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# Equivalence, recursive negation and invariance of the mathematical uncertainty predicate.

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Key words: formalism, intuitionism, uncertainty.

Abstract: It is impossible to prove the equivalence of double negation for the Systeme I of Gödel. It is possible to deduce the equivalence of double negation with a three valued logic which is coherent with respect to symmetric implication and has the third value as invariant by negation. It is possible to annihilate the third value and switch back to the two valued boundary logic. Brouwer and Gödel provide the foundation for the theory of uncertainty.

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The existence of a multi language knowledge base and the precision required to address uncertainty, advise to write the following precautionary expression:

A phrase (phrasis in Greek) or expression $f$ is a collection of words of the dictionary. The dictionary itself is a phrase or expression. A proposition $p$ is a phrase or expression that *can be* evaluated or tested. $p$ *can* be tributed one of two (or more) exclusive values. False and true are represented, in the Basic notation, by $-1$ and 1. A theorem $t$ is a proposition that is judged *to be* true. $p$ is defined in the context of Aristotele's possibilism [20]; $t$ is defined in the context of Crisippus' determinism [20]. A theorem consists of an announced (enuntiare in Latin) proposition that specifies hypothesis and thesis together with a proof or judgment. Evaluation and judgment are formally expressed in the modalities of necessity, possibility and impossibility. A proposition that is not judged is undecided. A proof that a proposition cannot be judged is an impossibility theorem and the proposition is undecidible.

Two phrases, propositions, theorems are *equal* if they are written in the same way: identical words in the same order.

1. Classic negation is the operation that consists of transforming a proposition with it's possible logical values into a proposition with opposite logical values. It is stated by the expression "it is not true that" and formally represented by the symbol ¬. The model for this operation is a table consisting of 3 lines: the first line identifies the propositions and the next two form a matrix:

Table 1

| $p$ | $\neg p$ |
|---|---|
| 1 | $-1$ |
| $-1$ | 1 |

Notice that $p$ and $\neg p$ are written differently.

Definition: Two propositions $p$ and $q$ are *equivalent*, written $p \equiv q$, in case their possible logical values identify:

Table 2

| $p$ | $q$ | $p \equiv q$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | $-1$ | $-1$ |
| $-1$ | 1 | $-1$ |
| $-1$ | $-1$ | 1 |

Equivalence is the relation that compares propositions by their logical values. In the defining table, $p$ and $q$ are written differently.

The operation of negation is indefinitely iterable: $\neg ... \neg\neg\neg p$ is a proposition if $p$ is a proposition. Iteration operates at the constructive level: it produces many propositions from a given one.

Proposition 1: $\neg\neg p$ is equivalent to $p$.

Brouwer theorem. The absurdity of absurdity of absurdity is equivalent to absurdity.
proved in Cambridge Lectures [2, chp.1].

The following textbook table judges proposition 1, or transforms proposition 1 into a theorem, or proves the theorem $\neg\neg p \equiv p$:

Table 3

| $p$ | $\neg p$ | $\neg\neg p$ | $p \equiv \neg\neg p$ |
|---|---|---|---|
| 1 | $-1$ | 1 | 1 |
| $-1$ | 1 | $-1$ | 1 |

The column of one's identifies the tautology. Notice that $p, \neg p, \neg\neg p$ are written differently.

Theorem 1. The negation of theorem $t$: $\neg t$ is not a theorem.
The logical value of $t$ *is* 1. The logical value of $\neg t$ *is* $-1$, and it is not possible to judge $\neg t$ *to be* true.

Theorem 2. The negation of theorem $t$: $\neg t$ is not a proposition.
The logical value of $\neg t$ *is* $-1$ and *cannot be* 1.

Theorem 3. The negation of theorem $t$: $\neg t$ is non logic.
From T2: $\neg t$ is not a proposition, therefore it is not a logical expression.

2. So far we have defined a generic structure based on evaluation, equality, negation, equivalence. Without rules of inference, collective structure nor quantifiers. Now we adopt:

I.1) The formal system of Principia Mathematica or of Hilbert-Ackermann as in Gödel [6][7] and

I.2) The system of axioms for abstract set theory, by Zermelo and Fraenkel, [23][18].

I.1) and I.2) are realization of Gödel's (unentsheidbare) Systeme I [8]. As Peano arithmetic and von Neumann classical mathematics.

Whenever a set $X$ is defined, $x \in X$ is to be read as "x is an element of the set $X$ and obeys the ZF axioms defining $X$". Generic $f$, $p, t$ require no collective structure.

**Definitions**: $P, T$ are the sets of propositions and theorems.

Equivalence is a binary relation between elements in $P$, in $T$, which is reflexive, symmetric and transitive.

Theorem 4. Theorems in $T$ are equivalent one to the other.
$t \in T$ has logical value 1. So it is for any other theorem in $T$ differing from $t$. Each differing theorem is equivalent to $t$ and, by transitivity, they are so one to the other.

In the next sections 3 and 4 we come to the impossibility theorem stating that it is impossible to extend the generic proof of the equivalence of double negation to the first order incomplete systems examined by Gödel. By generic we mean that we are considering elementary propositions one by one and not as a whole or totality. Negation (the "contradictory function" of Principia Mathematica) acts individually and a formal problem of recovering $p$ from $\neg\neg p$ is not apparent. It is believed that the recovery is possible for all propositions considered as a whole by the ZF axioms ruling $P$. I contend the belief. I will define 3 subsets of $P$ with respect to negation and examine the implications of relating them biunivocally. It turns out that there is not a formally explicit rule or formula to do so. Finally I show the existence of a proposition that cannot be equivalent to it's double negation and show that this proposition makes the biunivocal correspondence impossible. Particular care is given to the issue of difference and equality. It is important to understand that if the extension or generalization of the generic proof passes through the implication from difference to equivalence it can be contradictory. If it passes through the implication from equality to equivalence then the linguistic, semantic and syntactic problems are present that lead to the impossibility. Brouwer had already proved the impossibility with species. These are collective structures supported by 2 main axioms: the existence of the empty species and a species cannot be element of itself.

3. The next ordered steps are an examination of negation, iteration of negation and equivalence. $P$ is not empty: the announcing proposition of T4 is there. Consider the model, where $p$, $q$ are propositions and $(p = q)$, $(p \equiv q)$ have been substituted to the domain variables *x*,*y*, in $x \Rightarrow y$ and $y \Rightarrow x$:

Table 4

| $p = q$ | $p \equiv q$ | $(p = q) \Rightarrow (p \equiv q)$ | $(p \equiv q) \Rightarrow (p = q)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | $-1$ | $-1$ | 1 |
| $-1$ | 1 | 1 | $-1$ |
| $-1$ | $-1$ | 1 | 1 |

3.1. hypothesis: $p \neq \neg\neg p$.
$\neg p, \neg\neg p$ are in $P$ and are different. The first question we ask is if it is possible to infer the equivalence of double negation from the inequality between $p$ and $\neg\neg p$. The answer is no: by hypothesis, we need to consider only the last two lines: 4th and 5th of Table. 4. To infer the equivalence from inequality we need the first 3 elements of line $4$, and we rewrite them by the assumption $p \neq q$ to obtain the inferential form of modus ponens $((p \neq q) \wedge ((p \neq q) \Rightarrow (p \equiv q))) \Rightarrow (p \equiv q)$. As we substitute $\neg\neg p$ to q, we establish $p \equiv \neg\neg p$. As $\neg\neg p = \neg(\neg p)$, from the iterative property of the operation, we also substitute $\neg p$, establishing $p \equiv \neg p$. Therefore the circuit from difference to equivalence can be contradictory. This is also intuitive: many different propositions are equivalent, not all different propositions are equivalent.
If we add a collective structure specifying a domain with the substitution rule and modus ponens then equivalence of double negation is not deductible from inequality, and this carries over, for finite sets of formulae, to all higher order levels of the information semi lattice as proved in Urquhart [26]. This means that the extension of the proof of the equivalence of double negation given for elementary propositions (Principia Mathematica [24, p.101]), encounters the problem that $p$ and $\neg\neg p$ are written differently and from the difference it is not possible to deduce equivalence over the specified domain $P$.
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3.2. hypothesis: $\forall p \in P,\ p = \neg\neg p$. Assume that there is a magic formula that identifies each proposition in $P$ with it's double negation: Is it possible to imply the equivalence of double negation from the equality between $p$ and $\neg\neg p$? The answer is yes:
From the axiom of power sets, exists $\mathcal{P}(P)$: $\{p, \neg p, \neg\neg p\} \in \mathcal{P}(P)$. Apply the hypothesis to the elements $p$ of $P$. Apply the axiom of extensionality. By hypothesis $\{p, \neg p, \neg\neg p\} = \{p, \neg p, p\} = \{p, \neg p\}$. In the right hand side of the equality only $p \equiv p$ is expressible and obviously deductible by the reflexive property.

