Speed of convergence of the threshold estimator of integrated variance

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Abstract

In this paper we consider a semimartingale model for the evolution of the price of a financial asset, driven by a Brownian motion (plus drift) and possibly infinite activity jumps. Given discrete observations, the threshold estimator is able to separate the integrated variance from the sum of the squared jumps. This has importance in measuring and forecasting the asset risks. The exact convergence speed was found in the literature only when the jumps are of finite variation. Here we give the speed even in presence of infinite variation jumps, as they appear e.g. in some cgmy plus diffusion models.

Keywords: Integrated variance, threshold estimator, convergence speed, infinite activity stable Lévy jumps.

1 Definitions and notation

We consider a semimartingale $(X_t)_{t\in[0,T]}$, defined on a (filtered) probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, P)$ with paths in $D([0,T],\mathbb{R})$, driven by a (standard) Brownian motion W and a pure jump Lévy process L:

$$X_t = x_0 + \int_0^t a_s ds + \int_0^t \sigma_s \, dW_s + L_t, \ t \in]0,T], \tag{1}$$

where a, σ are any adapted càdlàg processes such that (1) admits a unique strong solution X on $[0,T]$ which is adapted and càdlàg [4]. L has Lévy measure ν and may be decomposed as $L_t = J_t + M_t$, where

$$J_t := \int_0^t \int_{|x|>1} x\mu(dx\ ds) = \sum_{\ell=1}^{N_t} \gamma_\ell, \quad M_t := \int_0^t \int_{|x|\le 1} x[\mu(dx\ ds) - \nu(dx)dt]. \tag{2}$$

J is a compound Poisson process representing the "large" jumps of L (and X), i.e. with absolute value larger than one, μ is a Poisson random measure on $[0,T]\times\mathbb{R}$ with intensity measure $\nu(dx)dt$, N is a Poisson process with intensity $\nu(\{x, |x|>1\}) < \infty$, γ_ℓ are IID and independent of N and the martingale M is the compensated sum of small jumps of L. We will denote $\mu(dx,dt) - \nu(dx)dt =: \tilde{\mu}(dx,dt)$ the compensated Poisson random measure associated to μ. We allow for *infinite activity* (IA) jumps, where small jumps of L occur infinitely often, i.e. $\nu(\mathbb{R}) = \infty$. This work contributes to the existing literature ([2], [**?**]) precisely in the case where the jumps have also infinite variation. For a semimartingale Z we denote $\Delta_i Z = Z_{t_i} - Z_{t_{i-1}}$ its increments and $\Delta Z_t = Z_t - Z_{t-}$ its jump at time t.

The *Blumenthal-Getoor (BG) index* of L, defined as

$$\alpha := \inf\{\delta \ge 0, \ \int_{|x|\le 1} |x|^\delta \nu(dx) < +\infty\} \le 2,$$

measures the degree of *activity* of small jumps.

We call $IV = \int_0^T \sigma_u^2 du$ the *integrated variance* of X and denote

$$X_{0t} = \int_0^t a_s ds + \int_0^t \sigma_s dW_s, \qquad X_{1t} = X_{0t} + J_t$$

We will work under the following assumption, which allows us to control the behavior of the small jumps and of the large jumps (remark 1.1 and lemma 2 in [1]).

Assumption A1 L is symmetric α stable.

A1 means that (see [3]) ν has a density of the form $\frac{A}{|x|^{1+\alpha}}$, for some constants $A \in \mathbb{R}$, $\alpha \in]0,2[$. α is the BG index of L.

We observe X_t on a time grid $t_i = ih$, for a given resolution $h = T/n$. Since X is a semimartingale, the *Realized Variance* (RV) $\sum_{i=1}^n (\Delta_i X)^2$ converges in probability (see [11]) to

$$[X]_T := \int_0^T \sigma_t^2 dt + \int_0^T \int_{\mathbb{R}-\{0\}} x^2 \mu(dx, ds).$$

The *threshold estimator* ([7, 8]) of the *integrated variance* $IV = \int_0^T \sigma_t^2 dt$ is based on the idea of summing only some of the squared increments of X, those whose absolute value is smaller than some *threshold* r_h:

$$\hat{IV}_h := \sum_{i=1}^n (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \le r_h\}}. \tag{3}$$

The term $\int_0^T \int_{\mathbb{R}-\{0\}} x^2 \mu(dx,\ ds)$ due to jumps is asymptotically eliminated as $h \to 0$ by an appropriate choice of the threshold. Paul Lévy's law for the modulus of continuity of the Brownian motion paths ([12], p.10) implies

$$P\left(\lim_{h\to 0} \sup_{i\in\{1..n\}} \frac{|\Delta_i W|}{\sqrt{2h \ln \frac{1}{h}}} \le 1 \right) = 1$$

and allows to properly choose a threshold. It is shown in ([8], Cor. 2 and Thm 4) that, under the above assumptions, if we choose a deterministic threshold r_h such that

$$\lim_{h\to 0} r_h = 0 \text{ and } \lim_{h\to 0} \frac{h \ln h}{r_h} = 0 \tag{4}$$

then $\hat{IV}_h \xrightarrow{P} IV$, as $h \to 0$.

