv) From (10) we have that $n - m$ eigenvalues take values in $\{b_i\}_{i=1, \dots, n}$. Also, from (9), it follows that $m - 1$ eigenvalues belong to (b_n, b_1) . Finally, as $P(b_n) < 0$ and $\lim_{\lambda \rightarrow -\infty} P(\lambda) = +\infty$, the remaining root of $P(\lambda)$ must be smaller than b_n and therefore it is greater than -1 if and only if $P(-1) > 0$. ■

Proof of Proposition 2. Recalling that Jacobian (4) is a particular specification of matrix (5) with $a_i = -\alpha_i$, $b_i = 1 - \alpha_i$, $c_i = \phi_i \delta$, Lemma 1 part *iv)* states that local stability is equivalent to having the characteristic polynomial positive when evaluated at -1 , $P(-1) > 0$. In the case of matrix J_n we have

$$\begin{aligned} P_n(-1) &= \prod_{j=1}^n (2 - \alpha_j) - \delta \sum_{i=1}^n \alpha_i \phi_i \prod_{j \neq i} (2 - \alpha_j) \\ &= \prod_{j=1}^n (2 - \alpha_j) \left(1 - \delta \sum_{i=1}^n \phi_i \frac{\alpha_i}{2 - \alpha_i} \right) \\ &= \prod_{j=1}^n (2 - \alpha_j) (1 - \delta \bar{\beta}_n) \end{aligned}$$

which is positive if and only if $\delta \bar{\beta}_n < 1$. ■

Proof of Proposition 3. Stability conditions for the two cases (see Proposition 2) imply:

$$\begin{aligned} \sum_{i=1}^n \phi_i \frac{\alpha_i}{2 - \alpha_i} &< \frac{1}{\delta} \text{ for the heterogeneous market} \\ \frac{\sum_{i=1}^n \phi_i \alpha_i}{2 - \sum_{i=1}^n \phi_i \alpha_i} &< \frac{1}{\delta} \text{ for the homogeneous market} \end{aligned}$$

Observe that the function $f(x) = \frac{x}{2-x}$ is strictly convex in $[0, 1]$ so necessarily

$$\frac{\sum_{i=1}^n \phi_i \alpha_i}{2 - \sum_{i=1}^n \phi_i \alpha_i} < \sum_{i=1}^n \phi_i \frac{\alpha_i}{2 - \alpha_i}$$

which gives the desired result. ■

Proof of Lemma 4. Consider the derivative with respect to α_i of the characteristic polynomial,

$$\begin{aligned}\frac{\partial P(\lambda)}{\partial \alpha_i} &= \frac{\partial}{\partial \alpha_i} \left(\prod_{j=1}^n (1 - \alpha_j - \lambda) - \sum_{k=1}^n \alpha_k \phi_k \delta \prod_{j \neq k} (1 - \alpha_j - \lambda) \right) \\ &= - \prod_{j \neq i} (1 - \alpha_j - \lambda) - \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda) + \sum_{k \neq i} \alpha_k \phi_k \delta \prod_{j \neq k, i} (1 - \alpha_j - \lambda)\end{aligned}$$

Recall that, given Lemma 1 iii), for all i , $\lambda_{\min} < 1 - \alpha_i$ while $\lambda_{\max} \leq 1 - \alpha_{\min}$. Whenever $\lambda \neq 1 - \alpha_i$

$$\begin{aligned}\frac{\partial P(\lambda)}{\partial \alpha_i} &= - \frac{\prod_{j=1}^n (1 - \alpha_j - \lambda)}{(1 - \alpha_i - \lambda)} - \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda) + \frac{\sum_{k \neq i} \alpha_k \phi_k \delta \prod_{j \neq k} (1 - \alpha_j - \lambda)}{(1 - \alpha_i - \lambda)} \\ &= - \frac{\prod_{j=1}^n (1 - \alpha_j - \lambda) - \sum_{k=1}^n \alpha_k \phi_k \delta \prod_{j \neq k} (1 - \alpha_j - \lambda)}{(1 - \alpha_i - \lambda)} - \left(1 + \frac{\alpha_i}{1 - \alpha_i - \lambda} \right) \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda) \\ &= - \frac{P(\lambda)}{(1 - \alpha_i - \lambda)} - \left(1 + \frac{\alpha_i}{1 - \alpha_i - \lambda} \right) \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda)\end{aligned}$$

