A note on the law of large numbers in economics

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Abstract

Let $(S,\mathcal{B},\Gamma)$ and $(T,\mathcal{C},Q)$ be probability spaces, with Q nonatomic, and $\mathcal{H}=\{H\in\mathcal{C}:Q(H)>0\}$. In some economic models, the following conditional law of large numbers (LLN) is requested. There are a probability space $(\Omega,\mathcal{A},P)$ and a process $X=\{X_t:t\in T\}$, with state space $(S,\mathcal{B})$, satisfying

$$\text{for each } H\in\mathcal{H},\text{ there is } A_H\in\mathcal{A} \text{ with } P(A_H)=1 \text{ such that}$$
$$t\mapsto X(t,\omega) \text{ is measurable and } Q\big(\{t:X(t,\omega)\in\cdot\}\mid H\big)=\Gamma(\cdot)\text{ for } \omega\in A_H.$$

If Γ is not trivial and the σ-field $\mathcal{C}$ countably generated, the conditional LLN fails in the usual (countably additive) setting. Instead, as shown in this note, it holds in a finitely additive setting. Also, X can be taken to have any given distribution. In fact, for any consistent set $\mathcal{P}$ of finite dimensional distributions, there are a finitely additive probability space $(\Omega,\mathcal{A},P)$ and a process X such that $X\sim\mathcal{P}$ and the conditional LLN is satisfied.

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1 The problem

In various economic frameworks, there are three probability spaces

$$(T, \mathcal{C}, Q), \quad (\Omega, \mathcal{A}, P), \quad (S, \mathcal{B}, \Gamma)$$

and a stochastic process

$$X : T \times \Omega \to S.$$

Since X is a process, $X_t : (\Omega, \mathcal{A}) \to (S, \mathcal{B})$ is measurable for each $t \in T$, where

$$X_t(\cdot) = X(t, \cdot) \quad \text{and} \quad X^\omega(\cdot) = X(\cdot, \omega), \quad (t, \omega) \in T \times \Omega,$$

denote the X-sections with respect to t and ω.

For some reasons, to be recalled in Section 2, it is often asked that

$$X^\omega \text{ is measurable and } Q(X^\omega \in \cdot) = \Gamma(\cdot) \text{ for } P\text{-almost all } \omega \in \Omega. \tag{1}$$

Typically, the process X is identically distributed, in the sense that $X_a \sim X_b$ for all a, $b \in T$, and Γ is the distribution common to the X_t. Then, fixing $t_0 \in T$ (and neglecting measurability issues) condition (1) takes the more familiar form

$$Q(X^\omega \in \cdot) = P(X_{t_0} \in \cdot) \quad \text{for } P\text{-almost all } \omega \in \Omega.$$

This particular case of (1) is usually called *law of large numbers* (LLN) in economics. (We note that, incidentally, the term LLN is used with a different meaning in probability theory).

However, condition (1) is sometimes not enough and a conditional form of the LLN is asked. Let $\mathcal{H} = \{H \in \mathcal{C} : Q(H) > 0\}$. Then, X is requested to meet

$$\begin{gathered} \text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that} \\ X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for } \omega \in A_H. \end{gathered} \tag{2}$$

It is not hard to prove that, when the σ-field $\mathcal{C}$ is countably generated, condition (2) implies $\Gamma(B) \in \{0, 1\}$ for all $B \in \mathcal{B}$. See [13] and Remark 2. Thus, to get (2) with non trivial Γ and countably generated $\mathcal{C}$, an extension of $(T, \mathcal{C}, Q)$ is to be involved.

One (interesting) approach is to look for reasonable extensions, that is, extensions which grant (2) and some other properties, such as a form of Fubini's theorem. This route is followed by [9], [11] and [12]. In [11], assuming Q non atomic, condition (2) is shown to be true if X is *essentially pairwise independent* and measurable with respect to a *Fubini extension*. Conditions for such an X to exist are given in [9] and [12].

A different route, closer to the ideas of [8], is taken in this note. On one hand, we aim to avoid extensions of $(T, \mathcal{C}, Q)$ and to have

$$X \sim \mathcal{P}$$

where $\mathcal{P}$ is *any* given consistent set of finite dimensional distributions. We remark that both these goals are generally precluded in the approach of [11]. On the other hand, we content ourselves with proving consistency of (2) with $X \sim \mathcal{P}$.

