Endogenous household formation and inefficiency in a general equilibrium model

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Abstract

The main purpose of the paper is to show that the process of household formation in a competitive market does not necessarily lead to outcomes that are efficient at the economy level, even assuming that members of each household take efficient collective consumption decisions. To this end, we consider a generalization of the Arrow-Debreu exchange economy model in which endogenous household formation is introduced, we assume efficient household decision processes, and we show that if there are many households which can potentially be formed, then there is a not negligible set of economies admitting inefficient equilibrium allocations.

Keywords: General Equilibrium; Bargaining; Endogenous household formation; Efficiency; Pareto Optimality; Asymmetric Nash bargaining solution.

JEL classification: C71, C78, D13, D50.

1 Introduction

The main goal of the paper is to present a general equilibrium model of endogenous household formation in which members of each household take efficient collective consumption decisions but efficiency of market outcomes is not necessarily obtained. To this end, we consider a generalization of the Arrow-Debreu exchange economy model in which rational individuals compete both for resources and for membership to households. After having introduced suitable definitions of equilibrium, efficient allocation and efficient household demand, we show that if individuals have the possibility to choose the household they want to belong to among many, that is, individuals have many outside options, then efficiency of household decisions does not imply efficiency of equilibrium allocations.

Before describing in details the model, we observe that only few contributions about the analysis of endogenous household formation in a general equilibrium context are available in the literature. In the models of Club theory (Ellickson et al. (1999)), where individuals exchange private goods and memberships to clubs, equilibria exist and associated allocations are efficient. The general approach presented by Gersbach and Haller (2005) and related articles quoted therein, as well as the viewpoint proposed by Gori and Villanacci (2009), is very close to ours. A brief description of those works is useful for a better understanding and assessment of our results.

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The presence of positive externalities can be easily recognized as the basic source of household formation. Indeed, the formation of some households may affect individual utilities or allow the production of a certain amount of goods. Therefore, if some individuals realize that forming a household produces positive externalities, then they might reach an agreement on staying together in order to improve their welfare. Externalities are of three main types: utility of one individual is affected by some characteristic of the household she belongs to (group externality), utility of one individual depends on the consumption of other members of her household (consumption externality), and the aggregate endowment of one household is different from the sum of the endowments owned by its members before forming an alliance (endowment externality).

In their papers Gersbach and Haller present a general approach to the household formation process using all externalities listed above, but focusing in particular on the first two. In their models, they describe household choice rules by not specified and exogenously given demand correspondences and propose several notions of equilibrium. The common feature of all those notions is that dissatisfied individuals have the option to leave their household and form other households but nobody wants to exercise that option. The only difference among those notions of equilibrium consists in the different options individuals may choose from. More precisely, Gersbach and Haller consider three types of options: each individual can leave her household and stay alone (free exit), each individual can join a household (free joining) and each group of individuals can form a new household (free household formation)[1]. Then they consider different types of equilibria according to the stability properties satisfied by household allocations. Finally, they introduce a suitable notion of efficiency at the economy level, namely Full Pareto Optimality, which is a modified version of the very standard notion of Pareto Optimality taking into account the variable structure of households. In the described framework, the authors investigate whether endogenous household formation, increasing competition among individuals, makes efficiency easier to get.

The contributions by Gersbach and Haller provide interesting results. In particular, Gersbach and Haller (2003) propose a model with group externalities, households making efficient decisions and in which free exit is allowed and they present an example of an economy for which there are Pareto rankable equilibrium allocations associated with the same equilibrium price. In particular, at least one of them is not Fully Pareto Optimal. This result suggests that, for equilibria with free exit, "efficiency (at the household level) may beget inefficiency (at the economy level)". Gersbach and Haller (2005) prove instead that if household demand correspondences are efficient and other very mild conditions are assumed, then every equilibrium with free household formation is efficient at the economy level. In other words, for equilibria with free household formation, "efficiency (at the household level) begets inefficiency (at the economy level)". Moreover, in the same paper, they also give sufficient, strong conditions under which not necessarily efficient household decision rules lead to efficient choices in equilibrium.

Gori and Villanacci (2009) consider a model in which individuals in the market are exogenously divided in pairs and each pair can produce a given amount of goods beyond initial individual endowments. Production does take place if an agreement on the distribution of the potential surplus is reached by the two members as a result of a bargaining game. If an agreement on the distribution is (not) reached, the household is (not) formed. When bargaining is over, individuals are left with their personal initial endowments plus the, possibly zero, share of household production. At that stage, they behave as price taking consumers in a standard Arrow-Debreu exchange economy. The authors introduce then a suitable notion of equilibrium concept which is consistent with the following

[1]Note that the name we choose for the third stability condition is different from the one in Gersbach and Haller (2003) and (2005). In fact, in those papers the authors call *free household formation* the union of free exit and free joining conditions, and describe the third one saying that *no group of consumers can benefit from forming a new household*.

requirements: individuals are utility maximizers in the process of household formation and rules of that process allow households which are advantageous for their members not to be necessarily formed. Then, they show existence of equilibria in which decision rules of household formed in equilibrium are efficient but equilibrium allocations are inefficient at the economy level. In other words, "efficiency (at the household level) may beget inefficiency (at the economy level)" as already highlighted in one of their models by Gersbach and Haller. Indeed, Gori and Villanacci (2009) support that conclusion in a quite different equilibrium model, where a different type of externality is considered, household formation is the result of an explicitly modeled strategic interaction, and lack of efficiency at the economy level arises because of rational disagreement among individuals within households.

We can now describe the main features of our model. We consider a finite number of individuals who are allowed to form only households belonging to an exogenously given set, called constitution. In general, constitution may differ from the power set of the set of individuals because of physical constraints or laws preventing the formation of some households. Note that this aspect is not present in the quoted papers by Gersbach and Haller where no constraint on the household formation is considered. As in Gori and Villanacci (2009), we assume that each household having at least two members can produce a given amount of commodities beyond initial endowments of its members. Group and consumption externalities are not introduced and then the unique motive to form households is a positive endowment externality. As in Gersbach and Haller, we assume there are exogenously given household demand correspondences describing the outcomes of household decision processes.

The definition of efficiency at the economy level we consider agrees with the notion of Full Pareto Optimality introduced by Gersbach and Haller (2003). The notion of equilibrium we propose is instead different from the ones introduced in the models previously described. In fact, agreeing with Gersbach and Haller (2005), we think that every reasonable definition of equilibrium has to require a suitable stability condition for households[2] to hold true. However, in our opinion, definitions introduced by Gersbach and Haller present some conceptual difficulties. In fact, free exit and free joining conditions, even when considered together, are not enough to thoroughly describe the mechanism of household formation. Moreover, free joining and free household conditions implicitly assume that if members of a certain household realize that an affordable collective consumption is more advantageous than the candidate equilibrium consumption, then they always agree to demand it. However, in general, the set of collective consumptions on which household members really agree is much smaller than the set of affordable collective consumptions. In fact, because of disagreement among household members, even though an affordable collective consumption makes everybody better off, it might not be really chosen. That suggests that Gersbach and Haller's stability conditions need revision[3].

The definition of equilibrium we propose tries to overcome the just mentioned difficulties by requiring a modification of the free household formation property to be satisfied. Our equilibrium concept is simply described as follows. Given commodity prices, individuals divide into households belonging to the constitution and household members decide household demand. In equilibrium, total household demands do not exceed total supply, formed households have no incentive to modify their demands and individuals have no incentive to form new households. Of course, according to the above observations, members of a household can deviate from the equilibrium resource and household allocation only if they can really find an agreement on an advantageous collective consumption, that is, only if the advantageous collective consumption is found via the household demand

[2]Note that the definition of equilibrium in Gori and Villanacci (2009) requires no explicit stability condition because of the special structure of the set of potential households.

[3]For further comments on the topic, see Section 2.

correspondence.

The equilibrium concept we propose has interesting welfare implications. As for the equilibrium concept by Gersbach and Haller, efficiency may not occur in equilibrium unless we provide suitable assumptions on the way households make their choices. Then, we specialize on household demand correspondences satisfying the so called internal efficiency property. A household has an internally efficient demand correspondence if it cannot afford consumption vectors for its members that Pareto improve upon any household demand, that is, any possible outcome of the household decision process. Such property, first introduced by Haller (2000), is also the notion of efficiency at the household level used by Gersbach and Haller.

When internally efficient household demand correspondences are considered, the differences between Gersbach and Haller's equilibrium with free household formation and ours just turn out to be essential. Indeed, as already said, the free household formation is sufficient for efficiency at the economy level. On the contrary, our stability condition is not. That result is the major novelty of our paper. It shows that, unlike Gersbach and Haller's, our definition of equilibrium implies existence of inefficient equilibrium allocations even though households efficiently choose their demands. Moreover, in our opinion, many other remarkable welfare results regarding equilibria with free household formation, like the already quoted result about the property of equilibrium to make potential inefficient household choices efficient (Proposition 3, Gersbach and Haller (2005)) and the one showing that each equilibrium is a valuation equilibrium for a club model (Proposition 3, Gersbach and Haller (2010)), don't hold true anymore. Pointing out conditions on household demands such that those results can be proved seems to be an interesting research project.

The above described inefficiency result is in fact a byproduct of the analysis of a particular case of the general model discussed in the first part of the paper. That special model is characterized by demand correspondences that are explicitly built assuming specific decision processes for households. Moreover, differently from the model in Gori and Villanacci (2009), we suppose that members of each household always reach an agreement on the distribution of all the surplus that household can produce, that is, they efficiently choose their collective demand. More precisely, for every household, we exogenously associate individual bargaining powers with its members and we assume that household decision is always the outcome of the corresponding (possibly asymmetric) Nash bargaining within the household. We observe that the household decision rule we choose is the same as the one considered in Gersbach and Haller (2009), where the authors are interested in analysing the effects of a shift of bargaining power within households operating in a competitive market environment on resource allocations and welfare. We also note that, because of the above assumptions, inefficiency at the economy level cannot be a consequence of the disagreement in bargaining processes as the surplus is never wasted and household decisions are always efficient.

In that special framework, we prove that if the constitution has a very simple structure, that is, individuals are only allowed to stay alone or to belong at most to one household having more than one member, then existence of equilibria is always assured, and other nice properties of equilibria, such as generic finiteness and smooth dependence on the economy, hold true. On the other hand, we show that if the constitution is complex enough, that is, individuals have many outside options, then we can find robust examples of both economies admitting equilibria that are efficient at the economy level and economies admitting equilibria that aren't. In particular, as already said, we find economies where households behave efficiently but equilibrium allocations are not Fully Pareto Optimal. We finally note that those results definitely highlight the significant role that outside options may have in determining inefficiency of equilibria at the economy level.

The paper is organized as follows. In Section 2, we present the general set-up of the model, definitions of Full Pareto Optimality and equilibrium, and related observations. In Section 3, we describe the special model where individual bargaining powers are introduced to build household demand correspondences and we state and comment the results obtained. Section 4 contains some remarks about the generality of the main assumptions, the role of constitutions and the possibility to evaluate welfare implications caused by a change of constitution. In Section 4, we also discuss another equilibrium concept in which the stability condition takes into account the fact that some household objections might be not justified because of the presence of suitable counter-objections proposed by other households. Finally, Section 5 contains the proofs of the results stated in the previous sections. In order to make things clearer, many technical details are proved in the Appendix.