Of the two mutually exclusive hypothesis, only the first is true. All propositions are self equivalent and self imply. This leads to exploit reflexivity in establishing $p \equiv \neg\neg p$. But this follows from the second hypothesis, and we are in the case that a false proposition implies any proposition. Because of 3.1 and 3.2 we have to follow the line $(p = \neg\neg p) \Rightarrow (p \equiv \neg\neg p)$, but this line requires a careful construction.

4. Now the meaning (in Russell's sense of the objects extensionally defined) of $p = \neg\neg p$ is examined. Idealistically, the entire stream of propositions arising from negation is in $P$. This, as in 3.2, leads to use the axiom of power sets and the axiom of choice. Applying the negation operation to the elements of $P$ constructs $P'$. $\{\neg p, \neg\neg p\}$ is an element of the power set (from the axiom of power sets) $\mathcal{P}(P')$. Applying the negation operation to the elements of $P'$ constructs $P''$. $\{\neg\neg p\}$ is an element of the power set (from the axiom of power sets) $\mathcal{P}(P'')$.
Form the difference and define $P - P' = P_+$. From the axiom of power sets, $\{p\} \in \mathcal{P}(P_+)$. Form the difference and define $P' - P'' = \neg P_+$. From the axiom of power sets $\{\neg p\} \in \mathcal{P}(\neg P_+)$.

4.1. The operation of negation is recursive. In order to obtain $\neg\neg p$ it is necessity to negate twice and independently.

4.2. The operation of negation induces a map $P \mapsto P$.

Theorem 5. $p = \neg\neg p$ is a sufficient condition for the double negation to establish the identity for the map $P \mapsto P$.
$\forall p \in P$, if $p = \neg\neg p$ then the negation of the elements of $\neg P_+$ consists of elements $\neg\neg p$ in $P''$ and $P'' = P_+$. Conversely, if $P'' = P_+$, for each element of $P''$ there is a unique (by idempotency) element of $P_+$, but this does not rule out a simple permutation like $p = \neg\neg q$ and $q = \neg\neg p$. Therefore $P'' = P_+$ does not establish the necessity of $p = \neg\neg p$.

We lament a deficit of instructions for the recovery of $p$ from $\neg\neg p$. We implicitly believe that not not $p$ *is* $p$. Formally speaking this requires a formula that recovers $p$ from not not $p$. If it exists and is consistent with the elements of $P$ it has to be of the following kind:
Relatively to the Russell-Whitehead contradictory function, define the following operator $\chi$, such that $\chi(p) = \neg p$, $\chi(\neg p) = \neg\neg p$; $\chi$ acts on the propositions or 'fundamentals' . Negation is reversible if $\exists\, \chi^{-1}$ such that (read from right) $\chi^{-1} \circ \chi^{-1} \circ \chi \circ \chi(p)$ is $p$ and $\chi \circ \chi \circ \chi^{-1} \circ \chi^{-1}(\neg\neg p)$ is $\neg\neg p$. Therefore two identities are defined: one for $P_+$ and one for $P''$. Under these conditions it is possible to recover univocally $p$ from $\neg\neg p$. This implies idempotency between $P_+$ and $P''$ and the correspondence to be the identity.

4.3. The operation of negation determines the value function $\phi$: $P \mapsto \{1, -1\} \times \{1, -1\}$. The Basic notation simplifies the composition of $\phi$ by exploiting $\phi(\neg p) = -\phi(p)$:

$$\begin{array}{ccccc} P_+ & \mapsto & \neg P_+ & \mapsto & P'' \\ \downarrow & & \downarrow & & \downarrow \\ \phi(p) & \mapsto & -\phi(p) & \mapsto & -(-\phi(p)) \end{array}$$

The vertical arrow means that there is a correspondence between the sets in the first line and the composition of the value function, but the value function is independent: $\phi$ values $\neg\neg p$ even if $\neg\neg p$ is written differently from $p$. The vertical arrow imposes valuation coherence on $\chi$ and $\chi^{-1}$ and admits the interpretation of a projection. As negation is iterated the sign of $\phi$ switches, and so it is when moving back from $P''$ to $P_+$. The common and ancient view that not not p is p, that not crosses not and restores p, can be falsified. From T.5 I deduce the existence in $P$ of a proposition $\neg\neg\omega \neq \omega$ and from the above diagram I deduce the existence of a proposition for which $p = \neg\neg p \Rightarrow (p \equiv \neg\neg p)$ cannot establish $(p \equiv \neg\neg p)$ because in passing from $\neg\neg p$ to $p$ the signs of $\phi$ do not alternate as in the diagram. How do I deduce it? From the axiom of extensionality and the characteristic function: as the magic formula is insufficiently formalized, it can happen that a characteristic proposition conflicts with the objects defined in a way that the signs of $\phi$ do not alternate.

4.4. In the 1929 dissertation [6][11,p.67], Gödel adds the axioms (7) $x = x$ and (8) $x = y \Rightarrow (F(x) \Rightarrow F(y))$. In this extended sense one obtains $x = y \Rightarrow (F(x) \equiv F(y))$, by symmetry. By choosing the identity for $F$ on the domain $x$, $y$ and by theorem 5, we obtain, for propositions $p$ and $q$ in $P$:

$$p = q \Rightarrow (p \equiv q)$$
$$p = \neg\neg p \Rightarrow (p \equiv \neg\neg p).$$

The last we interpret: if in $P$ is specified, by negation, a domain of $p$-objects and a domain of $\neg\neg p$-objects, for which the equality applies, then the respective values coincide.

Gödel's extended system (axioms 7 and 8) captures the issues and opens to intuitionist logic: in intuitionist (Heyting) logic [13, chp.7], axiomatization leads to the recovery of $p \Rightarrow \neg\neg p$ and $(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$. With the constructive modality for $p \Rightarrow q$, translatable in Lewis's $p \subset q$ [19]: the construction of $p$ is part of the construction of $q$. On the other hand the inverse implications, $\neg\neg p \Rightarrow p$ and $(\neg q \Rightarrow \neg p) \Rightarrow (p \Rightarrow q)$, are denied universal validity. Heyting supports both $p$ and $q$ by a third construction.

There is plenty of evidence that one needs a formal conversion of $\neg\neg p$ into $p$. Consider the list of symbols ( $\in$ , $\notin$ ), ( $=$ , $\neq$ ), ( $<$ , $>$ ). One notices the relativity and symmetry of each couple. In mathematics, it is possible to recover $p$ by directly operating on the system of symbols and definitions. At this point we could proceed as Kuratovsky and Mostovsky and restrict $P$ to mathematical propositions. But how do you trace this fine boundary in $P$? Given that, with Kronecker, most of mathematics consists of words?

In order to extend the generic proof of equivalence to a system for the total domain one needs the axioms of Zermelo to specify two related operations and one condition: The first consists in choosing $\neg\neg p$ out of the totality of propositions. The second consists in choosing the couple $(p, \neg\neg p)$ whenever $\neg\neg p$ is chosen. The condition states that the choice of $(p, \neg\neg p)$ is unique whenever $\neg\neg p$ is fished out of totality. I doubt that this condition is verified. The short circuit on difference, the sufficiency only of equality, the functions $\chi$ and $\phi$ make the doubt solid as a rock. But I would not insist on such a difficult and controversial matter if I did not have also a counterfactual proof:

**Impossibility theorem: In general, $\forall p \in P$, it is impossible to prove $\neg\neg p \equiv p$.**

Begin by generically considering "those propositions consisting of ten words" and add the systemic structure for collecting over the domain $P$. Consider these specimens of Russell's contradiction of circularity:

$e =$ "the set of all propositions that consist of ten words". $e$ consists of 10 words and is self referential.

