Stable Determination of the Discontinuous Conductivity Coefficient of a Parabolic Equation *

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November 20, 2009

Abstract

We deal with the problem of determining a time varying inclusion within a thermal conductor. In particular we study the continuous dependance of the inclusion from the Dirichlet–to–Neumann map. Under a priori regularity assumptions on the unknown defect we establish logarithmic stability estimates.

AMS 2000 Mathematics Subject Classification Primary 35R30. Secondary 35B60, 33C90.

 ${\bf Keywords}$ inverse problems, stability, parabolic equations, unique continuation.

1 Introduction

In this paper we study the stability issue for the inverse problem of recovery the discontinuous conductivity coefficient of a parabolic equation from infinitely many boundary mesurements.

First let us give a coarse formulation of the problem which we are going to study. Let T be a given positive number. Let Ω be a bounded domain of \mathbb{R}^n , $n \geq 2$, with a sufficiently smooth boundary and let Q be a domain contained in $\Omega \times (0, T)$. Assume that for every $\tau \in (0, T)$ the intersection $D(\tau)$ of Q with the hyperplane $t = \tau$ is a nonempty set and $\Omega \setminus \overline{D(\tau)}$ is connected and denote by k, $k \neq 1$ a positive constant. Let u be the weak solution to the following parabolic initial-boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div}((1+(k-1)\chi_Q)\nabla u) = 0 & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = 0 & \text{in } \overline{\Omega}, \\ u = g & \text{on } \partial\Omega \times (0,T) \end{cases}$$

where g is a prescribed function on $\partial\Omega \times (0,T)$. The inverse problem we are addressing to is to determine the region Q when infinitely many boundary measurements $\left\{g, \frac{\partial u}{\partial \nu}_{|\partial\Omega \times (0,T)}\right\}$ are available. The problem formulated above arises in nondestructive testing evaluation ([Ca-Mo], [Pa-La-Al]).

^{*}Work supported by MIUR, PRIN n. 2006014115.

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A uniqueness result for the problem introduced above has been proven in 1997 by Elayyan and Isakov [El-Is]. The main tools on which the approach of [El-Is] is based are the Runge approximation property and the use of solutions with Green's function type singularity. For the nonconstructive character of the Runge property, such an argument does not seem suitable for our purpose of proving an accurate stability estimate of Q under some a priori information. Thus, along the line of previous elliptic and parabolic inverse problems [Al-DC], [Al-Ve], [Ve], we abandon such an approach and we choose to use arguments based on quantitative estimates of unique continuation [Al-Be-Ro-Ve], [DC-Ro-Ve], [Mo-Ro03, Mo-Ro04]. We also make use of singular solutions of Green's type, but more quantitative information are necessary in order to obtain stability estimates. In particular we need an accurate study of the asymptotic behaviour when the singularity gets close to the interface ∂Q .

In the present paper we prove that, under mild a priori assumption on the topology and the regularity of Q, such an inclusion depends continuously on the boundary data with a rate of continuity of logarithmic type (see Theorem 2.7 for a the precise statement of the result). In the context of elliptic inverse problems, it has been shown that logarithmic stability estimates are optimal ([DC-Ro]). For parabolic inverse problems with unknown boundaries (and the whole Dirichlet–to–Neumann map) examples showing that the continuous dependance can be at most of logarithmic type, have been obtained in [DC-Ro-Ve] and [Ve]. Their proofs work in our situation as well. Namely in such papers the limit cases $k = +\infty$ and k = 0 are considered. Everything remains basically the same in the intermediate situation with $0 < k < \infty$, $k \neq 1$.

A crucial tool to obtain the logarithmic stability estimate is connected with a precise evaluation of smallness propagation based on the two-sphere one-cylinder inequality for solution to parabolic equations [Es-Fe-Ve], [Ve] (Theorem 3.10, in the present paper). Indeed, roughly speaking, such an inequality allows us to approach the boundary of the inclusion in any slice of time.

Finally we wish to mention here papers of Daido, Kang and Nakamura [Da-Ka-Na] and, more recently, Isakov, Kim and Nakamura [Is-Ki-Na] which are strictly related to the present one. In [Is-Ki-Na], the authors consider a similar inverse parabolic problem of detecting an inclusion, that does not depend on the time, by mean of infinitely boundary measurements and provide a reconstruction procedure to identify it.

In this paper we have decided not to deal with the case n = 1. Let us just observe that such a case is easier and it can be treated essentially with similar assumptions regarding the topology of the set $\Omega \setminus \overline{D(t)}$ (see Remark 2.6 below).

The plan of the paper is the following. In Section 2 we state our main result. We first give the notations and definitions we need throughout the paper (Subsection 2.1) and then in Subsection 2.2 we state the hypothesis and the stability theorem (Theorem 2.7). In Section 3 we provide a proof of Theorem 2.7. We derive first some identity which will be the starting point of our proof. Then we give some auxiliary result concerning the Hausdorff distance (Proposition 3.2), 3.3 and 3.5), fundamental solutions (Proposition 3.6) and unique continuation properties (Theorem 3.10). Afterward, using the assumptions on the regularity of the inclusion, we derive some further property related to the distance of two inclusions and state Proposition 3.9 which provides lower bounds for the solution of the problem. Finally we prove Theorem 2.7. Proof of auxiliary propositions are given in Section 4. Proposition 3.3 is proven in Subsection 4.1. In Subsection 4.2 we prove Proposition 3.6 and we also give an asymptotic estimate for the fundamental solution (Theorem 4.3) which will be used in the next *Subsection* 4.3 for the proof of Proposition 3.9.

2 The Main Result

2.1 Notations and Definitions

For every $x \in \mathbb{R}^n$, with $n \ge 2$, $x = (x_1, \ldots, x_n)$, we shall set $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. We shall use X = (x, t) to denote a point in \mathbb{R}^{n+1} , where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For every $x \in \mathbb{R}^n$ and $X = (x, t) \in \mathbb{R}^{n+1}$, we shall set

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \qquad |X| = \left(|x|^2 + |t|\right)^{1/2}.$$

Let r be a positive number. For $x_0 \in \mathbb{R}^n$ we shall denote $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and $B'_r(x'_0) = \{x' \in \mathbb{R}^{n-1} : |x' - x'_0| < r\}$. We generally set $B_r = B_r(0)$ and $B'_r = B'_r(0)$. We denote by $B^+_r(0) = \{x \in B_r : x_n > 0\}$ and $B^-_r(0) = \{x \in B_r : x_n < 0\}$. For a point $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ we shall set $Q_r(X_0) = B_r(x_0) \times (t_0 - r^2, t_0)$.

Given a subset A of \mathbb{R}^n , we shall denote by

$$[A]_{\varepsilon} = \{x \in \mathbb{R}^{n} : \operatorname{dist}(x, A) \le \varepsilon\},\$$
$$(A)_{\varepsilon} = \{x \in A : \operatorname{dist}(x, \partial A) > \varepsilon\},\$$
$$[\partial A]_{\varepsilon} = \{x \in \mathbb{R}^{n} : \operatorname{dist}(x, \partial A) \le \varepsilon\}.$$

Let I be an interval of \mathbb{R} and let $\{D(t)\}_{t\in I}$ be a family of subsets D(t) of \mathbb{R}^n , we shall denote

$$D(I) = \bigcup_{t \in I} D(t) \times \{t\},\$$

and $Q = D(\mathbb{R})$.

Given a sufficiently smooth function u of variable $(x,t) \in \mathbb{R}^{n+1}$, we shall denote by $\partial_i u = \frac{\partial u}{\partial x_i}$, $\partial_{ij}^2 u = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, n$ and $\partial_t u = \frac{\partial u}{\partial t}$. For a multiindex $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}$, $i = 1, \ldots, n$ and $k \in \mathbb{N}$, we shall denote, as usual, $\partial_x^\beta \partial_t^k u = \frac{\partial^{|\beta|+k} u}{\partial x_1^{\beta_1} \ldots \partial_{x_n}^{\beta_n} \partial_t^k}$, where $|\beta| = \sum_{i=1}^n \beta_i$. Also we shall write $\nabla = \nabla_x$, div = div_x. For a matrix A we shall denote by A^* the transposed matrix of A. We denote by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$.

We shall use letters C, C_0, C_1, \ldots to denote constants. The value of the constant can change from line to line, but we shall specify the dependance everywhere they appear. Sometimes we have dropped the dependance on n which is fixed $(n \ge 2)$.

Functional Spaces

Let D be a subset of \mathbb{R}^{n+1} , f a function defined on D with values in \mathbb{R} or \mathbb{R}^n and $\alpha \in (0, 1]$. We shall set

$$[f]_{\alpha;D} = \sup\left\{\frac{|f(x,t) - f(y,s)|}{(|x-y|^2 + |t-s|)^{\alpha/2}} \, : \, (x,t), (y,s) \in D, \, (x,t) \neq (y,s)\right\}.$$

If $\alpha \in (0, 2]$ we shall set

$$< f >_{\alpha;D} = \sup \left\{ \frac{|f(x,t) - f(y,s)|}{|t - s|^{\alpha/2}} : (x,t), (y,s) \in D, t \neq s) \right\}$$

Let k be a positive integer number, f a sufficiently smooth function and $\alpha \in (0, 1]$. We shall denote

$$[f]_{k+\alpha;D} = \sum_{|\beta|+2j=k} [\partial_x^\beta \partial_t^j f]_{\alpha;D}, \qquad \langle f \rangle_{k+\alpha;D} = \sum_{|\beta|+2j=k-1} [\partial_x^\beta \partial_t^j f]_{1+\alpha;D}.$$

The following Sobolev spaces will be used (we refer to [Li-Ma] for further details). We denote by Ω a bounded domain in \mathbb{R}^n . The space $H = H_{,0}^{3/2,3/4}(\partial\Omega \times (0,T))$, its dual $H' = H_1 = H^{-3/2,-3/4}(\partial\Omega \times (0,T))$, and $H_0 = H^{1/2,1/4}(\partial\Omega \times (0,T))$. We consider now the interpolation spaces between H_0 and H_1 . For any θ , $0 \leq \theta \leq 1$, we define H_{θ} as $[H_0, H_1]_{\theta}$, where the latter denotes the interpolation at level θ between the two spaces H_0 and H_1 . The norm in H_{θ} will be denoted by $\|\cdot\|_{\theta}$. First, we notice that for any θ , $0 \leq \theta \leq 1$, there exists a constant C_{θ} , which depends on θ only, such that the following interpolation inequality holds for any $\psi \in H_0$

(2.1)
$$\|\psi\|_{\theta} \le C_{\theta} \|\psi\|_{0}^{1-\theta} \|\psi\|_{1}^{\theta}.$$

We also make use of the following notation

$$W(\Omega \times (0,T)) = \left\{ v : v \in L^2((0,T), H^1(\Omega)), \, \partial_t v \in L^2((0,T), H^{-1}(\Omega)) \right\}.$$

Boundary Regularity

Let us give the following definitions

Definition 2.1 Let Ω be a domain in \mathbb{R}^n . Given α , $\alpha \in (0, 1]$, we shall say that $\partial \Omega$ is of class $C^{1,\alpha}$ with constants $\rho_0, E > 0$ if for any $P \in \partial \Omega$, there exists a rigid transformation of \mathbb{R}^n under which we have $P \equiv 0$ and

$$\Omega \cap B_{\rho_0}(0) = \{ x \in B_{\rho_0}(0) : x_n > \varphi(x') \},\$$

where φ is a $C^{1,\alpha}$ function on $B'_{\rho_0}(0)$ which satisfies the following conditions $\varphi(0) = |\nabla_{x'}\varphi(0)| = 0$ and $\|\varphi\|_{C^{1,\alpha}(B'_{\rho_0}(0))} \leq E\rho_0$.

Remark 2.2 We have chosen to normalize all norms in such a way that their terms are dimensional homogeneous and coincide with the standard definition when $\rho_0 = 1$. For instance, for any $\varphi \in C^{1,\alpha}(B'_{\rho_0}(0))$ we set

$$\|\varphi\|_{C^{1,\alpha}(B'_{\rho_0}(0))} = \rho_0 \|\varphi\|_{L^{\infty}(B'_{\rho_0}(0))} + \rho_0 \|\nabla_{x'}\varphi\|_{L^{\infty}(B'_{\rho_0}(0))} + \rho_0^{1+\alpha} [\nabla_{x'}\varphi]_{\alpha;B'_{\rho_0}(0)}$$

Similarly we shall set

$$||u||_{L^2(\Omega)} = \rho_0^{-(n+1)/2} \left(\int_{\Omega} u^2 dX\right)^{1/2},$$

where dX = dxdt.

Definition 2.3 Let Q be a domain in \mathbb{R}^{n+1} . We shall say that Q (or equivalently ∂Q) is of class \mathcal{K} with constants ρ_0 , E if for all $P_0 \in \partial Q$ there exists a rigid transformation of space coordinates under which we have $P_0 = (0,0)$ such that

$$Q \cap \left(B_{\rho_0}(0) \times (-\rho_0^2, \rho_0^2)\right) = \{X \in B_{\rho_0}(0) \times (-\rho_0^2, \rho_0^2) : x_n > \varphi(x', t)\},\$$

where φ is endowed with second derivatives with respect to x_i , $i = 1, \dots, n$, with the t-derivative and with second derivatives with respect to x_i and t and it satisfies the following conditions $\varphi(0,0) = |\nabla_{x'}\varphi(0,0)| = 0$ and

$$\begin{split} \rho_0 \| \nabla_{x'} \varphi \|_{L^{\infty}(B'_{\rho_0} \times (-\rho_0^2, \rho_0^2))} + \rho_0^2 \| D_{x'}^2 \varphi \|_{L^{\infty}(B'_{\rho_0} \times (-\rho_0^2, \rho_0^2))} \\ + \rho_0^2 \| \partial_t \varphi \|_{L^{\infty}(B'_{\rho_0} \times (-\rho_0^2, \rho_0^2))} + \rho_0^3 \| \nabla_{x'} \partial_t \varphi \|_{L^{\infty}(B'_{\rho_0} \times (-\rho_0^2, \rho_0^2))} \leq E \rho_0. \end{split}$$

Definition 2.4 (relative graphs) Let $\beta \in (0, 1]$. We shall say that two bounded domains Ω_1 and Ω_2 in \mathbb{R}^n of class $C^{1,\beta}$ with constants R_0, E are relative graphs if for any $P \in \partial \Omega_1$ there exists a rigid transformation of coordinates under which we have $P \equiv 0$ and there exist $\varphi_{P,1}, \varphi_{P,2} \in C^{1,\beta}(B'_{r_0}(0))$, where $\frac{r_0}{R_0} \leq 1$ depends on E and β only, satisfying the following conditions

 $\begin{aligned} & R_{0} \\ & i) \varphi_{P,1}(0) = 0 , |\varphi_{P,2}(0)| \leq \frac{r_{0}}{2}, \\ & ii) \|\varphi_{P,i}\|_{C^{1,\beta}(B'_{r_{0}}(0))} \leq ER_{0}, \quad i = 1, 2, \\ & iii) \Omega_{i} \cap B_{r_{0}}(0) = \{x \in B_{r_{0}}(0) : x_{n} > \varphi_{P,i}(x')\}, \, i = 1, 2. \\ & We \ shall \ denote \end{aligned}$

(2.2)
$$\gamma\left(\Omega_{1},\Omega_{2}\right) = \sup_{P\in\partial\Omega_{1}} \left\|\varphi_{P,1} - \varphi_{P,2}\right\|_{L^{\infty}\left(B'_{r_{0}}(0)\right)}.$$

The Dirichlet-to-Neumann map

For any $g \in H$, let $u \in W(\Omega \times (0,T))$ be the weak solution to the initialboundary value problem

- (2.3a) $\partial_t u \operatorname{div}((1+(k-1)\chi_Q)\nabla u) = 0$ in $\Omega \times (0,T)$,
- (2.3b) u(x,0) = 0 for $x \in \overline{\Omega}$,

(2.3c)
$$u(x,t) = g(x,t)$$
 for $(x,t) \in \partial \Omega \times (0,T)$,

where χ_Q is the characteristic function of the set Q. Then, for any $g \in H$, we set

(2.4)
$$\Lambda_Q g = \frac{\partial u}{\partial \nu}|_{\partial \Omega \times (0,T)}, \quad u \text{ solution to } (2.3).$$

We have that there exists a unique solution $u \in W(\Omega \times (0, T))$ to problem (2.3) [Ev]. In addition, by standard regularity theorems [Li], [La-So-Ur] and by trace theorem [Li-Ma, Chapter 4, Theorem 2.1], we conclude that $\Lambda_Q g$ belongs to the space H_0 and that the operator $\Lambda_Q : H \to H_0$ is bounded. We can also consider Λ_Q as a linear and bounded operator between H and $H' = H_1$, by setting (2.5)

$$\langle \Lambda_Q g, \phi \rangle_{H',H} = \langle \frac{\partial u}{\partial \nu} |_{\partial \Omega \times (0,T)}, \phi \rangle_{H',H} = \int_{\partial \Omega \times (0,T)} \frac{\partial u}{\partial \nu} \phi, \quad \text{for any } g, \phi \in H,$$

where u solves (2.3) and $\langle \cdot, \cdot \rangle_{H',H}$ is the duality pairing between H' and H.