Note that the functions $r_h = ch^\beta$ satisfy condition (4) for any $\beta \in]0,1[$ and any constant c.

Remark 1.1. A1 *implies that*

$$\int_{|x|\le c\sqrt{r_h}} x^k \nu(dx) \sim r_h^{\frac{k-\alpha}{2}}, \ \ k = 2,3,4$$

$$\int_{2\sqrt{r_h}<|x|\le 1} x\nu(dx) \sim \left[c + c r_h^{\frac{1-\alpha}{2}}\right] I_{\{\alpha\ne 1\}} + c\left[\ln \frac{1}{2\sqrt{r_h}}\right] I_{\{\alpha=1\}}$$

$$\int_{2\sqrt{r_h}<|x|\le 1} \nu(dx) \sim r_h^{-\alpha/2},$$

where c indicates a generic constant and $f(h) \sim g(h)$ *means that both* $f(h) = O(g(h))$ *and* $g(h) = O(f(h))$ *as* $h \to 0$.

Further notation.

$f(\omega,h)\sim_P g(\omega,h)$ means that $f(\omega,h)=O_P(g(\omega,h))$ and $g(\omega,h)=O_P(f(\omega,h))$ as $h\to 0$.

$$X_{0t}=\int_0^t a_s ds+\int_0^t \sigma_s dW_s, \qquad X_{1t}=X_{0t}+J_t.$$

2 Speed of convergence of $\hat{IV}_h$

Theorem 2.1. *Take $r_h=ch^\beta$, $\beta\in]0,1[$, $c\in\mathbb{R}$. Under* **A1***, as $h\to 0$*

$$\hat{IV}_h-IV\sim_P\sqrt{h}Z_h+r_h^{1-\alpha/2}, \tag{5}$$

where $Z_h\stackrel{st}{\to}\mathcal{N}$. The first term in the rhs is due to the presence of a Brownian component within X, while the last term is led by the sum of the jumps of X smaller in absolute value than $\sqrt{r_h}$.

Proof. Since $X=X_1+M$, we decompose

$$\hat{IV}_h-IV=\textstyle\sum_{i=1}^n(\Delta_iX)^2I_{\{(\Delta_iX)^2\le r_h\}}-IV$$

$$=[\textstyle\sum_{i=1}^n(\Delta_iX_1)^2I_{\{(\Delta_iX_1)^2\le 4r_h\}}-IV]+\sum_{i=1}^n(\Delta_iX_1)^2\big(I_{\{(\Delta_iX)^2\le r_h\}}-I_{\{(\Delta_iX_1)^2\le 4r_h\}}\big)+ \tag{6}$$

$$2\textstyle\sum_{i=1}^n\Delta_iX_1\Delta_iMI_{\{(\Delta_iX)^2\le r_h\}}+\sum_{i=1}^n(\Delta_iM)^2I_{\{(\Delta_iX)^2\le r_h\}}:=\sum_{j=1}^4 I_j(h).$$

Inspection of the proof of Theorem 2 in [8] shows that $I_1(h)/\sqrt{h}$ converges stably in law to a standard Gaussian random variable.

We now show that $I_2(h)=o_P(\sqrt{h})+o_P(r_h^{1-\alpha/2})$. In [2] (Proof of theorem 2.5) it is shown that $I_2(h)/\sqrt{h}$ has the same limit in probability as

$$\frac{\sum_{i=1}^n\big(\int_{t_{i-1}}^{t_i}\sigma_udW_u\big)^2I_{\{(\Delta_iX)^2>r_h,(\Delta_iX_1)^2\le 4r_h\}}}{\sqrt{h}}.$$

Note that this last term equals

$$\frac{1}{\sqrt{h}}\sum_{i=1}^n\Big(\int_{t_{i-1}}^{t_i}\sigma_udW_u\Big)^2I_{\{(\Delta_iX)^2>r_h\}}-\frac{1}{\sqrt{h}}\sum_{i=1}^n\Big(\int_{t_{i-1}}^{t_i}\sigma_udW_u\Big)^2I_{\{(\Delta_iX)^2>r_h,(\Delta_iX_1)^2>4r_h\}}.$$

