Over-exploitation of open-access natural resources and global indeterminacy in an economic growth model

Angelo Antoci, University of Sassari*
Marcello Galeotti, University of Florence
Paolo Russu, University of Sassari

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Abstract

In this paper we use global analysis techniques to analyze an economic growth model with environmental negative externalities, giving rise to a three-dimensional dynamic system (the framework is the one introduced by Wirl (1997)). The dynamics of our model admits a locally attracting steady state P_1^*, which is, in fact, a *poverty trap*, coexisting with another steady state P_2^* possessing saddle-point stability. Global dynamical analysis shows that, under some conditions on the parameters, if the economy state variables are close enough to those of P_1^*, then there exists a continuum of equilibrium orbits approaching P_1^* and one orbit approaching P_2^*. Therefore, our model exhibits global indeterminacy, since either P_1^* or P_2^* can be selected according to agent expectations. Furthermore, by numerical simulations, we show that some orbits approaching P_1^* pass *very close* to the locally determinate stationary state P_2^*. So, our results suggest that, in perfect foresight dynamical models, local stability analysis can be misleading if it is not accompanied by global analysis.

Keywords: environmental externalities; indeterminacy; history versus expectations; global analysis of dynamic systems

JEL classification: C61, C62, E13, E32, O13

1 Introduction

As Krugman (1991) and Matsuyama (1991) pointed out in their seminal papers, equilibrium selection in dynamic optimization models with externalities depends on expectations; that is, given the initial values of the state variables (history), the path followed by the economy is determined by the choice of the initial

*Corresponding author; address: Angelo Antoci, Dipartimento di Economia Impresa e Regolamentazione, Università di Sassari, via Torre Tonda 34, 07100, Sassari, Italy. E-mail: angelo.antoci@virgilio.it

values of the jumping variables. This implies that expectations play a key role in equilibrium selection and in fact global indeterminacy may occur: that is, starting from the same initial values of the state variables, different equilibrium paths can approach different ω-limit sets (for example, different steady states). In this context, local stability analysis may be misleading, in that it refers to a *small* neighbourhood of a stationary state, whereas the initial values of jumping variables do not have to belong to such a neighbourhood. For example (see Coury and Wen (2009)), it may happen that a locally determinate (i.e. saddle-point stable) stationary state is surrounded by stable periodic orbits, so that the economy can approach either the stationary state or the periodic orbits, starting from the same initial values of the state variables.

Although some works on indeterminacy focus on global dynamics and stress the relevance of global analysis (see, among the others, Christiano and Harrison (1999), Pintus et al. (2000), Benhabib and Eusepi (2005), Wirl and Feichtinger (2006), Benhabib et al. (2008), Mattana et al. (2008), Coury and Wen (2009)), the literature on indeterminacy is almost exclusively based on local analysis, due to the fact that dynamic models exhibiting indeterminacy are often highly nonlinear and difficult to be analyzed globally.

The objective of our paper is to highlight the relevance of global indeterminacy in a context in which economic activity depends on the exploitation of a free access environmental resource.

We analyze a growth model with environmental externalities, giving rise to a three-dimensional nonlinear dynamic system (the framework was introduced by Wirl (1997)). In particular, we study the equilibrium growth dynamics of an economy constituted by a continuum of identical agents. At each instant of time t, the representative agent produces the output $Y(t)$ by labour $L(t)$, by the accumulated physical capital $K(t)$ and by the stock $E(t)$ of a free-access renewable environmental resource. The economy-wide aggregate production $\overline{Y}(t)$ negatively affects the stock of the environmental resource; however, the value of $\overline{Y}(t)$ is considered as exogenously determined by the representative agent, so that economic dynamics is affected by negative environmental externalities.[1]

We assume that the representative agent's instantaneous utility, depending on leisure $1 - L(t)$ and consumption $C(t)$ of the output $Y(t)$, is represented by the function $\frac{[C(1-L)^{\varepsilon}]^{1-\eta}-1}{1-\eta}$, similar to those used, for example, by Bennett and Farmer (2000), Mino (2001) and Itaya (2008). Moreover, we assume that the production technology is represented by the Cobb-Douglas function $[K(t)]^{\alpha}\,[L(t)]^{\beta}\,[E(t)]^{\gamma}$, with $\alpha+\beta<1$ and $\gamma>0$.

In this context, we show that, if $\alpha+\gamma<1$, the dynamics can admit a locally attracting stationary state $P_1^* = (K_1^*, E_1^*, L^*)$, in fact a *poverty trap*, coexisting with another stationary state $P_2^* = (K_2^*, E_2^*, L^*)$, where $K_1^* < K_2^*$ and $E_1^* < E_2^*$, possessing saddle-point stability.

[1] Environmental externalities can affect economic activities especially in developing countries, where property rights tend to be ill-defined and ill-protected, environmental institutions and regulations are weak and natural resources are more fragile than in developed countries, which are located in temperate areas instead than in tropical and sub-tropical regions (see e.g. López (2003, 2007)).

Global dynamical analysis shows that, under some conditions on the parameters, if the economy starts from initial values K_0 and E_0 sufficiently close to K_1^* and E_1^*, then there exists a continuum of initial values L_0^1 such that the trajectory from (K_0, E_0, L_0^1) approaches P_1^* and a locally unique initial value L_0^2 such that the trajectory from (K_0, E_0, L_0^2) approaches P_2^*. Therefore our model exhibits local indeterminacy (i.e. there exists a continuum of trajectories leading to P_1^*), but also global indeterminacy, since either P_1^* or P_2^* may be selected according to agent expectations. Along the trajectories belonging to the basin of attraction of P_1^*, over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario.

Furthermore, by means of numerical simulations, we show that the stable manifold of the locally determinate point P_2^* *bounds* the basin of attraction of the locally indeterminate point P_1^*, so that some orbits approaching P_1^* pass *very close* to P_2^*. This implies that even if the economy starts near the locally determinate point P_2^*, it may approach the poverty trap P_1^*; this result is analogous to the one obtained in Coury and Wen (2009), where global indeterminacy is observed in a dynamic system with a unique locally determinate stationary state.

Although our analysis focuses on global indeterminacy of dynamics, it also gives sufficient conditions for local indeterminacy. There exists an enormous literature on local indeterminacy in economic growth models. We don't have room in this article for a review, however we point out the place that our results occupy in the current research.

Even if the main body of the literature on local indeterminacy concerns economies with increasing social returns[2], a growing proportion of articles deals with models where indeterminacy is obtained under the assumption of social constant return technologies. For example, Benhabib and Nishimura (1998 and 1999) find that indeterminacy may occur in multi-sector growth models with social constant returns. They require relative factor intensities of the social technologies involving externalities to be opposite to those of the private technologies. Mino (2001) emphasizes the role of preference structure rather than the one of production technologies and shows that local indeterminacy of a stationary state may emerge even in the absence of increasing returns to scale in two-sector endogenous growth models with physical and human capital[3]. Inde-

[2]See, for example, early studies of Benhabib and Farmer (1994) and Boldrin and Rustichini (1994) where the degree of increasing returns are assumed sufficiently large to produce indeterminacy. Subsequent works have shown that indeterminacy may also emerge without assuming strong degree of increasing returns (see, among the others, Benhabib and Farmer (1996), Perli (1998), Bennett and Farmer (2000), Itaya (2008)). Nishimura et al. (2008) provide a unified analysis of local indeterminacy within an aggregate model with small externalities and illustrate the main sufficient conditions for local indeterminacy, stressing the role of the complex interplay between preferences and technology (for a review of the literature on local indeterminacy in models with externalities see Benhabib and Farmer (1999) and Mino et al. (2008)).

[3]In such a paper sector-specific externalities are considered in a framework of social constant returns. It is worth observing that Mino finds, as in our work, that two stationary states may coexist, one determinate and the other indeterminate (Corollary 1, p.11). However the analysis is bounded to local stability properties of stationary states.

terminacy results with social constant returns are also obtained by Nishimura and Shimomura (2002) in a small open economy context with production externalities and endogenous time preferences, by Zhang (2008) in a two-sector small open economy model and by Mino et al. (2008) in a discrete-time framework.

In our paper, local indeterminacy can occur with social constant or decreasing returns and is generated by the interplay between environmental externalities of production activity negatively affecting natural capital and the positive externalities of natural capital on the production activity.

Other works focus on the role played by negative externalities in producing local indeterminacy. For example, Chen and Lee (2007) consider a social constant returns economy where a congestible public good exerts positive sector-specific externalities, while a congestion effect generates negative aggregate externalities. Itaya (2008) shows how pollution may affect indeterminacy results in a one-sector growth model with social increasing returns. In Meng and Yip (2008) indeterminacy is produced by negative capital externalities. In Antoci et al. (2005) and in Antoci and Sodini (2009) negative externalities may generate indeterminacy in an economy where private goods can be consumed as substitutes for free access environmental goods.

The present paper has the following structure. Sections 2 and 3 define the set-up of the model and the associated dynamic system. Section 4 deals with the existence and local stability of stationary states and with Hopf bifurcations arising from stability changes. Section 5 is devoted to global analysis of dynamics and provides the main results of the paper. Section 6 contains the conclusions.