So evaluating the derivative at $\lambda_{\min}$, we have

$$\left. \frac{\partial P(\lambda)}{\partial \alpha_i} \right|_{\lambda=\lambda_{\min}} = - \left(1 + \frac{\alpha_i}{1 - \alpha_i - \lambda_{\min}} \right) \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda_{\min}) < 0$$

as $P(\lambda_{\min}) = 0$ and, due to part iii) of Lemma 1, $(1 - \alpha_j - \lambda_{\min}) > 0$ for all j . Finally, because $\lim_{\lambda \rightarrow -\infty} P(\lambda) = +\infty$ the result follows from the intermediate value theorem. Besides, at $\lambda_{\max}$, we have

$$\left. \frac{\partial P(\lambda)}{\partial \alpha_i} \right|_{\lambda=\lambda_{\max}} = - \left(1 + \frac{\alpha_i}{1 - \alpha_i - \lambda_{\max}} \right) \phi_i \delta \prod_{j \neq i} (1 - \alpha_j - \lambda_{\max}) \begin{cases} > 0 & \text{if } n \text{ is even} \\ < 0 & \text{if } n \text{ is odd} \end{cases}$$

due to Lemma 1 iii) and $\lambda_{\max} \neq 1 - \alpha_i$. Again, because $\lim_{\lambda \rightarrow +\infty} P(\lambda) = (-1)^n \infty$ the result follows from the intermediate value theorem. It remains to consider the case $\lambda_{\max} = 1 - \alpha_{\min}$. If $\alpha_i \neq \alpha_{\min}$ then trivially

$$\left. \frac{\partial P(\lambda)}{\partial \alpha_i} \right|_{\lambda=\lambda_{\max}} = 0$$

Otherwise if $\alpha_i = \alpha_{\min}$ the result directly follows from part iii) of Lemma 1. ■

Proof of Proposition 5. Notice: $\frac{\Psi_A}{\Theta_A} > \frac{\Psi_B}{\Theta_B} \Rightarrow \frac{\Psi_B}{\Theta_B} < \frac{\Psi_A + \Psi_B}{\Theta_A + \Theta_B} < \frac{\Psi_A}{\Theta_A}$; also $p_A^* < p^* < p_B^*$. Assumptions in (6) imply that

$$\left(-\frac{s'(p)}{d'(p)} \right)' = \frac{-s''(p) d'(p) + s'(p) d''(p)}{(d'(p))^2} \geq 0$$

As a result

$$\begin{aligned}
\delta \bar{\beta} &= -\frac{\left(\frac{\Psi_A + \Psi_B}{\Theta_A + \Theta_B}\right) s'(p^*)}{d'(p^*)} \left(\frac{\Psi_A}{\Psi} \bar{\beta}_A + \frac{\Psi_B}{\Psi} \bar{\beta}_B\right) \\
&\leq \left(\frac{\Psi_A + \Psi_B}{\Theta_A + \Theta_B}\right) \left(-\frac{s'(p_B^*)}{d'(p_B^*)}\right) \left(\frac{\Psi_A}{\Psi} \bar{\beta}_A + \frac{\Psi_B}{\Psi} \bar{\beta}_B\right) \\
&= \left(\frac{\Psi_A}{\Theta_A + \Theta_B} \bar{\beta}_A\right) \left(-\frac{s'(p_B^*)}{d'(p_B^*)}\right) + \left(\frac{\Psi_B}{\Theta_A + \Theta_B} \bar{\beta}_B\right) \left(-\frac{s'(p_B^*)}{d'(p_B^*)}\right) \\
&= \frac{\Theta_A}{\Theta_A + \Theta_B} \left(\frac{\Psi_A}{\Theta_A} \bar{\beta}_A\right) \left(-\frac{s'(p_B^*)}{d'(p_B^*)}\right) + \delta_B \bar{\beta}_B \frac{\Theta_B}{\Theta_A + \Theta_B} \\
&< \frac{\Theta_A}{\Theta_A + \Theta_B} + \frac{\Theta_B}{\Theta_A + \Theta_B} = 1
\end{aligned}$$