However the last term is negligible since if $(\Delta_iX_1)^2>4r_h$ then (by (18) in [2]: for any fixed $c>0$ a.s. for sufficiently small h we have $\sup_{i=1..n}|\Delta_iX_0|<c\sqrt{r_h}$) $\Delta_iN\ne 0$ and thus, by (7) in [2] and assuming wlg σ bounded on $\Omega\times[0,T]$,

$$\frac{1}{\sqrt{h}}E[\sum_{i=1}^n\Big(\int_{t_{i-1}}^{t_i}\sigma_udW_u\Big)^2I_{\{(\Delta_iX)^2>r_h,(\Delta_iX_1)^2>4r_h\}}]\le cE[N_T]\sqrt{h}\ln\frac{1}{h}\to 0,$$

as $h\to 0$. So now we deal with $\sum_{i=1}^n\big(\int_{t_{i-1}}^{t_i}\sigma_udW_u\big)^2I_{\{(\Delta_iX)^2>r_h\}}$. If $|\Delta_iJ+\Delta_iM|-|\Delta_iX_0|>|\Delta_iX|>\sqrt{r_h}$ then $|\Delta_iJ+\Delta_iM|>\sqrt{r_h}+|\Delta_iX_0|>\sqrt{r_h}$ and then either $\Delta_iJ\ne 0$ or $\Delta_iM>\sqrt{r_h}/2$. However

$$\frac{\sum_i(\int_{t_{i-1}}^{t_i}\sigma_udW_u)^2I_{\{|\Delta_iJ|\ne 0\}}}{r_h^{1-\alpha/2}}\le\frac{N_Th\ln\frac{1}{h}}{r_h^{1-\alpha/2}}\to 0,$$

and, by (20) in [2],

$$\frac{E[\sum_i(\int_{t_{i-1}}^{t_i}\sigma_u dW_u)^2 I_{\{|\Delta_i M|>\sqrt{r_h}/2\}}]}{r_h^{1-\alpha/2}} \le h\ln\frac{1}{h}\frac{\sum_i P\{|\Delta_i M|>\sqrt{r_h}/2\}}{r_h^{1-\alpha/2}} \sim \frac{h^{1-\alpha\beta/2}}{r_h^{1-\alpha/2}}\ln\frac{1}{h}\to 0.$$

Then our result on $I_2(h)$ behavior is reached.

In [2] (Proof of theorem 2.5) it is shown that $I_3(h)/\sqrt{h}\xrightarrow{P} 0$.

We now show that $I_4(h)$ has the same asymptotic behavior as $r_h^{1-\alpha/2}$. Fix any $q>1$ and define

$$\tilde N_s = \sum_{u\le s} I_{\{|\Delta X_u|>\frac{\sqrt{r_h}}{q}\}},$$

$$\xi_{ni} := \left(\int_{t_{i-1}}^{t_i}\int_{|x|\le\frac{\sqrt{r_h}}{q}} x\tilde\mu(dx,dt) - h\int_{\frac{\sqrt{r_h}}{q}<|x|\le 1} x\nu(dx)\right)^2.$$

We can write

$$I_4(h)=\sum_{i=1}^n(\Delta_i M)^2 I_{\{(\Delta_i X)^2\le r_h\}} = \sum_{i=1}^n(\Delta_i M)^2\left[I_{\{\Delta_i\tilde N=0\}} - I_{\{\Delta_i\tilde N=0,(\Delta_i X)^2>r_h\}} + I_{\{\Delta_i\tilde N\ge1,(\Delta_i X)^2\le r_h\}}\right].$$

On $\{\Delta_i\tilde N=0\}$ the squared increment $(\Delta_i M)^2$ equals ξ_{ni}, so we can write the rhs term above as

$$\sum_i\xi_{ni}-\sum_i\xi_{ni}I_{\{\Delta_i\tilde N\ge1\}}-\sum_i\xi_{ni}I_{\{\Delta_i\tilde N=0,(\Delta_i X)^2>r_h\}}+\sum_i(\Delta_i M)^2 I_{\{\Delta_i\tilde N\ge1,(\Delta_i X)^2\le r_h\}} \doteq \sum_{k=1}^4 I_{4,k}(h).$$

We are now going to show that

$$I_{4,1}(h)=\sum_i\xi_{ni}$$

is the leading term of $I_4(h)$ and that it has the same asymptotic behavior as

$$nE[\xi_{n1}]\sim r_h^{1-\alpha/2}.$$

In fact theorem 2.4 in [2] states the following CLT

$$\frac{\sum_i\xi_{ni}-nE[\xi_{n1}]}{\sqrt{nVar[\xi_{n1}]}}\xrightarrow{d}\mathcal N.$$

Since $nE[\xi_{n1}]\sim r_h^{1-\alpha/2}+h(1-r_h^{\frac{1-\alpha}{2}})^2 I_{\{\alpha\ne1\}}+h\ln^2\frac{1}{\sqrt{r_h}}I_{\{\alpha=1\}}\sim r_h^{1-\alpha/2}\to 0$, and $\sqrt{nVar[\xi_{n1}]}\sim r_h^{1-\alpha/4}\to 0$ we reach that $I_{4,1}(h)=\sum_i\xi_{ni}$ tends to zero in probability at speed $r_h^{1-\alpha/2}$.