2 Set-up of the model

The economy we analyze is constituted by a continuum of identical economic agents; the size of the population of agents is normalized to unity. At each instant of time $t \in [0, \infty)$, the representative agent produces an output $Y(t)$ by the following Cobb-Douglas technology:

$$Y(t) = [K(t)]^{\alpha} [L(t)]^{\beta} [E(t)]^{\gamma}, \text{ with } \alpha + \beta < 1 \text{ and } \gamma > 0 \tag{1}$$

where $K(t)$ is the stock of physical capital accumulated by the representative agent, $L(t)$ is the agent's labour input and $E(t)$ is the stock of a free access renewable environmental resource.

We assume that the representative agent's instantaneous utility function depends on leisure $1 - L(t)$ and consumption $C(t)$ of the output $Y(t)$; precisely, we consider the following non-separable function (a function of this type is used, among others, by Bennet and Farmer (2000), Mino (2001) and Itaya (2008)):

$$U(C(t), L(t)) = \frac{[C(t)(1 - L(t))^{\varepsilon}]^{1-\eta} - 1}{1 - \eta}$$

where ε, $\eta > 0$ and $\eta \neq 1$. Moreover, we assume that the utility function is concave in C and in $1 - L$, i.e. $\eta > \frac{\varepsilon}{1+\varepsilon}$. The parameter η denotes the inverse of the intertemporal elasticity of substitution in consumption. Our function displays a constant intertemporal elasticity of substitution and possesses the property that income and substitution effects exactly balance each other in the labour supply equation.

The evolution of $K(t)$ (assuming, for simplicity, the depreciation of K to be zero) is represented by the differential equation

$$\dot{K} = K^{\alpha} L^{\beta} E^{\gamma} - C \tag{2}$$

where $\dot{K}$ is the time derivative of K. In order to model the dynamics of E we start from the well-known *logistic equation*

$$\dot{E} = E(\overline{E} - E)$$

where the parameter $\overline{E} > 0$ represents the carrying capacity of the natural resource (i.e. the value that E *reaches* as $t \to +\infty$), and we augment it by considering the negative impact due to the production process

$$\dot{E} = E(\overline{E} - E) - \delta \overline{Y} \tag{3}$$

where $\overline{Y} = \overline{K}^{\alpha} \overline{L}^{\beta} E^{\gamma}$ is the economy-wide average output and the parameter $\delta > 0$ measures the negative impact of $\overline{Y}$ on E.

The logistic function is a standard specification, extensively used as a growth function of renewable resources (see e.g. Brown (2000), Koskela et al. (2002), Elíasson and Turnovsky (2004)).

Under the specification (3) of the environmental dynamics, the production process in our economy can be interpreted as extractive activity. Its impact on the natural resource is given by the rate of harvest which depends on the economy-wide average labour input $\overline{L}$ and physical capital $\overline{K}$ and on the stock of natural capital E. For example, we can identify production activity with forestry or fishery. As in all simplifications of reality, some scenarios are not captured by our model. In particular, the model cannot describe contexts where the production of output is linked to E, but its environmental impact is not positively correlated to E, for instance the tourism industry.

We assume that the representative agent chooses the functions $C(t)$ and $L(t)$ in order to solve the following problem

$$\underset{C,\, L}{MAX} \int_0^{\infty} \frac{[C(1-L)^{\varepsilon}]^{1-\eta} - 1}{1-\eta} e^{-\theta t} dt \tag{4}$$

with the constraints (2) and (3), where $\theta > 0$ is the subjective rate of time preference. Furthermore, we assume that in solving problem (4), the representative agent considers $\overline{Y}$ as exogenously determined, since, being economic agents a continuum, the impact on $\overline{Y}$ of each one is null. However, since agents are identical, ex post $\overline{Y} = Y$ holds. This implies that the trajectories resulting from our model are not optimal (i.e. they do not describe the social optimum).

However they represent Nash equilibria in the sense that, along them, no agent has an incentive to modify his choices if the others don't modify theirs.

3 Dynamics

The current value Hamiltonian function associated to problem (4) is

$$H(\bullet) = \frac{\left[C(1-L)^{\varepsilon}\right]^{1-\eta} - 1}{1-\eta} + \Omega\left[K^{\alpha}L^{\beta}E^{\gamma} - C\right] + \Gamma\left[E(\overline{E} - E) - \delta\overline{Y}\right]$$

where Ω and Γ are the co-state variables associated to K and E, respectively. Since $\overline{Y}$ is considered as exogenously determined, the evolution of Γ doesn't affect the representative agent's choices and, consequently, economic dynamics. This implies (following Wirl, 1997) that, by applying the Maximum Principle, the dynamics of the economy is described by the system

$$\begin{aligned} \dot{K} &= \frac{\partial H}{\partial \Omega} = K^{\alpha}L^{\beta}E^{\gamma} - C \\ \dot{E} &= \frac{\partial H}{\partial \Gamma} = E(\overline{E} - E) - \delta\overline{Y} \\ \dot{\Omega} &= \theta\Omega - \frac{\partial H}{\partial K} = \Omega\left[\theta - \alpha K^{\alpha-1}L^{\beta}E^{\gamma}\right] \end{aligned} \tag{5}$$

where C and L satisfy the following conditions[4]

$$\frac{\partial H}{\partial C} = C^{-\eta}\left(1-L\right)^{\varepsilon(1-\eta)} - \Omega = 0$$

$$\frac{\partial H}{\partial L} = 0 \text{ i.e. } \beta(1-L)\Omega K^{\alpha}L^{\beta-1}E^{\gamma} - \varepsilon C^{1-\eta}\left(1-L\right)^{\varepsilon(1-\eta)} = 0$$

Since our system meets Mangasarian sufficient conditions, the above conditions are necessary and sufficient for solving problem (4)[5]. This is the case also if $\alpha + \beta + \gamma > 1$ (remember we assumed $\alpha + \beta < 1$), because the stock E is considered as a positive externality in the decision problem of the representative agent.

By replacing $\overline{Y}$ with $K^{\alpha}L^{\beta}E^{\gamma}$, the Maximum Principle conditions yield a dynamic system with two state variables, K and E, and one jumping variable, Ω. Notice that, from $\varepsilon C\frac{\partial H}{\partial C} + \frac{\partial H}{\partial L} = 0$, one obtains

$$\begin{aligned} C &= \tfrac{\beta}{\varepsilon}\left(1-L\right)L^{\beta-1}K^{\alpha}E^{\gamma} \\ f(L) &= \tfrac{\varepsilon}{\beta}\left(1-L\right)^{\frac{\varepsilon-\eta(1+\varepsilon)}{\eta}}L^{1-\beta} = K^{\alpha}E^{\gamma}\Omega^{\frac{1}{\eta}} \end{aligned}$$

[4]Notice that the utility function we adopted implies $C > 0$ and $0 < L < 1$.

[5]Furthermore, since our model does not exhibit unbounded growth of the state variables, the usual transversality conditions are always met along the orbits approaching a stationary state or a limit cycle, whose state variables lie in the positive quadrant of the plane (K, E).

Hence one can write the following system, equivalent to (5)

$$\begin{aligned}
\dot{K} &= \frac{1}{\varepsilon} K^{\alpha} E^{\gamma} L^{\beta-1} \left(L \left(\beta + \varepsilon \right) - \beta \right) \\
\dot{E} &= E(\overline{E} - E) - \delta K^{\alpha} L^{\beta} E^{\gamma} \\
\dot{L} &= \frac{f(L)}{f'(L)} \left[\frac{\alpha}{\varepsilon} K^{\alpha-1} E^{\gamma} L^{\beta-1} (L(\beta+\varepsilon) - \beta) + \right. \\
&\left. \gamma(\overline{E} - E - \delta K^{\alpha} L^{\beta} E^{\gamma-1}) + \frac{1}{\eta}(\theta - \alpha K^{\alpha-1} L^{\beta} E^{\gamma}) \right]
\end{aligned} \tag{6}$$

In such a context, the jumping variable becomes L. Given the initial values of the state variables, K_0 and E_0, the representative agent has to choose the initial value L_0 of L.

4 Fixed points, stability and Hopf bifurcations

We recall the conditions on the parameters: they are all positive, with $\alpha + \beta < 1$ and $1 \neq \eta > \frac{\varepsilon}{1+\varepsilon}$. The following proposition deals with the problem of the existence and numerosity of fixed points (stationary states) of the dynamic system (6).

Proposition 1 *System* (6) *has one fixed point if* $\alpha + \gamma > 1$*; one or zero fixed points if* $\alpha + \gamma = 1$*; zero, one or two fixed points if* $\alpha + \gamma < 1$.

Proof. A fixed point $P^* = (K^*, E^*, L^*)$ of (6) must satisfy

$$\begin{aligned}
&L^* = \tfrac{\beta}{\beta+\varepsilon} \\
&K^* = \tfrac{\alpha}{\theta\delta} E^* (\overline{E} - E^*) \\
&g\left(E^*\right) = E^* + \delta \left(\tfrac{\beta}{\beta+\varepsilon} \right)^{\frac{\beta}{1-\alpha}} \left(\tfrac{\alpha}{\theta} \right)^{\frac{\alpha}{1-\alpha}} \left(E^*\right)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \overline{E}
\end{aligned} \tag{7}$$

Hence the graph of $g\left(E\right)$ intersects the line $E = \overline{E}$ exactly at one point if $\alpha + \gamma > 1$, at most at one point if $\alpha + \gamma = 1$, at zero, one or two points if $\alpha + \gamma < 1$. ∎

Observe that, if $\alpha + \gamma < 1$, then there exists one fixed point only if the minimum of the function $g\left(E^*\right)$ coincides with the value $\overline{E}$; so, generically, the fixed points are zero or two.