Let us remark that the operator Λ_Q is usually referred to as the *Dirichlet*to-Neumann map associated to the equation (2.3a).

2.2 Assumptions and Statement of the Main Result

Assumptions on the domain

Let ρ_0, M, E be given positive numbers. We assume that Ω is a bounded domain in \mathbb{R}^n satisfying

(2.6a)
$$|\Omega| \le M\rho_0^n,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω . We also assume that

(2.6b) $\partial \Omega$ is of class $C^{1,1}$ with constants ρ_0, E .

A priori information on the inclusion

Denoting by $Q=\bigcup_{t\in\mathbb{R}}D(t)\times\{t\}$ $(Q=D((-\infty,+\infty))),$ we assume the following conditions

(2.7a) ∂Q is of class \mathcal{K} with constant ρ_0, E ,

(2.7b)
$$\operatorname{dist}(D(t), \partial \Omega) \ge \rho_0, \qquad \overline{D(t)} \subset \Omega, \quad \forall t \in [0, T],$$

(2.7c) $\Omega \setminus \overline{D(t)}$ is connected $\forall t \in [0, T]$.

Remark 2.5 Let t be any number in [0, T]. Observe that (2.7a) automatically implies a lower bound on the diameter of every connected component of D(t)and $\Omega \setminus \overline{D}(t)$. In addition, combining (2.6a) with (2.6b), we have an upper bound of the diameter of Ω and thus of D(t). Note also that (2.7a) and (2.7b) implicitly comprise an a priori upper bound on the number of connected components of D(t).

Remark 2.6 For n = 1, it is possible to obtain Theorem 2.7 replacing assumption (2.7) by considering $\Omega = (0, L)$ and $D(t) = \{x \in \mathbb{R} : s_1(t) < x < s_2(t)\}$, where s_i , i = 1, 2, are C^1 functions such that for all $t \in (0, T)$

$$L - s_2(t) \ge \rho_0, \quad s_1(t) \ge \rho_0, \quad s_2(t) - s_1(t) \ge \rho_0$$

and

$$||s_i||_{L^{\infty}((0,T))} + \rho_0^2 ||s_i'||_{L^{\infty}((0,T))} \le E\rho_0, \qquad i = 1, 2,$$

here $3\rho_0 < L$.

Theorem 2.7 Let $\Omega \subset \mathbb{R}^n$ satisfying (2.6). Let k > 0, $k \neq 1$ be given. Let $\{D_1(t)\}_{t\in\mathbb{R}}, \{D_2(t)\}_{t\in\mathbb{R}}$ be two families of domains satisfying (2.7). Assume that for $\varepsilon > 0$,

(2.8)
$$\|\Lambda_{Q_1} - \Lambda_{Q_2}\|_{\mathcal{L}(H,H')} \le \varepsilon,$$

where $Q_i = D_i((-\infty, +\infty)), i = 1, 2$. Then

(2.9)
$$d_{\mathcal{H}}(D_1(t), D_2(t)) \le \omega_t(\varepsilon), \qquad t \in (0, T],$$

where $\omega_t(s)$ is such that

(2.10)
$$\omega_t(s) \le C\rho_0 |\log s|^{-\eta}, \quad 0 < s < 1,$$

with C = C(t), depending on t, M, E, k only, and $0 < \eta \leq 1$ depending on M, E, k only. In addition we have that C(t) tends to $+\infty$ as t tends to 0.

Here $d_{\mathcal{H}}$ denotes the Hausdorff distance.

3 Proof of Theorem 2.7

For the sake of brevity we name $a_j = 1 + (k-1)\chi_{Q_j}$, j = 1, 2. We fix $g \in H$. We shall denote by u_j , j = 1, 2 the solution of (2.3) when $Q = Q_j$. For $\psi \in H^{1,1}(\Omega \times (0,T))$ such that

(3.1)
$$\psi(\cdot, T) = 0 \quad \text{in } \Omega,$$

using the weak formulation of (2.3) we have

$$\int_{\partial\Omega\times(0,T)} a_j \frac{\partial u_j}{\partial\nu} \psi dS + \int_{\Omega} u_j(x,0)\psi(x,0)dx \\ - \int_{\Omega\times(0,T)} (a_j \nabla u_j \cdot \nabla \psi - u_j \partial_t \psi) \, dx dt = 0 \quad \text{for } j = 1, 2.$$

Subtracting the two equations we obtain

(3.2)
$$\int_{\Omega \times (0,T)} \left(a_1 \nabla (u_1 - u_2) \cdot \nabla \psi - (u_1 - u_2) \partial_t \psi \right) dx dt + \int_{\Omega \times (0,T)} (a_1 - a_2) \nabla u_2 \cdot \nabla \psi = \langle (\Lambda_{Q_1} - \Lambda_{Q_2}) g, \psi \rangle_{H',H},$$

(we notice here that in these identities it is possible to have $u_i(\cdot, 0) \neq 0$ for i = 1, 2). Taking ψ such that it satisfies (3.1) and

(3.3)
$$\partial_t \psi + \operatorname{div}(a_1 \nabla \psi) = 0 \quad \text{in } \Omega \times (0, T),$$

by (3.2) we have (recalling that on $\partial \Omega \times (0,T) u_1 = u_2 = g$)

$$\int_{\Omega \times (0,T)} (a_1 - a_2) \nabla u_2 \cdot \nabla \psi = \langle (\Lambda_{Q_1} - \Lambda_{Q_2})g, \psi \rangle_{H',H}, \qquad \forall g \in H$$

or, equivalently,

(3.4)
$$\int_0^T \int_\Omega (\chi_{Q_1} - \chi_{Q_2}) \nabla u_2 \cdot \nabla \psi dx dt = \frac{1}{k-1} < (\Lambda_{Q_1} - \Lambda_{Q_2}) u_2, \psi >_{H',H} .$$

Let us denote by $\Gamma_2(x, t; y, s)$ and $\Gamma_1^*(x, t; y, s)$ the fundamental solutions of the operator $\partial_t - \operatorname{div}(a_2 \nabla)$ and $\partial_t + \operatorname{div}(a_1 \nabla)$ respectively $(\Gamma_1^*(x, t; y, s) = 0$ if $t \ge s$ and $\Gamma_2(x, t; y, s) = 0$ if $t \le s$), that is

$$\begin{split} &\int_{\mathbb{R}^{n+1}} [-\Gamma_2(x,t;y,s)\partial_t \phi(x,t) + a_2 \nabla_x \Gamma_2(x,t;y,s) \cdot \nabla_x \phi(x,t)] \, dx dt = \phi(y,s), \\ &\int_{\mathbb{R}^{n+1}} [\Gamma_1^*(x,t;y,s)\partial_t \phi(x,t) + a_1 \nabla_x \Gamma_1^*(x,t;y,s) \cdot \nabla_x \phi(x,t)] \, dx dt = \phi(y,s), \end{split}$$

for every $\phi \in C_0^1(\mathbb{R}^{n+1})$, that is using the δ Dirac symbol, we have respectively

$$\partial_t \Gamma_2(x,t;y,s) - \operatorname{div}(a_2 \nabla_x \Gamma_2(x,t;y,s)) = \delta(x-y,t-s)$$

and

 $-\partial_t \Gamma_1^*(x,t;y,s) - \operatorname{div}(a_1 \nabla_x \Gamma_1^*(x,t;y,s)) = \delta(x-y,t-s).$

Choosing in (3.4) $u_2(x,t) = \Gamma_2(x,t;y,s)$ and $\psi(x,t) = \Gamma_1^*(x,t;\xi,\tau)$, with (y,s) and $(\xi,\tau) \notin \Omega \times (0,T), 0 \le s < \tau \le T$, we obtain

(3.5)
$$\int_{0}^{T} \int_{\Omega} (\chi_{Q_{1}} - \chi_{Q_{2}}) \nabla_{x} \Gamma_{2}(x,t;y,s) \cdot \nabla_{x} \Gamma_{1}^{*}(x,t;\xi,\tau) dx dt = \frac{1}{k-1} < (\Lambda_{Q_{1}} - \Lambda_{Q_{2}}) \Gamma_{2}(\cdot,\cdot;y,s), \Gamma_{1}^{*}(\cdot,\cdot;\xi,\tau) >_{H',H}.$$

For $t \in [0, T]$ we shall define $\mathcal{G}(t)$ as the connected component of $\Omega \setminus (D_1(t) \cup D_2(t))$ that contains $\partial\Omega$, $\tilde{\mathcal{G}}(t) = (\mathbb{R}^n \setminus \Omega) \cup \mathcal{G}(t)$ and $\tilde{\mathcal{G}}((0, T)) := \bigcup_{t \in (0, T)} \tilde{\mathcal{G}}(t) \times \{t\}$. For (y, s), $(\xi, \tau) \in \tilde{\mathcal{G}}((0, T))$ with $0 \leq s < \tau \leq T$, we set

$$S_1(y,s;\xi,\tau) := \int_{Q_1} \nabla_x \Gamma_2(x,t;y,s) \cdot \nabla_x \Gamma_1^*(x,t;\xi,\tau) dx dt,$$

$$S_2(y,s;\xi,\tau) := \int_{Q_2} \nabla_x \Gamma_2(x,t;y,s) \cdot \nabla_x \Gamma_1^*(x,t;\xi,\tau) dx dt$$

$$\mathcal{U}(y,s;\xi,\tau) := S_1(y,s;\xi,\tau) - S_2(y,s;\xi,\tau).$$

Remark 3.1 Let us observe here that for $\tau < s$, S_1 and S_2 can be defined identically zero since for (y,s) and $(\xi,\tau) \in \tilde{\mathcal{G}}((0,T))$ we have $S_1(y,s;\xi,s) = S_2(y,s;\xi,s) = 0$.

By (3.5) we have

(3.6)
$$\mathcal{U}(y,s;\xi,\tau) = \frac{1}{k-1} < (\Lambda_{Q_1} - \Lambda_{Q_2})\Gamma_2(\cdot,\cdot;y,s), \Gamma_1^*(\cdot,\cdot;\xi,\tau) >_{H',H},$$

for all $y, \xi \notin \Omega$, $0 \leq s < \tau \leq T$. Denoting by $\Omega_D(t) := \Omega \setminus \overline{\mathcal{G}(t)}, t \in [0, T]$, we introduce a variation of the Hausdorff distance that, even though it is not a metric, we call it *modified distance*

(3.7)
$$d_{\mu}(t) = d_{\mu}(D_{1}(t), D_{2}(t))$$
$$= \max \left\{ \sup_{x \in \partial D_{1}(t) \cap \partial \Omega_{D}(t)} \operatorname{dist}(x, D_{2}(t)), \sup_{x \in \partial D_{2}(t) \cap \partial \Omega_{D}(t)} \operatorname{dist}(x, D_{1}(t)) \right\},$$

 $t \in [0, T]$. We point out here that trivially $d_{\mu}(D_1(t), D_2(t)) \leq d_{\mathcal{H}}(\overline{D_1(t)}, \overline{D_2(t)})$. The following proposition shows the relation between the Hausdorff distance and d_{μ} , provided the priori assumptions (2.7) hold. We refer to [Al-DC, Proposition 3.3] for the proof.

Proposition 3.2 Let $D_1(t)$ and $D_2(t)$ be two sets satisfying (2.7) then for any $t \in [0,T]$

(3.8)
$$d_{\mathcal{H}}(\partial D_1(t), \partial D_2(t)) \le C d_{\mu}(t),$$

where C depends on E and M only.

We now give a proposition which connects the Hausdorff distance between the boundary of the inclusions and between the inclusions.

Proposition 3.3 Let D_1 and D_2 be two domains of class C^2 with constants E, ρ_0 such that $\mathbb{R}^n \setminus D_j$, j = 1, 2, is connected. There exists a positive constant C depending on E only such that

(3.9)
$$d_{\mathcal{H}}(\overline{D}_1, \overline{D}_2) \le C d_{\mathcal{H}}(\partial D_1, \partial D_2).$$

Proof. See Section 4.1

Remark 3.4 By Propositions 3.2 and 3.3 we have

$$d_{\mathcal{H}}(\overline{D}_{1}(t),\overline{D}_{2}(t)) \leq Cd_{\mathcal{H}}(\partial D_{1}(t),\partial D_{2}(t)) \leq C'd_{\mathcal{H}}(\overline{D}_{1}(t),\overline{D}_{2}(t)) \leq C'd_{\mathcal{H}}(\overline{D}_{1}(t),\overline{D}_{2}(t)).$$

Thus it turns out that the distances $d_{\mathcal{H}}(\overline{D}_1(t), \overline{D}_2(t)), d_{\mathcal{H}}(\partial D_1(t), \partial D_2(t))$ and $d_{\mu}(D_1(t), D_2(t))$ are equivalent.

Proposition 3.5 Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^n of class $C^{1,\beta}$ with constants R_0 , E and satisfying $|\Omega_i| \leq MR_0^n$. There exist numbers $\overline{d}, \overline{\rho} \in (0, R_0]$ such that $\frac{\overline{d}}{R_0}$ and $\frac{\overline{\rho}}{R_0}$ depend on β and E only, such that if we have

(3.10)
$$d_{\mathcal{H}}\left(\overline{\Omega_1},\overline{\Omega_2}\right) \le \overline{d},$$

then the following facts hold true

i) Ω_1 and Ω_2 are relative graphs and

(3.11)
$$\gamma(\Omega_1, \Omega_2) \le C d_{\mathcal{H}}(\Omega_1, \Omega_2) ,$$

where C depends on β and E only,

iii) any connected component of $\Omega_1 \cap \Omega_2$ has boundary of Lipschitz class with constants ρ_0 , L, where ρ_0 is as above and L > 0 depends on E only.

Proof. See [Ve, Proposition 4.1.8].

A key ingredient for the proof of our stability theorem are fundamental solutions. We collect here some results we need.

We shall denote by $\Gamma_0(x-y,t-s)$ the standard fundamental solution of $\partial_t - \Delta$ which is

$$\Gamma_0(x-y,t-s) = \frac{1}{[4\pi(t-s)]^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}}, \qquad t > s.$$

We shall denote by $\Gamma(x, t; y, s)$ the fundamental solution of the operator $\partial_t - \operatorname{div}((1 + (k - 1)\chi_Q)\nabla_x)$ (see [Ar]). We recall that Γ satisfies the following properties

$$(3.12) \qquad \Gamma(x,t;y,s) = \Gamma^*(y,s;x,t) \qquad \forall \, (x,t), (y,s) \in Q, \, (x,t) \neq (y,s),$$

where Γ^* is the fundamental solution to $-\partial_t - \operatorname{div}((1 + (k-1)\chi_Q)\nabla_x))$, and

(3.13)
$$0 < \Gamma(x,t;y,s) \le \frac{C}{[4\pi(t-s)]^{n/2}} e^{-\frac{|x-y|^2}{C(t-s)}} \chi_{[s,+\infty)}(t),$$

where $C \ge 1$ depends on k only. Furthermore we have also the following estimate for the gradient of Γ .

Proposition 3.6 Let $\Gamma(x,t;y,s)$ be the fundamental solution of the operator $\partial_t - \operatorname{div}\left((1+(k-1)\chi_Q)\nabla_x\right)$. There exists $C \geq 1$, depending on k and E only such that

(3.14)
$$|\nabla_x \Gamma(x,t;y,s)| \le \frac{C}{(t-s)^{\frac{n+1}{2}}} e^{-\frac{|x-y|^2}{C(t-s)}},$$

for almost every $x, y \in \mathbb{R}^n$ and $t, s \in \mathbb{R}, t > s$.

Proof. See Section 4.2.

 \square

In the sequel we need the fundamental solution of the operator $\mathcal{L}_{+} = \partial_t - \partial_t$ $\operatorname{div}((1+(k-1)\chi_+)\nabla)$ where $\chi_+ = \chi_{\{(x,t)\in\mathbb{R}^{n+1}:x_n>0\}}$. We shall denote by Γ_+ such a fundamental solution. Also, we shall denote by Γ^*_+ the fundamental solution of the adjoint operator of \mathcal{L}_+ . Observe that $\Gamma_+(x,t;y,s) = \Gamma_+(x,t-t)$ s; y, 0 and $\Gamma^*_+(x, t; y, s) = \Gamma_+(x, s - t; y, 0)$. Here and in the sequel, for a given function $f(x', x_n)$, we shall denote by $\mathcal{F}_{\zeta'}(f(\cdot, x_n))$ the Fourier transform of f with respect to the variable x'. Thus

$$\mathcal{F}_{\zeta'}(f(\cdot, x_n)) = \int_{\mathbb{R}^{n-1}} f(x', x_n) \mathrm{e}^{-ix' \cdot \zeta'} dx',$$

for every $\zeta' \in \mathbb{R}^{n-1}$.