$s =$ "the set of all propositions consisting of ten words". $s$ consists of 9 words and is not self referential.

Now consider $\neg\neg e$. This proposition consists of at least 12 words and is not self referential. Notice that we can formally write, say, $e = \{p\text{: } p \in P \wedge g(p) = 10\}$ where $g$ is a function that counts the words in $p$. We could even observe that with $n$ words n$^{10}$ expressions qualify for admittance in $s$. Therefore we can write $e = s = \neg\neg e$ by the axiom of extensionality because "that consist" and "consisting" identify the same $g$ function and the double negation does not alter it. Nothwistanding that $e$, $s$, $\neg\neg e$ are different propositions, mathematics identifies them by the $g$ constituent and states that the three differing propositions are equivalent in defining the same set indifferently $named$ $e, s, \neg\neg e$. In $P$ the issue of the intended concept of a proposition and the extensive meaning matters in determining the possible values.

$\omega =$ "the set of all propositions that characterize the set $s$". $\omega$ consists of ten words and is an element of $s$, while this would not happen replacing "that characterize" with "characterizing". That is

$\omega' =$ "the set of all propositions characterizing the set $s$".

As propositions: $\omega \neq \omega', \ \omega \in s \wedge \omega' \notin s$. As sets, the axiom of extensionality applies: $\forall p((p \in \omega \equiv p \in \omega') \Rightarrow \omega = \omega')$. Leading to the first contradictory node: $\omega \in s \wedge \omega \notin s, \ \omega' \in s \wedge \omega' \notin s$. Furthermore, the extension of $\omega$ contains also $s$, but $s \notin s$, therefore we also have $\omega \notin s$, leading also to contradiction. $\neg\omega$ and $\neg\omega$' are different propositions and equal sets by extensionality, but $\neg\omega \notin s \wedge \neg\omega' \in s$, second contradictory node. $\neg\neg\omega$ and $\neg\neg\omega$' are different propositions and equal sets by extensionality, $\neg\neg\omega \notin s \wedge \neg\neg\omega' \notin s$ First non contradictory node. Along this line we have found that $\omega$ and $\neg\neg\omega$, $\omega'$ and $\neg\neg\omega'$ cannot be equally valued expressions being the double negations non contradictory relatively to $s$. $\omega \Rightarrow \neg\neg\omega \wedge \neg(\neg\neg\omega \Rightarrow \omega)$, the same for $\omega'$.

This is sufficient to stop the extension of the generic proof once the extensive meaning of a proposition characterizing a set is defined as in Zermelo and Russell. This also explains why it is possible to prove the equivalence principle generically while it is not possible to prove it in the presence of the Zermelo and Russell systemic structure for collecting.

Furthermore, use axioms 7 and 8 and the ZF axioms to define in $P$ the identity $P'' = P_+$ as done before. Consider $\neg\neg\omega \in P''$. To which element in $P_+$ does it correspond? It cannot correspond to $\omega$ as $\neg\neg\omega$ is meaningful in specifying $s$ while $\omega$ is contradictory. The diagram (4.3) relating the map on $P$ to the value function imposes recursive coherence by projection. Does $\neg\neg\omega$ correspond to $\omega'$? If so, then to which element does $\neg\neg\omega' \in \ P''$ correspond in $P_+$? $P''$ is formed by $\neg P_+$, but now the last set contains an invariant by negation established by the second contradictory node. $\neg\neg\omega = \omega \Rightarrow (\neg\neg\omega \equiv \omega)$ transits through $\neg\omega \equiv \omega$ and this makes the identities on $P_+$ and $P''$ undefined and the correspondence impossible. The necessity (T.5) related conclusion is that there is no general extension of the generic proof of the equivalence of double negation.

All the defined sets are consistent with respect to the separation axiom $\forall x, \ \exists y, \forall z$ $(z \in y \equiv F(x) \wedge z \in x)$ and the separation theorem $\forall x, \ \exists y(y \notin x)$. 3.1 and T.5 are the proof of the impossibility though abstract, the rest is corollary. One deduces, 4.3, the existence of a proposition with the characteristics of $\omega$ within the systemic structure for collecting. Of course, actually writing the propositions one feels better, but that is all.

All the elements for the next move have been collected:

## 5. What is $p \equiv \neg\neg p$ ?

To answer this question it is better to begin by asking what it is not. Name ($p \equiv \neg\neg p$), for convenience, as $f_0$. The equivalence cannot be proved in $P$, therefore $f_0$ does not belong to $T$. On the other hand it is proved generically for propositions which can be valued on $\{-1, 1\}$. Therefore $f_0$ is judged. Being judged, it is not a formula like (a $\vee$ b) in the Whitehead and Russell sense [21, chp.5]. We conclude that $f_0$ is primitive.

Two possible avenues open: the idealist consists in refining the axioms for sets in order to cancel the previous contradictions in a similar way to the theory of ramified types and exploiting the axioms for small sets. If one were to stratify $P$ so that propositions of differing type cannot be compared in value, then already $\neg\neg e$ damages the 1 to 1 correspondence. A further refinement of Zermelo's axioms would be quite artificial. The root of the counterexample is only partially the contradiction of circularity for the total domain as in Burali Forti and Russell. The root actually is the major topic of the Principles: the duality between word and meaning, wholeness and parts, intention and extension, conceptual form and content. Unfortunately, as also the original is debated in a lengthy controversial way up to the theory of types, there is a serious risk of sliding into the pedantic and scholastic mode. Given that the I systems are proved incomplete, one wonders if the refinement stops to an ultimate system.

Furthermore, one notices the curious state of the equivalence principle: Given that it cannot extend to sets nor to species, there is only one use for it: to control the way the inferential system treats equality and inequality. The second avenue consists in asking the question if it is possible to control the inferential process by exploiting the knowledge of the properties of the object. This avenue is examined in the next part of the paper. The existence of invariants up to negation is the object of study. As we move through the nodes in the impossibility theorem we notice that $\neg\omega \equiv \omega \wedge \neg(\neg\neg\omega \equiv \omega)$. This is a special or partial type of invariance. The general type is examined in the next part.

6. The lack of regularities (for example finitely repeated finite cycles) in the decimal expansion of $\pi$ are known after the diffusion of fast computing machines. It is remarkable that Brouwer, long time ago, offered as example for a "fleeing property" an expression $\beta$: '(if) there exists a natural number n such that, in the decimal expansion of $\pi$, the nth, n+1th,...,n+9th cipher form the sequence 0123456789'. The definition of a fleeing property in the Cambridge Lectures is:

1) For each natural number n it is possible to decide if n has the property or not.
2) A way to compute n is unknown
3) It is not known if it is an absurdity that at least one natural number n has the property.

It is readily verified that this is a property characterizing irrational numbers.

Consider the negation of Brouwer's expression $\neg\beta$: the sequence 0123456789 does not occur in the decimal expansion of $\pi$: this is still a fleeing property of $\pi$. The negation of an expression predicate of a fleeing property is still an expression predicate of a fleeing property. We denominate this characteristic or quality of fleeing properties as modal invariance of the mathematical uncertainty predicate up to negation.

The partial invariance (of the undefined value) is discovered by Kleene in the 1938 study on the notation for ordinal numbers and partial recursive functions. And is recovered by Kripke in his 1975 paper on the theory of truth.

$\beta \equiv \neg\beta$ in the possibility modality is fundamental: it immediately follows that $\neg\beta \equiv \neg\neg\beta$ and $\beta \equiv \neg\beta \equiv \neg\neg\beta$. This peculiarity of fleeing properties and of irrational numbers predicate of fleeing properties we denominate general invariance of the mathematical uncertainty predicate. As far as I know, it goes unnoticed in Brouwer. With this limitation proceeds the discussion of many valued logic, from the early contribution by Barzin and Errera, through Kolmogorov and Glivenko, to Heyting, Gödel and Troelstra.