By (7), when two fixed points exist, $P_1^* = (K_1^*, E_1^*, L^*)$ and $P_2^* = (K_2^*, E_2^*, L^*)$, then $K_1^* < K_2^*$ and $E_1^* < E_2^*$; so P_2^* Pareto-dominates P_1^*. If the economy approaches the latter, then a *tragedy of commons* scenario emerges, characterized by over-exploitation of the natural resource and by low physical capital accumulation (labour input is equal to $L^* = \frac{\beta}{\beta+\varepsilon}$ at both fixed points). Notice that, in our model, multiplicity of fixed points may occur also in a context of social constant returns to scale, $\alpha + \beta + \gamma = 1$, whereas it is ruled out if the elasticity γ of the production function with respect to natural capital E is relatively *high*, that is if $\alpha + \gamma \geq 1$.

Now, let $P^* = (K^*, E^*, L^*)$ be a fixed point of (6) and consider the Jacobian matrix of system (6) evaluated at P^*

$$J^* = \begin{pmatrix} 0 & 0 & \frac{\partial \dot{K}}{\partial L} \\ \frac{\partial \dot{E}}{\partial K} & \frac{\partial \dot{E}}{\partial E} & \frac{\partial \dot{E}}{\partial L} \\ \frac{\partial \dot{L}}{\partial K} & \frac{\partial \dot{L}}{\partial E} & \frac{\partial \dot{L}}{\partial L} \end{pmatrix}$$

where, by straightforward computations

$$\begin{array}{l} \frac{\partial \dot{K}}{\partial L} = \frac{\beta+\varepsilon}{\delta\varepsilon} E^*(\overline{E} - E^*) \\ \frac{\partial \dot{E}}{\partial K} = -\delta\theta \\ \frac{\partial \dot{E}}{\partial E} = \overline{E}(1-\gamma) - E^*(2-\gamma) \\ \frac{\partial \dot{E}}{\partial L} = -(\beta+\varepsilon) E^*(\overline{E} - E^*) \\ \frac{\partial \dot{L}}{\partial K} = \frac{f(L^*)}{f'(L^*)} \frac{\delta\theta}{E^*} \left[-\gamma + \frac{\theta(1-\alpha)}{\alpha\eta\left(\overline{E}-E^*\right)} \right] \\ \frac{\partial \dot{L}}{\partial E} = \frac{f(L^*)}{f'(L^*)} \frac{\gamma}{E^*} \left[(1-\gamma)(\overline{E} - E^*) - E^* - \frac{\theta}{\eta} \right] \\ \frac{\partial \dot{L}}{\partial L} = \frac{f(L^*)}{f'(L^*)} (\beta+\varepsilon) \left[\frac{\theta(\beta+\varepsilon)}{\varepsilon} - \frac{\theta}{\eta} - \gamma(\overline{E} - E^*) \right] \end{array} \tag{8}$$

The following proposition holds:

Proposition 2 *If the fixed point is unique ($\alpha + \gamma \geq 1$) or, in case of two fixed points, is the one with the larger E^*, then J^* has an odd number of positive eigenvalues; instead, if, in case of two fixed points, P^* corresponds to the one with the smaller E^*, then J^* has an odd number of negative eigenvalues.*

Proof. By computing $\det(J^*)$, one can check that

$$\det(J^*) < 0 \text{ iff } E^* < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}\overline{E} \tag{9}$$

Clearly (9) implies that two fixed points exist ($\alpha+\gamma < 1$) and, moreover, as one can calculate, that $g'(E^*) < 0$ (see (7)). In fact it is easily seen that $\det(J^*)$ has the same sign of $g'(E^*)$, which proves the Proposition. ∎

It is easy to check that in the non generic case when a unique fixed point exists under the condition $\alpha+\gamma < 1$, then $\det(J^*) = 0$ holds and the fixed point is not hyperbolic (in fact, a saddle-node bifurcation occurs). Consequently, if we look for an attracting fixed point, we have to restrict our analysis to the case when, under the assumption $\alpha+\gamma < 1$, two fixed points exist, P_1^* and P_2^*, with $E_1^* < E_2^*$ and $K_1^* < K_2^*$. We aim to show that, in such a context, P_1^* can be attractive for suitable values of the parameters. Along the trajectories belonging to the basin of attraction of P_1^* the over-exploitation of the natural resource drives the economy towards a *tragedy of commons* scenario.

First of all, if $\alpha+\gamma < 1$, a necessary and sufficient condition for the existence of two fixed points is

$\overline{E} > g\left(\widetilde{E}\right) := E_A$, where $\widetilde{E}$ is the only positive value satisfying $g'\left(\widetilde{E}\right) = 0$.

Straightforward computations yield

$$E_A = \frac{2-2\alpha-\gamma}{1-\alpha-\gamma}\widetilde{E} = \\ = (2-2\alpha-\gamma)\left[\frac{\delta^{1-\alpha}}{(1-\alpha)^{1-\alpha}(1-\alpha-\gamma)^{1-\alpha-\gamma}}\left(\frac{\beta}{\beta+\varepsilon}\right)^{\beta}\left(\frac{\alpha}{\theta}\right)^{\alpha}\right]^{\frac{1}{2-2\alpha-\gamma}} \tag{10}$$

Hence $E_1^* < \widetilde{E} < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}\overline{E}$. From now on let us omit index 1.

The well-known Routh-Hurwitz Criterion (see Hurwitz (1964)) yields that J^*, the Jacobian matrix at P^*, has three eigenvalues with negative real part if and only if

$$\det(J^*) < 0 \tag{11}$$

$$\sigma(J^*) = \frac{\partial\dot{E}}{\partial E}\frac{\partial\dot{L}}{\partial L} - \frac{\partial\dot{E}}{\partial L}\frac{\partial\dot{L}}{\partial E} - \frac{\partial\dot{K}}{\partial L}\frac{\partial\dot{L}}{\partial K} > 0 \tag{12}$$

$$\rho(J^*) = -\sigma(J^*)\cdot trace(J^*) + \det(J^*) > 0$$

The last inequality, in particular, guarantees the non-existence of complex eigenvalues with non-negative real part. In fact, when $\rho(J^*)$ crosses the value 0, the real part of two complex conjugate eigenvalues changes sign, causing, generically, a Hopf bifurcation.

Remember that the condition (11) is always verified at P^* (see (9)). As for the condition (12), we state the following Lemma.

Lemma 3 *If*

$$\eta \geq \frac{\varepsilon}{\varepsilon+\alpha\beta} \text{ and } \overline{E} > E_B = \frac{\theta(\beta+\varepsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta} \tag{13}$$

then the condition $\sigma(J^*) > 0$ is verified.

Proof. By recalling (8), straightforward computations lead to

$$sign[\sigma(J^*)] = sign\left[\left(\frac{\beta+\varepsilon}{\varepsilon} - \frac{1}{\eta}\right)\left(\overline{E} - 2E^*\right) - \frac{\theta(1-\alpha)(\beta+\varepsilon)}{\alpha\varepsilon\eta}\right]$$

So, since $E^* < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}\overline{E}$, the assumptions of the Lemma imply $\sigma(J^*) > 0$. ■

Let us now compute $trace\,(J^*)$, observing that $\frac{f(L^*)}{f'(L^*)} = \frac{\beta\varepsilon\eta}{(\beta+\varepsilon)[\eta(\beta+\varepsilon)-\beta\varepsilon]}$. We obtain

$$\begin{array}{l} trace\,(J^*) = a\left(\overline{E} - E^*\right) - E^* + b; \\ a = \frac{\eta[(1-\gamma)(\beta+\varepsilon)-\beta\gamma\varepsilon]-\beta\varepsilon(1-\gamma)}{\eta(\beta+\varepsilon)-\beta\varepsilon}, \quad b = \frac{\beta\theta[\eta(\beta+\varepsilon)-\varepsilon]}{\eta(\beta+\varepsilon)-\beta\varepsilon} \end{array} \tag{14}$$

Then the results of our analysis, aimed at detecting an attractive fixed point, are summarized by the following Theorem.

Theorem 4 *Let* $\alpha + \gamma < 1$ *and* $\overline{E} > E_A$*, so that system (6) has two fixed points,* P_1^* *and* P_2^**, with* $E_1^* < E_2^*$ *and* $K_1^* < K_2^*$*. Then, for suitable values of the parameters,* P_1^* *is a sink, while* P_2^* *is a saddle with a bi-dimensional stable manifold. Moreover, in such a case, take* $\overline{E}$ *as a bifurcation parameter. As* $\overline{E}$ *is increased,* P_2^* *does not change its nature (i.e. it remains a saddle with a bi-dimensional stable manifold), whereas* P_1^* *can undergo one, two or no Hopf bifurcations.*

Proof. First of all, let $E_C := \frac{(2-2\alpha-\gamma)b}{1-\alpha-\gamma}$, where b is defined in (14). Assuming $\eta \geq \frac{\varepsilon}{\varepsilon+\alpha\beta}$, recall the expressions of E_A (10) and E_B (13). It is easily checked that, for θ sufficiently small or/and δ sufficiently large,

$$E_A > \max(E_B, E_C).$$

Let $\overline{E}$ be sufficiently close to E_A, $0 < \overline{E} - E_A << 1$. Then, if

$$a \leq 0 \text{ or } 0 < a < \frac{1-\alpha-\gamma}{1-\alpha} \text{and } E_A > \frac{1-\alpha-\gamma}{1-\alpha-\gamma-a\,(1-\alpha)} E_C,$$

$$trace\,(J^*) < 0$$

both at P_1^* and P_2^*.