In [Is-Ki-Na] it has been proved some formulae for $\mathcal{F}_{\zeta'}(\Gamma_+(.,x_n,t;y))$. The technique to prove such formulae is rather classical and lengthy. For this reason we display only the formulae that we need corresponding to the case in which $x_n > 0, y_n < 0.$

$\underline{\text{Case } k > 1}.$ Denote by

(3.15)
$$E(\zeta', x_n, t; \rho) = \exp\left[-t(k - (k - 1)\rho)|\zeta'|^2 - \sqrt{\frac{k - 1}{k}}x_n|\zeta'|\sqrt{\rho}\right],$$

(3.16)
$$F(\zeta', y_n; \rho) = \operatorname{Im}\left(A_1(\rho)e^{iy_n\sqrt{k - 1}\sqrt{1 - \rho}|\zeta'|}\right),$$

where, for complex number z = a + ib, Im(z) denotes the imaginary part b of z, and

(3.17)
$$A_1(\rho) = \frac{\sqrt{k-1}}{\pi} \frac{1}{i\sqrt{k-1}\sqrt{1-\rho} + \sqrt{k}\sqrt{\rho}}$$

Then

(3.18)
$$\mathcal{F}_{\zeta'}(\Gamma_+(\cdot, x_n, t; y, 0)) = \int_0^1 |\zeta'| \mathrm{e}^{-iy' \cdot \zeta'} E(\zeta', x_n, t; \rho) F(\zeta', y_n; \rho) d\rho,$$

for every $x_n > 0$, $y_n < 0$. Case 0 < k < 1. Denote by

$$G(\zeta', y_n, t; \rho) = \exp\left[-t(1 - (1 - k)\rho)|\zeta'|^2 + \sqrt{1 - k} y_n |\zeta'|\sqrt{\rho}\right],$$
$$H(\zeta', x_n; \rho) = \operatorname{Im}\left(A_2(\rho) e^{-ix_n \sqrt{\frac{1-k}{k}}\sqrt{1-\rho}|\zeta'|}\right),$$

where

$$A_{2}(\rho) = \frac{\sqrt{1-k}}{\pi} \frac{1}{\sqrt{k}\sqrt{\rho} - i\sqrt{1-k}\sqrt{1-\rho}}.$$

Then

$$\mathcal{F}_{\zeta'}(\Gamma_+(\cdot, x_n, t; y, 0)) = \int_0^1 |\zeta'| \mathrm{e}^{-iy' \cdot \zeta'} G(\zeta', y_n, t; \rho) H(\zeta', x_n; \rho) d\rho,$$

for every $x_n > 0$, $y_n < 0$.

Proposition 3.7 For every $\lambda_0 \in (0,1]$ there exist $\lambda_1, \lambda_2, \lambda_3 \in (0,\lambda_0]$ such that for every h > 0 the following inequality holds true

$$(3.19) \quad I^{(h)} := \left| \int_0^{\lambda_2 h^2} dt \int_{\mathbb{R}^n_+} \nabla_x \Gamma^*_+(x,t;-\lambda_1 he_n,\lambda_2 h^2) \right. \\ \left. \cdot \nabla_x \Gamma_0(x,t;-\lambda_3 he_n,0) dx \right| \ge \frac{1}{Ch^n},$$

where $C, C \geq 1$, depends on $\lambda_1, \lambda_2, \lambda_3$ and k only.

Proof. See Section 4.3.

Through the paper we shall fix the value of $\lambda_1, \lambda_2, \lambda_3$ in such a way that (3.19) is satisfied and we shall omit the dependence of various constants by $\lambda_1, \lambda_2, \lambda_3$. In the following we shall often make use of this technical lemma whose proof can be found in [Fr, Lemma 3 pg. 15].

Lemma 3.8 Let $\alpha, \beta < \frac{n}{2} + 1$ and a > 0. Then

$$\int_{s}^{t} \int_{\mathbb{R}^{n}} (t-\tau)^{-\alpha} e^{-\frac{a|x-\xi|^{2}}{4(t-\tau)}} (\tau-s)^{-\beta} e^{-\frac{a|\xi-y|^{2}}{4(t-\tau)}} d\xi d\tau$$
$$= \frac{C}{a^{n/2}} (t-s)^{\frac{n}{2}+1-\alpha-\beta} e^{-\frac{a|x-\xi|^{2}}{4(t-s)}}, \qquad \forall x, y \in \mathbb{R}^{n}, s < t.$$

where C depends on α , β and n only.

For $\overline{t} \in (0,T]$ fixed, we can assume, without loosing generality, that there exists $O \in \partial D_1(\overline{t}) \cap \partial \Omega_D(\overline{t})$ (for the sake of brevity we assume that O is the origin of \mathbb{R}^n) such that

(3.20)
$$d_{\mu}(\overline{t}) = \operatorname{dist}(O, D_2(\overline{t})).$$

Denote by

$$\rho = \min\{d_{\mu}(\overline{t}), \rho_0\}.$$

Furthermore, denote by $\nu(O, \bar{t})$ the exterior unit normal to $\partial D_1(\bar{t})$ in O pointing towards $\mathcal{G}(\bar{t})$. Now we introduce parameter $\delta \in (0, 1]$ that we shall choose later on. We set

(3.21)
$$t_1 = \overline{t} - \lambda_2 h^2, \qquad \overline{y} = \lambda_1 h \nu(0, \overline{t}), \qquad y_1 = \lambda_3 h \nu(0, \overline{t}),$$

where

(3.22)
$$0 < h \le \delta \min\{\rho, \sqrt{t}\}.$$

Notice that (3.22) implies that $t_1 \in (0, \bar{t})$. By using (2.7a) it is simple to check that there exists $C_1, C_1 \ge 1$, depending on E only such that if

$$(3.23) 0 < \delta \le \frac{\lambda_3}{C_1}$$

then, for every $t \in [t_1, \overline{t}]$, we have

(3.24)
$$\operatorname{dist}(\overline{y}, D_1(t)) \ge \frac{1}{2} \min \{\lambda_1, \lambda_2, \lambda_3\} h,$$

(3.25)
$$\operatorname{dist}(y_1, D_1(t)) \ge \frac{1}{2} \min \{\lambda_1, \lambda_2, \lambda_3\} h.$$

On the other side, using the inequality [Ve, Proposition 4.1.6]

(3.26)
$$\left|\operatorname{dist}(O, D_2(t)) - \operatorname{dist}(O, D_2(\overline{t}))\right| \le \frac{C_0}{\rho_0} |t - \overline{t}|,$$

where C_0 depends on E and M only, for $t \in [t_1, \overline{t}]$ and by using the triangle inequality we have that there exists $C_2, C_2 \ge 1$, depending on E and M only such that if

$$(3.27) 0 < \delta \le \frac{1}{C_2}$$

then for $t \in [t_1, \overline{t}]$

(3.28)
$$\operatorname{dist}(z, D_2(t)) \ge \frac{1}{2}\rho, \quad \text{with } z = \overline{y}, y_1.$$

Proposition 3.9 Let $\{D_1(t)\}_{t\in\mathbb{R}}, \{D_2(t)\}_{t\in\mathbb{R}}$ be two families of domains satisfying (2.7) and let $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ be such that the inequality (3.19) is satisfied. Then there exist $C, C \geq 1$, and $\tilde{C}, \tilde{C} \geq 1, C$ depending on k only and \tilde{C} depending on $k, E, M, \lambda_1, \lambda_2$ and λ_3 only such that

(3.29)
$$|\mathcal{U}(y_1, t_1; \overline{y}, \overline{t})| \ge \frac{1}{Ch^n}$$

for $0 < h \leq \frac{1}{\tilde{C}} \min\{\rho, \sqrt{\tilde{t}}\}$, where $y_1, t_1, \overline{y}, \overline{t}$, and ρ are defined in (3.21).

Proof. See Section 4.3

Theorem 3.10 (Two-spheres and one-cylinder inequality) Let λ , Λ and R positive numbers with $\lambda \in (0, 1]$. Let P be the parabolic operator

(3.30)
$$P = \partial_t - \partial_i \left(a^{ij} \partial_j \right),$$

where $\{a^{ij}(x,t)\}_{i,j=1}^n$ is a symmetric $n \times n$ matrix. For $\xi \in \mathbb{R}^n$ and $(x,t), (y,s) \in \mathbb{R}^{n+1}$ assume that

(3.31a)
$$\lambda |\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \le \lambda^{-1}|\xi|^2$$

and

(3.31b)
$$\left(\sum_{i,j=1}^{n} \left(a^{ij}(x,t) - a^{ij}(y,s)\right)^2\right)^{1/2} \le \frac{\Lambda}{R} \left(|x-y|^2 + |t-s|\right)^{1/2}.$$

Let u be a function in $H^{2,1}(B_R \times (0, \mathbb{R}^2))$ satisfying the inequality

(3.32)
$$|Pu| \le \Lambda \left(\frac{|\nabla u|}{R} + \frac{|u|}{R^2}\right) \quad in \ B_R \times (0, R^2]$$

Then there exist constants $\eta_1 \in (0,1)$ and $C \in [1, +\infty)$, depending on λ , Λ and n only such that for every $r_1, r_2, 0 < r_1 \leq r_2 \leq \eta_1 R$ we have

(3.33)
$$\|u(\cdot, R^2)\|_{L^2(B_{r_2})} \le \frac{CR}{r_2} \|u\|_{L^2(B_R \times (0, R^2))}^{1-\theta_1} \|u(\cdot, R^2)\|_{L^2(B_{r_1})}^{\theta_1},$$

where $\theta_1 = \frac{1}{C \log \frac{R}{r_1}}$.

Proof. See [Ve]

We can now start to prove our stability theorem. Before entering into details, we wish to warn the reader that, sometimes we use the previous auxiliary results (such as Lemma 3.8 or Proposition 3.6) omitting some computations that are similar to the one contained in the proofs of Section 4.

Proof of Theorem 2.7. We divide the proof of the theorem in two steps. In the first step we provide a rough stability estimate (see (3.49) below). In the second step we prove the logarithmic stability estimate (2.10).

Step 1.

We shall denote by

$$(3.34) \qquad \qquad \Xi_{\rho_0} = \{ x \in \mathbb{R}^n : \rho_0/2 < \operatorname{dist}(x, \Omega) < \rho_0 \}$$

and

(3.35)
$$\Omega_{\rho_0} = \{ x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) < \rho_0 \}.$$

Since $\mathcal{U}(y, s; \xi, \tau)$ is equal to 0 when $s \ge \tau$ (see Remark 3.1), for $(y, s) \in \Xi_{\rho_0} \times (0, T)$ we define

$$(3.36) v(\xi,\tau) := \mathcal{U}(y,s;\xi,\tau), (\xi,\tau) \in \tilde{\mathcal{G}}((0,T)) := \cup_{t \in (0,T)} \tilde{\mathcal{G}}(t) \times \{t\}.$$

Let *h* be the one defined in (3.22), with $\delta \leq \frac{1}{\tilde{C}}$ and let $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ be such that inequality (3.19) is satisfied. Let $\overline{x} \in \Xi_{\rho_0}$ be such that $\operatorname{dist}(\overline{x}, \mathbb{R}^n \setminus \Xi_{\rho_0}) \geq \rho_0/8$. Let us denote by γ a simple connected arc in $(\Omega_{\rho_0} \setminus \Omega_D(\overline{t}))_{\lambda_1 h/2} = \{x \in \Omega_{\rho_0} \setminus \Omega_D(\overline{t}) : \operatorname{dist}(x, \partial(\Omega_{\rho_0} \setminus \Omega_D(\overline{t}))) > \frac{\lambda_1 h}{2}\}$, connecting \overline{x} to \overline{y} , where \overline{y} is defined in (3.21). By (2.8) we have

(3.37)
$$\|v(\cdot,\overline{t})\|_{L^{\infty}(B_{\rho_0/2}(\overline{x}))} \leq \frac{C\varepsilon}{\rho_0^n}$$

and by Lemma 3.8 and Proposition 3.6 we have

(3.38)
$$\|v\|_{L^{\infty}(\mathcal{G}((0,T)))} \le \frac{C}{\rho_0^n}$$

where C depends on k only. It is easy to check that by (2.6) and (2.7) there exists $C, C \geq \tilde{C}$, depending on k, E, M such that for all $x \in (\Omega_{\rho_0} \setminus \Omega_D(\bar{t}))_{\lambda_1 h/2}$ and $0 < h \leq \frac{1}{C} \min\{\rho, \sqrt{\bar{t}}\},$

(3.39)
$$Q_{\lambda_1 h/4} := B_{\lambda_1 h/4}(x) \times \left(\overline{t} - \left(\frac{\lambda_1 h}{4}\right)^2, \overline{t}\right] \subset \tilde{\mathcal{G}}((-\infty, \overline{t}]).$$

Since v solves the heat equation, we can apply Theorem 3.10 along a chain of balls centered in points of γ . More precisely, let us define $\overline{\rho} = \eta_1 \lambda_1 h/12$, where $\eta_1 \in (0,1)$ is defined in Theorem 3.10, and x_i , $i = 1, \ldots, m_h$ as follows: $x_1 = \overline{x}$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t : |\gamma(t) - x_i| = 2\overline{\rho}\}$, if $|x_i - \overline{y}| > 2\overline{\rho}$, otherwise let $i = m_h$ and stop the process. We have $m_h \leq CM \left(\frac{\rho_0}{h}\right)^n$, where C > 0is an absolute constant. By construction the balls $B_{\overline{\rho}}(x_i)$ are pairwise disjoint and $|x_{i+1} - x_i| = 2\overline{\rho}$ for $i = 1, \ldots, m-1$ and $|x_m - \overline{x}| \leq 2\overline{\rho}$. By an iterated application of the two–sphere and one–cylinder inequality (Theorem 3.10) to vwith $R = \lambda_1 h/4$, $r_2 = 3\overline{\rho}$, $r_1 = \overline{\rho}$ over the chain of balls $B_{\overline{\rho}}(x_i)$, since we have $B_{r_1}(x_{i+1}) \subset B_{r_2}(x_i)$, $i = 1, \ldots, m_h$, by (3.37) and (3.38) we have

(3.40)
$$\left(\frac{1}{r_2^n} \int_{B_{r_2}(\bar{y})} v^2(\xi, \bar{t}) d\xi\right)^{1/2} \le \frac{C}{\rho_0^n} \varepsilon^{s_2^{m_h}} (C+\varepsilon)^{1-s_2^{m_h}},$$

where $s_2 \in (0, 1)$ is an absolute constant and C depends on a priori data only. From now on, in order to simplify the writing and since the case $\varepsilon \ge 1$ is trivial, we shall assume that $\varepsilon \in (0, 1)$. By standard regularity estimates [Li] and taking into account (3.37) and (3.38) we have

(3.41)
$$\|\nabla v(\cdot,\overline{t})\|_{L^{\infty}(B_{r_2/2}(\overline{y}))} \le \frac{C}{h\rho_0^n}$$

where C depends on k only. Recalling now the interpolation inequality (see [Al-Be-Ro-Ve, (5.29)])

$$(3.42) ||g||_{L^{\infty}(B_r)} \le C \left[||g||_{L^{\infty}(B_r)} + r ||\nabla g||_{L^{\infty}(B_r)} \right]^{\frac{n}{n+2}} \left(r^{-n} \int_{B_r} g^2 \right)^{\frac{1}{n+2}},$$

for every r > 0, where C is an absolute constant, by (3.41) and (3.40) we have

(3.43)
$$\|v(\cdot,\bar{t})\|_{L^{\infty}(B_{\frac{r_2}{2}}(\bar{y}))} \leq \frac{C}{\rho_0^n} \varepsilon^{s_2^{m_h}} := \omega_h^{(1)}(\varepsilon),$$

where C depends on the a priori data only. Now defining $w(y,s) = \mathcal{U}(y,s;\overline{y},\overline{t})$ and taking into account (3.43) we have

$$\begin{cases} \partial_s w(y,s) + \Delta_y w(y,s) = 0, & \text{in } \tilde{\mathcal{G}}((0,T)), \\ w(y,s)_{|\{s \ge \bar{t}\}} = 0, \\ |w(y,t)| \le \omega_h^{(1)}(\varepsilon), & (y,s) \in \Xi_{\rho_0} \times (0,T) \end{cases}$$

Now we want to estimate from above $|w(y_1, t_1)|$. In order to obtain such an estimate we argue as before, but here, instead of (3.38), we use the inequality

(3.44)
$$\|w\|_{L^{\infty}(\tilde{\mathcal{G}}^{(h)}([t_1,\overline{t}]))} \leq \frac{C}{h^n}$$

where $\tilde{\mathcal{G}}^{(h)}([t_1, \overline{t}]) = \{(x, t) \in \tilde{\mathcal{G}}((0, T)) : t_1 \leq t \leq \overline{t}, \operatorname{dist}(x, \Omega_D(t)) \geq \delta_2 h\}, \delta_2 = \frac{1}{8} \min\{\lambda_1, \lambda_2, \lambda_3\}$ and *C* depends on *k* only. Inequality (3.44) is a consequence of Proposition 3.6 and Lemma 3.8. Notice that by virtue of (3.24) and (3.25) we have $(y_1, t_1), (\overline{y}, \overline{t}) \in \tilde{\mathcal{G}}^{(h)}([t_1, \overline{t}])$. Therefore we obtain

(3.45)
$$|\mathcal{U}(y_1, t_1; \overline{y}, \overline{t})| = |w(y_1, t_1)| \le \frac{C}{h^n} \varepsilon^{\tilde{s}_2^{m_h}},$$

where $\tilde{s}_2 \in (0,1)$ is an absolute constant and C depends on the a priori data only. Now we introduce some notation. We set $\delta(\bar{t}) = \min\left\{\frac{\sqrt{\bar{t}}}{\rho_0}, 1\right\}$ and $h/\rho_0 = q$. By Proposition 3.9 and (3.45) we have that there exists $C_5, C_5 \ge 1$, depending on k, E and M only such that

(3.46)
$$1 \le C_5 \varepsilon^{s_3^{q^{-n}}}, \quad \text{for every } q \le \frac{1}{C_5} \min\left\{\frac{d_\mu(\bar{t})}{\rho_0}, \delta(\bar{t})\right\},$$

where $s_3, s_3 \in (0, 1)$, depends on M only. We distinguish two cases

i)
$$d_{\mu}(\overline{t}) \ge \min\{\sqrt{\overline{t}}, \rho_0\}$$
 ii) $d_{\mu}(\overline{t}) < \min\{\sqrt{\overline{t}}, \rho_0\}.$

If case i) occurs we choose q such that $s_3^{q^{-n}} = |\log \varepsilon|^{-1/2},$ that is

$$q = q_{\varepsilon} := \left(\frac{|\log s_3|}{|\log|\log \varepsilon|^{-1/2}|}\right)^{\frac{1}{n}}.$$

Denote by $\varepsilon_*(\bar{t})$ the least upper bound of the set $\{\varepsilon \in (0,1) : q_\varepsilon \leq \delta(\bar{t})\}$. By (3.46) we have

$$1 \le C_5 \exp\left\{-\left|\log\varepsilon\right|^{1/2}\right\},\,$$

which, for $0 < \varepsilon \le \varepsilon_{**}(t) := \min \left\{ \varepsilon_*(\bar{t}), e^{-(\log C_5)^2} \right\}$, yields to a contradiction. Thus, if $0 < \varepsilon \le \varepsilon_{**}(\bar{t})$, case i) cannot occur.