2 General set-up of the model

We consider a market in which there are $I \geq 2$ individuals, denoted by $i \in \mathcal{I} = \{1, ..., I\}$, and $C \geq 2$ types of different commodities, denoted by $c \in \mathcal{C} = \{1, ..., C\}$. Individuals in the market are allowed to divide into households and then commodities are exchanged by households and distributed among their members. We assume that each individual owns a certain amount of initial endowment. We assume further that, when at least two individuals decide to form a feasible household, a given amount of commodities beyond initial individual endowments is produced by that household. Each individual, consistently with the rational behaviour paradigm, compares the consumption she could get within each household and chooses the household she desires to belong to in order to maximize her own utility.

In our framework, a household is a nonempty subset of $\mathcal{I}$ while a household allocation is a partition of $\mathcal{I}$. Later on, we denote by $\mathbf{P}(\mathcal{I})$ the set of households and by $\mathfrak{P}(\mathcal{I})$ the set of household allocations. We define also[4]

$$\mathbf{P}_\sigma(\mathcal{I}) = \{\mathcal{H} \in \mathbf{P}(\mathcal{I}) : |\mathcal{H}| = 1\}, \quad \mathbf{P}_\nu(\mathcal{I}) = \{\mathcal{H} \in \mathbf{P}(\mathcal{I}) : |\mathcal{H}| \geq 2\}.$$

For every $i \in \mathcal{I}$, individual i is characterized by an initial endowment $\omega_i \in \mathbb{R}^C_{++}$ and a utility function

$$u_i : \mathbb{R}^C_{++} \to \mathbb{R}, \quad x_i \mapsto u_i(x_i),$$

representing her preferences over her consumption set $\mathbb{R}^C_{++}$ [5]. In what follows, we assume

$$u_i \in C^2; \tag{1}$$

$$\text{for every } x_i \in \mathbb{R}^C_{++},\ Du_i(x_i) \gg 0\,; \tag{2}$$

$$\text{for every } x_i \in \mathbb{R}^C_{++} \text{ and } v \in \mathbb{R}^C \setminus \{0\},\ v\, D^2 u_i(x_i)\, v < 0\,; \tag{3}$$

$$\text{for every } \underline{x}_i \in \mathbb{R}^C_{++},\ \left\{x_i \in \mathbb{R}^C_{++} : u_i(x_i) \geq u_i(\underline{x}_i)\right\} \text{ is a closed subset of } \mathbb{R}^C, \tag{4}$$

and we denote by $\mathcal{U}_C$ the set of such functions. For every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, household $\mathcal{H}$ is characterized by a vector $\eta^{\mathcal{H}} \in \mathbb{R}^C_+ \setminus \{0\}$, representing the additional endowment which is jointly owned by all the members of household $\mathcal{H}$. Finally, we define the set of economies as

$$\mathcal{E}(I, C) = \mathbb{R}^{CI}_{++} \times \mathcal{U}^I_C \times (\mathbb{R}^C_+ \setminus \{0\})^{|\mathbf{P}_\nu(\mathcal{I})|} \times \Sigma(I, C), \tag{5}$$

[4]In what follows, the cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$.

[5]For every positive integer N, we define the binary relations $\gg$, $\geq$ and $>$ over $\mathbb{R}^N$ as follows: given $v = (v_1, \ldots, v_N)$ and $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$, we write

$$\begin{array}{lll} v \gg w & \text{if} & v_i > w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v \geq w & \text{if} & v_i \geq w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v > w & \text{if} & v \geq w \ \text{ and } \ v \neq w. \end{array}$$

We also define the sets $\mathbb{R}^N_+ = \{v \in \mathbb{R}^N : v \geq 0\}$ and $\mathbb{R}^N_{++} = \{v \in \mathbb{R}^N : v \gg 0\}$.

with generic element $E = \big((\omega_i)_{i\in\mathcal{I}}, (u_i)_{i\in\mathcal{I}}, (\eta^{\mathcal{H}})_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}, \sigma\big)$, and where $\mathbb{R}^{CI}_{++}$ describes household endowments, $\mathcal{U}^I_C$ describes individual utility functions, $(\mathbb{R}^C_+ \setminus \{0\})^{|\mathbf{P}_\nu(\mathcal{I})|}$ describes additional endowments for households having at least two members and $\Sigma(I, C)$ contains further information about individuals and households. Different qualifications of $\Sigma(I, C)$ allow to consider different models each of them focusing on specific features of economic agents and then catching particular economic relations.

The presence of physical constraints and laws in the market may prevent individuals from forming some households. In order to formalize the effects of such constraints, we associate with the market a set $\mathbf{H} \subseteq \mathbf{P}(\mathcal{I})$ whose elements, called feasible households, are just the households that individuals can really form. Such a set is called constitution and we assume that every singleton belongs to it, that is, individuals cannot be prevented from staying alone. The set of constitutions in then defined as

$$\mathfrak{H}(\mathcal{I}) = \big\{\mathbf{H} \subseteq \mathbf{P}(\mathcal{I}) : \mathbf{P}_\sigma(\mathcal{I}) \subseteq \mathbf{H}\big\}.$$

For every $\mathbf{H} \in \mathfrak{H}(\mathcal{I})$, we define also the set

$$\mathfrak{P}_{\mathbf{H}}(\mathcal{I}) = \{\pi \in \mathfrak{P}(\mathcal{I}) : \pi \subseteq \mathbf{H}\},$$

whose elements, called feasible household allocations, are household allocations that are compatible with constitution $\mathbf{H}$. Of course, because of assumptions on $\mathbf{H}$, we have $\mathfrak{P}_{\mathbf{H}}(\mathcal{I}) \neq \varnothing$.

In what follows, we assume that market properties are thoroughly described by the pair $(E, \mathbf{H}) \in \mathcal{E}(I, C) \times \mathfrak{H}(\mathcal{I})$.

Definition 1. *Let us fix $I, C \geq 2$ and consider $(E, \mathbf{H}) \in \mathcal{E}(I, C) \times \mathfrak{H}(\mathcal{I})$. An* allocation *associated with $(E, \mathbf{H})$ is an element of the set $\mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++}$. An allocation $(\pi, x) = (\pi, (x_i)_{i\in\mathcal{I}})$ associated with $(E, \mathbf{H})$ is called* feasible *if*

1.1. $\pi \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I})$,

1.2. $\displaystyle\sum_{i\in\mathcal{I}} x_i \leq \sum_{i\in\mathcal{I}} \omega_i + \sum_{\mathcal{H}\in\pi\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}}.$

The set of feasible allocations *associated with $(E, \mathbf{H})$ is denoted by $A_f(E, \mathbf{H})$.*

In the definition below, we are introducing a suitable notion of efficiency at the economy level for allocations. Of course, such notion depends on how much power a social planner is granted to have. Here, we focus on the case in which the social planner is allowed both to force people to form households and to allocate resources among individuals. This notion, called Full Pareto Optimality, follow the same line in Gersbach and Haller (2001) and (2005).

Definition 2. *Let us fix $I, C \geq 2$ and consider $(E, \mathbf{H}) \in \mathcal{E}(I, C) \times \mathfrak{H}(\mathcal{I})$ and $(\pi, x) \in A_f(E, \mathbf{H})$. We say that (π, x) is a* Fully Pareto Optimal allocation *associated with $(E, \mathbf{H})$ if there is no $(\pi^*, x^*) \in A_f(E, \mathbf{H})$ such that $(u_i(x_i^*))_{i\in\mathcal{I}} > (u_i(x_i))_{i\in\mathcal{I}}$. The* set of Fully Pareto Optimal allocations *associated with $(E, \mathbf{H})$ is denoted by $P_f(E, \mathbf{H})$.*

The following proposition, whose proof is a simple modification of Propositions 1 and 2 of Gersbach and Haller (2001), says that there are always Fully Pareto Optimal allocations.

Proposition 3. *Let us fix $I, C \geq 2$ and consider $(E, \mathbf{H}) \in \mathcal{E}(I, C) \times \mathfrak{H}(\mathcal{I})$. Then $P_f(E, \mathbf{H}) \neq \varnothing$.*

Commodity prices are described by a vector $p \in \mathbb{R}^C_{++}$. For every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, the budget correspondence of household $\mathcal{H}$ is the correspondence

$$B^{\mathcal{H}} : \mathbb{R}^C_{++} \times \mathcal{E}(I,C) \rightrightarrows \mathbb{R}^{C|\mathcal{H}|}_{++}, \quad (p,E) \rightrightarrows B^{\mathcal{H}}(p,E),$$

where, for every $(p,E) \in \mathbb{R}^C_{++} \times \mathcal{E}(I,C)$,

$$B^{\mathcal{H}}(p,E) = \begin{cases} \left\{x_i \in \mathbb{R}^C_{++} : px_i \leq p\omega_i\right\}, & \text{if } \mathcal{H} \in \mathbf{P}_\sigma(\mathcal{I}), \mathcal{H} = \{i\} \\ \left\{(x_i)_{i\in\mathcal{H}} \in \mathbb{R}^{C|\mathcal{H}|}_{++} : p\sum_{i\in\mathcal{H}} x_i \leq p\sum_{i\in\mathcal{H}} \omega_i + p\eta^{\mathcal{H}}\right\}, & \text{if } \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \end{cases}$$

while a demand correspondence of household $\mathcal{H}$ is a correspondence

$$D^{\mathcal{H}} : \mathbb{R}^C_{++} \times \mathcal{E}(I,C) \rightrightarrows \mathbb{R}^{C|\mathcal{H}|}_{++}, \quad (p,E) \rightrightarrows D^{\mathcal{H}}(p,E),$$

such that, for every $(p,E) \in \mathbb{R}^C_{++} \times \mathcal{E}(I,C)$, $D^{\mathcal{H}}(p,E) \subseteq B^{\mathcal{H}}(p,E)$. Finally, a demand correspondence profile is a vector $D = (D^{\mathcal{H}})_{\mathcal{H}\in\mathbf{P}(\mathcal{I})}$ such that, for every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, $D^{\mathcal{H}}$ is a demand correspondence of household $\mathcal{H}$. We denote the set of demand correspondence profiles by $\mathfrak{D}(\mathbb{R}^C_{++}\times \mathcal{E}(I,C))$.

When the commodity price vector is p and the properties of economic agents are described by E, $B^{\mathcal{H}}(p,E)$ represents the set of aggregate consumption vectors that household $\mathcal{H}$ can afford while $D^{\mathcal{H}}(p,E)$ represents the set of aggregate consumption vectors that members of household $\mathcal{H}$ agree to demand, that is, the set of all the possible outcomes of household $\mathcal{H}$'s decision process. Of course, such outcomes has to be affordable for that household. Moreover, there might be affordable aggregate consumption vectors of a household that cannot be obtained as outcome of the decision process of that household. In fact, disagreement might prevent household members from obtaining specific internal distributions of commodities. We note also that household decision is a complex procedure influenced by many factors like, for instance, personal relationships among household members and their relative bargaining powers. Qualifying $\Sigma(C,I)$ means just to provide a formalization of some of the factors entering in household decision processes and to allow demand correspondences to explicitly depend on such factors.

We are now ready to give the definition of equilibrium.