Furthermore, since $\det(J^*) = 0$ for $E^* = E_A$, also $\rho\,(J^*)$, in addition to $\sigma\,(J^*)$, is positive. It follows that J^* has three eigenvalues with negative real part at P_1^* and two with negative real part and one positive eigenvalues at P_2^*, which proves the first statement of the theorem.

Now, let $\overline{E}$ increase, *coeteris paribus*. Since E_2^* increases and $\overline{E} - E_2^* = \delta\left(\frac{\beta}{\beta+\varepsilon}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{\alpha}{\theta}\right)^{\frac{\alpha}{1-\alpha}} (E_2^*)^{\frac{\alpha+\gamma-1}{1-\alpha}}$ decreases, it follows that, no matter what the sign of a is, P_2^* remains a saddle with a bi-dimensional stable manifold.

Instead, since E_1^* decreases as $\overline{E}$ increases, P_1^* undergoes (generically) one Hopf bifurcation if $a \geq 0$ (see Figure 1): the real part of two complex eigenvalues turns from negative into positive and eventually $trace\,(J^*)$ becomes positive.

On the contrary, if $a < 0$, it happens that, when $\overline{E}$ is large enough, $trace\,(J^*) < 0$, while $\sigma\,(J^*)$ and $\rho\,(J^*)$ are both positive: in fact it is easily checked that $\rho\,(J^*)$ is a second degree polynomial in $\overline{E}$ with a positive coefficient of $\overline{E}^{\,2}$. So P_1^* is a sink for large values of $\overline{E}$.

In this case, we can detect possible Hopf bifurcations as follows. Remember that Hopf bifurcations correspond, generically, to equilibria for which $\rho(J^*) = 0$. Observe that, by our assumptions and notations

$$\overline{E} - 2E_1^* - \frac{\theta(1-\alpha)(\beta+\varepsilon)}{\alpha[\eta(\beta+\varepsilon)-\beta\varepsilon]} > \frac{\gamma}{2-2\alpha-\gamma}(E_A - E_B) > 0$$

Pose

$$x := E_1^* \text{ and } y := \overline{E} - 2E_1^* - \frac{\theta(1-\alpha)(\beta+\varepsilon)}{\alpha[\eta(\beta+\varepsilon)-\varepsilon]}$$

So x and y are both positive.

By straightforward computations it is seen that $\rho(J^*) = 0$ corresponds to the following curve in the positive quadrant of the (x, y) plane (see Figure 1)

$$C_1)\ \ x = \frac{-my^2 + ny + l}{my + q}, \text{ with } l, m, q > 0,$$

while the equilibrium condition $g(E_1^*) = \overline{E}$ gives place to the curve, in the positive quadrant of the (x, y) plane

$$C_2)\ \ y = rx^{\frac{\alpha+\gamma-1}{1-\alpha}} - x - s, \text{ with } r, s > 0 \text{ and } \alpha + \gamma - 1 < 0$$

Hence Hopf bifurcations correspond generically to the intersections of the two curves C_1 and C_2. Now it is easily checked that the curve C_1 intersects the positive vertical semi-axis ($x = 0$) at an ordinate $y_1 > 0$ and the positive horizontal semi-axis ($y = 0$) at an abscissa x_1, $0 < x_1 < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}E_A$. Consider, instead, C_2: then $y \to +\infty$ when $x \to 0^+$ and the intersection with the positive horizontal semi-axis takes place at $(x_2, 0)$, with $x_2 > \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}E_A$. As a consequence, it can be proved that the intersections of C_1 and C_2 are generically zero or two (see Figure 1). ■

Notice that the sufficient conditions for local indeterminacy given above depend on the intertemporal elasticity of substitution and can be satisfied in both cases $\eta < 1$ (i.e. elasticity of substitution greater than 1) and $\eta > 1$ (i.e. elasticity of substitution lower than 1): in fact, we assumed $\eta \geq \frac{\varepsilon}{\varepsilon+\alpha\beta}$. Furthermore, those conditions require that the impact of the production process of output (measured by δ) be high enough and/or the subjective discount rate θ be low enough. Finally, the elasticity γ of the production function with respect to the natural capital E must be not *too high*, that is $\alpha + \gamma < 1$, while social returns to scale can be constant or decreasing, that is, $\alpha + \beta + \gamma \leq 1$.

Figure 2 shows how the fixed point values of K and E change, by varying the value of $\overline{E}$ (the carrying capacity of the environmental resource). The coordinates of P_1^* are indicated by a bold line if P_1^* is a sink and by a dash-dot line if it is a saddle with a one-dimensional stable manifold; the coordinates of P_2^* (which, in the numerical example, is a saddle with a two-dimensional stable manifold) are indicated by a dotted line. Notice that a Hopf bifurcation occurs when the parameter $\overline{E}$ crosses the value 0.2 (the bifurcation point is indicated by H).

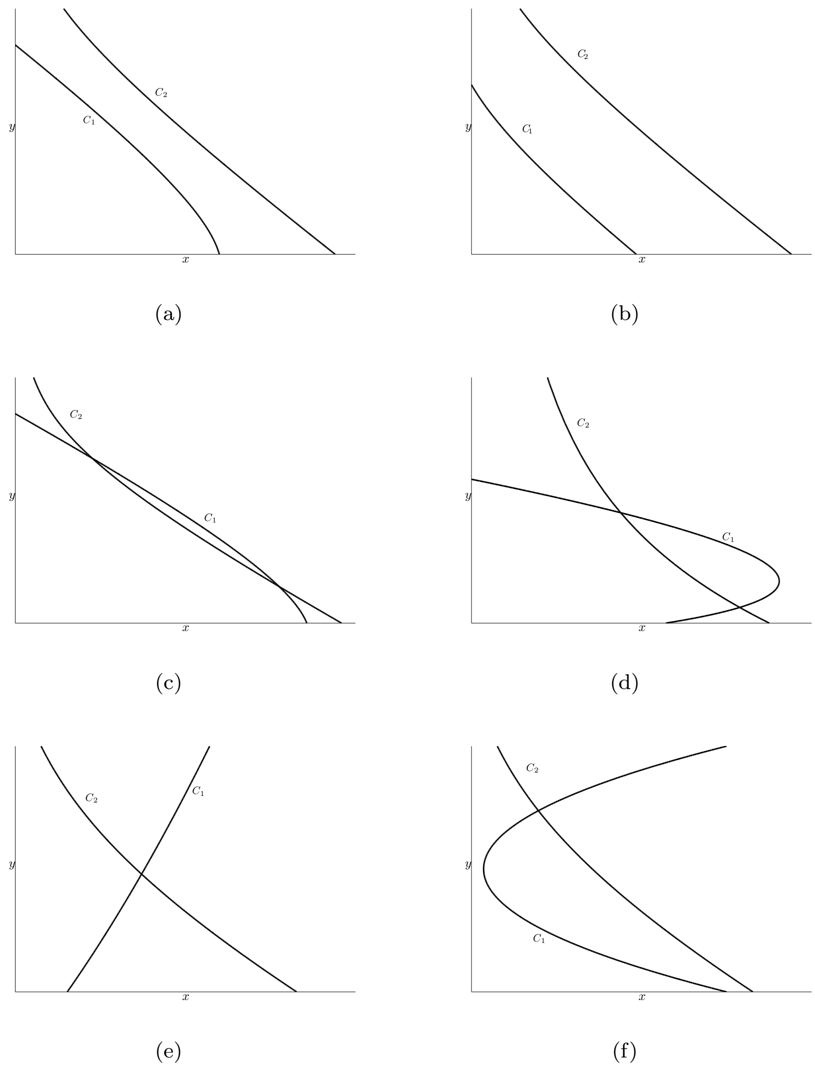
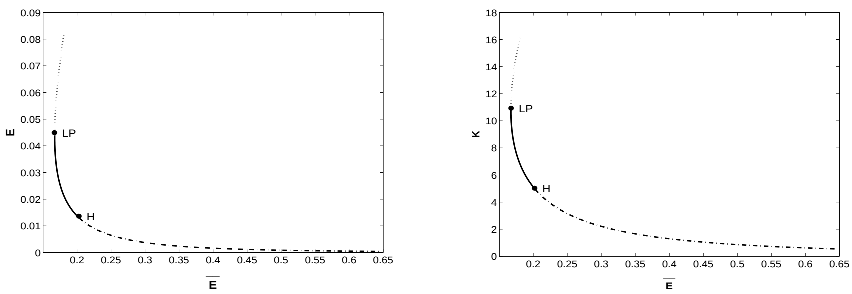


Figure 1: Dynamics and Stability. (a)-(b) $a < 0$, no bifurcation: P_1^* is a sink; (c)-(d) $a < 0$, two bifurcations: P_1^* is a sink when C_2 lies to the right of C_1, a saddle when C_2 lies to the left of C_2; (e)-(f) $a \geq 0$, one bifurcation: P_1^* is a sink when C_2 lies to the left of C_1, a saddle when C_2 lies to the right of C_2

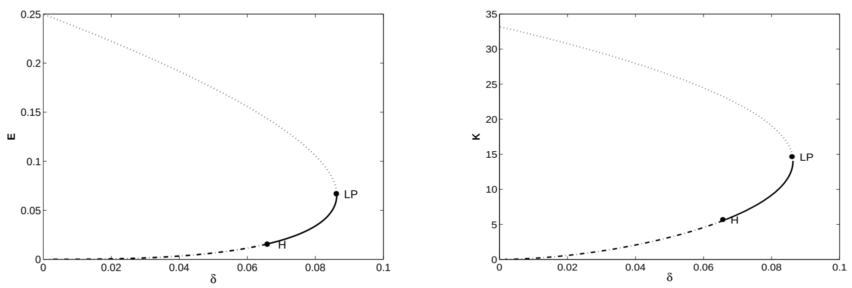
Adopting the same symbology, Figure 3 draws the fixed points coordinates when varying the parameter δ, which measures the environmental impact of the production process. Notice that a Hopf bifurcation occurs also in this example and that indeterminacy is observed when δ is *high enough*.