Glivenko [5] introduces a hypothetical third value $p'$. From $p \Rightarrow \neg p'$ and $\neg p \Rightarrow \neg p'$, by Brouwer's theorem $((\neg p \vee p) \Rightarrow \neg q) \Rightarrow \neg q$, deduces $((\neg p \vee p) \Rightarrow \neg p') \Rightarrow \neg p'$. What is surprisingly questionable in Glivenko's proof is the hypothetical reduction of the third value to true or false, when this value defines an invariant property characterizing irrational numbers. The proof, if any, must be given in the constructive modality, by establishing the construction of irrational numbers as independent and, in the Lewis modality, establishing the construction of $p$ as part of the construction of $\neg p'$. Glivenko should have noticed that the hypothetical statement is incomplete: by the same line, admitting hypothetically $p'$, $p' \Rightarrow \neg p$ and $p' \Rightarrow \neg\neg p$, I obtain $p' \Rightarrow \neg(\neg p \vee p)$ and $p' \Rightarrow (\neg p \wedge \neg\neg p)$, the negation of tertium non datur and the deductibility of contradiction.

Gödel's proof [9], [10], that intuitionistic (Heyting) logic is no n-valued logic, also ignores the invariance under negation of fleeing properties. Brouwer himself does not give the slightest hint of abandoning a two valued logic in the Cambridge lectures.

We notice that $\beta$ *is* fleeing, but *can be* true or false and *can be tested*. We allowed for this opportunity in the preliminary expression of the paper. We proceed by tailoring a three valued logic to the invariance of the uncertainty predicate and then, within the theory of irrational numbers we construct the elimination of the third value. We use $f_0$ as indicator of the deductive modality.

Subjectivity in this matter cannot be excluded. Long ago theology and philosophy played important roles in selecting between two logics (falsity of true implies false and the complementary symmetric logic) which had the equivalence theorems as invariants: $(p \vee \neg p) \equiv \neg(p \wedge \neg p)$, $\neg(p \equiv \neg p)$. The classic choice we denominate decision node 1.

7. We need to reconsider the value function $\phi$. Redefine it as the test function $\phi^3$: $f \mapsto \{1, 0, -1\}^3$. We choose $\phi^3$ to be the composable mapping on generic testable expressions such that the logical value 0 of a fleeing expression like $\beta$ is invariant under composition up to double negation. By negation, expressions transform in a testable way according to the model:

Table 5

| $f$ | $\neg f$ | $\neg\neg f$ | $f \equiv \neg\neg f$ |
|---|---|---|---|
| 1 | $-1$ | 1 | 1 |
| 0 | 0 | 0 | 1 |
| $-1$ | 1 | $-1$ | 1 |

The values are represented in basic notation and 0 stands for fleeing. Notice the tautology (decision node 2). This tautology is not in the Kleene and Kripke model. The intuition on how to model the undefined value is different. In this 'fleeing logic' the primitive proposition is maintained, in the logic of undefined value it is not. The first two nodes yield the model for negation:

Table 6

| $f$ | $\neg f$ | $\neg\neg f$ | $f \equiv \neg\neg f$ | $f \vee \neg f$ | $f \wedge \neg f$ | $f \Rightarrow \neg f$ | $\neg f \Rightarrow f$ | $f \equiv \neg f$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ | 1 | $-1$ |
| 0 | 0 | 0 | 1 | 1 | $-1$ | 1 | 1 | 1 |
| $-1$ | 1 | $-1$ | 1 | 1 | $-1$ | 1 | $-1$ | $-1$ |

We continue by determining the connectives for expressions $f$,$g$, $f \wedge g$, $f \vee g$ by adopting (node 3) the linguistic intuitive definitions (true for both $f, g$ true; false for both $f, g$ false). Here is the second difference with Kleene. We come to the implication $f \Rightarrow g$. Here we can still determine the rows involving the values (1, 1) and ($-1$, $-1$) but four couples involving 0 are undecidible (node 4). The third difference with Kleene: A methodology of constraints by coherence $\left[\text{if } -1 \Rightarrow 1 \text{ then } 0 \Rightarrow 1\text{; if } \neg(1 \Rightarrow -1) \text{ then } \neg(1 \Rightarrow 0)\right]$ can be adopted in reducing the number of the possible matrices to one for generic testable expressions $f, g$.

Table 7

| $f$ | $g$ | $f \equiv g$ | $f \vee g$ | $f \wedge g$ | $f \Rightarrow g$ | $g \Rightarrow f$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | $-1$ | 1 | $-1$ | $-1$ | 1 |
| 1 | $-1$ | $-1$ | 1 | $-1$ | $-1$ | 1 |
| 0 | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| 0 | 0 | 1 | 1 | $-1$ | 1 | 1 |
| 0 | $-1$ | $-1$ | 1 | $-1$ | $-1$ | 1 |
| $-1$ | 1 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $-1$ | 0 | $-1$ | 1 | $-1$ | 1 | $-1$ |
| $-1$ | $-1$ | 1 | $-1$ | $-1$ | 1 | 1 |

Coherence of the model is proved by deducing the third column from the last three in table 7. Of the theorems of classical logic, the exception is (table 6) $\neg(f \equiv \neg f)$. The notions of tertium, quartum... and contradiction are relative. The annihilation of the third value leads back to classical logic (erase the rows where a 0 occurs). Therefore classical logic acts as the boundary of the fleeing logic.

Annihilating and hypothetically eliminating are not equivalent, as noted in Glivenko's case. Khintchine [15] noticed that there was inconsistency in the two valued logic proposed by Barzin and Errera [1].

As $f_0$ is maintained also with fleeing properties, one needs to examine the correspondence, if there is one, between theorems in $T$ and theorems deduced with the new properties. If equivalence of double negation could be extended as a theorem to $T$, then the new properties would occur as an undecidible that invalidates tertium non datur. Brouwer and the intuitionists surely do not spare themselves. But, as a consequence of what stated so far it is a simple deduction that with three values tertium datur and, eventually, what non datur is quartum.

In the points (2) and (3) of the definition of a fleeing property (pgf. 6) notice the "It is not known". If it was known, then two simple alternatives would follow: evaluate two valued in $P$ and transfer to $T$, or cannot evaluate two valued and maintain $f$, $p$, $t$. In both cases $T$ cannot be contradicted. It is because "it is not known" if the research of a rule to determine the critical number is absurd that it looks like it is possible to contradict $T$. How? In an indirect way: as the fleeing property is related to irrationals, it is possible to maneuver a critical number $k_f$ over a sequence of binary fractions so that the usual properties of the continuum are undermined.

8. $f$, $p$, $t$ support the definition of Brouwerian species: by theorems 1,2,3 they are exactly in the relationship that leads a fleeing property to invalidate tertium non datur [by $f =$ c, $p =$ b, $t =$ a; Cambridge lectures, chp. 1, 1 page before the absurdity theorem]. Therefore, with $f, p, t$ we can operate as generic mathematical objects as well as species of mathematical objects.

**With a fleeing logic, three valued testable expressions $f$ allow to infer the equivalence of double negation.**

Consider the first four lines of table 7. In the first column substitute $f \neq g$ and in the second column substitute $f \equiv g$, then, in both columns, substitute $\neg\neg f$ to $g$. Obtain:

Table 8

| $f \neq \neg\neg f$ | $f \equiv \neg\neg f$ | $(f \neq \neg\neg f) \Rightarrow (f \equiv \neg\neg f)$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | $-1$ |
| 1 | $-1$ | $-1$ |

of the four lines, consider the second line: set true the implication, by assumption $f \neq \neg\neg f$ is true, establishing equivalence from difference (the second element of the first line). Substitute $\neg f$ to $\neg\neg f$ in the first line of table 8. Go to the last column of table 6. $f \equiv \neg f$ can be true only in correspondence to the invariant. The second line of 8 is maintained after substituting $\neg f$ and $f_0$ is provable by this three valued inferential system based solely on modus ponens and the substitution rule.