Let us consider now case ii), that is $d_{\mu}(\bar{t}) < \min\left\{\sqrt{\bar{t}}, \rho_0\right\}$. By (3.46) we have

$$1 \le C_5 \exp\left\{-s_3^{q^{-n}} \left|\log \varepsilon\right|\right\},\,$$

for every $q \leq \frac{C_5^{-1} d_\mu(\bar{t})}{\rho_0}$. Now, if

(3.47)
$$d_{\mu}\left(\overline{t}\right) \leq 2C_{5}\rho_{0} \left|\log\left(\left|\log\varepsilon\right|^{\frac{\left|\log s_{3}\right|^{-1}}{2}}\right)\right|^{-\frac{1}{n}}$$

then we are done. On the other side, if

(3.48)
$$d_{\mu}\left(\bar{t}\right) > 2C_{5}\rho_{0} \left|\log\left(\left|\log\varepsilon\right|^{\frac{\left|\log s_{3}\right|^{-1}}{2}}\right)\right|^{-\frac{1}{n}}$$

let us denote by

$$\widetilde{q}_{\varepsilon} = \left| \log \left(\left| \log \varepsilon \right|^{\frac{\left| \log s_{3} \right|^{-1}}{2}} \right) \right|^{-\frac{1}{n}}$$

and by $\varepsilon_0(\bar{t})$ the least upper bound of the set $\{\varepsilon \in (0, \varepsilon_*(\bar{t})) : \tilde{q}_{\varepsilon} \leq \delta(\bar{t})\}$. Now, for $0 < \varepsilon \leq \varepsilon_0(\bar{t})$, we choose $q = \tilde{q}_{\varepsilon}$ and by (3.46) we have

$$1 \le C_5 \exp\left\{-\left|\log\varepsilon\right|^{1/2}\right\}.$$

Since the last inequality yields to a contradiction whenever $0 < \varepsilon \leq \varepsilon_0(\bar{t})$, we have that if $0 < \varepsilon \leq \varepsilon_0(\bar{t})$ then (3.48) cannot occur, so inequality (3.47) holds true. Finally, by using Proposition 3.2 and 3.3 and Remark 3.4, we have

(3.49)
$$d_{\mathcal{H}}(\overline{D_1(\bar{t})}, \overline{D_2(\bar{t})}) \le 2C_5\rho_0 \left| \log\left(\left| \log \varepsilon \right|^{\frac{|\log s_3|^{-1}}{2}} \right) \right|^{-1/n} := \sigma(\varepsilon),$$

for $0 < \varepsilon \leq \varepsilon_0(\overline{t})$.

Step 2.

In order to prove estimate (2.9) we apply Proposition 3.5 to $\Omega_i := \Omega \setminus D_i(\bar{t})$, i = 1, 2, and $R_0 = \rho_0$. Indeed by (3.49) we have that, for ε small enough, Ω_1 and Ω_2 are relative graphs. More precisely if $0 < \varepsilon \leq \min\{\varepsilon_0(\bar{t}), \bar{d}\}$ (where \bar{d} is defined in Proposition 3.5) then there exists $r_0 > 0$ $(r_0/\rho_0 \leq 1$ depending on Eonly) such that

(3.50)
$$B_{r_0}(0) \cap D_i(\overline{t}) = \{ x \in B_{r_0}(0) : x_n > \varphi_i(x') \}, \quad i = 1, 2,$$

and $\|\varphi_1 - \varphi_2\|_{L^{\infty}(B'_{r_0})} \leq C\rho_0\sigma(\varepsilon)$, where *C* depends on *E* only. By (2.7a) and an interpolation inequality [Al-Be-Ro-Ve, (5.30)] we have that $\|\varphi_1 - \varphi_2\|_{C^1(B'_{r_0})} \leq C\rho_0(\sigma(\varepsilon))^{\overline{\beta}}, \overline{\beta} \in (0, 1)$. Thus, with eventually a rigid transform of coordinates, provided we pick a smaller r_0 , there exists $\varepsilon_0 > 0$, depending on *E* only, such that for $\varepsilon \leq \varepsilon_0$ we can assume that $|\nabla\varphi_1(0)| = 0$. In the sequel we continue to denote by $\varepsilon_0(\overline{t})$ the number $\min\{\varepsilon_0(\overline{t}), \varepsilon_0\}$. Let us define, for a unit vector ζ and $0 < \alpha < \pi/2$

$$\mathcal{C}(z,\zeta,\alpha,r_0) = \left\{ x \in B_{r_0}(z) : \frac{(x-z) \cdot \zeta}{|x-z|} > \cos \alpha \right\}.$$

By (3.50) we have that $\mathcal{C}(0,\nu,\alpha,r_0) \subset \mathcal{G}(\overline{t})$, where $\alpha, \alpha \in (0,\pi/2)$, depends on E only. Let us denote $\overline{\mu} = \frac{\rho_0}{1+\sin\alpha}$, $\delta^* = \frac{\sin\alpha}{\sqrt{2E+1}}$ and $\overline{\rho}_0 = \overline{\mu}\cos\frac{\alpha}{2}$. We have that $\mathcal{S}((0,\overline{t}),\nu,\frac{\alpha}{2},\delta^*,\overline{\rho}) \subset \tilde{\mathcal{G}}((-\infty,\overline{t}])$, where we set $\nu = \nu(0,\overline{t})$, for the sake of brevity, and

$$\mathcal{S}\left((0,\bar{t}),\nu,\frac{\alpha}{2},\delta^*,\bar{\rho}\right)$$
$$=\left\{(z,t)\in\mathbb{R}^{n+1}\,:\,z\in\mathcal{C}(0,\nu,\frac{\alpha}{2},\bar{\rho}),\,\bar{t}-\delta^*(x\cdot\nu)^2< t\leq\bar{t}\right\}$$

and $\nu = \nu(0, \bar{t})$. We want to estimate $v(\bar{y}, \bar{t}) = \mathcal{U}(y, s; \bar{y}, \bar{t})$ when $(y, s) \in \Xi_{\rho_0} \times (0, T), s < \bar{t}$, where v solves

(3.51)
$$\begin{cases} \partial_{\tau} v - \Delta_{\xi} v = 0, & \text{for } (\xi, \tau) \in \tilde{\mathcal{G}}((0, T)) \\ v(\xi, \tau)_{|\tau \leq s} = 0, \\ \|v\|_{L^{\infty}(\Xi_{\rho_0} \times (0, T))} \leq \tilde{\varepsilon}, \end{cases}$$

where $\tilde{\varepsilon} = C\varepsilon/\rho_0^n$, where C depends on k only. Let us fix (y, s) such that

(3.52)
$$(y,s) \in \tilde{\Xi}_{\rho_0} \times (0,T) = \{x \in \mathbb{R}^n : \frac{5}{8}\rho_0 \le d(x,\Omega) \le \frac{7}{8}\rho_0\} \times (0,T).$$

By Proposition (3.6) and Lemma (3.8), taking into account the last relation of (3.51) and by (3.52) we have

(3.53)
$$\|v\|_{L^{\infty}(\tilde{\mathcal{G}}(0,T))} \leq \frac{C}{\rho_0^n} + \tilde{\varepsilon} := H.$$

In the sequel we continue to denote by v the trivial extension of v. Notice that, by (3.53), we have

$$\|v\|_{L^{\infty}(\mathcal{S}((0,\overline{t}),\nu,\alpha/2,\delta^*,\overline{\rho})} \leq H.$$

Denote by

$$\alpha_1 = \arcsin\left(\min\left\{\sin\frac{\alpha}{2}, \delta^*\left(1-\sin\frac{\alpha}{2}\right)\right\}\right),$$
$$\mu_1 = \frac{\overline{\rho}}{1+\sin\alpha_1}, w_1 = z + \mu_1\nu, \rho_1 = \frac{1}{4}\mu_1\eta_1\sin\alpha_1,$$

where $\eta_1 \in (0, 1)$ is defined in Theorem 3.10. We have

(3.54)
$$\operatorname{dist}\left(w_{1},\partial\mathcal{G}\left(\overline{t}\right)\right) \geq \min\left\{\rho_{0}-\left|w_{1}-z\right|,\left|w_{1}-z\right|\sin\alpha\right\}=\rho_{0}\widetilde{\eta},$$

where

$$\widetilde{\eta} = \min\left\{1 - \frac{\cos\frac{\alpha}{2}}{1 + \sin\alpha} \frac{1}{1 + \sin\alpha_1}, \frac{\sin\alpha}{1 + \sin\alpha_1} \frac{\cos\frac{\alpha}{2}}{1 + \sin\alpha}\right\}.$$

Now $(\mathcal{G}(\bar{t}))_{\frac{\rho_0 \bar{\eta}}{2}}$ is connected and, by (3.54), $w_1 \in (\mathcal{G}(\bar{t}))_{\frac{\rho_0 \bar{\eta}}{2}}$. Therefore by an iterated application of the two-sphere and one-cylinder inequality (see also [Ve, Proposition 4.1.1]) we get

(3.55)
$$\left(\rho_1^{-n} \int_{B_{\rho_1(w_1)}} v^2\left(\xi, \bar{t}\right) d\xi\right)^{1/2} \le C\tilde{\varepsilon}^{s_4} H^{1-s_4},$$

where $s_4, s_4 \in (0, 1)$, is an absolute constant and C depends on the a priori data only. Denote

$$\mu_k = a^{k-1}\mu_1, \quad w_k = \mu_k \nu \quad \rho_k = a^{k-1}\rho_1$$
$$d_k = \mu_k - \rho_k = a^{k-1}\mu_1(1 - \frac{1}{4}\eta_1 \sin \alpha_1),$$

where $a = \frac{1 - \frac{1}{4}\eta_1 \sin \alpha_1}{1 + \frac{1}{4}\eta_1 \sin \alpha_1}$. For every $k \ge 1$, the following inclusions hold true

$$(3.56) B_{\rho_{k+1}}(w_{k+1}) \subset B_{3\rho_k}(w_k) \subset B_{4\eta_1^{-1}\rho_k}(w_k) \subset \mathcal{C}(0,\nu,\alpha_1,r_0)$$

(3.57)
$$B_{4\eta_1^{-1}\rho_k}(w_k) \times (\bar{t} - (4\eta_1^{-1}\rho_k)^2, \bar{t}] \subset \mathcal{S}((0,\bar{t}), \nu, \alpha, \delta^*, r_0).$$

Let us consider h defined in (3.22). We further assume $\lambda_1 h \in (0, d_1]$. Let \overline{k} be the smallest positive integer such that $d_k \leq \lambda_1 h$. We have

(3.58)
$$\frac{|\log(\lambda_1 h/d_1)|}{|\log a|} \le \overline{k} - 1 \le \frac{|\log(\lambda_1 h/d_1)|}{|\log a|} + 1.$$

Denote

$$\sigma_j = \left(\rho_0^{-n} \int_{B_{\rho_j}(w_j)} v^2\left(\xi, \bar{t}\right) d\xi\right)^{1/2}, \quad j = 1, ..., \bar{k}.$$

By the Theorem 3.10, (3.56), (3.57) and since

$$\sigma_{j+1} \le \left(\rho_0^{-n} \int_{B_{3\rho_j}(w_j)} v^2\left(\xi, \bar{t}\right) d\xi\right)^{1/2}, \quad j = 1, ..., \bar{k} - 1,$$

we obtain

(3.59)
$$\sigma_{j+1}^2 \le C_1 H^{2(1-\theta_*)} \sigma_j^{2\theta_*}, \quad j = 1, \dots, \overline{k} - 1,$$

where $\theta_* = \frac{1}{C_0 \log(4/\eta_1)}$. By iterating (3.59) we get

(3.60)
$$\sigma_{\overline{k}}^{2} \leq C^{\frac{1}{1-\theta_{*}}} H^{2(1-\theta_{*}^{\overline{k}})} \sigma_{1}^{2\theta_{*}^{\overline{k}}}.$$

By Lemma 3.8 and Proposition 3.6 we have

(3.61)
$$\|v\|_{L^{\infty}(Q_{\rho_{\overline{k}}})} \le \frac{C}{\rho_0^n}$$

where $Q_{\rho_{\overline{k}}} = B_{\rho_{\overline{k}}}(w_{\overline{k}}) \times (\overline{t} - \rho_{\overline{k}}^2, \overline{t}]$. By standard regularity estimate and (3.61) we get

(3.62)
$$\|\nabla v(\cdot, \overline{t})\|_{L^{\infty}(B_{\rho_{\overline{k}}/2}(w_{\overline{k}}))} \leq \frac{C}{\rho_{\overline{k}}\rho_{0}^{n}}.$$

Now by using interpolation inequality (3.42), (3.55) (3.60) and (3.62), taking into account that $\overline{y} \in B_{\rho_{\overline{k}}/2}(w_{\overline{k}})$, we have

(3.63)
$$|v(\overline{y},\overline{t})| \le C \left(\frac{\rho_0}{\rho_{\overline{k}}}\right)^{n/2} H\left(\left(\frac{\tilde{\varepsilon}}{H}\right)^{s_5}\right)^{\theta_*^{\overline{k}}},$$

where $\theta_* \in (0,1)$ and $s_5 \in (0,1)$ (absolute constant) and C depends on the a priori data only. Now evaluating \overline{k} in terms of h and recalling that $w(y,s) = \mathcal{U}(y,s;\overline{y},\overline{t}) = v(\overline{y},\overline{t})$ we have, for every $(y,s) \in \widetilde{\Xi}_{\rho_0} \times (0,T)$

(3.64)
$$\|w\|_{L^{\infty}(\tilde{\Xi}_{\rho_0}\times(0,T))} \leq \frac{C}{h^{n/2}} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{C}\left(\frac{h}{\rho_0}\right)^{\frac{|\log \theta_*|}{|\log a|}}}$$

where C depends on the a priori data only. Arguing as above to estimate $|w(y_1, t_1)|$ and recalling that $w(y_1, t_1) = \mathcal{U}(y_1, t_1; \overline{y}, \overline{t})$ we have

,

(3.65)
$$|\mathcal{U}(y_1, t_1; \overline{y}, \overline{t})| \le \frac{C}{h^n} \varepsilon^{\frac{1}{C} \left(\frac{h}{\rho_0}\right)^B},$$

where C and B depend on the a priori data only. Finally, using Proposition 3.9 and proceeding as in Step 1 we obtain (2.10).

4 Proof of the Auxiliary Results

4.1 **Proof of Proposition 3.3**

Proof of Proposition 3.3. We recall that, for a given a subset A of \mathbb{R}^n , we denote by $[A]_{\varepsilon} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leq \varepsilon\}$, $(A)_{\varepsilon} = \{x \in A : \operatorname{dist}(x, \partial A) > \varepsilon\}$ and $[\partial A]_{\varepsilon} = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial A) \leq \varepsilon\}$. We remark that $[A]_{\varepsilon} \setminus (A)_{\varepsilon} = [\partial A]_{\varepsilon}$ and $(A)_{\varepsilon} \subset A \subseteq [A]_{\varepsilon}$.