Figure 4 shows a locally attracting limit cycle around P_1^* (which has a one-dimensional stable manifold) arisen via the Hopf bifurcation shown in Figure 2. In such a case, local indeterminacy occurs, since for every initial point (K_0, E_0) *close* to the projection of the cycle in the plane (K, E), there exists a continuum of initial values L_0 of L such that the orbit starting from (K_0, E_0, L_0) approaches the cycle.



(a) (b)

Figure 2: The fixed point values of K and E, varying $\overline{E}$; values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.8$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$



(a) (b)

Figure 3: The fixed point values of K and E, varying δ; values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.8$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\overline{E} = 0.25$

Let us now complete the local analysis by discussing the case $\alpha + \gamma > 1$, when the fixed point P^* is unique. Since $\det(J^*) > 0$ (see Proposition 2), P^* is not attractive. If, for some value of $\overline{E}$, $trace\,(J^*) = a\left(\overline{E} - E^*\right) - E^* + b < 0$, with a and b defined in (14), then P^* is a saddle with a bi-dimensional stable

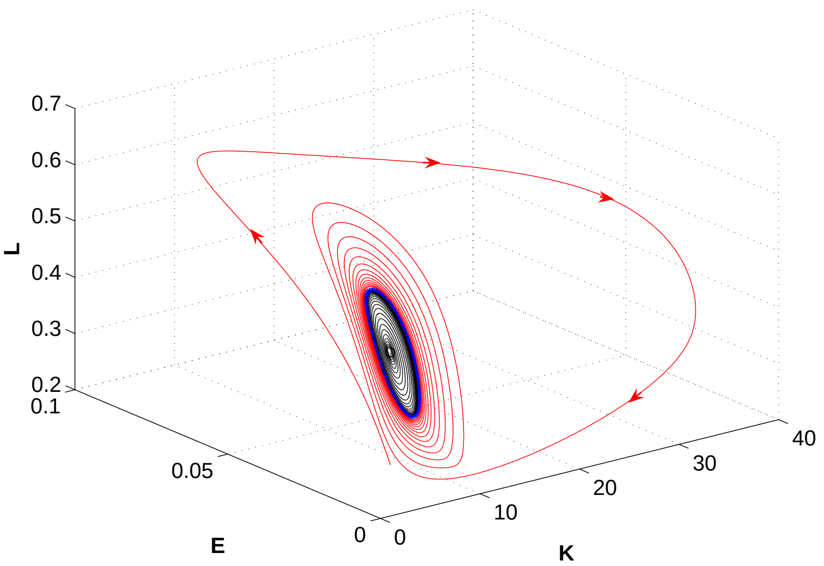


Figure 4: Locally attracting limit cycle around P_1^*; values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.8$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\overline{E} = 0.21$

manifold. Let us increase $\overline{E}$. Correspondingly E^* increases as well. By the equilibrium condition $g(E^*) = \overline{E}$, we can write $trace\,(J^*)$ as

$$trace\,(J^*) = r\,(E^*)^{\frac{\alpha+\gamma-1}{1-\alpha}} - E^* + b, \text{ with } b > 0,\ \alpha+\gamma-1 > 0 \text{ and } sign(r) = sign(a)$$

By the same arguments developed in Theorem 4, the following Proposition is easily proved.

Proposition 5 *Let $\alpha + \gamma > 1$ and P^* denote the only fixed point of system* (6)*. Write $trace\,(J^*) = a\left(\overline{E} - E^*\right) - E^* + b$, with a and b defined in* (14)*. Assume $trace\,(J^*) < 0$ for some value of $\overline{E}$ and let $\overline{E}$ increase. Then: if $a \leq 0$, no bifurcation occurs and P^* remains a saddle with a bi-dimensional stable manifold; if $a > 0$ and $2\alpha + \gamma > 2$, eventually P^* becomes a source and one Hopf bifurcation generically takes place; if $a > 0$ and $2\alpha + \gamma < 2$, P^* is a saddle with a bi-dimensional stable manifold for sufficiently large values of $\overline{E}$ and Hopf bifurcations are, generically, zero or two.*

According to the above proposition, indeterminacy cannot be observed in the context of a unique fixed point: i.e., the fixed point can possess saddle-type stability (two eigenvalues with negative real part) but cannot be a sink.

5 Global analysis

In this Section, for the sake of convenience in representing the Figures, we will change the order of the variables from (K, E, L) into (E, K, L).

In the following we take into consideration the case where, for $\alpha+\gamma<1$, two fixed points exist, $P_1^* = (E_1^*, K_1^*, L^*)$ and $P_2^* = (E_2^*, K_2^*, L^*)$, with $E_1^* < E_2^*$, $K_1^* < K_2^*$, $L^* = \frac{\beta}{\beta+\varepsilon}$, and P_1^* is a sink.

Hence the basin of attraction of P_1^* can be considered a *poverty trap* and we wonder if, given a point $P_0 = (E_0, K_0, L_0)$ belonging to such a basin, it is possible to modify the initial choice of labour in such a way that the positive semi-trajectory starting from the *new* point $\widetilde{P}_0 = \left(E_0, K_0, \widetilde{L}_0\right)$ can tend to the saddle P_2^* (having a bi-dimensional stable manifold).

In fact we will give a (partially) affirmative answer to the above question (Theorem 9) and, moreover, we will suggest how to conduct numerical experiments aimed at detecting trajectories leading to the desirable equilibrium, i.e. to the saddle P_2^* (Lemma 7).

Let us start from the following Proposition.

Proposition 6 *Consider a point $P_0 = (E_0, K_0, L_0)$, $0 < E_0 < \overline{E}$, $0 < K_0$, $0 < L_0 < 1$. Then, if L_0 is small enough, the positive semi-trajectory from P_0 tends to $\left(\overline{E}, 0, 0\right)$.*

Proof. Assume $L_0 < \min\left(\frac{\beta}{2(\beta+\varepsilon)}, \left(\frac{\frac{\alpha\beta}{2\varepsilon}K_0^{\alpha-1}E_0^{\gamma}}{\gamma\left(\overline{E}-E_0\right)+\frac{\theta}{\eta}}\right)^{\frac{1}{1-\beta}}, \left(\frac{\overline{E}-E_0}{\delta K_0^{\alpha}E_0^{\gamma-1}}\right)^{\frac{1}{\beta}}\right)$. Then it is easily checked that, along the trajectory from P_0, $\dot{K}$, $\dot{L} < 0$ as $t > 0$. It follows that E keeps increasing for $t > 0$, remaining, though, smaller than $\overline{E}$. In fact, suppose $\dot{E} = 0$ at some $\bar{t} > 0$. Then

$$\ddot{E}(\bar{t}) = \frac{\partial \dot{E}}{\partial K}\dot{K} + \frac{\partial \dot{E}}{\partial L}\dot{L} > 0,$$

so that , for $t > \bar{t}$, $\dot{E} > 0$ again.

Hence the statement of the Proposition follows. ■

Let us start, now, from an initial point P_0 belonging to the basin of attraction of P_1^*. For example, let us consider P_1^* itself. Moving downward along the half-line $E = E_1^*$, $K = K_1^*$, $L < L^*$, we cross, as shown in the above Proposition, the basin of attraction of P_1^* at a certain point, say $\widetilde{P} = \left(E_1^*, K_1^*, \widetilde{L}\right)$, $\widetilde{L} < L^*$. We wonder if the positive semi-trajectory starting from $\widetilde{P}$ tends to the saddle P_2^*.

For that to happen, since $E_1^* < E_2^*$, $K_1^* < K_2^*$ and, along the positive semi-trajectory from $\widetilde{P}$, K decreases until the trajectory crosses the plane $L = L^*$, it is necessary that the trajectory crosses the plane $L = L^*$ at a point where $K < K_1^*$ and $\dot{L} > 0$. Furthermore, observe that $\dot{E} > 0$ at $\widetilde{P}$. Should the trajectory *go back*, before crossing $L = L^*$, to a point where $E = E_1^*$, at such a point it would be again $\dot{E} > 0$, since $K < K_1^*$ and $L < L^*$. So our hypothetical trajectory must cross $L = L^*$ at a point where $E > E_1^*$ and $K < K_1^*$.

The following Lemma allows, precisely, to detect the points with the features described above belonging to the stable manifold of P_2^*.