We now show that $I_{4,2}(h)=-\sum_i\xi_{ni}I_{\{\Delta_i\tilde N\ge1\}}$ is negligible wrt $r_h^{1-\alpha/2}$. In fact, by the independence of ξ_{ni} on $\{\Delta_i\tilde N\ge1\}=\Big\{\mu\big(\{|x|>\sqrt{r_h}/q\}\times]t_{i-1},t_i]\ \big)\ge1\Big\}$, we have

$$\frac{E[\Big|\sum_i\xi_{ni}I_{\{\Delta_i\tilde N\ge1\}}\Big|]}{r_h^{1-\alpha/2}}\le\frac{nE[|\xi_{ni}|]P\{\Delta_i\tilde N\ge1\}}{r_h^{1-\alpha/2}}\sim\frac{n\sqrt{E[\xi_{ni}^2]}\ \theta}{r_h^{1-\alpha/2}}$$

$$\sim\frac{n\sqrt{hr_h^{2-\alpha/2}}\theta}{r_h^{1-\alpha/2}}=(hr_h^{-\alpha/2})^{1/2}\to 0,$$

where $\theta = h^{1-\alpha\beta/2} = hr_h^{-\alpha/2}$.

Now we prove that also $I_{4,3}(h) = -\sum_i \xi_{ni} I_{\{\Delta_i \tilde{N}=0,(\Delta_i X)^2>r_h\}}$ is negligible wrt $r_h^{1-\alpha/2}$.
First note that we take $\eta > 0 : 1/q + \eta < 1$. Now on $\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}$ we necessarily have that $|\Delta_i M| > \sqrt{r_h}/q$, because otherwise, for sufficiently small h we would have $|\Delta_i X_0 + \Delta_i M| < \eta\sqrt{r_h} + \sqrt{r_h}/q \doteq \sqrt{r_h}(1-\gamma)$ and

$$|\Delta_i J| = |\Delta_i X - \Delta_i M - \Delta_i X_0| \geq |\Delta_i X| - |\Delta_i M + \Delta_i X_0| > \sqrt{r_h}\gamma > 0,$$

implying that $|\Delta_i J| \geq 1$, which is impossible since J only moves by jumps bigger than 1, while $\Delta_i \tilde{N} = 0$ indicates that no jumps bigger than $\sqrt{r_h}/q < 1$ happened.
Second, note that on the set where X has no jumps bigger than $\sqrt{r_h}/q$, the same is for M and for L, and $P\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\} \leq P\{\Delta_i \tilde{N} = 0, |\Delta_i M| > \sqrt{r_h}/q\} = P\{\tilde{N}_h = 0, |M_h| > \sqrt{r_h}/q\}$, by the Lévy property of M, and this equals $P\{\tilde{N}_h = 0, |L'_h| > \sqrt{r_h}/q\}$, where L' is the L process deprived of its jumps bigger in absolute value than $\sqrt{r_h}/q$, since $M_0 = L'_0 = 0$ and M, L' move only by jumps, but on the given set they made no jumps bigger than $\sqrt{r_h}/q$, so they made the same jumps and $M_h = L'_h$. Moreover last probability is dominated by $P\{|L'_h| > \sqrt{r_h}/q\} \sim \theta^{4/3}$, by [1], end of proof of Lemma 2 (with β there in place of α here, $\delta/2$ there in place of $\sqrt{r_h}/q$ here, Y there in place of L here, Y'' there in place of L' here[1]). We then reach that $P\{|\Delta_i L'| > \sqrt{r_h}/4\} \leq K\theta^{4/3}$, and thus

$$\frac{E[|I_{4,3}(h)|]}{r_h^{1-\alpha/2}} \leq \frac{E[\sum_i |\xi_{ni}| I_{\{\Delta_i \tilde{N}=0,(\Delta_i X)^2>r_h\}}]}{r_h^{1-\alpha/2}}$$

$$\leq \frac{\sum_i \sqrt{E[\xi_{ni}^2]}\sqrt{P\{\Delta_i \tilde{N} = 0, (\Delta_i X)^2 > r_h\}}}{r_h^{1-\alpha/2}} \leq c\frac{n\sqrt{hr_h^{2-\alpha/2}}\theta^{2/3}}{r_h^{1-\alpha/2}} \to 0$$

Finally we show that $I_{4,4} = \sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}\geq 1,(\Delta_i X)^2\leq r_h\}}$ is also negligible wrt $r_h^{1-\alpha/2}$. In fact we decompose

$$\frac{\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}\geq 1,(\Delta_i X)^2\leq r_h\}}}{r_h^{1-\alpha/2}} =$$

$$\frac{1}{r_h^{1-\alpha/2}} \sum_i (\Delta_i M)^2 \left[I_{\{\Delta_i \tilde{N}=1,|\Delta X_{\bar{s}}|\leq 1,(\Delta_i X)^2\leq r_h\}} + I_{\{\Delta_i \tilde{N}=1,|\Delta X_{\bar{s}}|>1,(\Delta_i X)^2\leq r_h\}} + I_{\{\Delta_i \tilde{N}\geq 2,(\Delta_i X)^2\leq r_h\}} \right], \tag{7}$$

where $\bar{s}$ is the time instant of the unique jump of X bigger than $\sqrt{r_h}/q$ within $]t_{i-1}, t_i]$ when $\Delta_i \tilde{N} = 1$.