Our main result (Theorem 1, part (b)) is the following. As in most economic models, suppose $(S, \mathcal{B}, \Gamma)$ and $(T, \mathcal{C}, Q)$ are given with Γ tight, Q nonatomic and $\mathcal{C}$ including the singletons. Also, fix a consistent set $\mathcal{P}$ of finite dimensional distributions. Then, there are a *finitely additive* probability space $(\Omega, \mathcal{A}, P)$ and a process X such that $X \sim \mathcal{P}$ and condition (2) holds.

Two other results, both obtained for a countably additive P, should be mentioned. First, the conditional LLN is shown to be true provided it is only asked for each H in a finite subclass $\mathcal{H}_0 \subset \mathcal{H}$ (and not for each $H \in \mathcal{H}$). Second, an alternative version of (2), with possible economic meaning, is proved in Section 5.

Dating from de Finetti, the finitely additive theory of probability is well founded and developed, even if not prevailing. It finds applications in various fields, ranging from statistics and number theory to economics. A brief discussion is in Section 3.

The spirit of Theorem 1 is that one can always assume condition (2) and $X \sim \mathcal{P}$, for any consistent $\mathcal{P}$, but at the price of allowing for finitely additive probabilities. Whether or not this price is expensive is basically an axiomatic point.

2 Motivations

Several economic models deal with agents bearing a certain risk. In this framework, T denotes the set of agents and the random variable X_t the individual risk of agent $t \in T$. So, typically, $S = \mathbb{R}$ and $\mathcal{B}$ is the Borel σ-field. In addition, T is viewed as a "very large" set, and this is formalized by equipping T with a σ-field $\mathcal{C}$ and a nonatomic probability Q on $\mathcal{C}$. Suppose, for instance, that T is a Polish space (i.e., it is metric separable and complete). Then, $\text{card}\,(T) = \text{card}\,(\mathbb{R})$ whenever T supports a nonatomic Borel probability Q, that is, a probability Q on the Borel σ-field of T satisfying $Q\{t\} = 0$ for all $t \in T$.

The informal idea underlying most economic models is that, for a large set of agents, individual risks disappear in the aggregate. This looks more convincing if the individual risks X_t are independent and identically distributed (i.i.d.). The LLN (1), with Γ the distribution common to the X_t, makes this intuition precise. The conditional LLN (2) is a strengthening of (1), with the same interpretation, but conditionally on every non negligible subset H of agents. Roughly speaking, individual risks should cancel not only when the aggregate is T, but also when it is an arbitrary non negligible $H \subset T$.

It is usual to assume the process X i.i.d., in the sense that $X_{t_1}, \ldots, X_{t_n}$ are i.i.d. for all $n \geq 1$ and all distinct $t_1, \ldots, t_n \in T$. Even if strong, this assumption makes sense in the setting sketched above. However, some other assumptions on X are both more general and economically sound. Just to fix ideas, instead of i.i.d., it could be reasonable having X exchangeable, or Markov, or stationary, and so on. In this note, as remarked in Section 1, we aim to be distribution-free as regards X. Accordingly, we assume $X \sim \mathcal{P}$ where $\mathcal{P}$ is any consistent set of finite dimensional distributions.

We refer to [1], [2], [8], [9], [11], [12], [13] and references therein for more on the economic motivations of the LLN.

3 Finitely additive probabilities

In de Finetti's view, a probability assessment is a *coherent* function P on an (arbitrary) class $\mathcal{E}$ of events. Letting I_A denote the indicator function of the event A, coherence of $P : \mathcal{E} \rightarrow \mathbb{R}$ means that

$$\max \sum_{i=1}^{n} c_i \left\{ I_{A_i} - P(A_i) \right\} \geq 0$$

for all $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{E}$. Heuristically, suppose P describes your previsions on the members of $\mathcal{E}$. If you are coherent, it is impossible to make you a sure looser, whatever outcome turns out to be true, by some finite combination of bets (on $A_1, \ldots, A_n$ with stakes $c_1, \ldots, c_n$).

Suppose events are viewed as subsets of a set Ω and let $l^\infty(\Omega)$ denote the set of real bounded functions on Ω. By Hahn-Banach theorem, P is coherent if and only if P can be extended to a linear functional E on $l^\infty(\Omega)$ satisfying $E(f) \leq \sup f$ for all $f \in l^\infty(\Omega)$. When $\mathcal{E}$ is a field, it follows that P is coherent if and only if it is a *finitely additive probability* (f.a.p.), in the sense that $P \geq 0$, $P(\Omega) = 1$ and $P(A \cup B) = P(A) + P(B)$ whenever $A, B \in \mathcal{E}$ and $A \cap B = \emptyset$.