Lemma 7 *Let $\alpha+\gamma < 1$ and assume that two equilibria exist, $P_1^* = (E_1^*, K_1^*, L^*)$ and $P_2^* = (E_2^*, K_2^*, L^*)$, $E_1^* < E_2^*$, $K_1^* < K_2^*$, $L^* = \frac{\beta}{\beta+\varepsilon}$. Moreover, assume that the conditions of Lemma 3 of the previous Section (i.e. $\eta \geq \frac{\varepsilon}{\varepsilon+\alpha\beta}$, $\overline{E} > E_B = \frac{\theta(\beta+\varepsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta}$) are satisfied and that P_1^* is a sink. Consider, in the plane $L = L^*$, the open set*

$$A = \left\{ P = (E, K, L^*) : E > E_1^*,\ K < K_1^*,\ \dot{L}(P) > 0 \right\}$$

Then A can be partitioned into three non-empty pair-wise disjoint subsets, $A = A_1 \cup A_2 \cup A_3$, where A_1 and A_2 are open, while A_3 is an unidimensional set belonging to the stable manifold of P_2^* (see Figure 5). More precisely

$$\begin{aligned} A_1 &= \Big\{ P \in A : \text{ the positive semi-trajectory starting from } P \text{ crosses } \dot{E} = 0 \\ &\qquad \text{before crossing again } L = L^* \Big\}, \\ A_2 &= \Big\{ Q \in A : \text{ the positive semi-trajectory starting from } Q \text{ crosses again} \\ &\qquad L = L^* \text{ before crossing } \dot{E} = 0 \Big\}. \end{aligned}$$

Proof. First of all we want to describe the region of $L = L^*$ where $\dot{L} > 0$. By straightforward computations we can check that the curve $\dot{L} = 0$, lying in the plane $L = L^*$, is crossed at most twice by each line $E = E_0$ or $K = K_0$. In fact such a curve, say Γ, is an *oval* contained in a rectangle $\left[E', E''\right] \times \left[K', K''\right]$, where $0 < E' < E_1^* < E_2^* < E''$, $0 < K' < K_1^* < K_2^* < K''$ (in fact it can be proved that $E'' < \overline{E}$). Thus the region of $L = L^*$ where $\dot{L} > 0$ is the open region, say C, bounded by Γ.

More specifically let us consider the sub-region B of C, lying below the line joining P_1^* and P_2^*, where K is taken as the vertical coordinate. Again we can check that, at a point $P = (E, K, L^*) \in B$, $\dot{E} > 0$ and $\dot{\Omega} < 0$ (i.e. $\alpha K^{\alpha-1} (L^*)^\beta E^\gamma > \theta$), as shown in Figure 5.

Moreover, we claim that the conditions of Lemma 3 of the previous Section (in particular $\overline{E} > E_B$), which are supposed to hold, imply that the tangent line to Γ at P_1^* has a positive slope. In fact, write the equation of Γ on $L = L^*$

$$\Gamma)\ F(E, K) = \gamma \left(\overline{E} - E - \delta K^\alpha (L^*)^\beta E^{\gamma-1} \right) + \frac{1}{\eta} (\theta - \alpha K^{\alpha-1} (L^*)^\beta E^\gamma) = 0$$

and compute $\frac{\partial F}{\partial E}$ and $\frac{\partial F}{\partial K}$ at (E_1^*, K_1^*). After easy steps, we obtain

$$\frac{\partial F}{\partial E} = \frac{\gamma}{E_1^*}\left[-E_1^* + (1-\gamma)\left(\overline{E} - E_1^*\right) - \frac{\theta}{\eta}\right]$$

By recalling $E_1^* < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma}\overline{E}$ and the expression of E_B, it can be checked that our assumptions imply $\frac{\partial F}{\partial E} > 0$. Analogously it is proved that $\frac{\partial F}{\partial K} < 0$ at (E_1^*, K_1^*). So the claim follows.

Finally, let us consider $A \subset B$ and $A_1, A_2 \subset A$, as defined in the statement of the Lemma. First of all, let us see that A_1 and A_2 are non-empty.

In fact, if $P = (E, K, L^*) \in A$ belongs to the basin of attraction of P_1^* and is sufficiently close to P_1^*, then the positive semi-trajectory starting from P must remain close to P_1^*, because P_1^*, being a sink, is in particular Lyapunov stable. However, since $E > E_1^*$, this implies (see again Figure 5) that the positive semi-trajectory from P crosses $\dot{E} = 0$ before possibly crossing again $L = L^*$.

As to A_2, consider the intersections of Γ with the line $K = K_1^*$ (on $L = L^*$). One intersection is, of course, P_1^*, while the other is a point $R = \left(\widetilde{E}, K_1^*, L^*\right)$ with $\widetilde{E} > E_1^*$. Then $\dot{L}(R) = 0$ and we want to show that $\ddot{L}(R) < 0$. In fact, E_1^* and $\widetilde{E}$ are the solutions of the equation

$$G(E) = F(E, K_1^*) = 0$$

Since we have seen that $G'(E_1^*) = \frac{\partial F}{\partial E}(E_1^*, K_1^*) > 0$ and is easily checked that $G'(E)$ has only one zero in the interval $\left[E_1^*, \widetilde{E}\right]$ (corresponding to the intersection of one increasing and one decreasing graph), it follows that $G'\left(\widetilde{E}\right) < 0$.

Hence

$$\ddot{L}(R) = G'\left(\widetilde{E}\right)\dot{E}(R) < 0$$

This means that, along the trajectory through R, $L(t)$ has a relative maximum at R. Therefore, by the continuous dependence of trajectories on initial conditions (see V.I. Arnold (1978)), it follows that positive semi-trajectories from points of A sufficiently close to R cross again $L = L^*$ *near* R, and thus before reaching $\dot{E} = 0$ (see Figure 5).

Once we have proved that A_1 and A_2 are non-empty, their openness is just a consequence of the continuous dependence of trajectories on initial conditions, as in the definitions of A_1 and A_2 we have required trajectories to cross, and not merely touch, respectively, $\dot{E} = 0$ and $L = L^*$.

So A_1 and A_2 are two non-empty disjoint open subsets of A (in the plane $L = L^*$). But, since A is connected, there exist points of A which don't lie either in A_1 or A_2. Let $A_3 = A - A_1 \cup A_2$. Clearly A_3 has no isolated point.

Consider, now, the positive semi-trajectory starting from some $Q_0 \in A_3$. Along it E initially increases. May the trajectory cross at the same instant of time, say $t_0 > 0$, both $\dot{E} = 0$ and $L = L^*$? In such a case, as t_0 is finite, $\dot{E}(t_0) = \dot{K}(t_0) = 0$, while $\dot{L}(t_0) < 0$. Hence

$$\ddot{E}(t_0) = \frac{\partial \dot{E}}{\partial L}\dot{L}(t_0) > 0 \tag{15}$$

Therefore $E(t)$ has a (relative) minimum at t_0 and thus cannot increase for $t < t_0$.

So, along the positive semi-trajectory from Q_0, E keeps increasing and so does K, as $L > L^*$. However L cannot tend to 1 as $t \to +\infty$, since, along any positive semi-trajectory on the invariant plane $L = 1$, $E \to 0$, as it is easily checked. This implies that, if along a trajectory L and E both increase and L approaches 1, at a certain point the trajectory must cross the surface $\dot{E} = 0$.

As a conclusion the positive semi-trajectory starting from $Q_0 \in A_3$ must tend to the saddle P_2^*.

It follows that P_2^* has a bi-dimensional stable manifold, that A_3 is unidimensional and finally that trajectories tending to P_2^* do not spiral. ■

An immediate consequence of the previous Lemma is the following:

Corollary 8 *Let $\alpha + \gamma < 1$ and two equilibria, P_1^* and P_2^*, exist, P_1^* being a sink. Moreover, assume $\eta \geq \frac{\varepsilon}{\varepsilon + \alpha\beta}$ and $\overline{E} > E_B = \frac{\theta(\beta+\varepsilon)(2-2\alpha-\gamma)}{\alpha\beta\gamma\eta}$. Then P_2^* is a saddle with a bi-dimensional stable manifold and the eigenvalues of its Jacobian matrix are all real (one positive and two negative).*

We are now able to prove our main Theorem

Theorem 9 *Given the assumptions of Lemma 7, there exists a neighborhood* N *of the sink P_1^*, such that, for any $(E_0, K_0, L_0) \in \mathrm{N}$, the half line $\left\{E = E_0, K = K_0, L < L_0\right\}$ intersects the stable manifold of P_2^*.*

Proof. Recalling the notations of Lemma 7, we see that the set A, in the plane $L = L^*$, looks like (the interior of) a *triangoloid* with one curvilinear and two *straight* sides. So, let P_1^*, Q and R be the *vertices* of A, where P_1^* is the sink and Q and R are, respectively, the further intersections of $E = E_1^*$ and $K = K_1^*$ with the *oval* $\Gamma = \left\{\dot{L} = 0\right\} = \{F(E,K) = 0,\, L = L^*\}$. We also recall that, at P_1^*, $\frac{\partial F}{\partial E} > 0$ and $\frac{\partial F}{\partial K} < 0$. In particular this implies that, at Q, $\frac{\partial F}{\partial K} > 0$ (see Figure 5).

Our first step is to prove the following:

<u>Claim</u> The set $\overline{A_3}$ (the closure of A_3), contained in the stable manifold of P_2^*, meets the open segment (Q, P_1^*), i.e. the *vertical* side of $\overline{A}$ (beyond the *horizontal* side, as shown in the Lemma).