Definition 4. *Let us fix $I, C \geq 2$, $(E,\mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$ and $D \in \mathfrak{D}(\mathbb{R}^C_{++}\times \mathcal{E}(I,C))$. An* equilibrium *associated with $(E,\mathbf{H},D)$ is a vector $(\pi,x,p) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ such that*

4.1. *for every $\mathcal{H} \in \pi$, $(x_i)_{i\in\mathcal{H}} \in D^{\mathcal{H}}(p,E)$;*

4.2. *$(\pi,x) \in A_f(E,\mathbf{H})$;*

4.3. *there is no $\mathcal{J} \in \mathbf{H}$ and $(y_i)_{i\in\mathcal{J}} \in D^{\mathcal{J}}(p,E)$ such that $(u_i(y_i))_{i\in\mathcal{J}} > (u_i(x_i))_{i\in\mathcal{J}}$.*

The set of equilibria *associated with $(E,\mathbf{H},D)$ is denoted by $W(E,\mathbf{H},D)$. We also define the* set of equilibrium allocations *associated with $(E,\mathbf{H},D)$ as*

$$A_e(E,\mathbf{H},D) = \left\{(\pi,x) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} : \exists p \in \mathbb{R}^C_{++} \text{ such that } (\pi,x,p) \in W(E,\mathbf{H},D)\right\}.$$

In intuitive terms, equilibria can be described proceeding backward in time as follows. Given commodity prices, individuals divide into households and members of each household compute the value of household initial endowment and choose one of the affordable aggregate consumption vectors they agree to demand (see Condition 4.1). Of course, the considered household allocation has to be feasible and total household demand does not have to exceed total household supply (see Condition

4.2). Moreover, formed households and their demands have to be stable with respect to all inside and outside options. In other words, members of no feasible household can find an agreement over an affordable aggregate consumption vectors which makes at least one of its members better off and does not make the other ones worse off, that is, no household objects to the alleged equilibrium allocation (see Condition 4.3).

As already explained in the introduction, the purpose of the paper is to carry on a welfare analysis. First of all, let us observe that, because of the very general definition of household demand correspondence, equilibrium allocations are not necessary efficient at the economy level. This can be shown by considering, for instance, $\widehat{D} \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ defined, for every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$ and $(p,E) \in \mathbb{R}^C_{++} \times \mathcal{E}(I,C)$, as

$$\widehat{D}^{\mathcal{H}}(p,E) = \left\{ (x_i)_{i\in\mathcal{H}} \in \mathbb{R}^{C|\mathcal{H}|}_{++} : \forall i \in \mathcal{H},\ x_i \in \arg\max_{y_i \in \mathbb{R}^C_{++}} \{u_i(y_i) : py_i \leq p\omega_i\} \right\}. \tag{6}$$

For every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, the unique element of the set $\widehat{D}^{\mathcal{H}}(p,E)$ can be interpreted as the aggregate consumption vector of household $\mathcal{H}$ when, as a consequence of the household decision process, household members disagree about the distribution of the produced surplus $\eta^{\mathcal{H}}$, decide not to use it and consume only what they can afford by their own initial endowments. Using well-known results about standard exchange economy models, it is immediate to prove the following proposition.

Proposition 5. *Let us fix $I, C \geq 2$, $(E,\mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$ and consider $\widehat{D} \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ defined in* (6). *The following properties hold true.*

5.1. *$W(E,\mathbf{H},\widehat{D}) \neq \varnothing$.*

5.2. *For every $\pi \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I})$, there exists $(x,p) \in \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ such that $(\pi,x,p) \in W(E,\mathbf{H},\widehat{D})$.*

5.3. *If $\mathbf{H} \neq P_\sigma(\mathcal{I})$, then there exists $(\pi,x,p) \in W(E,\mathbf{H},\widehat{D})$ such that $(\pi,x) \notin P_f(E,\mathbf{H})$.*

In the above example, inefficiency at the economy level is due to the fact that the value of the aggregate consumption vector demanded by each household having at least two individuals is less then the value of household endowments, that is, to inefficiency of household decision processes. Moving from these observations, we have that further assumptions on demand correspondence profiles are needed in order to get efficiency of equilibria at the economy level. Moreover, such assumptions have to capture the idea of efficient household decision. The following property, first introduced by Haller (2000), just is fit for purpose.

Given $D \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ and $E \in \mathcal{E}(I,C)$, we say that D satisfies the internal efficiency property at E if, for every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, $p \in \mathbb{R}^C_{++}$ and $(x_i)_{i\in\mathcal{H}} \in D^{\mathcal{H}}(p,E)$, we have that

$$\text{there is no } (y_i)_{i\in\mathcal{H}} \in B^{\mathcal{H}}(p,E) \text{ such that } (u_i(y_i))_{i\in\mathcal{H}} > (u_i(x_i))_{i\in\mathcal{H}}. \tag{7}$$

Such a property simply means that, for each household, there is no affordable aggregate consumption vector that Pareto improves upon any possible outcome of the household decision process. The set of demand correspondence profiles satisfying the internal efficiency property at every economy is denoted by $\mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$.

The following proposition, which immediately follows from Corollary 1 in Haller (2000), says that internal efficiency property implies equilibrium allocations to satisfy a weaker notion of efficiency at the economy level.

Proposition 6. *Let us fix $I, C \geq 2$, $(E,\mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$ and $D \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$. If D satisfies the internal efficiency property at E, then, for every $(\pi,x) \in A_e(E,\mathbf{H},D)$, there is no $x^* \in \mathbb{R}^{CI}_{++}$ such that $(\pi,x^*) \in A_f(E,\mathbf{H})$ and $(u_i(x^*_i))_{i\in\mathcal{I}} > (u_i(x_i))_{i\in\mathcal{I}}$.*

Let us move on now to consider Full Pareto Optimality of equilibrium allocations. For a better understanding of the novelty of our results, some preliminary remarks are necessary.

The concept of equilibrium we propose in Definition 4 is similar to the notion of "D-equilibrium at which no group of consumers can benefit from forming a new household" by Gersbach and Haller (2005, p.114). In fact, the only difference is that the latter one considers, instead of Condition 4.3, the following stability condition

$$\text{there is no } \mathcal{J} \in \mathbf{H} \setminus \pi \text{ and } (y_i)_{i\in\mathcal{J}} \in B^{\mathcal{J}}(p, E) \text{ such that } (u_i(y_i))_{i\in\mathcal{J}} \gg (u_i(x_i))_{i\in\mathcal{J}}. \tag{8}$$

Of course, assumptions on utility functions imply that if the internal efficiency property is satisfied by the demand correspondence profile at the considered economy, then every equilibrium in the sense of Gersbach and Haller is an equilibrium in the sense of Definition 4, as well. Moreover, Condition (8) together with the internal efficiency property guarantees Full Pareto Optimality of equilibrium allocations. The following proposition, whose proof is a simple modification of Proposition 4 in Gersbach and Haller (2005), just states that interesting property.

Proposition 7. *Let us fix* $I, C \geq 2$, $(E, \mathbf{H}) \in \mathcal{E}(I, C) \times \mathfrak{H}(\mathcal{I})$ *and* $D \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I, C))$. *If* D *satisfies the internal efficiency property at* E *and* $(\pi, x, p) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ *satisfies Conditions 4.1, 4.2 and* (8)*, then* $(\pi, x) \in P_f(E, \mathbf{H})$.

Condition (8) means that in equilibrium there is no household outside the equilibrium household allocation that could afford consumption vectors for its members that make everybody better off. Of course, the use of such a condition in the definition of equilibrium implies that we are implicitly assuming that if members of a certain household realize that there is an affordable aggregate consumption vector which Pareto improves upon the alleged equilibrium allocation, then they always agree to demand it. However, in our opinion, this viewpoint is not appropriate and needs to be revised. The reason is that it does not take into account that, by definition, the decision process of each household is able to select, for a given price and a given economy, only elements belonging to the image of that price and that economy under its demand correspondence. As a result, there might be affordable aggregate consumption vectors that are advantageous for a certain household but that household members cannot agree on. In fact, Gersbach and Haller's stability condition seems to suggest the existence of a social planner that forces households to choose a Pareto improving aggregate consumption vector, if any, despite relationships and interactions among members of those households could make impossible reach that outcome.

Moreover, when property (7) is not satisfied, equilibria in the sense of Gersbach and Haller might have the property that members of a formed household also agree to demand an affordable aggregate consumption vector which is Pareto superior to the aggregate consumption vector demanded in equilibrium. However, in our opinion, a stability condition should assure not only that individuals don't want to break the equilibrium household allocation by forming new and more advantageous households but, at the same time, it should assure that households formed in equilibrium don't want to modify their own demands.

On the basis of the above observations, we propose Condition 4.3 in the definition of equilibrium, that is, we require that in equilibrium there is no household whose decision process can lead to an outcome that makes at least one of the members better off and does not make the other ones worse off. In particular, for every household belonging to the equilibrium household allocation, the household decision process cannot lead to an affordable aggregate consumption vector which is Pareto superior to the one demanded in equilibrium.

Considering our stability condition instead of Condition (8) has important welfare implications. Indeed, equilibrium allocations in the sense of Definition 4 are not necessarily Fully Pareto Optimal, even assuming the internal efficiency property for demand correspondence profiles. This result is obtained as a byproduct of Theorem 10 stated in Section 3. In that section, we focus on a very special demand correspondence profile obtained by assuming that each household chooses its demand via a specific bargaining process among its members based on their individual relative bargaining powers.

3 Bargaining power and household decisions

As already explained, we are going to build a specific demand correspondence profile explicitly depending on relative bargaining powers of individuals in each household. We assume then that each individual chooses the household she desires to belong to and decides the consumption of commodities in order to maximize her own utility without spending more than her own wealth. Individual wealth depends on the household she is member of, on initial endowments and preferences of all the members of that household, and on prices. If an individual is alone, then her wealth is simply the value of her initial endowment. If instead an individual belongs to a household having more than one member, then her wealth is equal to the value of her initial endowment plus a certain share of the value of the additional endowment produced by that household. Within each household, such individual shares of the produced surplus are the outcome of a bargaining process performed by household members before entering the market and whose outcome depend on their relative bargaining powers. Following the approach in Gersbach and Haller (2009), we assume that household decisions are Nash-bargained.

The formalization of what above described requires to qualify $\Sigma(I, C)$ in order to make it contain information on relative bargaining powers of household members. For this purpose, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, we associate with household $\mathcal{H}$ a vector $\theta^{\mathcal{H}} = (\theta_i^{\mathcal{H}})_{i\in\mathcal{H}} \in \Delta^{\mathcal{H}}$, where

$$\Delta^{\mathcal{H}} = \left\{ (\theta_i^{\mathcal{H}})_{i\in\mathcal{H}} \in (0,1)^{|\mathcal{H}|} : \sum_{i\in\mathcal{H}} \theta_i^{\mathcal{H}} = 1 \right\},$$

and, for every $i \in \mathcal{H}$, $\theta_i^{\mathcal{H}}$ measures the relative bargaining power of individual i within household $\mathcal{H}$. We define

$$\Sigma(I, C) = \left(\bigtimes_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})} \Delta^{\mathcal{H}} \right).$$

The set of economies is

$$\mathcal{E}(I, C) = \mathbb{R}^{CI}_{++} \times \mathcal{U}^I_C \times (\mathbb{R}^C_+ \setminus \{0\})^{|\mathbf{P}_\nu(\mathcal{I})|} \times \left(\bigtimes_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})} \Delta^{\mathcal{H}} \right) \tag{9}$$

with generic element $E = \big((\omega_i, u_i)_{i\in\mathcal{I}}, (\eta^{\mathcal{H}}, \theta^{\mathcal{H}})_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\big)$. We define also, for every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, the set

$$\mathcal{E}^{\mathcal{H}}(I, C) = \begin{cases} \mathbb{R}^C_{++} \times \mathcal{U}_C, & \text{if } \ \mathcal{H} \in \mathbf{P}_\sigma(\mathcal{I}) \\ \mathbb{R}^{C|\mathcal{H}|}_{++} \times \mathcal{U}^{|\mathcal{H}|}_C \times (\mathbb{R}^C_+ \setminus \{0\}) \times \Delta^{\mathcal{H}}, & \text{if } \ \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \end{cases}$$

with generic element

$$e^{\mathcal{H}} = \begin{cases} (\omega_i, u_i), & \text{if } \ \mathcal{H} \in \mathbf{P}_\sigma(\mathcal{I}), \mathcal{H} = \{i\} \\ \big((\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}}\big), & \text{if } \ \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \end{cases}$$

Finally, given $E \in \mathcal{E}(I, C)$ and $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, we denote by $E^{\mathcal{H}}$ the obvious projection of E on $\mathcal{E}^{\mathcal{H}}(I, C)$.