As the circuit from difference to equivalence is admissible, indirect reasoning based on $f_0$ can be both consistent and contradictory. As long as the third value is posed, the reliable deductive modality $can\,be$ only direct and constructive. Also intuitionist logic refrains from indirect reasoning.

9. We establish a correspondence between $T$ and Brouwer's fleeing expression: consider the following proved proposition in $T$: "a real irrational number is characterized by the property that it's binary expression does not fix in the end on 0, on 1, or on any sequence on 0 and 1". The quoted sentence comes from number theory: the theorem proving that periodic numbers are rational.

These sequences are constructible and several examples of construction rules can be written. For example, reduce the $\beta$ expression 0123456789 to a binary sequence: it consists of 26 digits. Start with 0, then continue with the 26 digit subsequence alternating with 0,0: 0,0,0 and so on. The n-th repetition of the 26 digits is preceded by n zeros This builds a sequence not fixing in the end in which the corresponding form of 0123456789 occurs.

10. Simplify the (Brouwer) notion of $\lambda$(n)-interval with this ordered couple: $2^{-n}(a, 1+\frac{a}{2})$ [2, chp. 2]. Ordinarily, this couple converges as n diverges. But if n is disturbed by associating a fleeing property, convergence is no more ensured:

$$s_f = \begin{cases} 2^{-n} & \text{if } n \leq k_f \\ 2^{-k_f} & \text{if } n > k_f \end{cases}$$

where $k_f$ is the critical number of which a proof of the absurdity or of the absurdity of absurdity is unknown. By critical number is meant the (hypothetical) smallest number that has the fleeing property. Brouwer thinks that this leads to invalidate tertium non datur and that $s_f$ can hold apart the two ordered elements of the couple.

What I do not know can contradict what I know? No: if it contradicts what I know, I would know it, violating not knowing.

The construction of $s_f$ reveals its indirect nature: from "not knowing" $k_f$ Brouwer shifts to 'as if it was not known' $k_f$. 'As if it was not known' is a counterfactual model. As a consequence of not knowing the absurdity or the absurdity of the absurdity, $k_f$ is not a number: it cannot be deduced from the Peano postulates nor constructed by adding the unity. A number that is not cannot enter into the formation of a sequence. The theory arising from $f_0$ predicted that if indirect reasoning was applied in the three valued fleeing context a contradiction could be deduced. $k_f$ in a binary relation with $n \in N$ is the quartum. $k_f$ is absurd and non contradictory and unknown and exists hypothetically in $N$.

We identify two questionable items in the Cambridge Lectures: (1) The invariance by negation of the fleeing value is ignored; (2) $k_f$ in a binary relation with the elements of the set of natural numbers is quartum. We proceed as follows: first we restate (2) and then discuss Brouwer's number architecture.

11. $k$ and $n$ are both natural numbers. Define $\pi(k)$ as the initial segment, of length $k$, of the decimal sequence of $\pi$. By the invariance property under negation, at every step in the computation of $\pi$ we can establish two regions in the natural numbers. The following table makes it clear:

Table 9

| | $\beta$ | $\neg\beta$ |
|---|---|---|
| $n \leq k$ | $-1$ | 1 |
| $n > k$ | 0 | 0 |

As a value of $k$ is reached for which the $\beta$ property holds, the second line of the three switches to $(1, -1)$ and the third to $(-1, 1)$. The $\beta$ expression does not mention reoccurrence. Otherwise the third line stays put. Not more than this can be stated.

12. Consider $3 \leq 3.1 \leq 3.9$; $3.1 \leq 3.14 \leq 3.19$ and so on. Of each disequation consider just the right side. Notice that $\pi(k) \leq \pi(k) + 9/10^{k+1}$, convergent to $\pi$ as $k$

diverges. The sequence of intervals for $\pi(k)$ is telescoping or conical on the limit point $\pi$ because $\pi$ is the name of the ratio of the circumference and the diameter, it is proved irrational and an algorithm and the limit exist. To appreciate this consider, between 3 and 4, a number $x$ which is unknown. This number $x$ is bounded between 3 and 4 and we are forced to use an expanding cylinder around $x$ : 3,000...01 $\leq x \leq$ 3,999...9.

13. In the following we limit consideration to the decimal expansion of a real number in an interval. In our case $[3,4]$. A binary sequence $\sigma$ is a function f : $N \mapsto \{0,1\}$. With domain of definition $N - \{0\}$. The stationary sequence on 0 we denote $o$, the stationary sequence on 1, $\iota$. The set of binary sequences $\Sigma$. $\forall\sigma \in \Sigma$, $\sigma = \sigma$, $o + \sigma = \sigma$. From the natural ordering, we define the initial segment of $\sigma$, $k \in N$, as $\sigma(k)$. For example $\iota(3) = 1,1,1$. We turn a binary sequence into a binary decimal expansion by $\sum_1^\infty \sigma_n\, 10^{-n}$. This expression is bounded by 0 ($\sigma = o$) and 1 ($\sigma = \iota$ and $\lim_{n\to\infty} \sum_1^n 10^{-i}$), all computations carried in binary form: 10 is 2 in the decadic. The binary decimal expansion corresponding to an initial segment are the numbers following the decimal comma up to the last one written under the pin of the printer attached to those fast computing machines. It takes math imagination to see the rest. It is *known* that $\sum_1^\infty \pi_n\, 10^{-n} = \pi$ by definition. $\pi$ has a proof of irrationality (Lambert) and several algorithms of approximation (geometric with $\sqrt{2}$ recurrent, continuing fraction, Leibniz series), other numbers do not.

We define the following (intuitionist) operation: $\sigma D\sigma(k)$, consisting of deleting from a sequence $\sigma$ it's initial segment $\sigma(k)$. The operation is well defined from the properties of linearly ordered sets and corresponds to the following Brouwer notions: species, equal species, deletable subspecies, congruent species. This operation between a sequence and an initial segment $\sigma D\sigma(k)$, is closed in $\Sigma$, yielding an element not necessarily equal to $\sigma$. Therefore, in the decimal expansion, $\sigma$ and $\sigma D\sigma(k)$ not necessarily coincide, nor do their limits. In fact the first term of $\sigma D\sigma(k)$ is the $k+1$ term of $\sigma$. We also define the operation of replacing the initial segment of $\sigma$ by the initial segment of $\sigma$. $\sigma \mathrm{R}\sigma(k)$. Of course $\sigma\mathrm{R}\sigma(k) = \sigma$. This identity allows the extension $\sigma\mathrm{R}\pi(k)$: the replacement of the initial segment of $\sigma$ by the initial segment of $\pi$. Of course $\sigma\mathrm{R}\pi(k) \in \Sigma$.

Define the distance between two binary sequences as the sum of the weighted ordered deviations: $d_k(s,\, z) = \sum_1^k \left|\mathrm{s}_n - \mathrm{z}_n\right| 10^{-n}$, bounded by 0 (for $s = z$) and $\sum_1^k 10^{-n} \leq \lim_{\mathrm{k}\to\infty} \sum_1^{\mathrm{k}} 10^{-\mathrm{n}} =$ 1. For the triangle inequality, assign $\{0,1\}$ to 3 points a,b,c in 8 ways and verify $(a-b)+(b-c) \geq (a-c)$ for each assignment. Then, being the inequality verified for each addendum, it is verified for the sum. Unfortunately in the decadic the reader has to control 1000 assignments.

It is not margi$n$al to observe that by defining the distance we have turned $\Sigma$ into a group, closed under addition with symmetric elements and neutral element: ($\forall\, k \in N$, $d_k(s,z) = 0) \Rightarrow (\sigma + (-\sigma) = o)$.

14. For a sequence $s \in \Sigma,\ k \in N,$ as before, let $s(k)$ be defined as the initial segment of length $k:\ s(k) = s_1, s_2, ..., s_k.$ Let $g$ be defined as a function that counts the length of an expression. For example, for $\beta\ = (0,1,2,3,4,5,6,7,8,9),\ g(\beta) = 10.$ Of course this implies that a particular alphanumeric expression -as a fleeing property- has been digitized in the decadic or binary system and transformed to an n-ple. We deal with expressions $f$ for which the reduction is possible, like Brouwer's $\beta$.