As we have seen in the proof of Lemma 7, the Claim follows if we show that, along the trajectory through Q, $L(t)$ reaches a relative maximum in Q.

Because $\dot{K}(Q) = \dot{L}(Q) = 0$, it is enough to prove that

$$\ddot{L}(Q) = \frac{\partial F}{\partial E}(Q)\, \dot{E}(Q) < 0,$$

which is equivalent, being $\dot{E}(Q) > 0$, to

$$\frac{\partial F}{\partial E}(Q) < 0$$

Since it is easily checked that $\frac{\partial F}{\partial E}$ (like $\frac{\partial F}{\partial K}$) changes sign and has exactly one zero in the interval $(0, K_1^*)$ of the line$\{E = E_1^*,\ L = L^*\}$, it suffices to show that at the point $S = \left(E_1^*, \widetilde{K}, L^*\right)$, where $\frac{\partial F}{\partial E} = 0$,

$$\dot{L}(S) = F(E_1^*, \widetilde{K}) > 0$$

Now it is easily computed that $\frac{\partial F}{\partial E}(S) = 0$ is equivalent to

$$E_1^* + \frac{\alpha}{\eta}\widetilde{K}^{\alpha-1}(L^*)^{\beta}(E_1^*)^{\gamma} - (1-\gamma)\delta\widetilde{K}^{\alpha}(L^*)^{\beta}(E_1^*)^{\gamma-1} = 0,$$

whereas $E_1^* + \frac{\alpha}{\eta}\widetilde{K}^{\alpha-1}(L^*)^{\beta}(E_1^*)^{\gamma} - (1-\gamma)\delta\widetilde{K}^{\alpha}(L^*)^{\beta}(E_1^*)^{\gamma-1} > 0$ implies $\frac{\partial F}{\partial E}(E_1^*, K) < 0.$

If we suppose, by contradiction, that $\widetilde{K} \leq K_Q$ (the K-coordinate of Q) and therefore $F(E_1^*, \widetilde{K}) \leq 0$, this implies, in particular, $\frac{\partial F}{\partial K}(E_1^*, \widetilde{K}) > 0$, i.e.

$$\frac{1-\alpha}{\eta}\widetilde{K}^{\alpha-1}(L^*)^{\beta}(E_1^*)^{\gamma} > \gamma\delta\widetilde{K}^{\alpha}(L^*)^{\beta}(E_1^*)^{\gamma-1},$$

that is

$$\widetilde{K} < \widehat{K} := \widehat{\lambda}E_1^* := \frac{1-\alpha}{\gamma\delta\eta}E_1^*$$

Vice-versa, in the following we will construct an increasing sequence $\{K_n\}$ approximating $\widetilde{K}$ from below.

First of all, define K_0 by

$$0 = (1-\gamma)\delta K_0^{\alpha}(L^*)^{\beta}(E_1^*)^{\gamma-1} - \frac{\alpha}{\eta}K_0^{\alpha-1}(L^*)^{\beta}(E_1^*)^{\gamma} < E_1^* \tag{16}$$

Hence

$$K_0 = \lambda_0 E_1^* := \frac{\alpha}{(1-\gamma)\delta\eta}E_1^* < \widetilde{K},$$

since the first member of the inequality (16) is clearly increasing in K.

On the other hand, $\frac{\partial F}{\partial K}(E_1^*, K) > 0$ for $K < \widehat{K}$ implies

$$F(E_1^*, \widetilde{K}) > F(E_1^*, K_0)$$

Thus, if $F(E_1^*, K_0) \geq 0$, the Claim is proven.

Suppose, instead, $F(E_1^*, K_0) < 0$. Recalling the equilibrium conditions (7), this is easily seen to be equivalent to

$$\varphi\left(\overline{E} - E_1^*\right) < \psi\left(\lambda_0\right),$$

where

$$\varphi(x) = \left(\frac{\alpha}{\theta}\right)^{\alpha} \delta^{1-\alpha} \left(\gamma x^{\alpha} + \frac{\theta}{\eta} x^{\alpha-1}\right)$$

and

$$\psi\left(\lambda\right) = \gamma\delta\lambda^{\alpha} + \frac{\alpha}{\eta}\lambda^{\alpha-1}$$

Therefore, if $K = \lambda E_1^*$ with $\lambda < \widehat{\lambda}$, $\frac{\partial F}{\partial K}(E_1^*, K) > 0$ implies $\psi'\left(\lambda\right) < 0$; whereas $\varphi(x)$ has exactly one minimum at $\overline{x} = \frac{\theta(1-\alpha)}{\alpha\gamma\eta}$, as is easily checked.

However, the conditions we posed (see (10) and (13)), i.e.

$$E_1^* < \frac{1-\alpha-\gamma}{2-2\alpha-\gamma} E_A \quad \text{and} \quad E_A < E_B,$$

imply, through easy computations,

$$\overline{E} - E_1^* > \overline{x}$$

that is

$$\varphi'\left(\overline{E} - E_1^*\right) > 0$$

Hence, considering $\varphi(x)$ as defined in $(\overline{x}, +\infty)$, we can write

$$\overline{E} - E_1^* < M_1 := \varphi^{-1}\left(\psi\left(\lambda_0\right)\right)$$

Therefore, by applying the equilibrium condition

$$g\left(E_1^*\right) = E_1^* + \rho\left(E_1^*\right)^{\frac{\alpha+\gamma-1}{1-\alpha}} = \overline{E} \tag{17}$$

($\rho > 0$ being defined in (7)), it follows, through easy computations,

$$\left(E_1^*\right)^{2-\alpha-\gamma} > N_1 = h\left(\lambda_1\right) := \left((1-\gamma)\delta\lambda_1^{\alpha} - \frac{\alpha}{\eta}\lambda_1^{\alpha-1}\right)\left(L^*\right)^{\beta},$$

where N_1 is calculated from (17), $h\left(\lambda\right)$ is clearly increasing and $K_1 := \lambda_1 E_1^*$ satisfies

$$K_0 < K_1 < \widetilde{K}$$

This way we can construct an increasing sequence $K_n = \lambda_n E_1^* < \widetilde{K}$. So, if it happens, for some K_n, that $F(E_1^*, K_n) \geq 0$, the Claim is proven.

Otherwise $K_n \to \overline{K} = \overline{\lambda} E_1^* \leq \widetilde{K}$. However, if $F(E_1^*, \overline{K}) < 0$, i.e. $\varphi\left(\overline{E} - E_1^*\right) < \psi\left(\overline{\lambda}\right)$, the algorithm can start again. In fact we obtain

$$\left(E_1^*\right)^{2-\alpha-\gamma} > \overline{N} = h\left(\overline{\lambda}\right),$$

so that we can replace, for example, in the above construction $\left(E_1^*\right)^{2-\alpha-\gamma}$ by $\left(E_1^*\right)^{2-\alpha-\gamma} - \overline{N} = \overline{\sigma}\left(E_1^*\right)^{2-\alpha-\gamma}$, $0 < \overline{\sigma} < 1$, and $h(\lambda)$ by $\overline{h}(\lambda) := h(\lambda) - \overline{N}$.

Hence, in the *worst* case, we can construct an increasing sequence $K_n \to \overline{K}$ $= \overline{\lambda} E_1^* \leq \widetilde{K}$ such that

$$F(E_1^*, \overline{K}) = 0$$

and

$$\frac{\partial F}{\partial E}(E_1^*, \overline{K}) \leq 0$$

If $\frac{\partial F}{\partial E}(E_1^*, \overline{K}) < 0$, the Claim is proven. Otherwise, being $Q = (E_1^*, \overline{K}, L^*)$, it holds

$$\dot{K}(Q) = \dot{L}(Q) = \frac{\partial F}{\partial E}(Q) = \ddot{L}(Q) = 0, \tag{18}$$

Hence Q is, anyway, in the boundary of A_2 and $\overline{A_3}$ could *reach* the vertical side of $\overline{A}$ precisely at Q. However (18) implies, as is easily checked,

$$\dddot{L}(Q) = \frac{\partial^2 F}{\partial E^2}(Q)\left(\dot{E}(Q)\right)^2 < 0$$

Therefore, along the trajectory starting at $t = 0$ from Q,

$$L(t_0) < L^* \tag{19}$$

for some $t_0 > 0$. Should Q be a *terminal point* of $\overline{A_3}$, (19) would hold also for points of A_3 near Q. But this contradicts what we have proved in Lemma 7, namely that, along trajectories meeting A_3 at $t = 0$, $L(t) > L^*$ for all $t > 0$.

This concludes the proof of the Claim.

The above argument, in fact, implies that $\frac{\partial F}{\partial E} < 0$ along the whole *curvilinear side* of $\overline{A}$, i.e. the *arc* $[Q, R]$ (see Figure 5). As a consequence this arc has no intersection with $\overline{A}_1$ and thus with $\overline{A}_3$ (see the proof of Lemma 7).