We assume that the consumption of an individual belonging to a certain feasible household depends only on the components of the economy related to that household and prices. Given a household $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, $e^{\mathcal{H}} \in \mathcal{E}^{\mathcal{H}}(I,C)$, and $p \in \mathbb{R}^C_{++}$, we have that individual i belonging to $\mathcal{H}$ can compute her consumption vector as follows.

If $\mathcal{H} \in \mathbf{P}_\sigma(\mathcal{I})$, $\mathcal{H} = \{i\}$, then the wealth of individual i is defined as the value of her initial endowment ω_i, that is, $p\omega_i$. Consequently, individual i consumption vector is the unique solution $\chi_i^{\mathcal{H}}(p, e^{\mathcal{H}}) \in \mathbb{R}^C_{++}$ to the problem

$$\max_{x_i \in \mathbb{R}^C_{++}} u_i(x_i) \quad \text{subject to} \quad px_i \leq p\omega_i.$$

If $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, we assume that the household decision is Nash-bargained and that, for every $i \in \mathcal{H}$, individual i's threat point is individual i's indirect utility in the household $\{i\}$ at the price vector p. Consequently, consumption vectors of individuals in $\mathcal{H}$ are obtained by considering the unique solution $(\chi_i^{\mathcal{H}}(p, e^{\mathcal{H}}))_{i \in \mathcal{H}} \in \mathbb{R}^{C|\mathcal{H}|}_{++}$ to the problem

$$\max_{(x_i)_{i\in\mathcal{H}} \in \mathbb{R}^{C|\mathcal{H}|}_{++}} \prod_{i \in \mathcal{H}} \left(u_i(x_i) - V_i^{p,\mathcal{H}}(0)\right)^{\theta_i^{\mathcal{H}}} \quad \text{subject to} \quad \begin{cases} u_i(x_i) \geq V_i^{p,\mathcal{H}}(0), \quad i \in \mathcal{H} \\ p\sum_{i\in\mathcal{H}} x_i \leq p \sum_{i\in\mathcal{H}} \omega_i + p\eta^{\mathcal{H}} \end{cases} \tag{10}$$

where

$$V_i^{p,\mathcal{H}}(0) = \max\left\{u_i(x_i) : x_i \in \mathbb{R}^C_{++}, px_i \leq p\omega_i\right\}.$$

First of all, note that Problem (10) is exactly the same as the household decision problem considered in Gersbach and Haller (2009, p.683). Moreover, we stress that considering $V_i^{p,\mathcal{H}}(0)$ as individual i's threat point in the bargaining process presumes that individual i has indeed the possibility to stay alone, that is, the option to form the household $\{i\}$. As a result, the assumption that constitutions contain the set of singletons is essential in the definition of household decision processes we are dealing with.

From the above discussion, for every $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, we obtain the demand function of household $\mathcal{H}$

$$\chi^{\mathcal{H}} : \mathbb{R}^C_{++} \times \mathcal{E}^{\mathcal{H}}(I,C) \to \mathbb{R}^{C|\mathcal{H}|}_{++}, \quad \left(p, e^{\mathcal{H}}\right) \to \chi^{\mathcal{H}}(p, e^{\mathcal{H}}) = \left(\chi_i^{\mathcal{H}}(p, e^{\mathcal{H}})\right)_{i\in\mathcal{H}}. \tag{11}$$

Of course, the vector $\chi = (\chi^{\mathcal{H}})_{\mathcal{H}\in\mathbf{P}(\mathcal{I})}$ can be also interpreted as an element of $\mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$, and it is immediate to verify that, in particular, $\chi \in \mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$.

Moreover, for every $(E, \mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$, an equilibrium associated with $(E, \mathbf{H}, \chi)$ is a vector $(\pi, x, p) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ such that

$$\text{for every } \mathcal{H} \in \pi, \ (x_i)_{i\in\mathcal{H}} = \chi^{\mathcal{H}}(p, E^{\mathcal{H}}); \tag{12}$$

$$(\pi, x) \in A_f(E, \mathbf{H}); \tag{13}$$

$$\text{there is no } \mathcal{J} \in \mathbf{H} \text{ such that } \left(u_i\left(\chi_i^{\mathcal{J}}(p, E^{\mathcal{J}})\right)\right)_{i\in\mathcal{J}} > \left(u_i\left(x_i\right)\right)_{i\in\mathcal{J}}. \tag{14}$$

Note that, equilibria associated with the demand correspondence profile χ are "competitive equilibria with free exit" in the sense of Gersbach and Haller (2003). Moreover, such equilibria are invariant under nominal changes in prices. Then, without loss of generality, we can assume the C-th commodity to be the numeraire commodity, that is, we can normalize its price. The following proposition, whose proof is straightforward, states this property.

Proposition 8. *Let us fix $I, C \geq 2$, $(E, \mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$ and $\chi \in \mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ defined in* (11). *Then, we have*

$$W(E, \mathbf{H}, \chi) = \left\{(\pi, x, \lambda p) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++} : (\pi, x, p) \in W_n(E, \mathbf{H}, \chi), \lambda \in \mathbb{R}_{++}\right\},$$

$$A_e(E,\mathbf{H},\chi) = \left\{(\pi,x) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} : \textit{there exists } p \in \mathbb{R}^C_{++} \textit{ such that } (\pi,x,p) \in W_n(E,\mathbf{H},\chi)\right\},$$

where $W_n(E,\mathbf{H},\chi) = \left\{(\pi,x,p) \in W(E,\mathbf{H},\chi) : p^C = 1\right\}$ *is called* set of normalized equilibria *associated with* $(E,\mathbf{H},\chi)$.

Let us consider now the topological space

$$\mathcal{V}(I,C) = \mathbb{R}^{CI}_{++} \times \left(C^2(\mathbb{R}^C_{++})\right)^I \times \mathbb{R}^{C|\mathbf{P}_\nu(\mathcal{I})|} \times \left(\bigtimes_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})} \mathbb{R}^{|\mathcal{H}|} \right), \tag{15}$$

endowed with the product topology of the natural topologies on each of the spaces in the cartesian product. In particular, following among others Allen (1981), we consider on $\left(C^2(\mathbb{R}^C_{++})\right)^I$ the C^2 compact-open topology. In what follows, we endow $\mathcal{E}(I,C) \subseteq \mathcal{V}(I,C)$ with the topology induced by $\mathcal{V}(I,C)$.

Finally, let us consider the following subsets of $\mathfrak{H}(\mathcal{I})$

$$\mathfrak{H}^f(\mathcal{I}) = \{\mathbf{H} \in \mathfrak{H}(\mathcal{I}) : \mathcal{H},\mathcal{K} \in \mathbf{H} \cap \mathbf{P}_\nu(\mathcal{I}),\ \mathcal{H} \cap \mathcal{K} \neq \varnothing \ \Rightarrow\ \mathcal{H} = \mathcal{K}\}, \quad \mathfrak{H}^v(\mathcal{I}) = \mathfrak{H}(\mathcal{I}) \setminus \mathfrak{H}^f(\mathcal{I}).$$

The set $\mathfrak{H}^f(\mathcal{I})$ represents the set of constitutions allowing each individual to belong at most to one household having more than one member. For every $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$, let us define

$$\widehat{\pi}(\mathbf{H}) = (\mathbf{P}_\nu(\mathcal{I}) \cap \mathbf{H}) \cup \left\{ \{i\} : i \in \mathcal{I} \setminus \bigcup_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})\cap\mathbf{H}} \mathcal{H} \right\},$$

and note that,

$$\widehat{\pi}(\mathbf{H}) \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I}) \quad \text{and} \quad \mathbf{H} = \widehat{\pi}(\mathbf{H}) \cup \mathbf{P}_\sigma(\mathcal{I}). \tag{16}$$

We are now ready to present the main results of the paper. We stress that Theorem 9 and Theorem 10 are both about properties of equilibria when demand correspondence profile χ is considered.

Theorem 9. *Let us fix* $I, C \geq 2$, $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$ *and* $\chi \in \mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ *defined in* (11). *The following statements hold true.*

9.1. *For every* $E \in \mathcal{E}(I,C)$, $W_n(E,\mathbf{H},\chi) \neq \varnothing$.

9.2. *For every* $E \in \mathcal{E}(I,C)$, *if* $(\pi,x,p) \in W_n(E,\mathbf{H},\chi)$, *then* $\pi = \widehat{\pi}(\mathbf{H})$.

9.3. *For every* $E \in \mathcal{E}(I,C)$, $A_e(E,\mathbf{H},\chi) \subseteq P_f(E,\mathbf{H})$.

9.4. *There exists an open and dense subset* $\mathcal{D}(I,C,\mathbf{H}) \subseteq \mathcal{E}(I,C)$ *such that, for every* $E^* \in \mathcal{D}(I,C,\mathbf{H})$, *there exists a positive integer* K *such that*

$$|W_n(E^*,\mathbf{H},\chi)| = K \quad \textit{and} \quad W_n(E^*,\mathbf{H},\chi) = \{(\widehat{\pi}(\mathbf{H}), x^{k*}, p^{k*})\}_{k=1}^K, \tag{17}$$

and there exist an open neighborhood $\mathcal{O}(E^*) \subseteq \mathcal{E}(I,C)$ *of* E^* *and, for every* $k \in \{1,\ldots,K\}$, *an open neighborhood* $\mathcal{O}(x^{k*},p^{k*}) \subseteq \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ *of* (x^{k*},p^{k*}) *and* $g_k : \mathcal{O}(E^*) \to \mathcal{O}(x^{k*},p^{k*})$ *such that:*

$$g_k \in C^0,\ g_k(E^*) = (x^{k*},p^{k*}) \textit{ and } \mathcal{O}(x^{k*},p^{k*}) \cap \mathcal{O}(x^{h*},p^{h*}) = \varnothing,\ \textit{for } k \neq h; \tag{18}$$

$$\left\{(E,x,p) \in \mathcal{O}(E^*) \times \mathcal{O}(x^{k*},p^{k*}) : (\widehat{\pi}(\mathbf{H}),x,p) \in W_n(E,\mathbf{H},\chi)\right\} = \operatorname{graph}(g_k); \tag{19}$$

$$\left\{(E,x,p) \in \mathcal{O}(E^*) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++} : (\widehat{\pi}(\mathbf{H}),x,p) \in W_n(E,\mathbf{H},\chi)\right\} = \bigcup_{k=1}^K \operatorname{graph}(g_k). \tag{20}$$