Let $d := d_{\mathcal{H}}(\overline{D}_1, \overline{D}_2)$ and $r := d_{\mathcal{H}}(\partial D_1, \partial D_2)$. If d = 0 then (3.9) holds trivially. Assume d > 0. Without loss of generality, we can assume that there exists $\overline{x} \in \overline{D}_1$ such that $d = \operatorname{dist}(\overline{x}, \overline{D}_2)$. Since d > 0 we have that $\overline{x} \notin \overline{D}_2$ and therefore $d = \operatorname{dist}(\overline{x}, \partial D_2)$. If $\overline{x} \in \partial D_1$, then (3.9) is trivially true. Assume $\overline{x} \in D_1 \setminus \overline{D}_2$. We have for every $x \in \overline{D}_1 \setminus \overline{D}_2$

$$\operatorname{dist}(x, \partial D_2) = \operatorname{dist}(x, \overline{D}_2) \le \operatorname{dist}(\overline{x}, \overline{D}_2) = \operatorname{dist}(\overline{x}, \partial D_2).$$

Thus for every $x \in \overline{D}_1 \setminus \overline{D}_2$ we have

$$\operatorname{dist}(x, \partial D_2) \le \operatorname{dist}(\overline{x}, \partial D_2),$$

that is \overline{x} is a maximum point in the set $\overline{D}_1 \setminus \overline{D}_2$ for the function dist $(\cdot, \partial D_2)$. In the set $\tilde{A} = \text{Int} \left([\partial D_2]_{\rho_0/E} \right) \setminus \partial D_2$, the function dist $(\cdot, \partial D_2)$ is C^2 and

(4.1)
$$|\nabla_x \operatorname{dist}(x, \partial D_2)| > 0 \quad \forall x \in \tilde{A}$$

Since \overline{x} is a maximum point and $\overline{x} \notin \partial D_2$, by (4.1) we have

(4.2)
$$\operatorname{dist}(\overline{x}, \partial D_2) \ge \frac{\rho_0}{E}$$

Otherwise, recalling that x is a maximum point of dist $(x, \partial D_2)$ interior to $D_1 \setminus \overline{D}_2$, if dist $(\overline{x}, \partial D_2) < \rho_0/E$ we should have $\nabla_x \text{dist}(x, \partial D_2) = 0$ contradicting (4.1). First let us assume r be such that

(4.3)
$$r < \min\left\{\frac{\rho_0}{E}, \frac{\rho_0}{2}\right\}.$$

We can write $\mathbb{R}^n = (D_2)_r \cup [\partial D_2]_r \cup (\mathbb{R}^n \setminus [D_2]_r)$. By (4.2) and (4.3) we have $\overline{x} \notin [\partial D_2]_r$. Since $(D_2)_r \subset D_2$ and $\overline{x} \notin \overline{D}_2$ we have that $\overline{x} \in \mathbb{R}^n \setminus [D_2]_r$. Recalling that $r < \rho_0/2$ and $\mathbb{R}^n \setminus D_2$ is connected, we have that $\mathbb{R}^n \setminus [D_2]_r$ is connected. Thus there exists a continuous path

$$\gamma : [0,1) \to \mathbb{R}^n$$

such that

(4.4a)
$$\gamma([0,1)) \subset \mathbb{R}^n \setminus [D_2]_r,$$

(4.4b)
$$\gamma(0) = \overline{x} \qquad \lim_{t \to 1^{-}} \gamma(t) = \infty$$

Since $d_{\mathcal{H}}(\partial D_1, \partial D_2) = r$ and $\partial D_1 \subset [\partial D_2]_r \subset [D_2]_r$, by (4.4a) we have

$$\gamma([0,1)) \cap \partial D_1 = \emptyset$$

which is a contradiction since $\overline{x} \in D_1$ and D_1 is bounded. Thus we cannot connect \overline{x} and ∞ with a path that does not intersect ∂D_1 . Hence $\overline{x} \in \partial D_1$. Thus if $d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \delta \rho_0$, with $\delta = \min\left\{\frac{1}{E}, \frac{1}{2}\right\}$, (4.3) is satisfied and we have

$$d_{\mathcal{H}}(\overline{D}_1, \overline{D}_2) = \operatorname{dist}(x, \partial D_2) \le r.$$

On the other side, if $d_{\mathcal{H}}(\partial D_1, \partial D_2) > \delta \rho_0$ we have trivially

(4.5)
$$d_{\mathcal{H}}(\overline{D}_1, \overline{D}_2) \le 2 \operatorname{diam}(\Omega) \le \frac{2 \operatorname{diam}(\Omega)}{\delta \rho_0} d_{\mathcal{H}}(\partial D_1, \partial D_2)$$

and the proposition is proven.

We shall make use of the following regularity theorem, whose proof can be found in [La-Ri-Ur], [La-So-Ur, Ch. III, Sec. 13].

Theorem 4.1 Let λ , M and r be positive numbers with $\lambda \in (0,1]$. Let $u \in H^{1,\frac{1}{2}}(B'_r \times (-r,r) \times (-r^2,r^2))$ be solution to

(4.6)
$$\operatorname{div}\left(A(x,t)\nabla_{x}u\right) + b(x,t)\cdot\nabla_{x}u - \partial_{t}u = 0,$$

where A(x,t) and b(x,t) are respectively a symmetric $n \times n$ matrix and a vector valued function satisfying the following conditions

(4.7a)
$$\lambda |\xi|^2 \le A(x,t)\xi \cdot \xi \le \lambda^{-1} |\xi|^2$$

for all $(x,t) \in B'_r \times (-r,r) \times (-r^2,r^2)$ and for all $\xi \in \mathbb{R}^n$,

(4.7b)
$$r \sum_{i=1}^{n} \|\partial_{i}A\|_{L^{\infty}(B'_{r} \times (-r,0) \times (-r^{2},r^{2}))} + r \sum_{i=1}^{n} \|\partial_{i}A\|_{L^{\infty}(B'_{r} \times (0,r) \times (-r^{2},r^{2}))} r^{2} \|\partial_{t}A\|_{L^{\infty}(B'_{r} \times (-r,r) \times (-r^{2},r^{2}))} \le M_{2}$$

(4.7c)
$$r \|b\|_{L^{\infty}(B'_r \times (-r,r) \times (-r^2,r^2))} \le M.$$

Then there exist positive constants $\beta \in (0,1)$ and C such that for every $\rho < \frac{r}{2}$ and all $(x,t) \in B'_{r-2\rho} \times (-(r-2\rho), (r-2\rho)) \times (-r^2 + 4\rho^2, r^2)$ the following inequality holds

$$(4.8) \qquad \rho \| \nabla_{x} u \|_{L^{\infty}(B'_{\rho}(x') \times (-\rho + x_{n}, \rho + x_{n}) \times (-\rho^{2} + t, t))} + \rho^{\beta + 1} [\nabla_{x} u]_{\beta;(B'_{\rho}(x') \times (-\rho + x_{n}, \rho + x_{n}) \times (-\rho^{2} + t, t)) \cap (B'_{\rho} \times (-r, 0) \times (-r^{2}, r^{2}))} + \rho^{\beta + 1} [\nabla_{x} u]_{\beta;(B'_{\rho}(x') \times (-\rho + x_{n}, \rho + x_{n}) \times (-\rho^{2} + t, t)) \cap (B'_{\rho} \times (0, r) \times (-r^{2}, r^{2}))} \\ \leq \frac{C}{\rho^{\frac{n}{2} + 1}} \left\{ \int_{B'_{2\rho}(x') \times (-2\rho + x_{n}, 2\rho + x_{n}) \times (-4\rho^{2} + t, t)} u^{2}(\xi, \tau) d\xi d\tau \right\}^{1/2}.$$

Here β depends on n only and C depends on λ , M and n only.

Before proving Proposition 3.6 we give the following estimate which is needed in the proof. We recall that $Q_{\rho}(x_0, t_0) = B_r(x_0) \times (t_0 - \rho^2, t_0)$.

Proposition 4.2 For every δ_1 , $0 < \delta_1 < 1$, there exist constant $C \ge 1$ depending on k, δ_1 and n only such that the following inequality holds.

(4.9)
$$\int_{Q_{\rho}(x_{0},t_{0})} |\Gamma(x,t;\xi,\tau)|^{2} dx \, dt \leq C \frac{\rho^{n}}{(t_{0}-\tau)^{n-1}} e^{-\frac{|x_{0}-\xi|^{2}}{C(t_{0}-\tau)}},$$

where $\rho = \delta_1 [|x_0 - \xi|^2 + t_0 - \tau]^{1/2}$.

Proof. From the inequality (3.13) we have (4.10)

$$\int_{Q_{\rho}(x_{0},t_{0})}^{\cdot} |\Gamma(x,t;\xi,\tau)|^{2} dx \, dt \leq C \int_{Q_{\rho}(x_{0},t_{0})} \frac{1}{(t-\tau)^{n}} \mathrm{e}^{-\frac{|x-\xi|^{2}}{C_{1}(t-\tau)}} \chi_{[\tau,+\infty)} dx \, dt,$$

where C_1 depends on k and n only. In what follows we denote by I the integral at the right-hand side of (4.10). We distinguish two cases

i)
$$t_0 - \rho^2 < \tau < t_0$$

ii) $\tau < t_0 - \rho^2$.

Let us consider case i). It is easy to see that there exists an absolute constant $C \geq 1$ such that

(4.11)
$$C^{-1}\rho \le |x-\xi| \le C\rho \qquad \forall x \in B_{\rho}(x_0)$$

By (4.11) we have

(4.12)
$$I \le c_n \rho^n \int_0^{t_0 - \tau} s^{-n} \mathrm{e}^{-\frac{\rho^2}{C_{2s}}} ds$$

where c_n is an absolute constant depending on n only and C_2 depends on k and n only. Now if $0 < t_0 - \tau < \frac{\rho^2}{nC_2}$, being $s \to s^{-n} e^{-\frac{\rho^2}{C_2 s}}$ an increasing function in $(0, \frac{\rho^2}{nC_2})$, by (4.12) we get

(4.13)
$$I \le \frac{\rho^n}{(t_0 - \tau)^{n-1}} e^{-\frac{\rho^2}{C(t_0 - \tau)}}.$$

Otherwise, if $\frac{\rho^2}{nC_2} < t_0 - \tau < \rho^2$ then since

$$\max_{(0,+\infty)} \{ s^{-n} \mathrm{e}^{-\rho^2/(C_2 s)} \} = \frac{(nC_2)^n}{\rho^{2n}} \mathrm{e}^{-1/n}$$

and now $t - \tau$ is of the same order of ρ^2 we have

$$\rho^{-n}I \le C_1 \int_0^{\rho^2} s^{-n} \mathrm{e}^{-\frac{\rho^2}{C_2 s}} ds \le \frac{C}{(t_0 - \tau)^{n-1}} \mathrm{e}^{-\frac{\rho^2}{C_2(t_0 - \tau)}}.$$

By the last inequality and (4.13) we get the Proposition in case i). Let us consider now case ii). It is easy to see that

(4.14)
$$6\rho^2 \le |x - \xi|^2 + t - \tau \le 60\rho^2,$$

for every $(x,t) \in Q_{\rho}(x_0,t_0)$. Moreover, denoting

$$M_{\rho} = \max\left\{\frac{\mathrm{e}^{-\frac{|x-\xi|^2}{C_1(t-\tau)}}}{(t-\tau)^n} : (x,t) \in Q_{\rho}(x_0,t_0)\right\}$$

and taking into account (4.14) we get

(4.15)
$$M_{\rho} \le C \left(\frac{C_1}{\rho^2}\right)^n,$$

where C depends on n only. Now, since $\tau < t_0 - \rho^2$ we have

(4.16)
$$\frac{|x_0 - \xi|^2}{t_0 - \tau} \le 4.$$

Therefore by (4.15) and (4.16) we get the Proposition in case ii) as well.

Proof of Proposition 3.6. Let U be a solution of the equation $\mathcal{L}U = 0$, where $\mathcal{L} = \partial_t - \operatorname{div}(1 + (k-1)\chi_Q \nabla)$. We recall the following regularity estimate (see [La-So-Ur])

(4.17)
$$\|\nabla U\|_{L^{\infty}(Q_{r}^{\pm}(\overline{x},\overline{t}))} \leq \frac{C}{r^{\frac{n+4}{2}}} \left(\int_{Q_{2r}(\overline{x},\overline{t})} U^{2}(x,t) dx dt \right)^{1/2},$$

where $Q_r^+(\overline{x},\overline{t}) = Q_r(\overline{x},\overline{t}) \cap Q$ (we recall $Q = D(\mathbb{R})$) and $Q_r^-(\overline{x},\overline{t}) = Q_r(\overline{x},\overline{t}) \setminus Q^+(\overline{x},\overline{t})$. Applying (4.17) to the function $\Gamma(\cdot,\cdot;\xi,\tau)$ we get

$$(4.18) \quad \|\nabla\Gamma(\cdot,\cdot;\xi,\tau)\|_{L^{\infty}(Q^{\pm}_{\rho}(x_{0},t_{0}))} \leq \frac{C}{\rho^{\frac{n+4}{2}}} \left[\int_{Q_{2\rho}(x_{0},t_{0})} |\Gamma(x,t;\xi,\tau)|^{2} dx dt \right]^{1/2},$$

where

(4.19)
$$\rho = \frac{1}{4} \left[|x_0 - \xi|^2 + t_0 - \tau \right]^{1/2}.$$

Applying Proposition 4.2 to the right hand side of (4.18) we have

$$\left\|\nabla\Gamma(\cdot,\cdot;\xi,\tau)\right\|_{L^{\infty}(Q^{\pm}_{\rho}(x_{0},t_{0}))} \leq \frac{C}{\rho^{\frac{n+4}{2}}} \left[\frac{\rho^{n}}{(t_{0}-\tau)^{n-1}} e^{-\frac{|x_{0}-\xi|^{2}}{C(t_{0}-\tau)}}\right]^{1/2}.$$

Since

$$\frac{1}{16\rho^2} \le \frac{1}{t_0 - \tau}$$

we obtain (3.14).

In order to state the next theorem we introduce some notations. Let φ : $B'_{\rho_0} \times (-\rho_0^2, \rho_0^2) \to \mathbb{R}$ such that it is differentiable with respect to t and x_i , $i = 1, \ldots, n-1$, it is twice differentiable with respect to x_i , $i = 1, \ldots, n-1$, and $\partial_t \varphi$ is differentiable with respect to x_i , $i = 1, \ldots, n-1$. We assume that

(4.20)
$$\varphi(0,0) = |\nabla_{x'}\varphi(0,0)| = 0$$

$$(4.21) \quad \rho_0^2 \| D_{x'}^2 \varphi \|_{L^{\infty}(B_{\rho_0}^{\prime} \times (-\rho^2, \rho^2))} + \rho_0^2 \| \partial_t \varphi \|_{L^{\infty}(B_{\rho_0}^{\prime} \times (-\rho^2, \rho^2))} + \rho_0^3 \| \partial_t \nabla_{x'} \varphi \|_{L^{\infty}(B_{\rho_0}^{\prime} \times (-\rho^2, \rho^2))} \le E\rho_0.$$

We shall denote by

$$\tilde{Q}_{\varphi,\rho_0}^+ = \{ x \in B_{\rho_0} \times (-\rho_0^2, \rho_0^2) : x_n > \varphi(x', t) \},\$$

and by $\Gamma_{\tilde{Q}^+_{\varphi,\rho_0}}(x,t;y,s)$ the fundamental solution of the operator $\partial_t - \operatorname{div}((1+(k-1)\chi_{\tilde{Q}^+_{\varphi,\rho_0}}\nabla))$, that is

$$\partial_t \Gamma_{\tilde{Q}^+_{\varphi,\rho_0}}(x,t;y,s) - \operatorname{div}\left((1+(k-1)\chi_{\tilde{Q}^+_{\varphi,\rho_0}})\nabla\Gamma_{\tilde{Q}^+_{\varphi,\rho_0}}(x,t;y,s)\right) = -\delta(x-y,t-s),$$

where $(y,s) \in \mathbb{R}^{n+1}$.

Theorem 4.3 (Asymptotic Estimate) Let φ and $\Gamma_{\bar{Q}^+_{\varphi,\rho_0}}(x,t;y,s)$ as above. Then there exists a constant $C \geq 1$ depending on n and E only such that

(4.22)
$$\left|\Gamma_{\tilde{Q}^+_{\varphi,\rho_0}}(x,t;y,0) - \Gamma_+(x,t;y,0)\right| \le C \frac{[|x-y|^2+t]^{1/2}}{\rho_0} \frac{\mathrm{e}^{-\frac{|x-y|^2}{Ct}}}{t^{n/2}},$$

(4.23)
$$\left| \nabla_x \Gamma_{\tilde{Q}_{\varphi,\rho_0}}(x,t;y,0) - \nabla_x \Gamma_+(x,t;y,0) \right|$$

 $\leq C \frac{[|x-y|^2+t]^{\frac{1}{2}(-1+\frac{\beta}{\beta+1})}}{\rho_0^{\frac{\beta}{1+\beta}}} \frac{\mathrm{e}^{-\frac{|x-y|^2}{Ct}}}{t^{n/2}},$

where β is the one defined in Theorem 4.1, depending on n only, for all

$$(x,t) \in \tilde{Q}_{\varphi,\frac{\rho_0}{C}} \cap \left\{ (x,t) \in \mathbb{R}^{n+1} : t > 0, \, x_n > \frac{1}{C\rho_0} (|x'|^2 + t) \right\}$$

and $y = y_n e_n, y_n \in (-\rho_0/C, 0).$

Remark 4.4 Theorem 4.3 provides an asymptotic estimate for the fundamental solution $\Gamma(x, t; y, s)$ when (x, t) and (y, s) stay on opposite sides of the interface (given by the graphic $x_n = \varphi(x', t)$). Our crucial requirement is that (y, s) approaches the interface in a nontangential way.