Next, define, $\forall \beta = (\beta_1,\ \beta_2, ..., \beta_g)$, the function $\psi(s(k), \beta) : \Sigma_k \mapsto \Sigma_n$, $n = k + g$, by shifting to the initial segment of length n, such that $s(k+g) = s_1, s_2, ..., s_k, \psi_{k+1}, \psi_{k+2}, ..., \psi_{k+g} \wedge\ \ (\psi_{k+1}, \psi_{k+2}, ..., \psi_{k+g}) = \beta.$ In words: $s(k+g)$ is the initial segment of length $k+g$ obtained by appending to $s(k)$ the finite expression $\beta$ of length $g$. Finally, let $s_\nu = s(k+g), s_{k+g+1}, s_{k+g+2}, ...$ In words: $s_\nu \in \Sigma$ is the sequence that differs from $s$ only for having the $g$ successors of the initial segment $s(k)$ replaced by $\beta$.

Now, for $x, s \in \Sigma$, define $\sum_{k+1}^{k+g} \left| x_n - s_n \right| 10^{-n} = d_{k,\,g}(x, s).$ Notice that $\sum_{k+1}^{k+g} \left| x_n - s_n \right| 10^{-n} \leq \ \sum_{k+1}^{k+g} 10^{-n} \leq \sum_{1}^{g} 10^{-n}$ this last term being the max of the possible distance for any binary sequence with a max of $g$ consecutive deviations from a given one. The max is attained by deviating in the initial terms. In the decadic this yields a formula with similar properties. I will provide the details by request. We have:

$$\lim_{k\to\infty} \ \ \sum_{k+1}^{k+g} \left| x_n - s_n \right| 10^{-n} = 0.$$

$\psi$ is the nihilating operator of any simple fleeing property. $s_\nu$ the nihilist sequence. Let $\pi_\beta$ be the number $\pi$ predicate of the $\beta$ expression and $\pi_{\nu\beta}$ the nihilist sequence of $\pi_\beta$. Observe the following: $\pi(k) = \pi_\beta(k) = \pi_{\nu\beta}(k)$ by construction. And also, from the previous limit:

$$\lim_{k\to\infty} \sum_{1}^{k} 10^{-n}\ \pi(n) = \ \lim_{k\to\infty} \sum_{1}^{k} 10^{-n} \pi_\beta(n) = \ \lim_{k\to\infty} \sum_{1}^{k} 10^{-n} \pi_{\nu\beta}(n) = \pi$$

By substituting the nihilist sequence, the fleeing property is annihilated and tertium is gone. Table 9 transforms to:

Table 10

| | $\beta$ | $\neg\beta$ |
|---|---|---|
| $n \leq k$ | $-1$ | $1$ |
| $n > k$ | $1$ | $-1$ |

Even if, contrary to empirical evidence, $\beta$ was to occur in $\pi$, row1 $= (1,\ -1) =$ row2 of the matrix. In the language of economists, the nihilist sequence is the certainty equivalent of the $\beta$ expression. Tertium is not hypothetically eliminated. The fleeing property is annihilated by means of a mathematically constructed sequence that cannot be distinguished from $\pi$ neither in the natural order of computation nor at infinity, and then table 9 changes and the third value is permanently eliminated. This route is a necessity, because we define a three valued logic with respect to which $f_0$ is not independent. As the three valued logic was built having the two valued one as boundary, one reconnects to all the indirect theorems and constructions in $T$. Neither $p' \Rightarrow \neg p$ nor $\neg p \Rightarrow p'$ make sense. We have established in the Lewis constructive modality $\nu p' \subset p \subset T$ : the annihilation of the uncertainty predicate is part of the construction of truth. And both are part of the construction of $T$.

We observe that any logic for which the tautology in table 5 holds (node 2), collapses back on the two valued boundary logic: the entire result is independent from our private intuitive preferences expressed in table 6 and 7 (nodes 3 and 4).

Theorem 6. The fleeing logic for the mathematical uncertainty predicate is infinitely valued and infinite is the set of annihilators.
Consider the $\beta$ expression and the construction for appending it to an initial segment. Reverse the construction: choose arbitrarily a sequence $\sigma$ in $\Sigma$ and replace the initial segment $\sigma(k)$ with the initial segment $\pi(k)$. $\sigma R \pi(k)$. This defines the general annihilator of $\pi_\sigma$: $\pi_{\nu\sigma}$. We underline two things: 1) as $k$ diverges, the initial segment of $\pi$ replaces the initial segment of $\sigma$, therefore $\pi_{\nu\sigma}(k) = \pi(k)$; by $k \in N$ and the linear ordering of the terms of $\sigma$, the replacement of $\sigma$ by $\pi$ is 1 to 1. $A$s we multiply by $10^{-k}$ to obtain the decimal binary expression we still obtain $\lim_{k\to\infty}\sum_1^k 10^{-n}\pi(n) = \lim_{k\to\infty}\sum_1^k 10^{-n}\pi_{\nu\sigma}(n) = \pi$, 2) in $\pi_{\nu\sigma}(k)$, $\sigma$ is variable: we are actually permuting over the elements of $\Sigma$ and, by Cantor's diagonal, this is not a denumerable set. As $k$ advances we are switching from one sequence of $\Sigma$ to another. The result is an infinitely valued logic which is realized (i) as a three valued logic for each finite specification $\beta$. These sequences are of type $o$, or $1, o$ (0 stationary after a 1), that is those that are reducible to a finite segment and then to a $\beta$ n-ple by deleting 0 stationary. $As$ we collect, we obtain $2^n$. While (ii) for the sequences not fixing in the end on 0 (periodic numbers included) we have a $2^N$ valued logic which is annihilated only globally.

15. The problem is uncertainty: the real numbers of which a proof of irrationality is unknown [12, chp. 4]. $\pi^e$ until 1985. The proposition (if they) do not fix in the end is undecided. Let $x(k)$ be the initial segment of a real number in the closed interval $[3, 4]$, such that $x(0) = 3,$ and $3 < x(1) < 4$. For each addition of the unity define the:

closed cone: $$\left[\, x(k) - \delta(k),\, x(k) + \delta(k)\right]$$

by the properties $\left[x(k+1) - \delta(k+1),\ x(k+1) + \delta(k+1)\right] \subset$
$\subset \left[x(k) - \delta(k), x(k) + \delta(k)\right]$ and $\delta(k+1) < \delta(k)$ and $\delta(k)$ convergent to 0 as $k$ diverges. And the :

open cylinder: $$\left(3,\, 3 + \delta(k)\right) \cup \left(4 - \delta(k),\, 4\right)$$

Notice that any cover of the interval $[3,\, 4]$ contains the following finite subcover: $[3 + \delta(k),\, 4] \cup [3,\, 4 - \delta(k)] = [3,\, 4]$. Define the:

inverse cone: $\left(3,\, x(k) \,-\, \delta(k)\right) \cup \left(x(k) + \delta(k),\, 4\right)$

Let the open sets be the open cylinders and the inverse cones, the closed sets be the closed cones. Both the closed sets and the closures of the opens are compact.

We proceed by excluding impossible alternatives: i) Can an initial segment of $\pi$ represent only $\pi$? No: an initial segment of $\pi$ is a donkey shouldering a huge number of real numbers. ii) Can a number which is unknown, with the same initial segment of $\pi$, have the same limit as $\pi(k)$ as $k \to \infty$? No: if it does it coincides with $\pi$, violating not knowing. iii) Can a number which is unknown, have a different initial segment from $\pi$ and then enter the $\pi$ cone and then exit in the end? No: it is sufficient to consider the initial segments 3.1 and 3.2 to see that continuing calculation makes it impossible for the second to enter the closed cone of the first. Then we can state that each conical sequence determines a region in the real interval to which relate sequences of which the formation rule is unknown. This region we call region of uncertainty and coincides with the inverse cone at $x(k)$: $\left(3, x(k) - \delta(k)\right) \cup \left(x(k) + \delta(k),\, 4\right)$. As $k$ diverges, uncertainty expands to $\left(3,\, x\right) \cup \left(x, 4\right)$.