Theorem 10. *Let us fix $I, C \geq 2$, $\mathbf{H} \in \mathfrak{H}^v(\mathcal{I})$ and $\chi \in \mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ defined in* (11). *The following statements hold true.*

10.1. *There exists a nonempty open set $\mathcal{O}_1 \subseteq \mathcal{E}(I,C)$ such that, for every $E \in \mathcal{O}_1$,*

$$W(E, \mathbf{H}, \chi) \neq \varnothing \quad \text{and} \quad A_e(E, \mathbf{H}, \chi) \cap P_f(E, \mathbf{H}) \neq \varnothing.$$

10.2. *There exists a nonempty open set $\mathcal{O}_2 \subseteq \mathcal{E}(I,C)$ such that, for every $E \in \mathcal{O}_2$,*

$$W(E, \mathbf{H}, \chi) \neq \varnothing \quad \text{and} \quad A_e(E, \mathbf{H}, \chi) \cap (A_f(E, \mathbf{H}) \setminus P_f(E, \mathbf{H})) \neq \varnothing.$$

Theorem 9 shows that if the constitution is simple enough, then there is existence of equilibria, equilibrium allocations are efficient and, in a suitable open and dense subset of the set of economies, the (normalized) equilibria are finite in number and smoothly depend on the economy. In particular, such a result demonstrates that, in fact under suitable and strong assumptions which make the model very similar to the Arrow-Debreu exchange economy one, the study of regular economies can be extended to models with multi-member households whose decision processes are described by a non-unitary models. Moreover, note that if $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$, then the singletons that do not belong to $\widehat{\pi}(\mathbf{H})$ are inessential in determining equilibria and their properties, even though, as already explained, they are fundamental to justify the specific household decision processes defining χ. As a consequence, Theorem 9 can be formally thought as a theorem analysing the case in which the household allocation $\widehat{\pi}(\mathbf{H})$ is fixed and then, in particular, it indeed adds to the other existence results related to fixed household allocations (see Gersbach and Haller (1999) and Sato (2009)).

Theorem 10 shows instead that if there is a wide range of outside options for individuals, then both efficiency and inefficiency of equilibrium allocations are not negligible in the set of economies. In fact, we prove that if the constitution is rich enough, then there are robust examples of both economies having at least an equilibrium whose associated allocation is efficient at the economy level and economies having at least an equilibrium whose associated allocation is not efficient at the economy level. The importance of Theorem 10 is due to the fact that it shows how the acceptance of the definition of equilibrium proposed in this paper, instead of the one proposed by Gersbach and Haller (2005), makes efficient choices of households be no more sufficient to beget efficiency of equilibrium allocations at the economy level for a lot of economies. In other words, in our framework, the inefficiency of household decisions becomes only one of the sources of inefficiency of equilibrium allocations.

4 Some final remarks

A. Definitions 1, 2 and 4 can be simply adapted to more general settings in which, for instance, individual consumption set is $\mathbb{R}^C_+$, individual preferences take into account consumption and group externalities and household endowments are elements in $\mathbb{R}^C_+$ without any further specification. Even though excluding group and consumption externalities from the model, the choice of $\mathbb{R}^C_{++}$ as individual consumption set and properties (1), (2), (3) and (4) on utility functions are certainly restrictions under an economic viewpoint, such assumptions are really used to get Theorems 9 and 10 and then we decided to introduce them from the beginning in order to simplify the notation. However, a discussion about the possibility to weaken them is certainly needed.

While stating and proving Theorems 9 and 10 when individual utility functions take into account group or consumption externalities seems to require substantial work, the choice of $\mathbb{R}^C_{++}$ as consumption set and properties (1), (2), (3) and (4) on utility functions are assumptions that can be relaxed. As in a large part of general equilibrium literature about regularity of equilibria, similar assumptions are introduced to employ differential techniques and use theorems from differential topology (see, Villanacci et al. (2002)). However, the same differential techniques can be employed even considering the case in which the consumption set is $\mathbb{R}^C_+$, provided the assumptions on the utility functions are obviously modified in order to take into account their new domain, and theorems similar to ours can be proved following analogous methods (see, Cass et al. (2001)).

Nevertheless, in our opinion, those more general conditions only imply technical discussions about many mathematical details and makes notation and proofs more complicated without gaining more economic insight. Indeed, in order to introduce our definition of equilibrium, explain its meaning and show it does not make efficient choices of households beget efficiency at the economy level of equilibrium allocations, the setting we have chosen just serves the purpose.

B. The model considered in this paper does not encompass group externality for individuals. An interesting issue is to understand if in that more general setting the concept of constitution gets redundant, as it seems possible to rule out certain households in equilibrium just imposing sufficiently negative group externalities to individuals forming those households. In our opinion, it is not the case. The main reason is that, under an interpretative viewpoint, imposing that a certain household is forbidden by physical constraints or laws is very different from imposing that individuals belonging to that household don't want to stay together because they can always find more advantageous households to belong to. Indeed, we can easily conceive situations in which people are forbidden to stay together but, if they could, they would.

Moreover, using constitutions instead of group externalities has another advantage. In fact, this viewpoint allows to study welfare implications of a social planner intervention on the constitution, that is, we can analyse what are the consequences for individuals if the set of feasible households changes. Of course, such an issue would look quite strange on the economic ground if we excluded households via group externalities. In fact, in that case, in order to make the social planner able to modify the set of feasible households we should assume he has the capability to change individual preferences. However, this assumption is very often judged too strong.

The proposition below, whose proof immediately follows from the definition of equilibrium, considers just those kind of questions and states that, for every equilibrium, the addition of new feasible households to the constitution cannot determine a Pareto inferior equilibrium allocation, provided prices are not changed. However, we guess that, dropping the assumption about prices, Proposition 11 does not hold true.

Proposition 11. *Let us fix $I, C \geq 2$, $E \in \mathcal{E}(I,C)$, $\mathbf{H}, \mathbf{H}^* \in \mathfrak{H}(\mathcal{I})$ and $D \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$ and consider $(\pi, x, p) \in W(E, \mathbf{H}, D)$ and $(\pi^*, x^*, p^*) \in W(E, \mathbf{H}^*, D)$. If $\mathbf{H} \subseteq \mathbf{H}^*$ and $p = p^*$, then $(u_i(x_i))_{i \in \mathcal{I}} \not> (u_i(x_i^*))_{i \in \mathcal{I}}$.*

C. The stability condition in Definition 4 requires that there is no household objecting to the alleged equilibrium allocation. By that condition, we are implicitly assuming that individuals are not very forward looking. In fact, if individuals would analyse more carefully household objections, they might understand that some of those objections are not justified as they are balanced by suitable counter-objections. Of course, in real world, some individuals are not interested in such an analysis. However, we think it would certainly be worth considering also equilibrium concepts in which individuals are more forward looking and evaluate the role of counter-objections to household objections, as well.

There are surely at least as many ways to address that issue as different notions of bargaining sets in cooperative game theory. Just to fix ideas, in what follows, we propose a definition of equilibrium whose stability condition is the same as the one in the solution concept for cooperative games introduced by Mas-Colell (1989).

Definition 12. *Let us fix* $I, C \geq 2$, $(E, \mathbf{H}) \in \mathcal{E}(I,C) \times \mathfrak{H}(\mathcal{I})$ *and* $D \in \mathfrak{D}(\mathbb{R}^C_{++} \times \mathcal{E}(I,C))$. *An* MC-equilibrium *associated with* $(E, \mathbf{H}, D)$ *is a vector* $(\pi, x, p) \in \mathfrak{P}(\mathcal{I}) \times \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ *such that*

12.1. *for every* $\mathcal{H} \in \pi$, $(x_i)_{i \in \mathcal{H}} \in D^{\mathcal{H}}(p, E)$*;*

12.2. $(\pi, x) \in A_f(E, \mathbf{H})$*;*

12.3. *for every* $\mathcal{J} \in \mathbf{H}$ *and* $(y_i)_{i \in \mathcal{J}} \in D^{\mathcal{J}}(p, E)$ *such that* $(u_i(y_i))_{i \in \mathcal{J}} > (u_i(x_i))_{i \in \mathcal{J}}$ *there exists* $\mathcal{K} \in \mathbf{H}$ *such that* $\mathcal{K} \cap \mathcal{J} \neq \varnothing$ *and* $(z_i)_{i \in \mathcal{K}} \in D^{\mathcal{K}}(p, E)$ *such that*

$$\forall i \in \mathcal{K} \cap \mathcal{J},\ u_i(z_i) \geq u_i(y_i)\,, \tag{21}$$

$$\forall i \in \mathcal{K} \setminus \mathcal{J},\ u_i(z_i) \geq u_i(x_i)\,, \tag{22}$$

where at least one of the inequalities in (21) *or* (22) *is strict.*

The set of MC-equilibria *associated with* $(E, \mathbf{H}, D)$ *is denoted by* $W^{MC}(E, \mathbf{H}, D)$.

Condition 12.3 simply requires that if a household is an advantageous outside option for its members, that is, it belongs to the constitution and has the property that at MC-equilibrium prices it makes at least one of its members better off and does not make the other ones worse off, then it is not really formed. This is due to the fact that a subset of its members find more convenient form another household, possibly together with other individuals who do not object to join them.

Of course, each equilibrium in the sense of Definition 4 is a MC-equilibrium. As a consequence, MC-equilibrium allocations are not necessarily Fully Pareto Optimal. Moreover, as shown by Proposition 13 below, welfare implications caused by changing the constitution are now different from the ones described in Proposition 11. Indeed, in the new framework, an enrichment of the set of feasible households may lead to a reduction of allocative efficiency even at the same equilibrium prices.

Proposition 13. *Let us fix* $I \geq 3$, $C \geq 2$ *and* $\mathbf{H} = \{\mathcal{I}\} \cup \{\{i\} : i \in \mathcal{I}\}$. *Then there exists* $E^* \in \mathcal{E}(I,C)$, $x^*, x^{**} \in \mathbb{R}^{CI}_{++}$ *and* $p^* \in \mathbb{R}^C_{++}$ *such that*

$$(\mathbf{P}_\sigma(\mathcal{I}), x^*, p^*) \in W^{MC}(E^*, \mathbf{P}(\mathcal{I}), \chi),\ (\{\mathcal{I}\}, x^{**}, p^*) \in W^{MC}(E^*, \mathbf{H}, \chi), \tag{23}$$

$$(u_i^*(x^{**}))_{i \in \mathcal{I}} > (u_i^*(x^*))_{i \in \mathcal{I}}. \tag{24}$$

Finally, we guess that similar results can be obtained also assuming definitions of equilibrium based on different notions of bargaining set.

5 Proofs

As already noted, Propositions 3, 5, 6, 7, 8 and 11 can be deduced by prior results or are straightforward. Then, we are left with proving Theorems 9, 10 and Proposition 13. In what follows, we refer to the notation introduced in Section 3.