Let us deal with the first term above. Note that, for small h, $\sqrt{r_h}/q < 1$ so on $\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1\}$ within $]t_{i-1}, t_i]$ we only have jumps less than one so that $\Delta_i J = 0$. Fix now any $p > 0$. If also $(\Delta_i X)^2 \leq r_h$ then for sufficiently small h we have $\sup_i |\Delta_i X_0| < p\sqrt{r_h}$ and $\sqrt{r_h} \geq |\Delta_i X| = |\Delta_i X_0 + \Delta_i M| > |\Delta_i M| - |\Delta_i X_0|$, so $|\Delta_i M| < \sqrt{r_h} + |\Delta_i X_0| \leq \sqrt{r_h}(1+p)$ uniformly in $i = 1..n$. Thus $\{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, |\Delta_i X| \leq \sqrt{r_h}\} \subset \{\Delta_i \tilde{N} = 1, |\Delta X_{\bar{s}}| \leq 1, |\Delta_i M| \leq \sqrt{r_h}(1+p)\}$, and the probability of this last set equals $P\{\tilde{N}_h = 1, |\Delta M_{\bar{s}}| \leq 1, |M_h| \leq \sqrt{r_h}(1+p)\}$ by the Lévy property of M, and in turn this equals $P\{\tilde{N}_h = 1, |\Delta L_{\bar{s}}| \leq 1, |L_h| \leq \sqrt{r_h}(1+p)\}$, since $M_0 = L_0 = 0$ and M, L move only by jumps, but on the given set they made only jumps smaller than one and so they made the same jumps. Moreover last probability is dominated by

$$P\{\tilde{N}_h = 1, |L_h| \leq \sqrt{r_h}(1+p), |\Delta L_{\bar{s}}| > \sqrt{r_h}(1+p)\} + P\{\tilde{N}_h = 1, |L_h| \leq \sqrt{r_h}(1+p), |\Delta L_{\bar{s}}| \leq \sqrt{r_h}(1+p)\}$$

[1]Within the last part of the proof of Lemma 2 in [1] we noticed a minor misprint which, however, is corrected by simply replacing $D' = \{|Y'| > \delta/2\}$ with $\tilde{D}' = \{|Y''| > \delta/2\}$, and does not substantially affect the statement in the Lemma.

$$\leq P\{\tilde{N}'_h = 1, |L_h| \leq \sqrt{r_h}(1+p)\} + P\{\tilde{N}''_h = 1\}, \tag{8}$$

where $\tilde{N}'_h \doteq \sum_{u\leq h} I_{\{|\Delta L_u|>\sqrt{r_h}(1+p)\}}$, and $\tilde{N}''_h \doteq \sum_{u\leq h} I_{\{|\Delta L_u|\in]\sqrt{r_h}/q,\sqrt{r_h}(1+p)]\}}$. The first term of (8), by lemma 2 in [1], is $O(\theta^{4/3})$. As for the second one we have

$$P\{\tilde{N}''_h = 1\} = h\int_{\sqrt{r_h}/q}^{\sqrt{r_h}(1+p)} 1\nu(dx) \sim \theta[q^\alpha - (1+p)^{-\alpha}],$$

so that

$$\frac{1}{r_h^{1-\alpha/2}} E\Big[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}=1, |\Delta X_{\bar{s}}|\leq 1, (\Delta_i X)^2\leq r_h\}}\Big] \leq$$

$$\frac{r_h(1+p)^2}{r_h^{1-\alpha/2}}\Big(nP\{\tilde{N}'_h = 1, |L_h| \leq \sqrt{r_h}(1+p)\} + nP\{\tilde{N}''_h = 1\}\Big)$$

$$\leq c r_h^{\alpha/2}\Big(n\theta^{4/3} + n\theta[q^\alpha - (1+p)^{-\alpha}]\Big) = o(1) + q^\alpha - (1+p)^{-\alpha}.$$

So we obtained that for any $q > 1, p > 0$, for sufficiently small h

$$\frac{1}{r_h^{1-\alpha/2}} E[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}=1, |\Delta X_{\bar{s}}|\leq 1, (\Delta_i X)^2\leq r_h\}}] \leq q^\alpha - (1+p)^{-\alpha}.$$

Letting $q \to 1$ and $p \to 0$ we find that

$$\lim_{h\to 0} \frac{1}{r_h^{1-\alpha/2}} E[\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}=1, |\Delta X_{\bar{s}}|\leq 1, (\Delta_i X)^2\leq r_h\}}] = 0.$$