Proof. of Theorem 4.3. Let θ be a C^{∞} function on \mathbb{R} such that $0 \leq \theta \leq 1$, $\theta(s) = 0$, for every $s \in \mathbb{R} \setminus (-2, 2)$, $\theta(s) = 1$ for every $s \in (-1, 1)$ and $|\theta'(s)| \leq 2$ for every $s \in \mathbb{R}$.

We define new variables by $(\xi, \tau) = \Psi(x, t)$, where $\Psi(x, t) = (\Phi(x, t), t)$ and

$$\begin{cases} \xi' = x', \\ \xi_n = x_n - \varphi(x', t) \theta\left(\frac{|x'|}{r_1}\right) \theta\left(\frac{x_n}{r_1}\right) \theta\left(\frac{t}{r_1^2}\right), \\ \tau = t, \end{cases}$$

and

where $r_1 = \rho_0 \min\{\frac{1}{4}, \frac{1}{32E}\}.$

Sometimes, for the sake of brevity for a fixed $t \in (-\rho_0^2, \rho_0^2)$ we denote by $\Phi^{(t)}(\cdot)$ the map $\Phi(\cdot, t)$ and by $G^{(t)}$ the graph of $\varphi(\cdot, t)$. It is not difficult to check that Ψ and Φ have the same regularity properties of φ and they are diffeomorphisms (that preserve orientation) of \mathbb{R}^{n+1} and \mathbb{R}^n respectively. We denote by $\Phi^{-1}(\cdot, t)$ the inverse of $\Phi^{(t)}(\cdot)$. The following properties hold:

(4.24a)
$$\Phi^{(t)}\left(G^{(t)}\cap (B'_{r_1}\times (-r_1,r_1))\right) = \{x\in B'_{r_1}\times (-r_1,r_1): x_n=0\},\$$

(4.24b)
$$\Psi(x,t) = (x,t),$$

$$\forall (x,t) \in \mathbb{R}^{n+1} \setminus \left(\left(B'_{2r_1} \times (-2r_1, 2r_1) \right) \times (-2r_1^2, 2r_1^2) \right),$$

(4.24c) $C^{-1}|x_1 - x_2| \le |\Phi^{(t)}(x_1) - \Phi^{(t)}(x_2)| \le C|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$

(4.24d)
$$|\Phi^{(t)}(x) - x| \le \frac{C}{\rho_0} |x|^2, \quad \forall x \in \mathbb{R}^n,$$

(4.24e)
$$|D_x \Phi^{(t)}(x) - I| \le \frac{C}{\rho_0} |x|, \quad \forall x \in \mathbb{R}^n,$$

where $C, C \geq 1$, depends on E only, I denotes the identity matrix and $D_x \Phi^{(t)}$ is the jacobian matrix with respect to variable x. For $y_n \in (-\frac{r_1}{2}, 0)$ and $\sigma \in (-r_1^2, r_1^2)$ we denote $y = y_n e_n$ and $\eta = \Phi(y, \sigma)$. Furthermore we shall use the following notation $\tilde{\Gamma}(\xi, \tau; \eta, \sigma) = \Gamma(\Psi^{-1}(\xi, \tau); \Psi^{-1}(\eta, \sigma))$, and $\gamma(\xi, \tau) = \det J(\xi, \tau)$, where $J(\xi, \tau) = (D_x \Phi)(\Psi^{-1}(\xi, \tau))$. We have that $\tilde{\Gamma}(\xi, \tau; \eta, \sigma)$ is a solution to

(4.25)
$$\operatorname{div}\left(\tilde{B}(\xi,\tau)\nabla_{\xi}\tilde{\Gamma}\right) + C(\xi,\tau)\nabla_{\xi}\tilde{\Gamma} - \partial_{\tau}\tilde{\Gamma} = -\gamma(\eta,\sigma)\delta(\xi-\eta,\tau-\sigma),$$

where $\tilde{B}(\xi,\tau) = (1 + (k-1)\chi^+)B(\xi,\tau), \ B(\xi,\tau) = (J(\xi,\tau))(J(\xi,\tau))^*$ and $C(\xi,\tau) = J(\xi,\tau)\frac{\partial\Phi^{-1}(\xi,\tau)}{\partial\tau} - \frac{\tilde{B}(\xi,\tau)}{\gamma(\xi,\tau)}\nabla_{\xi}\gamma(\xi,\tau).$

Since we want to study the asymptotic behaviour of $\tilde{\Gamma}(\xi, \tau; \eta, 0)$, we shall denote $\tilde{\Gamma}(\xi, \tau; \eta, 0)$ by $\tilde{\Gamma}(\xi, \tau; \eta)$.

By (4.24), we have that

(4.26)
$$B(0,0) = I$$
 and $||B||_{L^{\infty}(\Omega \times (0,T))} + \rho_0[B]_{1,\Omega \times (0,T)} \le C$,

where C depends on E only. Denote by

(4.27)
$$R(\xi,\tau;\eta) = \Gamma(\xi,\tau;\eta) - \gamma(\eta,0)\Gamma_{+}(\xi,\tau;\eta),$$

where $\Gamma_+(\xi,\tau;\eta) = \Gamma_+(\xi,\tau;\eta,0)$ is the fundamental solution to the operator div $((1+(k-1)\chi^+)\nabla_{\xi}) - \partial_{\tau}$. We have

$$\operatorname{div}\left((1+(k-1)\chi^{+})\nabla_{\xi}R\right) - \partial_{\tau}R = F(\xi,\tau;\eta),$$

where

$$F(\xi,\tau;\eta) = -C(\xi,\tau)\nabla_{\xi}\tilde{\Gamma}(\xi,\tau;\eta) + \operatorname{div}\left((1+(k-1)\chi^{+})(I-B(\xi,\tau))\nabla_{\xi}\tilde{\Gamma}(\xi,\tau;\eta)\right),$$

notice that $F(\xi, \tau; \eta) = 0$ if $\tau < 0$,

(4.28)
$$R(\xi, \tau; \eta) = 0, \text{ for } \tau < 0.$$

Therefore, [Ar]

$$R(\xi,\tau;\eta) = \int_0^\tau \int_{B_{2r_1}} F(\zeta,s;\eta) \Gamma_+(\xi,\tau;\zeta,s) d\zeta ds, \quad \text{if } \tau > 0.$$

We have

(4.29) $|R(\xi, \tau; \eta)| \le J_1 + J_2,$

where

(4.30a)
$$J_1 = \left| \int_0^\tau \int_{B_{2r_1}} C(\zeta, s) \nabla_{\zeta} \tilde{\Gamma}(\zeta, s; \eta) \Gamma_+(\xi, \tau; \zeta, s) d\zeta ds \right|$$

and

(4.30b)
$$J_2 = \left| \int_0^\tau \int_{B_{2r_1}} \left(1 + (k-1)\chi^+ \right) \left(I - B(\zeta, s) \right) \right. \\ \left. \times \nabla_{\zeta} \tilde{\Gamma}(\zeta, s; \eta) \cdot \nabla_{\zeta} \Gamma_+(\xi, \tau; \zeta, s) d\zeta ds \right|.$$

By Proposition 3.6 and Lemma 3.8 we have

(4.31)
$$J_1 \le \frac{C}{\rho_0 \tau^{\frac{n-1}{2}}} e^{-\frac{|\xi - \eta|^2}{C_1 \tau}}.$$

where $C, C_1, C \ge 1, C_1 \ge 1$, depend on E only. By Proposition 3.6 and (4.26) we get

$$(4.32) J_2 \le J_{2,1} + J_{2,2},$$

with

(4.33)
$$J_{2,1} = \frac{C}{\rho_0} \int_0^\tau \int_{\mathbb{R}^n} s^{-\frac{n}{2}} e^{-\frac{|\zeta - \eta|^2}{C_1 s}} (\tau - s)^{-\frac{n+1}{2}} e^{-\frac{|\xi - \zeta|^2}{C_1 (\tau - s)}} d\zeta ds,$$

and

(4.34)
$$J_{2,2} = \frac{C}{\rho_0} \int_0^\tau \int_{\mathbb{R}^n} |\zeta| s^{-\frac{n+1}{2}} e^{-\frac{|\zeta-\eta|^2}{C_1s}} (\tau-s)^{-\frac{n+1}{2}} e^{-\frac{|\xi-\zeta|^2}{C_1(\tau-s)}} d\zeta ds,$$

where $C, C_1, C \ge 1, C_1 \ge 1$, depend on E only. By Lemma 3.8 we obtain

(4.35)
$$J_{2,1} \le C \frac{\tau^{-\frac{n}{2} + \frac{1}{2}}}{\rho_0} \mathrm{e}^{-\frac{|\xi - \eta|^2}{C_1 \tau}}.$$

Let us consider now $J_{2,2}$. Performing a change of variables we get

$$J_{2,2} = \frac{C}{\rho_0} \left(\frac{\sqrt{C_1}}{2}\right)^n \frac{e^{-\frac{|\xi-\eta|^2}{C_1\tau}}}{\tau^{n/2}} \\ \times \int_0^1 \int_{\mathbb{R}^n} \left|\frac{C_1}{2} (\tau(1-\lambda)\lambda)^{1/2} z + \lambda(\xi-\eta) + \xi\right| \frac{e^{-|z|^2}}{\sqrt{(1-\lambda)\lambda}} dz d\lambda \\ \leq \frac{C}{\rho_0} \frac{e^{-\frac{|\xi-\eta|^2}{C_1\tau}}}{\tau^{n/2}} \left[(|\xi|^2 + \tau)^{1/2} + |\xi-\eta| \right],$$

where C depends on E only. Now, denoting by $C_2 = \max_{s \in (0, +\infty)} s^{1/2} e^{-\frac{1}{2C_1 s}}$, we have

$$\frac{\mathrm{e}^{-\frac{|\xi-\eta|^2}{C_1\tau}}}{\tau^{n/2}}|\xi-\eta| = \frac{1}{\tau^{\frac{n-1}{2}}} \left(\frac{|\xi-\eta|^2}{\tau}\right)^{1/2} \mathrm{e}^{-\frac{|\xi-\eta|^2}{2C_1\tau}} \mathrm{e}^{-\frac{|\xi-\eta|^2}{2C_1\tau}} \le \frac{C_2}{\tau^{\frac{n-1}{2}}} \mathrm{e}^{-\frac{|\xi-\eta|^2}{2C_1\tau}}.$$

Thus

(4.36)
$$J_{2,2} \le \frac{C}{\rho_0} \frac{\mathrm{e}^{-\frac{|\xi-\eta|^2}{2C_1\tau}}}{\tau^{n/2}} (|\xi|^2 + \tau)^{1/2}.$$

Now since $\eta = e_n \eta_n$, $\eta_n < 0$ and $\xi_n > 0$, we have $|\xi - \eta|^2 = |\xi|^2 - 2\eta_n \xi_n + |\eta|^2 \ge |\xi|^2$. Such an inequality and (4.28), (4.31), (4.35), (4.36) give

(4.37)
$$|R(\xi,\tau;\eta)| \le \frac{C}{\rho_0} \chi_{\mathbb{R}^n \times [0,+\infty)} \frac{\mathrm{e}^{-\frac{|\xi-\eta|^2}{C\tau}}}{\tau^{n/2}} (|\xi-\eta|^2+\tau)^{1/2},$$

for every $\xi \in B_{2r_1}^+$ and $\tau \in (0, 4r_1^2)$, where $C, C \ge 1$, depends on E only. Let δ_1 be the constant defined in Proposition 4.2 ($\delta_1 \in (0, 1)$) and, for fixed $\overline{\xi} \in B_{r_1/8}^+$, $\eta_n \in (-r_1/8, 0), \ \eta = e_n \eta_n, \ \overline{\tau} \in (0, (r_1/8)^2)$ denote by

$$h = \frac{\delta_1}{4} [|\overline{\xi} - \eta|^2 + \overline{\tau}]^{1/2}.$$

We have

$$\operatorname{div}\left(\tilde{B}(\xi,\tau)\nabla_{\xi}\tilde{\Gamma}\right) + C(\xi,\tau)\nabla_{\xi}\tilde{\Gamma} - \partial_{\tau}\tilde{\Gamma} = 0,$$

in $B'_{h/2}(\overline{\xi}') \times (\overline{\xi}_n - h/2, \overline{\xi}_n + h/2) \times (\overline{\tau} - (h/2)^2, \overline{\tau}]$, where \tilde{B} and C are defined above. Therefore by Theorem 4.1 and Proposition 4.2 we get

(4.38)
$$[\nabla_{\xi} \tilde{\Gamma}(\cdot, \overline{\tau}, \eta)]_{\beta,Q} \le \frac{C}{h^{2+\beta}} \frac{1}{\overline{\tau}^{\frac{n-1}{2}}} \mathrm{e}^{-\frac{|\overline{\xi}-\eta|^2}{C\overline{\tau}}},$$

where where $Q = B'_{h/4}(\overline{\xi}') \times (\overline{\xi}_n, \overline{\xi}_n + h/4)$. Since a similar inequality holds true for $\nabla_{\xi} \Gamma_+(\cdot, \overline{\tau}; \eta)$, by (4.27) we obtain

(4.39)
$$[\nabla_{\xi} R(\cdot, \overline{\tau}; \eta)]_{\beta;Q} \leq \frac{C}{h^{2+\beta}} \frac{1}{\overline{\tau}^{\frac{n-1}{2}}} \mathrm{e}^{-\frac{|\overline{\xi}-\eta|^2}{C\overline{\tau}}}.$$

In (4.38) and (4.39), $C,\,C\geq 1,$ depends on E only. Now we recall the following interpolation inequality

(4.40)
$$\|\nabla f\|_{L^{\infty}(Q)} \le C\left(\|f\|_{L^{\infty}(Q)}^{\frac{\beta}{1+\beta}} |\nabla f|_{\beta;Q}^{\frac{1}{1+\beta}} + \frac{1}{h}\|f\|_{L^{\infty}(Q)}\right).$$

Since (4.37) easily yields

(4.41)
$$\|R(\cdot,\overline{\tau};\eta)\|_{L^{\infty}(Q)} \leq \frac{C}{\rho_0} \frac{\mathrm{e}^{-\frac{|\xi-\eta|^2}{C\overline{\tau}}}}{\overline{\tau}^{n/2}}h,$$

where $C, C \ge 1$, depends on E only, we obtain by (4.39) and (4.40)

(4.42)
$$|\nabla_{\xi} R(\overline{\xi}, \overline{\tau}; \eta)| \leq \frac{C}{\rho_0} \left(\frac{h}{\rho_0}\right)^{-1 + \frac{\beta}{\beta+1}} \frac{\mathrm{e}^{-\frac{|\overline{\xi}-\eta|^2}{C_1 \overline{\tau}}}}{\overline{\tau}^{n/2}},$$

for every $\overline{\xi} \in B_{r_1/8}^+$, $\eta = e_n \eta_n$, $\eta_n \in (-r_1/8, 0)$, $\overline{\tau} \in (0, (r_1/8)^2]$, where $C, C \ge 1$, depends on E only.