Fixed $I, C \geq 2$, $p \in \mathbb{R}^C_{++}$, $\mathcal{H} \in \mathbf{P}(\mathcal{I})$, $e^{\mathcal{H}} \in \mathcal{E}^{\mathcal{H}}(I,C)$ and $i \in \mathcal{H}$, let us define the function[6]

$$V_i^{p,\mathcal{H}} : [0,1] \to \mathbb{R}, \quad a_i \mapsto V_i^{p,\mathcal{H}}(a_i^{\mathcal{H}}) = \max\left\{u_i(x_i) : x_i \in \mathbb{R}^C_{++}, px_i \leq p\omega_i + a_i p\eta^{\mathcal{H}}\right\},$$

[6]Note that, in order to simplify the notation and since no confusion should arise, we do not make $V_i^{p,\mathcal{H}}$ explicitly depend on $e^{\mathcal{H}}$.

representing the indirect utility for individual i in household $\mathcal{H}$ at prices p in the case in which she gets a share a_i of $p\eta^{\mathcal{H}}$. It is simple to verify that Assumptions (1)-(4) assure that $V_i^{p,\mathcal{H}}$ is C^1, strictly increasing and strictly concave on $[0,1]$.

If $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, we have that $\chi_i^{\mathcal{H}}(p, e^{\mathcal{H}})$ is the unique solution to the problem

$$\max_{x_i \in \mathbb{R}^C_{++}} u_i(x_i) \quad \text{subject to} \quad px_i \le p\omega_i + \alpha_i^{\mathcal{H}}(p, e^{\mathcal{H}})p\eta^{\mathcal{H}}, \tag{25}$$

where $\alpha_i^{\mathcal{H}}(p, e^{\mathcal{H}})$ is the share of the value of the additional endowment $\eta^{\mathcal{H}}$ she has obtained by the bargain within the household. The vector $\left(\alpha_i^{\mathcal{H}}(p, e^{\mathcal{H}})\right)_{i\in\mathcal{H}}$, representing individual shares of $p\eta^{\mathcal{H}}$ in household $\mathcal{H}$, is the unique solution to the problem

$$\max \prod_{i\in\mathcal{H}} \left(V_i^{p,\mathcal{H}}(a_i) - V_i^{p,\mathcal{H}}(0)\right)^{\theta_i^{\mathcal{H}}} \quad \text{subject to} \quad \begin{cases} a_i \ge 0, \quad i \in \mathcal{H} \\ \sum_{i\in\mathcal{H}} a_i = 1 \end{cases} \tag{26}$$

that is, the unique solution to the problem

$$\max \sum_{i\in\mathcal{H}} \theta_i^{\mathcal{H}} \ln\left(V_i^{p,\mathcal{H}}(a_i) - V_i^{p,\mathcal{H}}(0)\right) \quad \text{subject to} \quad (a_i)_{i\in\mathcal{H}} \in \Delta^{\mathcal{H}}. \tag{27}$$

The above discussion allows to define, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, the share function of household $\mathcal{H}$,

$$\alpha^{\mathcal{H}} : \mathbb{R}^C_{++} \times \mathcal{E}^{\mathcal{H}}(I, C) \to \Delta^{\mathcal{H}}, \quad (p, e^{\mathcal{H}}) \to \alpha^{\mathcal{H}}(p, e^{\mathcal{H}}) = \left(\alpha_i^{\mathcal{H}}(p, e^{\mathcal{H}})\right)_{i\in\mathcal{H}}.$$

Let us now recall some fundamental results from differential topology. The general formulation of the Implicit Function Theorem given in Theorem 14 can be found in Mas-Colell (1985). Theorem 15 and 16 can instead be found in Villanacci et al. (2002).

Theorem 14. *Let us consider $F : \mathcal{O} \times \mathcal{V} \to \mathbb{R}^n$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and $\mathcal{V}$ is a topological space. Assume that F is continuous, for every $(x, v) \in \mathcal{O} \times \mathcal{V}$, $D_xF(x, v)$ exists and the function[7] $D_xF : \mathcal{O} \times \mathcal{V} \to \mathbb{M}(n)$ is continuous. Let $(x^*, v^*) \in \mathcal{O} \times \mathcal{V}$ be such that $F(x^*, v^*) = 0$ and $\det D_xF(x^*, v^*) \ne 0$. Then there exist an open neighborhood $\mathcal{O}(x^*) \subseteq \mathcal{O}$ of x^*, an open neighborhood $\mathcal{O}(v^*) \subseteq \mathcal{V}$ of v^* and $\varphi : \mathcal{O}(v^*) \to \mathcal{O}(x^*)$ such that $\varphi \in C^0$, $\varphi(v^*) = x^*$ and*

$$\{(x, v) \in \mathcal{O}(x^*) \times \mathcal{O}(v^*) : F(x, v) = 0\} = \{(x, v) \in \mathcal{O}(x^*) \times \mathcal{O}(v^*) : x = \varphi(v)\}.$$

Theorem 15. *Let $\mathcal{M}$ and $\mathcal{N}$ be two C^2 boundaryless manifolds of the same dimension, $y \in \mathcal{N}$ and $F, G : \mathcal{M} \to \mathcal{N}$ be continuous functions. Assume that G is C^1 in an open neighborhood $\mathcal{O}$ of $G^{-1}(y)$, y is a regular value for G restricted to $\mathcal{O}$, $\left|G^{-1}(y)\right|$ is finite and odd and there exists a continuous homotopy $H : \mathcal{M} \times [0, 1] \to \mathcal{N}$ from F to G such that $H^{-1}(y)$ is compact. Then $F^{-1}(y) \ne \varnothing$.*

Theorem 16. *Let m, p, n, α be positive integers and $\mathcal{M}$, Ω and $\mathcal{N}$ be C^α manifolds of dimensions m, p and n, respectively. Let $F : \mathcal{M} \times \Omega \to \mathcal{N}$ be a C^α function. Assume $\alpha > \max\{m - n, 0\}$. If y is a regular value for F, then there exists a full measure subset Ω^* of Ω such that, for every $\omega \in \Omega^*$, y is a regular value for the function $F(\cdot, \omega) : M \to N$, $x \mapsto F(x, \omega)$.*

Proposition 17. *Let us fix $I, C \ge 2$, $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, $e^{\mathcal{H}} \in \mathcal{E}^{\mathcal{H}}(I, C, \mathbf{H})$ and $p \in \mathbb{R}^C_{++}$. Then the following statements hold true.*

[7] For every positive integer k, $\mathbb{M}(k)$ denotes the set of square matrices having real elements, k rows and k columns.

17.1. If $(a_i^*)_{i\in\mathcal{H}} = \alpha^{\mathcal{H}}(p, e^{\mathcal{H}})$ and $(x_i^*)_{i\in\mathcal{H}} = \chi^{\mathcal{H}}(p, e^{\mathcal{H}})$, then there exists a unique

$$\left((\lambda_i^*, \underline{x}_i^*, \underline{\lambda}_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*}\right) \in \left(\mathbb{R} \times \mathbb{R}^C_{++} \times \mathbb{R}\right)^{|\mathcal{H}|} \times \mathbb{R}$$

such that the vector

$$\left((x_i^*, \lambda_i^*, \underline{x}_i^*, \underline{\lambda}_i^*, a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*}\right) \in \left(\mathbb{R}^C_{++} \times \mathbb{R} \times \mathbb{R}^C_{++} \times \mathbb{R} \times (0,1)\right)^{|\mathcal{H}|} \times \mathbb{R} \tag{28}$$

is solution to the system

$$\begin{cases} i \in \mathcal{H}, \quad Du_i(x_i) - \lambda_i p = 0 & (29.1) \\ i \in \mathcal{H}, \quad -p(x_i - \omega_i - a_i \eta^{\mathcal{H}}) = 0 & (29.2) \\ i \in \mathcal{H}, \quad Du_i(\underline{x}_i) - \underline{\lambda}_i p = 0 & (29.3) \\ i \in \mathcal{H}, \quad -p(\underline{x}_i - \omega_i) = 0 & (29.4) \\ i \in \mathcal{H}, \quad \theta_i^{\mathcal{H}} \lambda_i p \eta^{\mathcal{H}} - \mu^{\mathcal{H}} \left(u_i(x_i) - u_i(\underline{x}_i)\right) = 0 & (29.5) \\ \quad -\sum_{i\in\mathcal{H}} a_i + 1 = 0 & (29.6) \end{cases} \tag{29}$$

17.2. System (29) has a unique solution and if (28) solves (29), then $(a_i^*)_{i\in\mathcal{H}} = \alpha^{\mathcal{H}}(p, e^{\mathcal{H}})$ and $(x_i^*)_{i\in\mathcal{H}} = \chi^{\mathcal{H}}(p, e^{\mathcal{H}})$.

Proof. First, let us present some useful remarks. Fix $i \in \mathcal{H}$ and $a_i \in (0,1)$. It is well known that if $x_i(a_i)$ is the unique element of the set

$$\arg\max\left\{u_i\left(x_i\right) : x_i \in \mathbb{R}^C_{++}, px_i \leq p\omega_i + a_i p\eta^{\mathcal{H}}\right\}, \tag{30}$$

then there exists a unique $\lambda_i(a_i) \in \mathbb{R}$ such that $(x_i(a_i), \lambda_i(a_i))$ is solution to

$$\begin{cases} Du_i(x_i) - \lambda_i p = 0 \\ -p(x_i - \omega_i - a_i\eta^{\mathcal{H}}) = 0 \end{cases} \tag{31}$$

and that if $(x_i(a_i), \lambda_i(a_i)) \in \mathbb{R}^C_{++} \times \mathbb{R}$ is solution to (31), then $x_i(a_i)$ is the unique element of the set (30). Then the two functions $x_i : (0,1) \to \mathbb{R}^C_{++}$ and $\lambda_i : (0,1) \to \mathbb{R}$ are well defined and, because of the assumptions on utility functions, they are C^1. Moreover, via the envelope theorem, we have also

$$\frac{d}{d\,a_i} u_i(x_i(a_i)) = \lambda_i(a_i) p\eta^{\mathcal{H}}. \tag{32}$$

Let us prove now Statement 17.1. If, for every $i \in \mathcal{H}$, $a_i^* = \alpha_i^{\mathcal{H}}\left(p, e^{\mathcal{H}}\right)$ and $x_i^* = \chi_i^{\mathcal{H}}\left(p, e^{\mathcal{H}}\right)$, then there exists a unique $\lambda_i^* \in \mathbb{R}$ such that (x_i^*, λ_i^*) solves the system

$$\begin{cases} Du_i(x_i) - \lambda_i p = 0 \\ -p(x_i - \omega_i - a_i^*\eta^{\mathcal{H}}) = 0 \end{cases}$$

and, as $(a_i^*)_{i\in\mathcal{H}}$ solves the maximization problem

$$\max \sum_{i\in\mathcal{H}} \theta_i^{\mathcal{H}} \ln\left(V_i^{p,\mathcal{H}}\left(a_i\right) - V_i^{p,\mathcal{H}}(0)\right) \quad \text{subject to} \quad (a_i)_{i\in\mathcal{H}} \in \Delta^{\mathcal{H}}, \tag{33}$$

there exists a unique $\mu^{\mathcal{H}*} \in \mathbb{R}$ such that $\left((a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*}\right)$ solves the system

$$\begin{cases} i \in \mathcal{H}, \quad \theta_i^{\mathcal{H}} \frac{d}{d\,a_i} V_i^{p,\mathcal{H}}(a_i) - \mu^{\mathcal{H}} \left(V_i^{p,\mathcal{H}}(a_i) - V_i^{p,\mathcal{H}}(0)\right) = 0 \\ \qquad\quad -\sum_{i\in\mathcal{H}} a_i + 1 = 0 \end{cases} \tag{34}$$

By the preliminary remarks and the definition of $V_i^{p,\mathcal{H}}$, we have $V_i^{p,\mathcal{H}}(a_i^*) = u_i(x_i^*)$,

$$\frac{d}{d\,a_i} V_i^{p,\mathcal{H}}(a_i^*) = \lambda_i^* p \eta^{\mathcal{H}},$$

and $V_i^{p,\mathcal{H}}(0) = u_i(\underline{x}_i^*)$, where $(\underline{x}_i^*, \underline{\lambda}_i^*)$ is the unique solution to the system

$$\begin{cases} Du_i(\underline{x}_i) - \underline{\lambda}_i p = 0 \\ -p(\underline{x}_i - \omega_i) = 0 \end{cases}$$

The desired result immediately follows from the above relations.