Let us now deal with the second term within (7). If $|\Delta X_{\bar{s}}| > 1$ then $\Delta_i J \neq 0$. If also $|\Delta_i X| \leq \sqrt{r_h}$ then for sufficiently small h we have $|\Delta_i M| > \sqrt{r_h}$ uniformly on i. In fact $|\Delta_i J + \Delta_i M| - |\Delta_i X_0| < |\Delta_i X| \leq \sqrt{r_h}$ so that $|\Delta_i J + \Delta_i M| < |\Delta_i X_0| + \sqrt{r_h}$, which, for any positive p, for sufficiently small h, is dominated by $\sqrt{r_h}(1+p)$. Moreover, since $|\Delta X_{\bar{s}}| > 1$ and for sufficiently small h in each $]t_{i-1}, t_i]$ at most one jump occurred, $|\Delta_i J| - |\Delta_i M| < |\Delta_i J + \Delta_i M| \leq \sqrt{r_h}(1+p)$ implies $|\Delta_i M| > |\Delta_i J| - \sqrt{r_h}(1+p) > 1 - \sqrt{r_h}(1+p) > \sqrt{r_h}$, for sufficiently small h, uniformly on i. As a consequence

$$P\left(\frac{1}{r_h^{1-\alpha/2}}\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}=1, |\Delta X_{\bar{s}}|>1, (\Delta_i X)^2\leq r_h\}} \neq 0\right) \leq nP(\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}) \to 0$$

by Lemma 6.1 ii) in [2].

Finally, we consider the last term in (7). On $|\Delta_i X| \leq \sqrt{r_h}$ either we have $\Delta_i J = 0$, and consequently $|\Delta_i M| \leq \sqrt{r_h}(1+p)$, or we have $\Delta_i J \neq 0$, and then as before $|\Delta_i M| > \sqrt{r_h}$. Therefore

$$\frac{1}{r_h^{1-\alpha/2}}\sum_i (\Delta_i M)^2 I_{\{\Delta_i \tilde{N}\geq 2, (\Delta_i X)^2\leq r_h\}} =$$

$$\sum_i \frac{(\Delta_i M)^2}{r_h^{1-\alpha/2}}\Big(I_{\{\Delta_i \tilde{N}\geq 2, (\Delta_i X)^2\leq r_h, \Delta_i J=0, |\Delta_i M|\leq (1+p)\sqrt{r_h}\}} + I_{\{\Delta_i \tilde{N}\geq 2, (\Delta_i X)^2\leq r_h, \Delta_i J\neq 0, |\Delta_i M|>\sqrt{r_h}\}}\Big).$$

The expectation of the first term is dominated by $\frac{r_h(1+p)^2}{r_h^{1-\alpha/2}} n\theta^2 \to 0$, being $P\{\Delta_i \tilde{N} \geq 2\} \leq c\theta^2$, while the probability that the second term differs from zero, similarly as before, is dominated by $nP\{\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}\} \to 0$.

Therefore, $I_4 \sim_P r_h^{1-\alpha/2}$ is proved.

We can summarize as follows

$$\hat{IV}_h - IV \sim_P \sqrt{h} Z_h + o_P(\sqrt{h}) + r_h^{1-\alpha/2} + o_P(r_h^{1-\alpha/2}),$$

where the first term in the rhs comes from I_1 and is due to the presence of a Brownian component within X, while the third term is determined by I_4, which is led by $nE[\xi_{n1}]$, where in turn the main term is the sum of the jumps of X smaller in absolute value than $\sqrt{r_h}$, and our theorem is proved. □

Corollary 2.2. *Under* **A1** *we have*

$$\begin{cases} \text{if } \sigma \equiv 0 \text{ then} & \hat{IV}_h - IV \sim_P r_h^{1-\alpha/2} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta > \frac{1}{2-\alpha} \text{ then} & \frac{\hat{IV}_h - IV}{\sqrt{h}} \stackrel{st}{\to} \mathcal{N} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha < 1, \beta \le \frac{1}{2-\alpha} \text{ then} & \hat{IV}_h - IV \sim_P r_h^{1-\alpha/2} \\ \text{if } \sigma \not\equiv 0 \text{ and } \alpha \ge 1 \text{ then} & \hat{IV}_h - IV \sim_P r_h^{1-\alpha/2}. \end{cases} \tag{9}$$