Consider $\pi(k)$ and deviate in the initial segment. This yields a sequence $\pi'(k)$, derived from $\pi(k)$, at an invariant distance $10^{-k^f}$, arbitrarily fixed the deviation at the number $k^f \leq k$. In this notation $k^f$ loses the quality of "unknown" and fleeing and acquires the quality of arbitrarily fixed. This construction we can extend by increasing the number of deviations and import it in the nihilist sequence. Whenever we choose a fixed value $k^f$ (or fixed values of $k$) we obtain an 'affine' or 'parallel' sequence. It is easy to show that we have produced a construction rule for sequences that stay apart from a given known one. Apartness is to be interpreted in the complementary sense of equality. It is an intuitionist relation [2, chp.3] [13, p:19,51,60].

At each step $k$, $\pi(k)$ and all the sequences, including those derived from $\pi(k)$ by deviating, determine a set of decimal approximations in $[3, 4]$, name it $\Lambda(k)$ or powders. $\Lambda(k) \subset [3, 4] \subset R$. It is convenient to summarize the following constructions: 1) a binary sequence, 2) an initial segment of a binary sequence, 3) The decimal representation of both 1) and 2) by the negative powers of 10 4) The decimal representation of a decadic sequence by the negative powers of 10. Now we remind Cantor's correspondence between the set of binary sequences and the continuum: this we represent as arbitrary instructions over a sequence of mid split intervals with 0 meaning 'go left' and 1 meaning 'go right'. We use the negative powers of the number 2, because there are 2 opportunities only. In Cantor's construction from $\Sigma$ to the continuum the sequence of intervals converges to a measureless point or measureless atom. It is natural to number the midpoints and the endpoints of the intervals, once assigned $k$: $\lambda_i \in \Lambda(k)$, $i = 1, ..., 2^k + 1$, $k \in N$. Then $\Lambda(k)$ constitutes a discrete grid of the continuum. This idea goes back to Leonardo's Treatise on Painting. Now, notice the correspondence between all the objects defined in this paragraph: announced the length of an initial segment $k$, the decimal expansions and representations are determined, the grid of midpoints $\Lambda(k)$ is determined, the sequence of midpoints is numbered, what lies between midpoints is blocked and consists of midpoints as $k$ is increased by induction. This simple and powerful intuition is the basement of Brouwer's bar induction theorem. It is advisable to read Brouwer with the support of

Heyting and Skolem. Brouwer's beautiful construction of the continuum is further supported with the image of the spread as a cascade of fans: from the well ordering of the terms of a sequence $\sigma$, it follows that at any term it is possible to develop a fan of 'length' $k$ and 'width' $2^k + 1$, and this extends by recursion all along.
(i) ∀s ∈ Σ, ∃z ∈ Σ: $\lim_{k\to\infty} d_k(s(k), z(k)) = 0$. By the previous construction. For example the stationary sequence on 0 and the sequence with $d_k(o(k), z(k)) = 10^{-k}$.
(ii) $\forall k \in N,\ \Lambda(k) \subset [3,4] \subset R,\ \forall s(k),\ \exists z(k):\ s(k) \neq z(k) \Rightarrow d_k(s(k), z(k) \geq 10^{-k}$.
This is a simple consequence of deviating in the initial segment. The decimal expansion of any $s(k)$ coincides with some endpoint $\lambda_i \in \Lambda(k),\ i = 1, ..., 2^k + 1,\ k \in N$. With ray $\delta < 10^{-k}$ do not exist elements of $\Lambda(k)$ different from the $\lambda_i$ correspondent to $s(k). T$hey exist inductively as $k$ ranges in $N$.

Theorem 7. $\forall k \in N$, any function $\gamma$: $\Lambda(k) \mapsto \Lambda(k)$, everywhere defined in $\Lambda(k)$, is uniformly continuous.
(ii) is true also for the respective image $\gamma(s(k))$.
(iii) Image and pre image of $\gamma$ are isolated points in $[3,4] \subset R$. By Darboux's definition, $\gamma$ is continuous. As $\Lambda(k) \subset R$ is closed and bounded in $[3,4]$, by the Heine-Cantor theorem, $\gamma$ is uniformly continuous.
(iv) In a neighborhood with ray $\epsilon < 10^{-k}$ of $\lambda_i$ correspondent to $\gamma(s(k))$, all images by $\gamma$ of segments $z(k)$ differing from $s(k)$ by less than $10^{-k}$, in $\Lambda(k) \subset [3,4] \subset R$, coincide with $\gamma(s(k))$, then there is $\delta(\epsilon) > 0$ such that, $\forall \lambda_i \in \Lambda(k)$, $|\,h\,| < \delta(\epsilon)$, $|\gamma(\lambda_i + h) - \gamma(\lambda_i)| < \epsilon$. Uniform continuity follows by (ii) and $10^{-1} \geq 10^{-k} > \delta(\epsilon) = \epsilon > 0$.

(iii) and (iv) are equivalent as the fleeing properties are annihilated. In (iv), $\Lambda(k)$ is a decimal grid indexed by $k$. $\gamma$ operates on the grid maintaining the index invariant between image and preimage.Cantor's correspondence from $\Sigma$ to the continuum is the transfinite conclusion. In Brouwer this takes place by using the well ordering, the fan theorem, the finitary fan, bar induction and the bar theorem [2, block, bunch, chp.5][13,chp.3][25, chp.16]. None of these depends on $k_f$ nor $s_f$. This is why Brouwer's mathematics is unaffected. To construct $\Lambda(k)$ we need two conditions, both verified for binary $\Sigma$: the fan of decisions to build an arbitrary sequence to be finite and a pivot from which to deviate like the null sequence or the nihilist $\pi$.

At each step $k$, by the well ordering, there is an unknown real number $\rho$ apart from the endpoints $\lambda(k)_i$ of a bracket in $\Lambda(k)$. We observe that $\rho$ respects the property of not entering the cones convergent to the endpoints of the bracket $[\lambda(k)_i\,, \lambda(k)_{i+1}]$, $\lambda(k)_{i+1} - \lambda(k)_i = 10^{-k}$ in $\Lambda(k)$. These are derived from $\pi_{\nu\beta}(k)$ and are known. $\rho$ is 'barred' or 'blocked' in $\Lambda(k)$. What we know bars or blocks what we do not know. We can extend $\Lambda(k)$ and make it close at will to an unknown number, while holding it apart by a decidible amount. Decidibility here is crucial and it comes from the well ordering and the apartness relation. An unknown number can be barred, but cannot be pinched [18, cut, gap, p.154, 206].

From the cones on the endpoints $[\lambda(k)_i\,, \lambda(k)_{i+1}]$ we derive the region of uncertainty $(\lambda(k)_i\,, \lambda(k)_{i+1})$, and pass to $\Lambda(k{+}1)$. Again we find endpoints $\lambda(k{+}1)_i$, $i = 1, ..., 2^{k+1}{+}1, k \in N,$ defining a closed interval containing an open interval containing

$\rho$. Once we assign the rule for the construction of $\Lambda(k)$, by specifying the deviations in the initial segment of $\pi_{\nu\sigma}$, the generalized annihilator, the fate of $\rho$ is marked. The measure of the region of uncertainty of $\rho$ is *continuously* related to $k$ and, by $k$, to the initial segments $\Sigma_k$, establishing the second part of Cantor's correspondence: from the continuum to $\Sigma$. By the construction of $\Lambda(k)$ follows the theorem "a function everywhere defined on a denumerable dense subset of the unitary continuum is uniformly continuous". By bar induction, Brouwer proves the full function theorem (a function everywhere defined on the unitary continuum is uniformly continuous), the non separation theorem and the topological invariance of dimension.

$$\begin{array}{ccccc} & \Sigma & \mapsto & \Sigma(k) & \\ \phi & \uparrow & & \downarrow & \sum_{1}^{k}\sigma_n 10^{-n} \\ & \Sigma(k) & \leftarrow & \Lambda(k),\gamma & \end{array}$$

where $\phi$ is multivocal for $k \in N$. $\gamma$ is defined on $\Lambda(k)$. As $k$ diverges, the preimage of $\gamma$ is dense in $[3,4]$. Consider the inverse cone as $k$ diverges: $\big(3,\ x\big) \cup \big(x,4\big)$. It is possible to form $\big(3,\ x\big) \cup x \cup \big(x,4\big)$, where $x$ is an irrational number both proved and known (or derived from $\pi$ by deviating). But if $x$ is assigned to any one of the classes, a similar ambiguity arises as for $k_f$. The inverse cone is the region of uncertainty and is kept disjoint from $x$. $x$ and $\Lambda(k)$ are mathematical certainty. Of the Cambridge Lectures, the chapter on order and the sections on connection, compactness, density, virtual ordering need careful reexamination. Without resorting to Brouwer's order relations, to well ordering and virtual ordering, one does not know how to order two unknown numbers, one unknown and one known number. Therefore, presently, it does not seem possible to pass from the powders to Dedekind's cut.