The proof of Statement 17.2 is a simple consequence of the preliminary remarks and of the fact that if

$$\left((a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*}\right) \in (0,1)^{|\mathcal{H}|} \times \mathbb{R}$$

is solution to (34), then $(a_i^*)_{i\in\mathcal{H}}$ is solution to (33). □

Proposition 18. *Let us fix $I, C \geq 2$ and $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$. Then the following statements hold true.*

18.1. The functions $\alpha^{\mathcal{H}}$ and $\chi^{\mathcal{H}}$ are continuous.

18.2. For every $p \in \mathbb{R}^C_{++}$, $(\omega_i, u_i)_{i\in\mathcal{H}} \in \mathbb{R}^{C|\mathcal{H}|}_{++} \times \mathcal{U}_C^{|\mathcal{H}|}$, $\eta^{\mathcal{H}} \in \mathbb{R}^C_+ \setminus \{0\}$, $\varepsilon > 0$ and $i^* \in \mathcal{H}$, there exists $\theta^{\mathcal{H}} \in \Delta^{\mathcal{H}}$ such that

$$\alpha_{i^*}^{\mathcal{H}}\left(p, (\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right) < \varepsilon.$$

Proof. In order to prove Statement 18.1, let us define

$$Z^{\mathcal{H}} = \left(\mathbb{R}^C_{++} \times \mathbb{R} \times \mathbb{R}^C_{++} \times \mathbb{R} \times (0,1)\right)^{|\mathcal{H}|} \times \mathbb{R}$$

with generic element $\zeta^{\mathcal{H}} = \left((x_i, \lambda_i, \underline{x}_i, \underline{\lambda}_i, a_i)_{i\in\mathcal{H}}, \mu^{\mathcal{H}}\right)$. Consider then the function

$$\mathcal{G}^{\mathcal{H}} : Z^{\mathcal{H}} \times \mathbb{R}^C_{++} \times \mathcal{E}^{\mathcal{H}}(I,C) \to \mathbb{R}^{|\mathcal{H}|(2C+3)+1}$$

$$\left(\zeta^{\mathcal{H}}, p, e^{\mathcal{H}}\right) \mapsto \text{ left hand side of System (29)}$$

and note that $\mathcal{G}^{\mathcal{H}}$ is continuous. From Proposition 17, we know that, for every $(p, e^{\mathcal{H}}) \in \mathbb{R}^C_{++} \times \mathcal{E}^{\mathcal{H}}(I,C)$, there exists a unique $\zeta^{\mathcal{H}} \in Z^{\mathcal{H}}$ such that $\mathcal{G}^{\mathcal{H}}(\zeta^{\mathcal{H}}, p, e^{\mathcal{H}}) = 0$. Then we can define

$$\Phi^{\mathcal{H}} : \mathbb{R}^C_{++} \times \mathcal{E}^{\mathcal{H}}(I,C) \to Z^{\mathcal{H}},$$

$$(p, e^{\mathcal{H}}) \mapsto \text{ the unique element of the set } \left\{\zeta^{\mathcal{H}} \in Z^{\mathcal{H}} : \mathcal{G}^{\mathcal{H}}(\zeta^{\mathcal{H}}, p, e^{\mathcal{H}}) = 0\right\}.$$

As proved in the Appendix, we have that

$$\Phi^{\mathcal{H}} \text{ is continuous,} \tag{35}$$

and then the continuity of $\alpha^{\mathcal{H}}$ and $\chi^{\mathcal{H}}$ follows.

In order to prove Statement 18.2, let us fix p, $(\omega_i, u_i)_{i\in\mathcal{H}}$, $\eta^{\mathcal{H}}$, ε, i^* and a sequence $(\theta^{\mathcal{H}[n]})_{n=1}^{\infty}$ in $\Delta^{\mathcal{H}}$ such that $\theta_{i^*}^{\mathcal{H}[n]} \to 0$ as $n \to \infty$, and prove there exists $n^* \in \mathbb{N}^*$ such that

$$\alpha_{i^*}^{\mathcal{H}}\left(p, (\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}[n^*]}\right) < \varepsilon.$$

Assume by contradiction that, for every $n \in \mathbb{N}^*$, $\alpha_{i^*}^{\mathcal{H}}\left(p, (\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}[n]}\right) \geq \varepsilon$. We know that, for every $n \in \mathbb{N}^*$, System (34) has a unique solution $((a_i^{[n]})_{i\in\mathcal{H}}, \mu^{\mathcal{H}[n]}) \in \Delta^{\mathcal{H}} \times \mathbb{R}$ and that such a solution satisfies $a_{i^*}^{[n]} \geq \varepsilon$. Note also that, for every $i \in \mathcal{H}$,

$$\mu^{\mathcal{H}[n]} = \frac{\theta_i^{\mathcal{H}[n]} \frac{d}{da_i} V_i^{p,\mathcal{H}}(a_i^{[n]})}{V_i^{p,\mathcal{H}}(a_i^{[n]}) - V_i^{p,\mathcal{H}}(0)}.$$

Assuming $i = i^*$, we deduce that $\mu^{\mathcal{H}[n]} \to 0$ as $n \to \infty$, for the denominator is uniformly bounded away from zero and the numerator goes to zero. Consider now any $i_* \in \mathcal{H}$ such that (up to a subsequence) $\theta_{i_*}^{\mathcal{H}[n]} \to L > 0$. Since

$$\mu^{\mathcal{H}[n]} = \frac{\theta_{i_*}^{\mathcal{H}[n]} \frac{d}{da_{i_*}} V_{i_*}^{p,\mathcal{H}}(a_{i_*}^{[n]})}{V_{i_*}^{p,\mathcal{H}}(a_{i_*}^{[n]}) - V_{i_*}^{p,\mathcal{H}}(0)} \geq \frac{\theta_{i_*}^{\mathcal{H}[n]} \frac{d}{da_{i_*}} V_{i_*}^{p,\mathcal{H}}(1)}{V_{i_*}^{p,\mathcal{H}}(1) - V_{i_*}^{p,\mathcal{H}}(0)},$$

we get

$$\liminf_{n\to\infty} \mu^{\mathcal{H}[n]} > 0,$$

and the contradiction is found. □

Let us introduce now further notation. Given $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$, let us define

$$\mathbf{H}_\sigma = \mathbf{H} \setminus \mathbf{P}_\nu(\mathcal{I}) \quad \text{and} \quad \mathbf{H}_\nu = \mathbf{H} \cap \mathbf{P}_\nu(\mathcal{I}).$$

Note also that $\mathbf{H}_\sigma = \mathbf{P}_\sigma(\mathcal{I}) \cap \widehat{\pi}(\mathbf{H})$ and $\mathbf{H}_\nu = \mathbf{P}_\nu(\mathcal{I}) \cap \widehat{\pi}(\mathbf{H})$. The notation used in the proofs of Propositions 19, 20 and 21 and Theorem 9 implicitly assumes that both $\mathbf{H}_\sigma \neq \varnothing$ and $\mathbf{H}_\nu \neq \varnothing$, but it can be easily adapted to treat the other cases.

Proposition 19. *Let us fix $I, C \geq 2$, $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$, $E \in \mathcal{E}(I, C)$ and $(x, p) \in \mathbb{R}_{++}^{CI} \times \mathbb{R}_{++}^{C}$. Then, the two following conditions are equivalent*[8].

19.1. $(\widehat{\pi}(\mathbf{H}), x, p) \in W_n(E, \mathbf{H}, \chi)$

19.2. $p^C = 1$, *for every* $\mathcal{H} \in \widehat{\pi}(\mathbf{H})$, $(x_i)_{i\in\mathcal{H}} = \chi^{\mathcal{H}}(p, E^{\mathcal{H}})$ *and*

$$\sum_{\{i\}\in\mathbf{H}_\sigma} x_i^{\setminus} + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \sum_{i\in\mathcal{H}} x_i^{\setminus} = \sum_{i\in\mathcal{I}} \omega_i^{\setminus} + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}\setminus}. \tag{36}$$

Proof. Assume at first that 19.2 holds true and prove $(\widehat{\pi}(\mathbf{H}), x, p) \in W_n(E, \mathbf{H}, \chi)$. Condition (12) is satisfied. By Assumption (2), for every $\{i\} \in \mathbf{H}_\sigma$, $px_i = p\omega_i$, and, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$,

$$px_i = p\omega_i + \alpha_i^{\mathcal{H}}(p, E^{\mathcal{H}}) p\eta^{\mathcal{H}}.$$

Then

$$\sum_{\{i\}\in\mathbf{H}_\sigma} px_i + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \sum_{i\in\mathcal{H}} px_i = \sum_{i\in\mathcal{I}} p\omega_i + \sum_{\mathcal{H}\in\mathbf{H}_\nu} p\eta^{\mathcal{H}}.$$

[8]For every $v = (v^c)_{c=1}^C \in \mathbb{R}^C$, we set $v^{\setminus} = (v^c)_{c=1}^{C-1} \in \mathbb{R}^{C-1}$.

Using now (36) we have

$$\sum_{\{i\}\in\mathbf{H}_\sigma} p^C x_i^C + \sum_{\mathcal{H}\in\mathbf{H}_\nu}\sum_{i\in\mathcal{H}} p^C x_i^C = \sum_{i\in\mathcal{I}} p^C \omega_i^C + \sum_{\mathcal{H}\in\mathbf{H}_\nu} p^C \eta^{\mathcal{H},C},$$

and since $p^C = 1$, we have indeed

$$\sum_{\{i\}\in\mathbf{H}_\sigma} x_i + \sum_{\mathcal{H}\in\mathbf{H}_\nu}\sum_{i\in\mathcal{H}} x_i = \sum_{i\in\mathcal{I}} \omega_i + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}}, \tag{37}$$

that is, Condition (13) is fulfilled. Condition (14) is trivially satisfied since $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$.