The proof under the alternative assumptions in (7) is identical. ■

Proof of Proposition 6. Following the same argument in the proof of part iv) of Lemma 1, $\lambda_{\min}$ is greater than 0 if and only if the characteristic polynomial, $P(\lambda)$, is positive at $\lambda = 0$. As

$$P(0) = \prod_{i=1}^n (1 - \alpha_i) - \sum_{i=1}^n \alpha_i \phi_i \delta \prod_{j \neq i} (1 - \alpha_j) = \prod_{i=1}^n (1 - \alpha_i) \left(1 - \sum_{i=1}^n \frac{\alpha_i \phi_i \delta}{(1 - \alpha_i)}\right)$$

we have $P(0) > 0$ if and only if

$$\sum_{i=1}^n \phi_i \frac{\alpha_i}{(1 - \alpha_i)} < \frac{1}{\delta}$$

■

Proof of Proposition 7. Suppose for example that $\lambda_{\max} > -\lambda_{\min}$. Then, using Lemma 4, for a small enough increase of a suitably chosen α_i both $\lambda_{\max}$ and $\lambda_{\min}$ decrease so that the speed of convergence actually rises. The case $\lambda_{\max} < -\lambda_{\min}$ is analogous. ■

Proof of Proposition 8. Consider first a homogeneous configuration $\alpha_1 = \dots = \alpha_n = \alpha$: in such case

$$\begin{aligned}
P(\lambda) &= (1 - \alpha - \lambda)^n - \alpha \delta (1 - \alpha - \lambda)^{n-1} \\
&= (1 - \alpha - \lambda)^{n-1} (1 - \alpha - \lambda - \alpha \delta) = 0 \\
\Rightarrow \lambda_{\max} &= 1 - \alpha, \quad \lambda_{\min} = 1 - \alpha - \alpha \delta
\end{aligned}$$

Due to Proposition 7, optimality requires $\lambda_{\max} = -\lambda_{\min}$ which implies $1 - \alpha = -1 + \alpha + \alpha \delta$, that is $\alpha = \frac{2}{\delta+2}$ and $\lambda_{\max} = -\lambda_{\min} = \frac{\delta}{\delta+2}$. Consider now a non-constant configuration for the α_i : surely it cannot be the speed of convergence maximizer if $\alpha_i \leq \frac{2}{\delta+2}$ ($\alpha_i \geq \frac{2}{\delta+2}$) for all i , since in that case, due to Lemma 4, it would be $\lambda_{\max} > \frac{\delta}{\delta+2}$ ($\lambda_{\min} < -\frac{\delta}{\delta+2}$). Also, there cannot be $\alpha_i \leq \alpha_j < \frac{2}{\delta+2}$ or $\alpha_i < \alpha_j \leq \frac{2}{\delta+2}$ for otherwise $\lambda_{\max} > \frac{\delta}{\delta+2}$, as a consequence of Lemma 1. So the only admissible configuration of non-constant α_i has the form $\alpha_1 < \frac{2}{\delta+2} < \alpha_2 \leq \dots \leq \alpha_n$. First

consider the case of only two different values of α , $\alpha_1 < \frac{2}{\delta+2} < \alpha_2 = \dots = \alpha_n$. In this case the characteristic polynomial is

$$P_n(\lambda) = (1 - \alpha_2 - \lambda)^{n-2} P_2(\lambda)$$

where

$$P_2(\lambda) = \left((1 - \alpha_1 - \lambda)(1 - \alpha_2 - \lambda) - \frac{\delta}{n}\alpha_1(1 - \alpha_2 - \lambda) - \frac{n-1}{n}\delta\alpha_2(1 - \alpha_1 - \lambda) \right)$$