Loosely speaking, there are two kinds of reasons for adopting de Finetti's view. One is of foundational type. F.a.p.'s have a solid motivation in terms of coherence and can be always extended to the power set. (Both things are not true for countably additive probabilities). The other reason is of applicative type. There are problems which can not be solved in the usual setting, while admit a finitely additive solution. Examples are in conditional probability, convergence in distribution of non measurable random elements, Bayesian statistics, game theory, stochastic integration and the first digit problem. See e.g. [3] and references therein.

Plainly, working with f.a.p.'s has a lot of technical disadvantages as well. Without countable additivity, various classical theorems fail and uniqueness results are quite unusual. A typical situation is that some f.a.p.'s allow to get a certain goal, but other f.a.p.'s are in contrast with such a goal and possibly lead to unreasonable conclusions.

This is certainly true, but two (non independent) remarks are in order. First, in the subjective approach to probability, existence of different f.a.p.'s (modelling different situations) should be viewed as a merit. Second, in the finitely additive theory, one can clearly use σ-additive laws. Merely, one is not obliged to do so.

We refer to [5], [7] and [10] for more on f.a.p.'s and coherence.

4 A finitely additive conditional law of large numbers

We first recall some definitions. Let T be a set and $(S, \mathcal{B})$ a measurable space. A set $\mathcal{P}$ of finite dimensional distributions is meant as

$$\mathcal{P} = \{\, \mu(t_1, \ldots, t_n) : n \geq 1,\, t_1, \ldots, t_n \in T\}$$

where each $\mu(t_1, \ldots, t_n)$ is a probability measure on $\mathcal{B}^n = \mathcal{B} \otimes \ldots \otimes \mathcal{B}$. We write $X \sim \mathcal{P}$ in case $X = \{X_t : t \in T\}$ is a process indexed by T, with state space $(S, \mathcal{B})$, satisfying

$$(X_{t_1}, \ldots, X_{t_n}) \text{ has probability distribution } \mu(t_1, \ldots, t_n)$$

for all $n \geq 1$ and $t_1, \ldots, t_n \in T$. We say that $\mathcal{P}$ is *consistent* in case $X \sim \mathcal{P}$ for some process X. Under mild assumptions on $(S, \mathcal{B})$, necessary and sufficient conditions for $\mathcal{P}$ to be consistent are given by Kolmogorov extension theorem; see e.g. Corollary 52, page 111, of [6].

Let S be a metric space and $\mathcal{B}$ the Borel σ-field. A probability measure Γ on $\mathcal{B}$ is *tight* in case, for each $\epsilon > 0$, there is a compact $K \subset S$ such that $\Gamma(K^c) < \epsilon$. If S is a Polish space, each probability measure on $\mathcal{B}$ is tight.

Let $(T, \mathcal{C}, Q)$ be a probability space. An atom of Q is a set $C \in \mathcal{C}$ such that $Q(C) > 0$ and $Q(\cdot \mid C)$ is 0-1 valued. If Q has no atoms, it is called *nonatomic*. When T is a separable metric space and $\mathcal{C}$ the Borel σ-field, Q is nonatomic if and only if $Q\{t\} = 0$ for all $t \in T$. Next, the outer and inner measures are

$$Q^*(H) = \inf\{Q(C) : H \subset C \in \mathcal{C}\}, \quad Q_*(H) = 1 - Q^*(H^c), \quad H \subset T.$$

Fix $H \subset T$ and a number $\alpha \in [Q_*(H), Q^*(H)]$. Then, Q can be extended to a probability measure Q_0 on $\sigma(\mathcal{C} \cup \{H\})$ such that $Q_0(H) = \alpha$.

We are now in a position to state our main result. Suppose

(i) S is a metric space, $\mathcal{B}$ the Borel σ-field and Γ a tight probability on $\mathcal{B}$;

(ii) $(T, \mathcal{C}, Q)$ is a nonatomic probability space such that $\{t\} \in \mathcal{C}$ for all $t \in T$;

(iii) $\mathcal{P}$ is a consistent set of finite dimensional distributions.

In addition, as in Section 1, let $\mathcal{H} = \{H \in \mathcal{C} : Q(H) > 0\}$.