Let us go back to the original coordinates (x, t). First of all let us estimate the function g defined by

(4.43)
$$g(x,t;y) := R(\Phi^{(t)}(x),t;\Phi^{(0)}(y)) = R(\Phi^{(t)}(x),t;e_ny_n).$$

To carry out the estimates, up to the end of the proof, we always consider xand y_n such that $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, where $\delta, \delta \in (0, 1)$, may change from line to line, but it shall depend on E only. Notice that for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$ we have $x_n > 0$. Also notice that

(4.44)
$$|x| \le |x - e_n y_n|, \quad x \in B_{\delta \rho_0}(\delta \rho_0 e_n), \, y_n \in (-\delta \rho_0, 0).$$

By such an inequality and (4.24d) we have

(4.45)
$$|\Phi^{(t)}(x) - x| \le \frac{C}{\rho_0} |x - e_n y_n|^2$$

for $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$ where C depends on E only. By (4.45), (4.44) and the triangle inequality we have

(4.46)
$$C^{-1}|x - e_n y_n| \le |\Phi^{(t)}(x) - e_n y_n| \le C|x - e_n y_n|,$$

for $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$ where $C, C \ge 1$, depends on E only. By (4.41), (4.42), (4.24e), (4.43), (4.44), (4.46) we obtain

(4.47)
$$|g(x,t,y)| \le C \frac{e^{-\frac{|x-e_ny_n|^2}{Ct}}}{t^{n/2}} \left[\frac{|x-e_ny_n|^2+t}{\rho_0^2}\right]^{1/2}$$

and

(4.48)

$$|\nabla_x g(x,t;y)| \le \frac{C}{\rho_0} \frac{\mathrm{e}^{-\frac{|x-e_n y_n|^2}{Ct}}}{t^{n/2} [|x-e_n y_n|^2+t]^{1/2}} \left(\frac{[|x-e_n y_n|^2+t]^{1/2}}{\rho_0}\right)^{-1+\frac{\beta}{1+\beta}},$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$ where $C, C \ge 1$, depends on E only. Recalling the definition of g we have that

$$(4.49) \quad \gamma \left(\Phi^{(0)}(y) \right) \left(\Gamma(x,t;y,0) - \Gamma_{+}(x,t;y,0) \right) = g(x,t;y) - \left(1 - \gamma(\Phi^{(0)}(y)) \right) \Gamma_{+}(x,t;y,0) - \gamma \left(\Phi^{(0)}(y) \right) \left(\Gamma_{+}(x,t;y,0) - \Gamma_{+}(\Phi^{(t)}(x),t;y,0) \right).$$

Now for $x \in B_{\delta\rho_0}(\delta\rho_0 e_n), y_n \in (-\delta\rho_0, 0)$ we have

$$(4.50) |y| \le |x - e_n y_n|,$$

so such an inequality, (4.26), (4.24c) and (3.13) give

(4.51)
$$\left| \left(1 - \gamma(\Phi^{(0)}(y)) \right) \Gamma_+(x,t;y,0) \right| \le C \left(\frac{|x - e_n y_n|^2 + t}{\rho_0^2} \right)^{1/2} \frac{\mathrm{e}^{-\frac{|x - e_n y_n|^2}{Ct}}}{t^{n/2}},$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$ where $C, C \ge 1$, depends on E only. In order to estimate from above the third term at the right hand side of (4.49) we use the mean value theorem. By such a theorem, (4.26) and Proposition 3.6 we get

(4.52)
$$\left| \gamma(\Phi^{(0)}(y)) \left(\Gamma_{+}(\Phi^{(t)}(x), t; y, 0) \right) - \Gamma_{+}(x, t; y, 0) \right| \leq C |x - \Phi^{(t)}(x)| \frac{\mathrm{e}^{-\frac{|\overline{x} - e_n y_n|^2}{Ct}}}{t^{\frac{n+1}{2}}},$$

where $\overline{x} = x + \lambda(\Phi^{(t)}(x) - x)$ for a suitable $\lambda \in (0, 1)$ and $C, C \ge 1$, depends on *E* only. Now, by triangle inequality, (4.24d), (4.44) we have

(4.53)
$$|\overline{x} - e_n y_n| \ge |x - e_n y_n| - |x - \overline{x}|$$

 $\ge |x - e_n y_n| \left(1 - \frac{C}{\rho_0} |x|\right) \ge \frac{1}{2} |x - e_n y_n|,$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, δ small enough and depending on E only. By inequality (4.53), (4.45), (4.47), (4.51) and (4.52) we obtain

(4.54) $|\Gamma_{+}(x,t;y,0) - \Gamma_{+}(\Phi^{(t)}(x),t;y,0)|$

$$\leq C \left(\frac{(|x - e_n y_n|^2 + t)}{\rho_0^2} \right)^{1/2} \frac{\mathrm{e}^{-\frac{|x - e_n y_n|^2}{Ct}}}{t^{n/2}},$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (r_1/8)^2)$, where $C, C \ge 1, \delta$, $0 < \delta < 1$, depend on E only.

We finally estimate $|\nabla_x \Gamma(x,t;y,0) - \nabla_x \Gamma_+(x,t;y,0)|$ for $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$. By (4.26), (4.50), (4.49), (4.48) and Proposition 3.6 we have

$$(4.55) \quad |\nabla_{x}\Gamma(x,t;y,0) - \nabla_{x}\Gamma_{+}(x,t;y,0)| \\ \leq \frac{C}{\rho_{0}} \frac{e^{-\frac{|x-e_{n}y_{n}|}{Ct}}}{t^{n/2}} \left(\frac{[|x-e_{n}y_{n}|^{2}+t]^{1/2}}{\rho_{0}}\right)^{-1+\frac{\beta}{\beta+1}} \\ + C \left|\nabla_{x}\Gamma_{+}(x,t;y,0) - \nabla_{x}\left(\Gamma_{+}(\Phi^{(t)}(x),t;y,0)\right)\right|,$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (r_1/8)^2)$, where $C, C \ge 1$, depends on E only. Let us consider now the last term at the right hand side of (4.55). We have

(4.56)
$$\left| \nabla_x \Gamma_+(x,t;y,0) - \nabla_x \left(\Gamma_+(\Phi^{(t)}(x),t;y,0) \right) \right| \le H_1(x,t;y) + H_2(x,t;y),$$

where

$$H_1(x,t;y) = C|\nabla_x \Gamma_+(x,t;y,0)| |I - D_x \Phi^{(t)}(x)|$$

and

$$H_2(x,t;y) = C|D_x\Phi^{(t)}(x)| | (\nabla_x\Gamma_+) (\Phi^{(t)}(x),t;y,0) - \nabla_x\Gamma_+(x,t;y,0)|,$$

where C depends on E only. By Proposition 3.6, (4.24e) and (4.44) we have

(4.57)
$$H_1(x,t,y) \le C|x - e_n y_n| \frac{e^{-\frac{|x - e_n y_n|^2}{Ct}}}{t^{\frac{n+1}{2}}} \le C' \frac{e^{-\frac{|x - e_n y_n|^2}{2Ct}}}{t^{n/2}},$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (r_1/8)^2)$, where C, C', depend on E only. To estimate from above the function H_2 we apply Theorem 4.1. Let δ_1 be defined as in Proposition 4.2 and let us denote

$$\rho = \frac{\delta_1}{4} [|x - e_n y_n|^2 + t]^{1/2}.$$

By (4.24d) and (4.44) we have that there exists $\delta \in (0, 1)$, depending on E only such that

(4.58)
$$|\Phi^{(t)}(x) - x| \le \frac{1}{2}\rho,$$

for every $x \in B_{\delta \rho_0}(\delta \rho_0 e_n)$. Now Theorem 4.1 and Proposition 4.2 yield

(4.59)
$$[\nabla_x \Gamma_+(\cdot,t;y,0)]_{\beta;B'_{\rho}(x')\times(x^0_n,x_n+\rho)} \le C\rho^{-(2+\beta)} \frac{e^{-\frac{|x-e_ny_n|^2}{Ct}}}{t^{\frac{n-1}{2}}},$$

where $x_n^0 = \max\{0, x_n^0 - \rho\}$ and *C* depends on *E* only, $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (\delta\rho_0)^2)$. By (4.24d), (4.59) and (4.58) we obtain

(4.60)
$$H_2(x,t;y) \le \frac{C}{\rho_0} |x|^2 \rho^{-(2+\beta)} \frac{e^{-\frac{|x-e_n y_n|^2}{Ct}}}{t^{\frac{n-1}{2}}},$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (\delta\rho_0)^2)$, where $C, C \ge 1$, depends on E only. Finally, (4.44) and (4.60) yield

$$H_2(x,t;y) \le \frac{C}{\rho_0} \frac{e^{-\frac{|x-e_n y_n|^2}{Ct}}}{t^{n/2}} \left(\frac{[|x-e_n y_n|^2+t]^{1/2}}{\rho_0}\right)^{-1+\beta}.$$

The last inequality, (4.57), (4.55), (4.56) give

$$\begin{aligned} |\nabla_x \Gamma(x,t;y,0) - \nabla_x \Gamma_+(x,t;y,0)| \\ &\leq \frac{C}{\rho_0} \frac{\mathrm{e}^{-\frac{|x-e_n y_n|}{Ct}}}{t^{n/2}} \left(\frac{[|x-e_n y_n|^2 + t]^{1/2}}{\rho_0}\right)^{-1 + \frac{\beta}{\beta+1}}. \end{aligned}$$

for every $x \in B_{\delta\rho_0}(\delta\rho_0 e_n)$, $y_n \in (-\delta\rho_0, 0)$, $t \in (0, (\delta\rho_0)^2)$, where C depends on E only.

4.3 **Proof of Proposition 3.9**

Proof of Proposition 3.7 First of all, let us observe that

(4.61)
$$\Gamma_0(\xi,\tau;-\lambda_3e_n,0) = h^n \Gamma_0(h\xi,h^2\tau;-\lambda_3he_n,0)$$

and

(4.62)
$$\Gamma_{+}^{*}(\xi,\tau;-\lambda_{1}e_{n},\lambda_{2}) = h^{n}\Gamma_{+}^{*}(h\xi,h^{2}\tau;-\lambda_{1}he_{n},\lambda_{2}h^{2}).$$

Indeed (4.61) is a trivial consequence of the definition of Γ_0 . Concerning (4.62), it can be proved as follows. Denote by $y^{(h)} = -\lambda_1 h e_n$, $s^{(h)} = \lambda_2 h^2$ we have

(4.63)
$$\int_{\mathbb{R}^{n+1}} \left(\Gamma_+^* \partial_t \varphi + (1 + (k-1)\chi_+(x)) \nabla_x \Gamma_+^* \cdot \nabla_x \varphi \right) dx dt = \varphi(y^{(h)}, s^{(h)}),$$

for every $\varphi \in C_0^{\infty}(\mathbb{R}^{n+1})$. In (4.63) Γ_+^* denotes the function $\Gamma_+^*(x,t;y^{(h)},s^{(h)})$. Now for an arbitrary function $\psi \in C_0^{\infty}(\mathbb{R}^{n+1})$ put $\varphi(x,t) := \psi\left(\frac{x}{h},\frac{t}{h^2}\right)$ in (4.63). In the obtained integral we perform the change of variables $x = h\xi$, $t = h^2\tau$. Thus, taking into account that $\chi_+(h\xi) = \chi_+(\xi)$, h > 0, and denoting by $W_h(\xi,\tau)$ the right-hand side of (4.62) we have

$$\int_{\mathbb{R}^{n+1}} \left(W_h \partial_\tau \psi + (1 + (k-1)\chi_+(\xi)) \nabla_\xi W_h \cdot \nabla_\xi \psi \right) d\xi d\tau = \psi(-\lambda_1 e_n, \lambda_2),$$

for every $\psi \in C_0^{\infty}(\mathbb{R}^{n+1})$. Therefore

(4.64)
$$\partial_{\tau} W_h + \operatorname{div}((1+(k-1)\chi_+(\xi))\nabla_{\xi} W_h) = -\delta(\xi + \lambda_1 e_n, \tau - \lambda_2)$$

and, by the definition of W_h ,

(4.65)
$$W_h(\cdot, \tau) = 0$$
 for every $\tau > \lambda_2$.

Finally, by the uniqueness for Cauchy problem [Ar], by (4.64) and (4.65) we obtain (4.62).

Now, performing the change of variable $x = h\xi$, $t = h^2 \tau$ in the integral at the left-hand side of (3.19), we get, by (4.61) and (4.62),

(4.66)
$$I^{(h)} = h^{-n} I^{(1)}.$$

Now, recall that

(4.67)
$$\mathcal{F}_{\zeta'}\left(\Gamma_0(\cdot, x_n, t; -\lambda_3 e_n, 0)\right) = \frac{\mathrm{e}^{-|\zeta'|^2 t}}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x_n + \lambda_3)^2}{4t}}$$

In the case k > 1, by Parseval formula, (3.18) and (4.67) we have

$$I^{(1)} = \left| \int_{Y} M_{\lambda_2,\lambda_3}(\zeta', x_n, t; \rho) F(\zeta', -\lambda_1; \rho) d\zeta' dx_n dt d\rho \right|,$$

where $Y = \mathbb{R}^{n-1} \times (0, +\infty) \times (0, \lambda_2) \times (0, 1)$ and

$$M_{\lambda_2,\lambda_3}(\zeta', x_n, t; \rho) = \frac{|\zeta'|^2}{(2\pi)^{n-1}} \frac{\mathrm{e}^{-t|\zeta'|^2}}{\sqrt{4\pi t}} \mathrm{e}^{-\frac{(x_n+\lambda_3)^2}{4t}} \times \left(|\zeta'| + \sqrt{\frac{k-1}{k}} \frac{(x_n+\lambda_3)}{2t} \sqrt{\rho}\right) E(\zeta', x_n, \lambda_2 - t; \rho).$$

For fixed $\lambda_2 > 0$, $\lambda_3 > 0$, taking into account (3.16) and (3.17) we have

$$\lim_{\lambda_1 \to 0^+} I^{(1)} = \left| \int_Y M_{\lambda_2,\lambda_3}(\zeta', x_n, t; \rho) \operatorname{Im}(A_1(\rho)) d\zeta' dx_n dt d\rho \right|.$$

$$\lim_{\lambda_1 \to 0^+} I^{(1)} > 0$$

and by (4.66) the thesis follows.

Concerning the case 0 < k < 1, we only give a sketch of the proof, indeed such a case can be treated similarly to the case k > 1. In the case 0 < k < 1, in order to have a suitable formula for $I^{(1)}$, first we evaluate the integrals

$$\operatorname{Im}\left(\int_{0}^{+\infty} e^{-i\mu x_{n}} K(x_{n} + \lambda_{3}, t) dx_{n}\right),$$
$$\operatorname{Im}\left(\int_{0}^{+\infty} e^{-i\mu x_{n}} \frac{\partial}{\partial x_{n}} K(x_{n} + \lambda_{3}, t) dx_{n}\right).$$

where

$$\mu = \sqrt{\frac{1-k}{k}}\sqrt{1-\rho}|\zeta'| \quad \text{and} \quad K(x_n,t) = \frac{\mathrm{e}^{-\frac{x_n^2}{4t}}}{\sqrt{4\pi t}}.$$

In order to carry out such an evaluation we may use formula 3.322 of [Gr]. Then we choose $\lambda_2 = \lambda_1^2$, we perform the change of variable $\zeta' = \frac{\xi'}{\lambda_1}$, $t = \lambda_1^2 \eta$ in the integral $I^{(1)}$ and we get

$$\lim_{\lambda_1 \to 0^+} \left(\lambda_1^n \lim_{\lambda_3 \to 0^+} I^{(1)} \right) > 0$$

and the thesis follows.

Proof of Proposition 3.9. By the triangle inequality we get

$$(4.68) \qquad |\mathcal{U}(y_1, t_1; \overline{y}, \overline{t})| = |S_1(y_1, t_1; \overline{y}, \overline{t}) - S_2(y_1, t_1; \overline{y}, \overline{t})| \\ \geq |S_1(y_1, t_1; \overline{y}, \overline{t})| - |S_2(y_1, t_1; \overline{y}, \overline{t})|.$$

Let us first estimate from below $|S_1(y_1, t_1; \overline{y}, \overline{t})|$. Recall that $\Gamma^*_+(x, t; \overline{y}, \overline{t})$ is the fundamental solution of the adjoint operator of $\mathcal{L}_+ = \partial_t - \operatorname{div}((1 + (k-1)\chi_+)\nabla)$. Denote by

$$Q_{\rho/2} = B_{\rho/2}(0) \times (t_1, \bar{t}), \quad Q_{\rho/2}^+ = B_{\rho/2}^+(0) \times (t_1, \bar{t}), \quad Q_{\rho/2}^- = B_{\rho/2}^-(0) \times (t_1, \bar{t}).$$

By the triangle inequality we have

$$(4.69) |S_1(y_1, t_1; \overline{y}, \overline{t})| \ge I_1 - R_1 - R_2,$$

where

(4.70)
$$I_1 = \left| \int_{Q_{\rho/2}^+} \nabla_x \Gamma_+^*(x,t;\overline{y},\overline{t}) \cdot \nabla_x \Gamma_0(x,t;y_1,t_1) dx dt \right|,$$

$$(4.71) \quad R_{1} = \int_{D_{1}(t_{1},\bar{t})\backslash Q_{\rho/2}} |\nabla_{x}\Gamma_{2}(x,t;y_{1},t_{1})| |\nabla_{x}\Gamma_{1}^{*}(x,t;\bar{y},\bar{t})| dxdt + \int_{Q_{\rho/2}^{+}\backslash (D_{1}(t_{1},\bar{t})\cap Q_{\rho/2}^{+})} |\nabla_{x}\Gamma_{+}^{*}(x,t;\bar{y},\bar{t})| |\nabla_{x}\Gamma_{0}(x,t;y_{1},t_{1})| dxdt,$$