□

Proof. If $\sigma \not\equiv 0$ note that as $h \to 0$

$$\frac{\sqrt{h}}{r_h^{1-\alpha/2}} = h^{\frac{1}{2}-\beta(1-\frac{\alpha}{2})} \to \begin{cases} 0 & \text{if } \alpha \ge 1 \\ +\infty & \text{if } \alpha < 1 \text{ and } \beta > \frac{1}{2-\alpha} \in]\frac{1}{2}, 1[\end{cases}$$

since the h exponent above is positive iff $\beta < \frac{1}{2-\alpha}$, which is always the case when $\alpha \ge 1$, since $\frac{1}{2-\alpha} \in]1, \infty[$, while the exponent is negative when $\alpha < 1$ and β is close to one, since $\frac{1}{2-\alpha} \in]1/2, 1[$. Therefore if $\alpha \ge 1$ we have $\sqrt{h} Z_h + o_P(\sqrt{h}) = o_P(r_h^{1-\alpha/2})$ and

$$\hat{IV}_h - IV \sim_P r_h^{1-\alpha/2}.$$

If $\alpha < 1$ and β is close to 1 ($\beta > \frac{1}{2-\alpha}$) then $r_h^{1-\alpha/2} + o_P(r_h^{1-\alpha/2}) = o_P(\sqrt{h})$ and

$$\hat{IV}_h - IV \sim_P \sqrt{h} Z_h.$$

If $\alpha < 1$ and $\beta \le \frac{1}{2-\alpha}$ then $\sqrt{h} = O(r_h^{1-\alpha/2})$ and

$$\hat{IV}_h - IV \sim_P r_h^{1-\alpha/2}.$$

We now consider the case $\sigma \equiv 0$. Recall decomposition (6). We have $IV \equiv 0$ and that $I_1(h) = O_p(h)$. In fact

$$I_1(h) = \sum_i (\Delta_i X_1)^2 I_{\{(\Delta_i X_1)^2 \le 4 r_h\}} \tag{10}$$

and, assuming wlg a bounded on $\Omega \times [0, T]$, we have that for sufficiently small h, for all $i = 1..n$, $I_{\{(\Delta_i X_1)^2 \le 4 r_h\}} = I_{\{\Delta_i N = 0\}}$, since if $|\Delta_i J| - |\int_{t_{i-1}}^{t_i} a_u du| < |\int_{t_{i-1}}^{t_i} a_u du + \Delta_i J| = |\Delta_i X_1| \le 2\sqrt{r_h}$ then $|\Delta_i J| \le 2\sqrt{r_h} + |\int_{t_{i-1}}^{t_i} a_u du| = O_P(\sqrt{r_h}) \to 0$ and then, for sufficiently small h, $\Delta_i J = 0$. If otherwise $|\Delta_i J| + |\int_{t_{i-1}}^{t_i} a_u du| \ge |\int_{t_{i-1}}^{t_i} a_u du + \Delta_i J| = |\Delta_i X_1| > 2\sqrt{r_h}$ then $|\Delta_i J| > 2\sqrt{r_h} - |\int_{t_{i-1}}^{t_i} a_u du| > 0$ and $\Delta_i J \ne 0$. Therefore in (10) we have

$$\sum_i (\Delta_i X_1)^2 I_{\{\Delta_i N = 0\}} = \sum_i (\int_{t_{i-1}}^{t_i} a_u du)^2 (1 - I_{\{\Delta_i N \ne 0\}}) = O_P(h).$$

Now we show that $I_2(h) = o_P(h)$. In fact, similarly as for I_2 in the proof of theorem 2.5 in [2] on $\{(\Delta_i X_1)^2 > 4r_h, (\Delta_i X)^2 \le r_h\}$ we have $\Delta_i N \neq 0$ and $|\Delta_i M| > \sqrt{r_h}$ so

$$P\Big\{\frac{1}{h}\sum_{i=1}^{n}(\Delta_i X_1)^2 I_{\{(\Delta_i X)^2 \le r_h, (\Delta_i X_1)^2 > 4r_h\}} \neq 0\Big\} \le nP\{\Delta_i N \neq 0, |\Delta_i M| > \sqrt{r_h}\} \to 0.$$

Moreover on $\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 \le 4r_h\}$ we have $\Delta_i N = 0$, so $\Delta_i X = \int_{t_{i-1}}^{t_i} a_u du + \Delta_i M$, $\Delta_i X_1 = \int_{t_{i-1}}^{t_i} a_u du$ and $|\int_{t_{i-1}}^{t_i} a_u du| + |\Delta_i M| > |\Delta_i X| > \sqrt{r_h}$ implying that, for sufficiently small h, $|\Delta_i M| > c\sqrt{r_h}$. Therefore

$$\frac{1}{h}\sum_{i=1}^{n}(\Delta_i X_1)^2 I_{\{(\Delta_i X)^2 > r_h, (\Delta_i X_1)^2 \le 4r_h\}} \le \frac{\sum_i (\int_{t_{i-1}}^{t_i} a_u du)^2 I_{\{|\Delta_i M| > c\sqrt{r_h}\}}}{h} = O_P(\theta) \to_P 0.$$

We then have $I_4(h) \sim_P r_h^{1-\alpha/2}$, as in the proof of the previous theorem.