Theorem 1. *Let $(S, \mathcal{B}, \Gamma)$, $(T, \mathcal{C}, Q)$ and $\mathcal{P}$ be given according to (i)-(iii). Then:*

(a) If $\mathcal{H}_0 \subset \mathcal{H}$ is finite, there are a probability space $(\Omega, \mathcal{A}, P)$ and a process $X = \{X_t : t \in T\}$ with state space $(S, \mathcal{B})$ such that $X \sim \mathcal{P}$ and

$$\text{for each } H \in \mathcal{H}_0, \text{ there is } A_H \in \mathcal{A},\ P(A_H) = 1, \text{ such that}$$
$$X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for } \omega \in A_H;$$

(b) There are a set Ω, a f.a.p. P on the power set of Ω, and a process $X = \{X_t : t \in T\}$ with state space $(S, \mathcal{B})$ such that $X \sim \mathcal{P}$ and condition (2) *holds.*

Proof. Part (a). Since $\mathcal{H}_0$ is finite, $\mathcal{H}_0 \subset \sigma(\Pi) \cap \mathcal{H}$ where Π is the finite partition of T formed by the constituents of the members of $\mathcal{H}_0$. Thus, it can be assumed $\mathcal{H}_0 \subset \sigma(\Pi) \cap \mathcal{H}$ where $\Pi \subset \mathcal{C}$ is any countable partition of T.

Let Ω be the set of functions $\omega : T \to S$ and X the canonical process

$$X(t, \omega) = \omega(t), \quad t \in T,\ \omega \in \Omega.$$

Also, let $\mathcal{G}$ be the σ-field on Ω generated by the maps $\omega \mapsto \omega(t)$ for all $t \in T$. Since $\mathcal{P}$ is consistent, there is a probability measure $\mathbb{P}$ on $\mathcal{G}$ such that $X \sim \mathcal{P}$ under $\mathbb{P}$. Define

$$L = \{\omega \in \Omega : \omega \text{ is measurable and } Q\big(\{t : \omega(t) \in \cdot\} \mid H\big) = \Gamma(\cdot) \text{ for all } H \in \mathcal{H}_0\}.$$

It is enough to prove that $\mathbb{P}^*(L) = 1$. In this case, in fact, it suffices to let $\mathcal{A} = \sigma(\mathcal{G} \cup \{L\})$ and to take P as the extension of $\mathbb{P}$ such that $P(L) = 1$.

To show that $\mathbb{P}^*(L) = 1$, we base on the following fact.

Claim. Let $A \subset \Omega$. Then, $\mathbb{P}^*(A) = 1$ whenever $A \neq \emptyset$ and

$$\omega \in A,\ \omega^* \in \Omega,\ \{t : \omega^*(t) \neq \omega(t)\} \text{ countable} \quad \Longrightarrow \quad \omega^* \in A. \tag{3}$$

To prove the Claim, let S^∞ denote the set of sequences $(x_1, x_2, \ldots)$, with $x_n \in S$ for all n, and $\mathcal{B}^\infty$ the σ-field on S^∞ generated by the canonical projections. Every $G \in \mathcal{G}$ can be written as

$$G = \{\omega \in \Omega : (\omega(t_1), \omega(t_2), \ldots) \in B\} \tag{4}$$

for some $B \in \mathcal{B}^\infty$ and some sequence $(t_1, t_2, \ldots) \subset T$. Fix $G \in \mathcal{G}$ with $G \neq \Omega$ and take B and $(t_1, t_2, \ldots)$ satisfying (4). Since $G \neq \Omega$, then $B \neq S^\infty$. Fix $\omega \in A$, $(x_1, x_2, \ldots) \in B^c$, and define

$$\omega^*(t_n) = x_n \text{ for all } n \geq 1 \text{ and } \omega^* = \omega \text{ on } T \setminus \{t_1, t_2, \ldots\}.$$

Then, $\omega^* \in A$ (by (3)) but $\omega^* \notin G$, so that A is not included in G. Thus, Ω is the only member of $\mathcal{G}$ including A, and this implies $\mathbb{P}^*(A) = \mathbb{P}(\Omega) = 1$.

Next, fix $\omega \in L$ and $\omega^* \in \Omega$ such that $\{t : \omega^*(t) \neq \omega(t)\}$ is countable. Since $\mathcal{C}$ includes the singletons and ω is measurable, ω^* is measurable as well. Since Q is nonatomic, $Q\{t : \omega^*(t) \neq \omega(t)\} = 0$ so that

$$Q\big(\{t : \omega^*(t) \in \cdot\} \mid H\big) = Q\big(\{t : \omega(t) \in \cdot\} \mid H\big) = \Gamma(\cdot) \quad \text{for all } H \in \mathcal{H}_0.$$

Thus, condition (3) holds with $A = L$.