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Thus

$$(4.72) R_2 = \int_{D_1(t_1,\overline{t})\cap Q_{\rho/2}} |\nabla_x \Gamma_+^*(x,t;\overline{y},\overline{t}) - \nabla_x \Gamma_1^*(x,t;\overline{y},\overline{t})| \\ \times |\nabla\Gamma_0(x,t;y_1,t_1)| dx dt \\ + \int_{D_1(t_1,\overline{t})\cap Q_{\rho/2}} |\nabla_x \Gamma_0(x,t;y_1,t_1) - \nabla_x \Gamma_2(x,t;y_1,t_1) \\ \times |\nabla_x \Gamma_1^*(x,t;\overline{y},\overline{t})| dx dt,$$

where $D_1(t_1, \bar{t}) = \bigcup_{t \in (t_1, \bar{t})} D_1(t) \times \{t\}$. Now we estimate from below the term I_1 . First we notice that if $0 < \delta \le \frac{1}{4\sqrt{2}}$ then

(4.73)
$$|x - \overline{y}|^2 \ge \frac{1}{32}(|x|^2 + \rho^2), \qquad |x - y_1|^2 \ge \frac{1}{32}(|x|^2 + \rho^2),$$

for every $x \in \mathbb{R}^n_+ \setminus B^+_{\rho/2}$. Also, we have trivially

$$(4.74) \quad I_1 \ge \left| \int_{\mathbb{R}^n_+ \times (t_1, \overline{t})} \nabla_x \Gamma^*_+(x, t; \overline{y}, \overline{t}) \cdot \nabla_x \Gamma_0(x, t; y_1, t_1) dx dt \right| \\ - \int_{(\mathbb{R}^n_+ \times (t_1, \overline{t})) \setminus Q^+_{\rho/2}} |\nabla_x \Gamma^*_+(x, t; \overline{y}, \overline{t})| |\nabla_x \Gamma_0(x, t; y_1, t_1)| dx dt.$$

We now use Proposition 3.6 and (4.73) to estimate from above the second integral of the right-hand side of (4.74). We have

$$(4.75) \quad \int_{(\mathbb{R}^{n}_{+}\times(t_{1},\bar{t}))\setminus Q^{+}_{\rho/2}} |\nabla_{x}\Gamma^{*}_{+}(x,t;\bar{y},\bar{t})| |\nabla_{x}\Gamma_{0}(x,t;y_{1},t_{1})| dxdt$$
$$\leq C_{0} \int_{\mathbb{R}^{n}_{+}\times(t_{1},\bar{t})} e^{-\frac{\rho^{2}}{C_{0}(t-t_{1})} - \frac{\rho^{2}}{C_{0}(\bar{t}-t)}} \frac{e^{-\frac{|x|^{2}}{C_{0}(\bar{t}-t_{1})} - \frac{|x|^{2}}{C_{0}(\bar{t}-t)}}}{(\bar{t}-t)^{\frac{n+1}{2}}(t-t_{1})^{\frac{n+1}{2}}} := \tilde{R},$$

where $C_0, C_0 \ge 1$, depends on k only. Now performing the change of variables

$$z = \left(\frac{\overline{t} - t_1}{(t - t_1)(\overline{t} - t)}\right)^{1/2} x,$$

we have

$$\begin{split} \tilde{R} &\leq C_0 \int_{\mathbb{R}^n \times (t_1, \bar{t})} \frac{\mathrm{e}^{-\frac{2\rho^2}{C_0(\bar{t} - t_1)}}}{(\bar{t} - t_1)^{n/2}} \frac{\mathrm{e}^{-\frac{|z|^2}{C_0}}}{\sqrt{(\bar{t} - t)(t - t_1)}} dz dt \\ &\leq \frac{C_3 C_0}{\rho^n} \left(\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|z|^2}{C_0}} dz \right) \left(\int_0^1 \frac{d\lambda}{\sqrt{\lambda(1 - \lambda)}} \right) + \frac{C_0 C_0}{\rho^n} dz dz dz \end{split}$$

where $C_3 = \max_{s \in (0,+\infty)} \{s^{n/2} e^{-2C_0 s}\}$. By the inequality obtained above, by (4.74) and by Proposition 3.7 we have

(4.76)
$$I_1 \ge \frac{1}{Ch^n} - \frac{C}{\rho^n},$$

where $C, C \geq 1$, depends on k only.

In order to complete the proof we have to estimate from above the terms R_1 and R_2 defined in (4.71), (4.72). Denote by R_{11} and R_{12} the first and the second integral at the right-hand side of (4.71) respectively. R_{11} can be estimate in the same way of the integral at the left-hand side of (4.75) and we have

$$(4.77) R_{11} \le \frac{C}{\rho^n},$$

where C depends on k only. Concerning R_{12} , by (2.7a) and Proposition 3.6 we have

$$(4.78) R_{12} \le C_4 \int_{t_1}^{\overline{t}} dt \int_{\mathbb{R}^{n-1}} dx' \int_{-\psi(x',t)}^{\psi(x',t)} \frac{\mathrm{e}^{-\frac{|x-y_1|^2}{C_4(\overline{t}-t_1)}}}{(t-t_1)^{\frac{n+1}{2}}} \frac{\mathrm{e}^{-\frac{|x-\overline{y}|^2}{C_4(\overline{t}-t)}}}{(\overline{t}-t)^{\frac{n+1}{2}}} dx_n,$$

where

$$\psi(x',t) = \frac{3}{2} \frac{E}{\rho_0} \left(|x'|^2 + |t - \bar{t}| \right)$$

and C_4 , $C_4 \ge 1$, depends on k only. Now we perform, in the integral at the right-hand side of (4.78) the following change of variables

$$t = t_1 + \tau(\bar{t} - t_1), \quad x' = (\tau(1 - \tau))^{1/2} z', \quad x_n = (\tau(1 - \tau))^{1/2} \xi - \lambda_1 h.$$

Thus, denoting by

$$\begin{split} \sigma(\tau) &= \frac{1}{\sqrt{\tau(1-\tau)}},\\ \phi_1(z',\tau) &= \frac{3}{2} \frac{E}{\rho_0} \left(|z'|^2 \sqrt{\lambda_2} \sqrt{\tau(1-\tau)} + \frac{\sqrt{\lambda_2} \sqrt{1-\tau}}{\sqrt{\tau}} \right),\\ A(z',\xi,\tau) &= |z'|^2 + \tau \xi^2 + (1-\tau) \left(\xi + \frac{(\lambda_3 - \lambda_1)h}{\sqrt{\lambda_2} \sqrt{\tau(1-\tau)}} \right)^2\\ \theta(h) &= \int_0^1 \int_{\mathbb{R}^{n-1}} \int_{\frac{\sigma(\tau)}{\sqrt{\lambda_2}} - h\phi_1(z',\tau)}^{\frac{\sigma(\tau)}{\sqrt{\lambda_2}} + h\phi_1(z',\tau)} \frac{e^{-\frac{A(z',\xi,\tau)}{C_4}}}{\sqrt{\tau(1-\tau)}} d\xi dz' d\tau, \end{split}$$

we get

(4.79)
$$R_{12} \le \frac{C}{h^n} \theta(h),$$

where C depends on k only. Now observing that

$$A(z',\xi,\tau) \ge |z'|^2 + \left(\xi + \frac{(\lambda_3 - \lambda_1)}{\sqrt{\lambda_2}\sqrt{\tau}}\sqrt{1-\tau}\right)^2$$

and applying the Hölder inequality, we obtain

(4.80)
$$\theta(h) \le \tilde{C}_p \left(\frac{h}{\rho_0}\right)^{1-\frac{1}{p}},$$

for every $p \in (1, +\infty)$, where \tilde{C}_p depends on p and E only. By (4.77), (4.80) and recalling that $R_1 = R_{11} + R_{12}$ we obtain

(4.81)
$$R_1 \le \frac{C}{\rho^n} + \tilde{C}_p \left(\frac{h}{\rho_0}\right)^{1-\frac{1}{p}} \frac{1}{h^n},$$

for every $h, 0 < h \leq \delta \min\{\rho, \sqrt{\overline{t}}\}$ and every $\delta, 0 < \delta \leq \min\left\{\frac{\lambda_3}{C_1}, \frac{1}{C_2}, \frac{1}{4\sqrt{2}}\right\}$, where C_1 and C_2 are defined in (3.23) and (3.27) respectively and C depends on k only.

In order to estimate R_2 , denote by R_{21} and R_{22} the first and the second integral at the right-hand side of (4.72) respectively. By Theorem 4.3 we have that there exists a constant C_5 , $C_5 \ge 1$, depending on E only such that if $0 < \delta \le \frac{1}{C_5}$ and $(x,t) \in \mathbb{K}_{\rho} := \{(x,t) \in B_{\rho/C_5} \times (t_1,\bar{t}) : x_n > \frac{1}{C_5\rho_0}(|x'|^2 + |t - \bar{t}|)\}$ then

(4.82)
$$|\nabla_x \Gamma_1^*(x,t;\overline{y},\overline{t}) - \nabla_x \Gamma_+^*(x,t;\overline{y},\overline{t})| \le \frac{C}{\rho_0^{\alpha}} \frac{\mathrm{e}^{-\frac{|x-\overline{y}|^2}{C(t-t)}}}{(\overline{t}-t)^{\frac{n}{2}+\frac{1}{2}-\frac{\alpha}{2}}},$$

where $C, C \ge 1$, depends on E and k only and $\alpha = \frac{\beta}{\beta+1}$, β being defined in Theorem 4.1. We have

(4.83)
$$R_{21} = J' + J'',$$

where

$$J' = \int_{D_1(t_1,\bar{t})\cap Q_{\rho/2}\cap\mathbb{K}_{\rho}} |\nabla_x\Gamma_+^*(x,t;\bar{y},\bar{t}) - \nabla_x\Gamma_1^*(x,t;\bar{y},\bar{t})| |\nabla_x\Gamma_0(x,t;y_1,t_1)| dxdt,$$
$$J'' = \int_{D_1(t_1,\bar{t})\cap Q_{\rho/2}\setminus\mathbb{K}_{\rho}} |\nabla_x\Gamma_+^*(x,t;\bar{y},\bar{t}) - \nabla_x\Gamma_1^*(x,t;\bar{y},\bar{t})| |\nabla_x\Gamma_0(x,t;y_1,t_1)| dxdt.$$

By (4.82) and Lemma 3.8 we have

(4.84)
$$J' \le C \frac{h^{\alpha - n}}{\rho_0^{\alpha}},$$

where C depends on E and k only. To estimate from above J'' we can arrange the method used to estimate R_{11} and R_{12} and we obtain that there exists $C_6 \ge C_5$, C_6 depending on E and k only such that if $0 < \delta \le \frac{1}{C_6}$ then, for $p \in (1, +\infty)$,

(4.85)
$$J'' \le \frac{C}{\rho^n} + \tilde{C}_p \left(\frac{h}{p}\right)^{1-\frac{1}{p}} \frac{1}{h^n}$$

where C depends on E and k only and \tilde{C}_p depends on p and E only. By choosing $p = \frac{1}{1-\alpha}$, (4.83), (4.84) and (4.85) yield

(4.86)
$$R_{21} \le \frac{C}{\rho^n} + \tilde{C} \frac{h^{\alpha - n}}{\rho_0^{\alpha}},$$

where C depends on E and k only and \tilde{C} depends on p and E only.

Now we estimate R_{22} . Denote by

$$w(x,t) := \Gamma_0(x,t;y_1,t_1) - \Gamma_2(x,t;y_1,t_1)$$

and recall (3.28). We have that w solves the heat equation in $B_{\rho/2}(y_1) \times (t_1, \bar{t})$ and, since w(x,t) = 0 for $(x,t) \in \mathbb{R}^n \times (-\infty, t_1]$ we can say that w solves the heat equation in $B_{\rho/2}(y_1) \times (\bar{t} - \rho^2, \bar{t})$. On $\partial B_{\rho/2}(y_1) \times (\bar{t} - \rho^2, \bar{t})$ we have

$$|w(x,t)| \le \frac{C}{(t-t_1)^{n/2}} e^{-\frac{\rho^2}{C(t-t_1)}} \chi_{[t_1,+\infty)} \le \frac{C'}{\rho^n} \chi_{[t_1,+\infty)}$$

where C, C' depend on k only. Therefore by maximum principle and by standard regularity estimates we get

(4.87)
$$|\nabla_x w(x,t)| \le \frac{C}{\rho^{n+1}} \quad \text{in } B_{\rho/4}(y_1) \times \left(\overline{t} - \frac{\rho^2}{4}, \overline{t}\right).$$

It is possible to have a similar estimate for w in $(\mathbb{R}^n \setminus B_{\rho/4}(y_1)) \times (\bar{t} - \rho^2, \bar{t})$, namely by Proposition 3.6 we have (4.88)

$$|\nabla_x w(x,t)| \le \frac{C \mathrm{e}^{-\frac{\rho^2}{C((t-t_1))}}}{(t-t_1)^{\frac{n+1}{2}}} \le \frac{C'}{\rho^{n+1}}, \qquad (x,t) \in \left(\mathbb{R}^n \setminus B_{\rho/4}(y_1)\right) \times (\bar{t}-\rho^2,\bar{t}).$$

By (4.87) and (4.88) we have

(4.89)
$$R_{22} \le \frac{C}{\rho^{n+1}} \int_{t_1}^{\overline{t}} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-\frac{|x-\overline{y}|^2}{\overline{t}-t}}}{(\overline{t}-t)^{\frac{n+1}{2}}} dx dt \le \frac{C'}{\rho^n}.$$

where C, C' depend on k only.

The estimate from above of $|S_2(y_1, t_1; \overline{y}, \overline{t})|$ can be carried out in a similar way of that used to estimate the integral in formula (4.75). Thus taking into account (3.28), we get there exists $C_7 \ge C_6$ such that if $0 < \delta \le \frac{1}{C_7}$ then

$$|S_2(y_1, t_1; \overline{y}, \overline{t})| \le \frac{C}{\rho^n},$$

where C depends on k only. This inequality and (4.89), (4.86), (4.81), (4.76), (4.69), (4.68) give (3.29). \Box

References

- [Al-Be-Ro-Ve] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4 (2000), pp. 755–806.
- [Al-DC] G. Alessandrini, M. Di Cristo, Stable determination of an inclusion by boundary measurements SIAM J. Math. Anal. 37 (2005), pp. 200–217.
- [Al-Ve] G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem Adv. in Appl. Math. 35 (2005), pp. 207–241.

- [Ar] D. G. Aronson, Non-negative solutions of linear parabolic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1968), pp. 607–694.
- [Ca-Mo] W. J. Cantel, J. Morton, The significance of damage and defects and theory detection in composite materials: a review. J. Strain. Anal. 27 (1992), pp. 29–42.
- [Da-Ka-Na] Y. Daido, H. Kang, G. Nakamura, Reconstruction of inclusions for the inverse boundary value problems for non-stationary heat equation, Algebraic Analysis of Differential Equations, Springer Japan (2009), pp. 89–99.
- [DC-Ro] M. Di Cristo, L. Rondi, Examples of exponential instability for inverse inclusion and scattering problems, Inverse Problems, 19 (2003), pp. 685– 701.
- [DC-Ro-Ve] M. Di Cristo, L. Rondi, S. Vessella, Stability properties of an inverse parabolic problem with unknown boundaries, Ann. Mat. Pura Appl. 4 (2006), pp. 223–255.
- [El-Is] A. Elayyan, V. Isakov, On uniqueness of recovery of the discontinuous conductivity coefficient of a parabolic equation, SIAM J. Math. Anal. 28 (1997), pp. 49–59.
- [Es-Fe-Ve] L. Escauriaza, F. J. Fernandez, S. Vessella, *Doubling properties of caloric functions*, Appl. Anal., 85, (2006), pp. 205–223. Special issue dedicated to the memory of Carlo Pucci, editors R. Magnanini and G. Talenti.
- [Ev] L. C. Evans, Partial differential equations, American Mathematical Society, Providence, Rhode Island 1998.
- [Fr] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.
- [Gr] I. S. Gradshteyn, I. M. Ryzhik Table of Integrals, Series and Product, Academic Press, Inc., New York 1980.
- [Is88] V. Isakov, On uniqueness of recovery of a discontinuous conductivity coefficient, Comm. Pure Appl. Math. 41 (1988), pp. 865–877.
- [Is-Ki-Na] V. Isakov, K. Kim, G. Nakamura Reconstruction of an unknown inclusion by thermography, Preprint.
- [La-Ri-Ur] O. A. Ladyzenskaja, V. J. Rivkind, N. N. Ural'ceva, The classical sovability of diffraction problems, Proc. Steklov Inst. Math. 98 92 (1966), pp. 132–166.
- [La-So-Ur] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1967.
- [Li] G. Lieberman, Second order parabolic differential equations, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
- [Li-Ma] J. L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Vol. II. Springer-Verlag, New York-Heidelberg, 1972.

- [Mo-Ro03] A. Morassi, E. Rosset, Detecting rigid inclusions, or cavities, in an elastic body, J. Elasticity 73 (2003), pp. 101–126.
- [Mo-Ro04] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, Inverse Problems 20 (2004), pp. 453–480.
- [Pa-La-Al] P. M. Patel, S. K. Lau, D. P. Aldmond, A review of image analysis techniques applied in transient termographic nondestructive testing. Nondestructive testing and eval. 6 (1992), pp. 343–364.
- [Ve] S. Vessella Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates, Inverse Problems 24 (2008), pp. 1–81.