Once critically restated $k_f$, I believe that now the reader realizes the full extent of Brouwer's contribution. Relatively to the study of real numbers not proven irrational nor rational. Numbers without being. With respect to their decidibility, even the use of the limit operator and the notion of limit point require particular care, as much as, though in the inconsistently hypothetical way, in $s_f$. The theory of uncertainty begins with and is founded by Brouwer and Gödel.

$f_0$ does not fail. In mathematics. Contradiction is language and no mathematics. This statement is true for Burali Forti, Russell and also for the impossibility theorem. I do not know if it is universally true. Language, the written object rich of overlapping properties, synonyms and context sensitivities. We saw that $f_0$ is primitive. If we postulate $f_0$, a screening of $P$ can be constructed. But, in order to prove $f_0$ one cannot presuppose the reduced form of $P$: $f_0$ can be postulated and cannot be proved. With this in mind I can give an outline of mathematical uncertainty. In the way of postulating:

1) $f_0$, 2) Gödel's Systeme I, or first order predicative systems, 3) the general invariance of the mathematical uncertainty predicate. Then, the impossibility theorem for the unknown to contradict the known, Brouwer's bar induction theorems and Cantor's correspondence.

## Appendix: A homological model of $f_0$ and irreversible nodes.

It has been common belief that axioms and theorems can be interchanged. We need a model for the geometry of $T$. Consider a circle C and a line R and place them in the position of the trigonometry textbook by identifying the 0 of the line with the East of the circle. Next wrap the line around the circle by wrapping the positive side counterclockwise and the negative clockwise. Consider the points of the line: 0, $\frac{1}{2}\pi$, $\pi$,$\frac{3}{2}\pi$,$2\pi$,... and ..., $-2\pi$, $-\frac{3}{2}\pi$, $-\pi$, $-\frac{1}{2}\pi$, 0. They are symmetric and constitute 4 points on C. Now, take out the West point of C. Notice that it is not possible to restore the line after the circle is broken in this way: One obtains an open interval and has to induce a correspondence.

As the impossibility theorem is proved the circle is broken by a gap, $p \equiv \neg\neg p$ is not in $T$. The existence of a proposition which cannot be proved in $P$, makes $T$ consistent and incomplete. This is the first consequence of the impossibility theorem and it brings a second result: logic and time are coherent, and time can be constructed mathematically. Theorems are like american indians: they walk in line and when they sit around the fire, they sit in a bent line.

A property characterizing some of the nodes is best described by Brouwer's fixed point theorem: A continuous function of the disc in itself has a fixed point. In some proofs, the main notions required are that the fundamental group of the disc is null and the fundamental group of the circle (or disc boundary) is different (relative numbers Z), plus the notion that the fundamental group is invariant (up to isomorphism) for continuos functions.

The proof is as follows: Suppose there are no fixed points, then it is possible to define the projection constructed as a semi line from f(x) through x intersecting the circle in g(x). This maps continuously the disc on the circle and their respective fundamental groups are different, leading to contradiction and establishing the thesis. This is an indirect proof.

Without resetting the memory, it is not possible to repeat the proof. The existence of fixed points inhibits the construction of the projection. But if one reaches anew all the premises, the theorem is proved. Perhaps it is useful to underline the difference between this proof and the proof of the irrationality of $\sqrt{2}$. It is possible to repeat the last one because the set of rational numbers is independently defined and constructed, and supports any absurd assumption involving it. Instead the everywhere definiteness of the projection is not independent from the thesis. In this version of the theorem, the expression "if there are fixed points there is nothing to prove", that implicitly precedes it, actually should be substituted, for the sake of precision, by: "assuming the existence of fixed points makes the proof of their existence impossible".

The quality of these nodes is that they are not only directed, they are also irreversibly oriented. In passing an irreversible node there is no turning back.

Consciousness, logic and time are coherent.

Bibliography

[1]Barzin M. and Errera A., 1927, "Sur la logique de M. Brouwer, Academie Royale de Belgique, Bulletin de la classe des sciences, 12: 56-71.
[2] Brouwer, L., 1981, Brouwer's Cambridge Lectures on Intuitionism. Cambridge (1983, Torino).
[3] Carnap R., 1958, Introduction to symbolic logic and it's applications, New York.
[4] Courant R and Robbins H., 1941, What is mathematics, Oxford-New York (Torino1971).
[5] Glivenko, V., 1928, "Sur la logique de M. Brouwer", Academie Royale de Belgique, Bulletin de la classe de sciences, 14: 225-228.
[6] Gödel, K., 1929, "Über die Vollständigkeit des Logikkalküls", in Gödel, K., 1986: 60-101.
[7] Gödel, K, 1930, "Die Vollständigkeit der Axiome des logishen Funktionenkalküls", Monatshefte für Mathematik und Physik, 37, in Gödel, K., 1986: 102-123.
[8] Gödel, K, 1931, "Über formal unentsheidbare Sätze der *Principia mathematica* und verwandter Systeme I", Monatshefte für Mathematik und Physik, 38, in Gödel, K., 1986: 144-195.
[9] Gödel, K.,1932, "Zum Intuitionistischen Aussagencalcul", Anzeigen der Academie des Wissenshaften in Wien, 69. in Gödel, K., 1986, 222-225.
[10] Gödel, K.,1932, "Heyting Arend: Die Intuitionistische Grundlegung der Mathematik", Zentralblatt für Mathematik und ihre Grenzgebreite", 2, in Gödel, K., 1986: 286-295.
[11] Gödel, K., 1986, Collected Works 1, Oxford.
[12] Hardy, G.H. and Wright, E.M., 1985, An introduction to the theory of numbers, Oxford.
[13] Heyting A., 1971, Intuitionism. An introduction, Amsterdam-London.
[14] Hilbert D. and Cohn-Vossen S., Geometry and the Imagination,( New York, 1990).
[15] Khintchine A., 1928, "Objection a une note de MM. Barzin et Errera.", Academie Royale de Belgique, Bulletin de la classe des sciences, 14:223-24.
[16] Kleene, $S$.C., 1938, "On notation for ordinal numbers", Journal of Symbolic Logic, 3, 150-155.
[17] Kripke, $S$.A., 1975, "Outline of a theory of truth", Journal of Philosophy, 72, 690-716.
[18] Kuratovski, K. and Mostovski, A., 1976, Set theory, Amsterdam,
[19] Lewis, C.I., 1918, A survey of symbolic logic, Berkeley.
[20] Mazzacua, G., 1973, Alle origini della logica matematica, Messina-Firenze.
[21] Nagel E. e Newman J., 1958, Gödel's proof, (1974, Torino).
[22] Quine W., 1965, Elementary logic, Harvard (1968, Roma).
[23] Russell B, 1903, The principles of mathematics (1974, Perugia).
[24] Russell B. and Whitehead A., 1910, Principia mathematica (1970, Cambridge)
[25] Skolem T.A., 1962, Abstract set theory, University of Notre Dame.
[26] Urquhart A.,1972, "Semantics for relevant logics", Journal of symbolic logic, 37, 159-169.
The concise Oxford dictionary, 1956, Oxford.