By the Claim, it remains only to check $L \neq \emptyset$. Let $U \in \Pi \cap \mathcal{H}$. Since Q is nonatomic, $Q(\cdot \mid U)$ is nonatomic as well. By Theorem 3.1 of [4], since Γ is tight and $\big(T, \mathcal{C}, Q(\cdot \mid U)\big)$ nonatomic, there is a measurable function $f_U : T \to S$ satisfying $Q\big(f_U \in \cdot \mid U\big) = \Gamma(\cdot)$. Define $\omega = f_U$ on U, for all $U \in \Pi \cap \mathcal{H}$, and ω constant otherwise. Then, $\omega : T \to S$ is measurable. Fix $H \in \mathcal{H}_0$. Since H is a countable union of elements of Π and $Q\big(f_U \in \cdot \mid U\big) = \Gamma(\cdot)$, one obtains

$$\begin{aligned} Q\big(\{t : \omega(t) \in \cdot\} \mid H\big) &= \sum_{U \in \Pi \cap \mathcal{H}} Q\big(U \cap \{t : \omega(t) \in \cdot\} \mid H\big) \\ &= \sum_{U \in \Pi \cap \mathcal{H}} Q\big(\{t : \omega(t) \in \cdot\} \mid U\big)\, Q(U \mid H) \\ &= \sum_{U \in \Pi \cap \mathcal{H}} Q\big(f_U \in \cdot \mid U\big)\, Q(U \mid H) = \Gamma(\cdot). \end{aligned}$$

Therefore, $\omega \in L$.

Part (b). Take $(\Omega, \mathcal{G}, \mathbb{P})$ and X as in Part (a). Also, let $\mathcal{Z}$ denote the set of $[0,1]$-valued functions defined on the power set of Ω. For $H \in \mathcal{H}$, define

$$\begin{aligned} A_H &= \{\omega \in \Omega : \omega \text{ is measurable and } Q\big(\{t : \omega(t) \in \cdot\} \mid H\big) = \Gamma(\cdot)\}, \\ F_H &= \{Z \in \mathcal{Z} : Z \text{ is a f.a.p., } Z = \mathbb{P} \text{ on } \mathcal{G},\ Z(A_H) = 1\}. \end{aligned}$$

Let $\mathcal{Z}$ be equipped with the product topology. Then, $\mathcal{Z}$ is compact. Since convergence of a net of elements of $\mathcal{Z}$ is setwise convergence, then F_H is closed. Let $\mathcal{H}_0 \subset \mathcal{H}$ be finite. By Part (a), there is a (countably additive) probability measure P_0 such that $P_0 = \mathbb{P}$ on $\mathcal{G}$ and $P_0\big(\bigcap_{H \in \mathcal{H}_0} A_H\big) = 1$. Then $Z \in \bigcap_{H \in \mathcal{H}_0} F_H$, where Z is any finitely additive extension of P_0 to the power set of Ω. Hence, $\{F_H : H \in \mathcal{H}\}$ has the finite intersection property, so that

$$\bigcap_{H \in \mathcal{H}} F_H \neq \emptyset.$$

To conclude the proof, it suffices to take any $P \in \bigcap_{H \in \mathcal{H}} F_H$. □

The conditional LLN (2) fails, in the usual (countably additive) setting, whenever Γ is not trivial and the σ-field $\mathcal{C}$ countably generated; see forthcoming Remark 2. Thus, Theorem 1 provides one more example of a problem admitting a finitely additive solution but not a countably additive one. A different opinion is in Subsection 6.2 of [11]. See also [1] and [2].

As apparent from the proof, Part (a) of Theorem 1 holds, even if $\mathcal{H}_0$ is not finite, provided $\mathcal{H}_0 \subset \sigma(\Pi) \cap \mathcal{H}$ where $\Pi \subset \mathcal{C}$ is a countable partition of T.

We finally give the remark announced some lines above.