Hence, let us indicate by $[T, V]$ a (possibly unique, as in Figure 5) connected component of $\overline{A_3}$, such that T and V belong, respectively, to the open segments (Q, P_1^*) and (P_1^*, R). It follows that there exist points of (T, V) whose *negative* (i.e. backward) semi-trajectories intersect the half-plane $H = \{E = E_1^*, L < L^*\}$. Furthermore, it is easily observed that, if such a a semi-trajectory intersects H at a point $\left(E_1^*, \overline{K}, \overline{L}\right)$, with $\overline{L} < L^*$ and $\overline{K} \leq K_1^*$, then along it (having exchanged t with $-t$) E continues to decrease, while K continues to increase. On the other hand, consider the negative semi-trajectory starting from V. Obviously it reaches some half-plane $\{K = K_1^* + \sigma, L < L^*\}$ with $\sigma > 0$. It follows, by the continuous dependence of trajectories on initial conditions, that there exists $Z \in (T, V)$ such that its negative semi-trajectory intersects H at some point $U = \left(E_1^*, K_1^* + \zeta, \overline{L}\right)$, with $\overline{L} < L^*$ and $\zeta > 0$.

Therefore, projecting on the plane $L = L^*$ the intersections with H of the negative semi-trajectories from the *arc* $[T, Z] \subset [T, V]$, we get a continuous map

π from $[T, Z]$ onto a closed segment $[T, U']$ of the line $\{E = E_1^*, L = L^*\}$, containing P_1^*. Moreover, a continuity argument, again, implies that the negative semi-trajectories from $[T, Z]$ also cross any half-plane $\{E = E_1^* - \nu, L < L^*\}$, if $\nu > 0$ is sufficiently small.

In fact this concludes the proof of the Theorem. ■

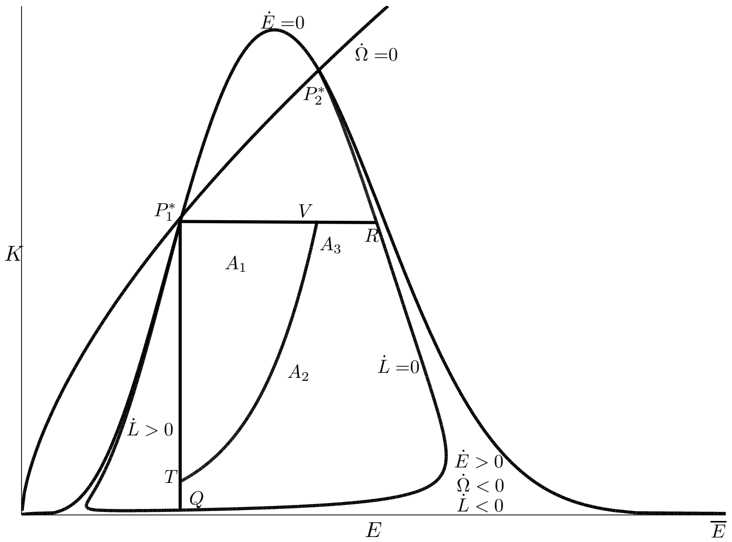


Figure 5: Configuration in the plane $L^* = \dfrac{\beta}{\epsilon + \beta}$; values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\overline{E} = 0.17$

Hence, under the assumptions of the Theorem, for any initial point (E_0, K_0) sufficiently close to (E_1^*, K_1^*), there exists a continuum of initial values L_0^1 such that the trajectory starting from $\big(E_0, K_0, L_0^1\big)$ approaches P_1^*, and a locally unique value L_0^2 such that the trajectory starting from $\big(E_0, K_0, L_0^2\big)$ converges to P_2^*. So global indeterminacy occurs, since, from the initial position (E_0, K_0), the economy may follow one of the trajectories belonging to the basin of attraction of the poverty trap P_1^* but it may also follow a trajectory lying on the stable manifold of the the stationary state P_2^*.

In Figure 6 a numerical simulation is shown. The starting points of the trajectories are chosen along the half line $\{E = E_1^*, K = K_1^*, L < L^*\}$. The trajectory starting from the lowest initial value of L (in bold) lies on the stable manifold of P_2^* and consequently converges to P_2^*, while all the others approach P_1^*. In Figure 7 the projections on the plane (K, L) of the trajectories in Figure 6 are drawn. Notice that some trajectories approaching P_1^* are characterized by an initial phase where the values of K and L are higher than along the trajectory converging to P_2^*; however, the higher values of K and L give rise to over-exploitation of the natural resource and the consequent reduction of the stock E drives the economy towards the undesirable equilibrium P_1^*, where the values of K and E are lower than in P_2^*.

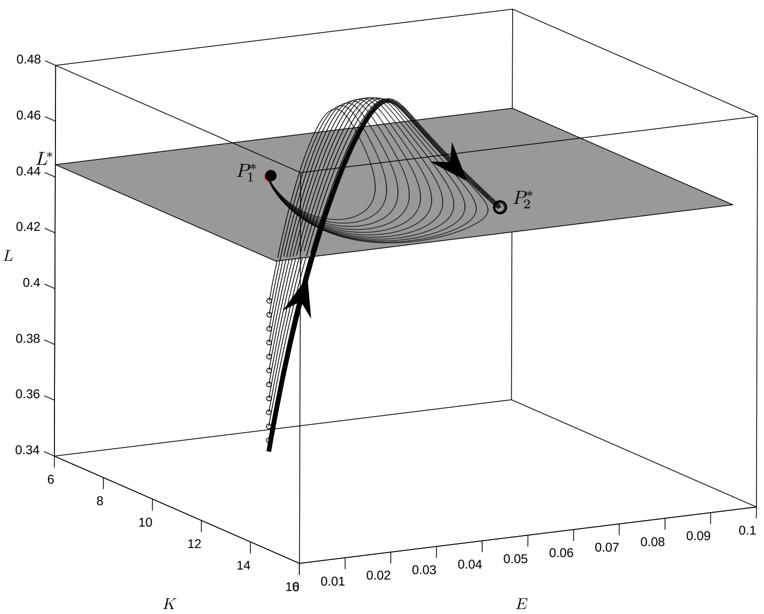


Figure 6: Global indeterminacy in the space (E, K, L); values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\overline{E} = 0.17$

6 Conclusions

We have analyzed an economic growth model where local indeterminacy -i.e. the existence of a continuum of equilibrium trajectories approaching the same stationary state- can occur also in a context of social constant returns to scale.

Such indeterminacy is due to the interplay between negative externalities (negatively affecting the stock E of the natural resource) and positive externalities (generated by E and augmenting the productivity of K and L in the production process of output). Our analysis has shown that local indeterminacy can be observed only when two fixed points exist: $P_1^* = (E_1^*, K_1^*, L^*)$ and $P_2^* = (E_2^*, K_2^*, L^*)$, with $E_1^* < E_2^*$, $K_1^* < K_2^*$ and $L^* = \frac{\beta}{\beta+\varepsilon}$. In such a case, we have shown that only the Pareto-dominated equilibrium P_1^* can be attractive.

Furthermore, we have seen that the conditions assuring the attractivity of

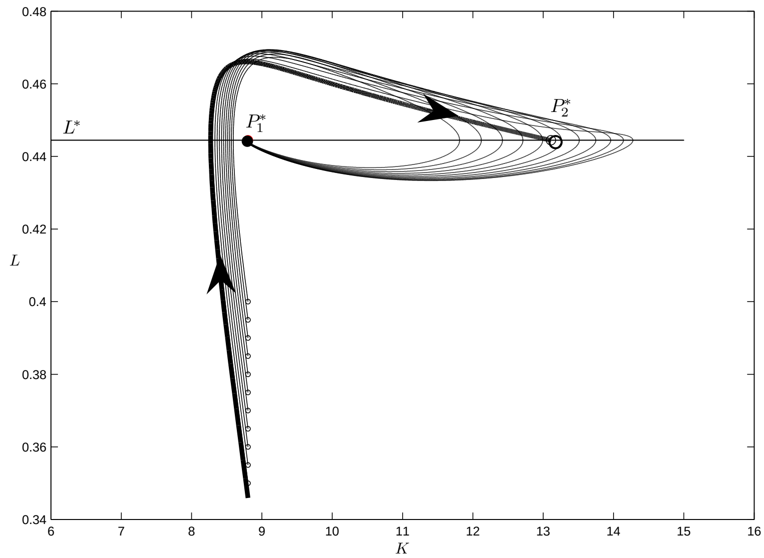


Figure 7: Global indeterminacy in the plane (K, L); values of parameters: $\alpha = 0.1$, $\beta = 0.8$, $\gamma = 0.58$, $\delta = 0.05$, $\epsilon = 1$, $\eta = 1.5$, $\theta = 0.001$, $\overline{E} = 0.17$

P_1^* also guarantee that the stable manifold of the desirable equilibrium P_2^* can be *reached* from initial state variables close to those of P_1^*. That is, for any initial pair (E_0, K_0) sufficiently close to (E_1^*, K_1^*), there exists some initial value L_0^2 of L such that the trajectory starting from $\left(E_0, K_0, L_0^2\right)$ converges to P_2^*. This represents a global indeterminacy result, since, from the initial position (E_0, K_0), the economy may follow a continuum of trajectories belonging to the basin of attraction of the poverty trap P_1^*, but it may also follow one trajectory reaching the desirable point P_2^*. Consequently, in this context expectations matter in equilibrium selection.

More generally, one may wonder if under our conditions P_2^* could be a *global saddle*. Precisely, having assumed the conditions of Lemma 7, we may wonder if, given any initial $(E_0, K_0) \in \left(0, \overline{E}\right) \times (0, \infty)$, there exists a unique $L_0 = L\left(E_0, K_0\right) \in (0, 1)$ such that the positive semi-trajectory from (E_0, K_0, L_0) converges to P_2^*. Clearly it is quite a difficult question, although numerical experiments seem to suggest some sort of positive answer.

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