Finally we see that $I_3(h) = \sum_{i=1}^n \Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \le r_h, \Delta_i J \neq 0\}} + \sum_{i=1}^n \Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \le r_h, \Delta_i J = 0\}}$ $= o_P(r_h^{1-\alpha/2})$. In fact, similarly as for I_3 in the proof of theorem 2.5 in [2] we have that if $\Delta_i J \neq 0$ then $|\Delta_i X_1| > \sqrt{r_h}$ and if further $|\Delta_i X| \le \sqrt{r_h}$ then $|\Delta_i M| > \sqrt{r_h}$ and then

$$P\left\{\frac{1}{r_h^{1-\alpha/2}}\sum_{i=1}^{n}\Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \le r_h, \Delta_i J \neq 0\}} \neq 0\right\} \le nP\{\Delta_i J \neq 0, |\Delta_i M| > \sqrt{r_h}\} \to 0.$$

Moreover

$$\sum_{i=1}^{n}\Delta_i X_1 \Delta_i M I_{\{(\Delta_i X)^2 \le r_h, \Delta_i J = 0\}} = \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} a_u du\ \Delta_i M I_{\{(\Delta_i X)^2 \le r_h, \Delta_i J = 0\}}$$

$$\le \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} a_u du\ \Delta_i M I_{\{(\Delta_i X)^2 \le r_h\}} \le \sqrt{\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i} a_u du)^2}\sqrt{\sum_i (\Delta_i M)^2 I_{\{(\Delta_i X)^2 \le r_h\}}}$$

$$= \sqrt{\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i} a_u du)^2}\sqrt{I_4(h)}$$

and

$$\frac{1}{r_h^{1-\alpha/2}}\sqrt{\sum_{i=1}^{n}(\int_{t_{i-1}}^{t_i} a_u du)^2}\sqrt{I_4(h)} = O_P\left(\sqrt{\frac{h}{r_h^{1-\alpha/2}}}\right) \to 0.$$

Summarizing, when $\sigma \equiv 0$ we have

$$\hat{IV}_h - IV = \sum_{j=1}^{4} I_j(h) \sim_P h + o_P(h) + o_P(r_h^{1-\alpha/2}) + r_h^{1-\alpha/2} \sim r_h^{1-\alpha/2}.$$

and the final behavior of the estimation error is determined by $I_4(h)$. □

Remarks.

i) When $\alpha < 1$ and $\beta > 1/(2-\alpha)$ result (9) is consistent with [2] and [5] where, under some different assumptions on X in the two cases, we find that in presence of a Brownian part within X and for threshold exponent β sufficiently close to 1, $\hat{IV}_h - IV/\sqrt{h} \stackrel{st}{\to} \mathcal{N}$.

ii) Result (9) is also consistent with [2] and [5] when $\alpha \ge 1$ and in presence of a Brownian component within X. In fact in [2] we have that $\frac{\hat{IV}_h - IV}{\sqrt{h}} \stackrel{P}{\to} +\infty$ and in [5] we have that $\frac{\hat{IV}_h - IV}{r_h^{1-s/2}} \stackrel{P}{\to} 0$, for all exponents s such that $\int 1 \wedge |x|^s \nu(dx) < \infty$, i.e. for all $s > \alpha$.

iii) The new feature here is giving the exact speed at which the estimation error $\hat{IV}_h - IV$ converges to zero when $\alpha \geq 1$, both in presence and in absence of a Brownian component, and when $\alpha < 1$ in absence of it. Such a speed depends both on the jump activity index α of X and on the threshold exponent β.

iv) In the bivariate case things are more complicated. Given two processes such that $dX_t^{(q)} = a_t^{(q)} dt + \sigma_t^{(q)} dW_t^{(q)} + dL_t^{(q)}, q = 1, 2,$ for $t \in [0, T]$, where $W_t^{(2)} = \rho_t W_t^{(1)} + \sqrt{1 - \rho_t^2} W_t^{(3)}$ with independent Brownian motions $W^{(1)}$ and $W^{(3)}$, the speed of convergence of the threshold estimator $\hat{IC}_h = \sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \leq r(h)\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \leq r(h)\}}$ to the integrated covariance $IC = \int_0^T \rho_t \sigma_t^{(1)} \sigma_t^{(2)} dt$ turns out to have some common features with the univariate case ([9]), but a complete framework has to separately account many different cases. More precisely: in presence of Brownian parts and $\alpha_1, \alpha_2 < 1$ we still have $\frac{\hat{IV}_h - IV}{\sqrt{h}} \xrightarrow{st} \mathcal{N}$ (see [10], [6]). Otherwise the speed still depends on the jumps of both $M^{(1)}$ and $M^{(2)}$ smaller in absolute value than the threshold, but now such a speed is different according to on different relations among $\alpha_1, \alpha_2, \beta$ and to the presence or absence of a new parameter γ measuring the degree of dependence among the jumps of the two components ([9]).

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