Remark 2. Let $(S, \mathcal{B}, \Gamma)$ and $(T, \mathcal{C}, Q)$ be probability spaces and $X = \{X_t : t \in T\}$ a process, with state space $(S, \mathcal{B})$, defined on the probability space $(\Omega, \mathcal{A}, P)$. *If condition*

(2) *holds and* $\mathcal{C}$ *is countably generated, then* $\Gamma(B) \in \{0,1\}$ *for all* $B \in \mathcal{B}$. In fact, since $\mathcal{C}$ is countably generated, $\mathcal{C} = \sigma(\mathcal{F})$ for some countable field $\mathcal{F}$. By (2) and since $\mathcal{F}$ is countable, there is $A \in \mathcal{A}$ such that $P(A) = 1$ and

$$X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma(\cdot) \text{ for all } H \in \mathcal{F} \cap \mathcal{H} \text{ and } \omega \in A.$$

Fix $\omega \in A$ and $B \in \mathcal{B}$. Since $\mathcal{F}$ is a field and $\mathcal{C} = \sigma(\mathcal{F})$, it follows that

$$Q\big(H \cap \{X^\omega \in B\}\big) = \Gamma(B)\, Q(H) \quad \text{for all } H \in \mathcal{C}.$$

For $H = T$, one obtains $Q(X^\omega \in B) = \Gamma(B)$. Hence, letting $H = \{X^\omega \in B\}$ yields

$$\Gamma(B) = Q(X^\omega \in B) = \Gamma(B)\, Q(X^\omega \in B) = \Gamma(B)^2.$$

5 An alternative solution

Suppose that, rather than a single probability Γ, we are given a collection

$$\{\Gamma_H : H \in \mathcal{H}\}$$

of probability measures on $\mathcal{B}$. Replacing Γ with $\{\Gamma_H : H \in \mathcal{H}\}$, condition (2) turns into

$$\begin{gathered} \text{for each } H \in \mathcal{H}, \text{ there is } A_H \in \mathcal{A} \text{ with } P(A_H) = 1 \text{ such that} \\ X^\omega \text{ is measurable and } Q\big(X^\omega \in \cdot \mid H\big) = \Gamma_H(\cdot) \text{ for } \omega \in A_H. \end{gathered} \tag{2*}$$

Condition (2*) looks (to us) a reasonable extension of (2). Loosely speaking, given a non negligible subset H of agents, the individual risks cancel on the aggregate H but in a way possibly depending on H; see Section 2. However, apart from the economic interpretation, condition (2*) has a merit with respect to (2). It can be realized in the usual (countably additive) setting.

Theorem 3. *Suppose* $(S, \mathcal{B})$ *is a measurable space,* $(T, \mathcal{C}, Q)$ *and* $\mathcal{P}$ *satisfy conditions (ii)-(iii), and* $\{\Gamma_H : H \in \mathcal{H}\}$ *is of the form*

$$\begin{gathered} \Gamma_H(\cdot) = Q\big(f \in \cdot \mid H\big), \quad H \in \mathcal{H}, \\ \text{for some measurable function } f : T \to S. \end{gathered} \tag{5}$$

Then, there are a probability space $(\Omega, \mathcal{A}, P)$ *and a process* $X = \{X_t : t \in T\}$ *with state space* $(S, \mathcal{B})$ *such that* $X \sim \mathcal{P}$ *and condition* (2*) *holds.*

Proof. We just apply the same argument used in the proof of Part (a) of Theorem 1. Let $(\Omega, \mathcal{G}, \mathbb{P})$ and X be as in such a proof and

$$L = \{\omega \in \Omega : \omega \text{ is measurable and } Q\big(\{t : \omega(t) \in \cdot\} \mid H\big) = \Gamma_H(\cdot) \text{ for all } H \in \mathcal{H}\}.$$

It suffices to prove $\mathbb{P}^*(L) = 1$. Since Q is nonatomic and $\mathcal{C}$ includes the singletons, condition (3) holds with $A = L$. Since $L \neq \emptyset$ (in fact, $f \in L$) an application of the Claim concludes the proof. □

As an example, if $(S, \mathcal{B}) = (T, \mathcal{C})$, one can take $\Gamma_H(\cdot) = Q(\cdot \mid H)$ for all $H \in \mathcal{H}$. This could be tempting in a few situations, for instance when $S = T = [0,1]$ and Q is Lebesgue measure.

A last point is to characterize those families $\{\Gamma_H : H \in \mathcal{H}\}$ satisfying (5). A necessary condition is

$$\Gamma_{\cup_n H_n}(\cdot) = \sum_j Q\big(H_j \mid \cup_n H_n\big)\, \Gamma_{H_j}(\cdot) \tag{6}$$

whenever $H_1, H_2, \ldots \in \mathcal{H}$ are pairwise disjoint. A conjecture is that $\{\Gamma_H : H \in \mathcal{H}\}$ admits representation (5) provided $(T, \mathcal{C}, Q)$ is nonatomic, each Γ_H is tight and condition (6) holds. But we have not a proof.

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