coincides with the characteristic polynomial of the matrix

$$\begin{pmatrix} 1 - \alpha_1 \left(1 + \frac{\delta}{n}\right) & -\alpha_1 \frac{\delta}{n} \\ -\alpha_2 \frac{n-1}{n}\delta & 1 - \alpha_2 \left(1 + \frac{n-1}{n}\delta\right) \end{pmatrix} \quad (11)$$

corresponding to a two-firms market with shares equal to $\frac{1}{n}$ and $\frac{n-1}{n}$. Due to the particular ordering of the α_i , the two roots of $P_2(\lambda)$ are $\lambda_{\min}$ and $\lambda_{\max}$ for $P_n(\lambda)$. To minimize the spectral radius, it is necessary that $\lambda_{\max} = -\lambda_{\min}$ (see Proposition 7) and hence the trace of the matrix in (11) must be equal to 0. So the problem (in the closure of the admissible region) reduces to

$$\begin{cases} \min_{\alpha_1, \alpha_2} - (1 - \alpha_1 \left(1 + \frac{\delta}{n}\right)) (1 - \alpha_2 \left(1 + \frac{n-1}{n}\delta\right)) + (\alpha_1 \frac{\delta}{n}) (\alpha_2 \frac{n-1}{n}\delta) \\ s.t. \\ 1 - \alpha_1 \left(1 + \frac{\delta}{n}\right) + 1 - \alpha_2 \left(1 + \frac{n-1}{n}\delta\right) = 0 \\ 0 \leq \alpha_1 \leq \frac{2}{\delta+2} \leq \alpha_2 \leq 1 \end{cases}$$

whose solution is $\alpha_1 = \alpha_2 = \frac{2}{\delta+2}$. We finally show that a configuration with three or more different values for the α_i cannot be optimal. Indeed let $\alpha_1 < \alpha_2 < \alpha_3$ and $\lambda_{\max} = -\lambda_{\min}$. Using the Implicit Function Theorem we have

$$\frac{\partial \lambda}{\partial \alpha_i} = - \frac{\frac{\partial P(\lambda)}{\partial \alpha_i}}{\frac{\partial P(\lambda)}{\partial \lambda}}$$

in which the denominator is the same for all i . Further, because

$$\frac{\partial P(\lambda)}{\partial \alpha_i} = - \frac{1 - \lambda}{(1 - \alpha_i - \lambda)^2} \frac{\delta}{n} \prod_j (1 - \alpha_j - \lambda)$$

it is

$$\left| \frac{\partial \lambda_{\min}}{\partial \alpha_1} \right| < \left| \frac{\partial \lambda_{\min}}{\partial \alpha_2} \right| < \left| \frac{\partial \lambda_{\min}}{\partial \alpha_3} \right| \quad (12)$$

$$\left| \frac{\partial \lambda_{\max}}{\partial \alpha_2} \right| > \left| \frac{\partial \lambda_{\max}}{\partial \alpha_3} \right| \quad (13)$$

From Proposition 4 we know that increasing (decreasing) an α_i reduces (increases) both $\lambda_{\min}$ and $\lambda_{\max}$: therefore a sufficiently small increase in α_2 and an equal decrease in α_3 will increase the speed of convergence. ■

Proof of Proposition 9. The strong law of large numbers shows that $\Pr(\lim_{n \rightarrow \infty} \bar{\beta}_n = \bar{\beta}) = 1$ so $\Pr(\lim_{n \rightarrow \infty} \delta \bar{\beta}_n = \delta \bar{\beta}) = 1$ and therefore, for any $\delta > 1$, $\Pr(\lim_{n \rightarrow \infty} \delta \bar{\beta}_n < 1) = \begin{cases} 1 & \text{if } 1 < \delta < 1/\bar{\beta} \\ 0 & \text{if } \delta \geq 1/\bar{\beta} \end{cases}$. ■

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