In order to prove the converse, assume $(\pi, x, p) \in W_n(E, \mathbf{H}, \chi)$. Of course, by definition of normalized equilibrium, $p^C = 1$ and, for every $\mathcal{H} \in \widehat{\pi}(\mathbf{H})$, $(x_i)_{i\in\mathcal{H}} = \chi^{\mathcal{H}}(p, E^{\mathcal{H}})$. Finally, following an argument similar to the one used in the first part of the proof, we have that Assumption (2) and Condition (13) imply (37) that immediately implies (36). □

Proposition 20. *Let us fix $I, C \geq 2$, $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$ and $E \in \mathcal{E}(I, C)$. Then the following statements hold true.*

20.1. If $(\widehat{\pi}(\mathbf{H}), x^*, p^*) \in W_n(E, \mathbf{H}, \chi)$, then there exists a unique

$$\left((\lambda_i^*)_{\{i\}\in\mathbf{H}_\sigma}, ((\lambda_i^*, \underline{x}_i^*, \underline{\lambda}_i^*, a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*})_{\mathcal{H}\in\mathbf{H}_\nu}\right) \in \mathbb{R}^{|\mathbf{H}_\sigma|} \times \bigtimes_{\mathcal{H}\in\mathbf{H}_\nu} \left(\left(\mathbb{R}\times\mathbb{R}^C_{++}\times\mathbb{R}\times(0,1)\right)^{|\mathcal{H}|}\times\mathbb{R}\right)$$

such that the vector

$$\xi^* = \left((x_i^*, \lambda_i^*)_{\{i\}\in\mathbf{H}_\sigma}, ((x_i^*, \lambda_i^*, \underline{x}_i^*, \underline{\lambda}_i^*, a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*})_{\mathcal{H}\in\mathbf{H}_\nu}, p^*\right) \in \tag{38}$$

$$\left(\mathbb{R}^C_{++}\times\mathbb{R}\right)^{|\mathbf{H}_\sigma|} \times \bigtimes_{\mathcal{H}\in\mathbf{H}_\nu} \left(\left(\mathbb{R}^C_{++}\times\mathbb{R}\times\mathbb{R}^C_{++}\times\mathbb{R}\times(0,1)\right)^{|\mathcal{H}|}\times\mathbb{R}\right)\times\mathbb{R}^C_{++} = \Xi$$

is solution to the system

$$\begin{cases} \{i\}\in\mathbf{H}_\sigma, & Du_i(x_i) - \lambda_i p = 0 & (39.1)\\ \{i\}\in\mathbf{H}_\sigma, & -p(x_i - \omega_i) = 0 & (39.2)\\ \mathcal{H}\in\mathbf{H}_\nu, i\in\mathcal{H}, & Du_i(x_i) - \lambda_i p = 0 & (39.3)\\ \mathcal{H}\in\mathbf{H}_\nu, i\in\mathcal{H}, & -p(x_i - \omega_i - a_i\eta^{\mathcal{H}}) = 0 & (39.4)\\ \mathcal{H}\in\mathbf{H}_\nu, i\in\mathcal{H}, & Du_i(\underline{x}_i) - \underline{\lambda}_i p = 0 & (39.5)\\ \mathcal{H}\in\mathbf{H}_\nu, i\in\mathcal{H}, & -p(\underline{x}_i - \omega_i) = 0 & (39.6)\\ \mathcal{H}\in\mathbf{H}_\nu, i\in\mathcal{H}, & \theta_i^{\mathcal{H}}\lambda_i p\eta^{\mathcal{H}} - \mu^{\mathcal{H}}\left(u_i(x_i) - u_i(\underline{x}_i)\right) = 0 & (39.7)\\ \mathcal{H}\in\mathbf{H}_\nu, & -\sum_{i\in\mathcal{H}} a_i + 1 = 0 & (39.8)\\ & \sum_{i\in\mathcal{I}} x_i^{\setminus} - \sum_{i\in\mathcal{I}} \omega_i^{\setminus} - \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}\setminus} = 0 & (39.9)\\ & p^C - 1 = 0 & (39.10) \end{cases} \tag{39}$$

and,

$$\forall\, \mathcal{H}\in\mathbf{H}_\nu,\, i\in\mathcal{H}, \quad a_i^* = \alpha_i^{\mathcal{H}}(p^*, E^{\mathcal{H}}). \tag{40}$$

20.2. If (38) *is solution to* (39)*, then* $(\widehat{\pi}(\mathbf{H}), x^*, p^*) \in W_n(E, \mathbf{H}, \chi)$ *and* (40) *holds true.*

Proof. The above relations easily follow from Proposition 17. $\square$

Proposition 21. *Let us fix* $I, C \geq 2$, $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$ *and* $E \in \mathcal{E}(I, C)$*. Then there exist* $\varepsilon > 0$, $\widetilde{p} \in \mathbb{R}^C_{++}$, $\widetilde{x} \in \mathbb{R}^{CI}_{++}$*, such that the following properties hold true:*

$$\begin{gathered} y \in \mathbb{R}^{CI}_{++}, \quad \sum_{i \in \mathcal{I}} y_i \leq \sum_{i \in \mathcal{I}} \widetilde{x}_i + \varepsilon \sum_{\mathcal{H} \in \mathbf{H}_\nu} \eta^{\mathcal{H}} \quad \Rightarrow \\ (u_i(y_i))_{i \in \mathcal{I}} \not> \left((u_i(\widetilde{x}_i))_{\{i\} \in \mathbf{H}_\sigma}, \left(\left(u_i\big(\widetilde{x}_i + \tfrac{\varepsilon}{|\mathcal{H}|} \eta^{\mathcal{H}}\big) \right)_{i \in \mathcal{H}} \right)_{\mathcal{H} \in \mathbf{H}_\nu} \right), \end{gathered} \tag{41}$$

$$\forall \{i\} \in \mathbf{H}_\sigma, \quad \widetilde{p} = \frac{Du_i(\widetilde{x}_i)}{D_{x_i^C} u_i(\widetilde{x}_i)}, \tag{42}$$

$$\forall \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, \quad \widetilde{p} = \frac{Du_i\big(\widetilde{x}_i + \frac{\varepsilon}{|\mathcal{H}|} \eta^{\mathcal{H}}\big)}{D_{x_i^C} u_i\big(\widetilde{x}_i + \frac{\varepsilon}{|\mathcal{H}|} \eta^{\mathcal{H}}\big)}. \tag{43}$$

Proof. Consider $z \in \mathbb{R}^{CI}_{++}$ and the problem

$$\max_{x \in \mathbb{R}^{CI}_{++}} u_1(x_1) \quad \text{s.t.} \quad \begin{cases} u_i(x_i) \geq u_i(z_i), & i \in \{2, \ldots, I\} \\ \sum_{i \in \mathcal{I}} x_i \leq \sum_{i \in \mathcal{I}} z_i \end{cases} \tag{44}$$

It is well known that (44) has a unique solution $\widehat{x} \in \mathbb{R}^{CI}_{++}$ satisfying the following properties:

$$y \in \mathbb{R}^{CI}_{++}, \quad \sum_{i \in \mathcal{I}} y_i \leq \sum_{i \in \mathcal{I}} \widehat{x}_i \quad \Rightarrow \quad \big((u_i(y_i))_{i \in \mathcal{I}}\big) \not> \big((u_i(\widehat{x}_i))_{i \in \mathcal{I}}\big), \tag{45}$$

$$\forall i, j \in \mathcal{I}, \quad \frac{Du_i(\widehat{x}_i)}{D_{x_i^C} u_i(\widehat{x}_i)} = \frac{Du_j(\widehat{x}_j)}{D_{x_j^C} u_j(\widehat{x}_j)}. \tag{46}$$

Define then

$$v = \left(\max \left\{ \frac{\eta^{\mathcal{H}, c}}{|\mathcal{H}|} : \mathcal{H} \in \mathbf{H}_\nu \right\} \right)_{c \in \mathcal{C}} \in \mathbb{R}^C_+ \setminus \{0\}, \quad w = (\min \{\widehat{x}_i^c : i \in \mathcal{I}\})_{c \in \mathcal{C}} \in \mathbb{R}^C_{++},$$

and let $\varepsilon > 0$ be such that $\varepsilon v \ll w$. Then, for every $\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}$, $\widehat{x}_i^{\mathcal{H}} - \frac{\varepsilon}{|\mathcal{H}|} \eta^{\mathcal{H}} \gg 0$, and the proof is complete by defining

$$\begin{aligned} &\forall \{i\} \in \mathbf{H}_\sigma, \ \widetilde{x}_i = \widehat{x}_i, \\ &\forall \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, \ \widetilde{x}_i = \widehat{x}_i^{\mathcal{H}} - \tfrac{\varepsilon}{|\mathcal{H}|} \eta^{\mathcal{H}} \end{aligned}$$

and

$$\widetilde{p} = \frac{Du_i(\widehat{x}_i)}{D_{x_i^C} u_i(\widehat{x}_i)},$$

where $i \in \mathcal{I}$ is arbitrarily chosen. $\square$

Proof of Statement 9.1 of Theorem 9. Using Proposition 20, we are going to prove via Theorem 15 that the function[9]

$$\mathcal{F} : \Xi \times \mathcal{E}(I, C) \to \mathbb{R}^{\dim \Xi}$$

[9] By $\dim \Xi$ we denote the dimension of the manifold Ξ.

$$(\xi, E) \mapsto \mathcal{F}(\xi, E) = \text{left hand side of the System (39)}$$

has the property that

$$\forall E \in \mathcal{E}(I, C), \quad \{\xi \in \Xi : \mathcal{F}(\xi, E) = 0\} \neq \varnothing. \tag{47}$$

Let us fix then

$$E = \Big((\omega_i, u_i)_{i \in \mathcal{I}}, \left(\eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right)_{\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})} \Big) \in \mathcal{E}(I, C),$$

and, from Proposition 21, consider $\varepsilon > 0$, $\widetilde{p} \in \mathbb{R}^C_{++}$, $\widetilde{x} \in \mathbb{R}^{CI}_{++}$ satisfying (41), (42) and (43). Moreover, for every $\mathcal{H} \in \mathbf{H}_\nu$, consider a function $\phi^{\mathcal{H}} : (0,1) \to \mathbb{R}$ such that $\phi^{\mathcal{H}} \in C^1$, $\phi^{\mathcal{H}}(|\mathcal{H}|^{-1}) = 0$, for every $s \in (0,1)$, $D\phi^{\mathcal{H}}(s) < 0$ and

$$\lim_{s \to 0^+} \phi^{\mathcal{H}}(s) = +\infty, \quad \lim_{s \to 1^-} \phi^{\mathcal{H}}(s) = -\infty.$$

Consider then the system in the unknowns $(\xi, \tau) \in \Xi \times [0,1]$ given by

$$\begin{cases}
\{i\} \in \mathbf{H}_\sigma, & Du_i(x_i) - \lambda_i p = 0 \quad (48.1)\\
\{i\} \in \mathbf{H}_\sigma, & -p(x_i - (1-\tau)\omega_i - \tau \widetilde{x}_i) = 0 \quad (48.2)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & Du_i(x_i) - \lambda_i p = 0 \quad (48.3)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p(x_i - (1-\tau)\omega_i - \tau \widetilde{x}_i - ((1-\tau) + \tau\varepsilon)\, a_i \eta^{\mathcal{H}}) = 0 \quad (48.4)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & Du_i(\underline{x}_i) - \underline{\lambda}_i p = 0 \quad (48.5)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p(\underline{x}_i - (1-\tau)\omega_i - \tau \widetilde{x}_i) = 0 \quad (48.6)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & (1-\tau)\theta^{\mathcal{H}}_i \lambda_i p \eta^{\mathcal{H}} + \tau \phi^{\mathcal{H}}(a_i) - \mu^{\mathcal{H}}\left(u_i(x_i) - u_i(\underline{x}_i)\right) = 0 \quad (48.7)\\
\mathcal{H} \in \mathbf{H}_\nu, & -\sum_{i \in \mathcal{H}} a_i + 1 = 0 \quad (48.8)\\
& \sum_{i \in \mathcal{I}} x_i^{\setminus} - \tau \sum_{i \in \mathcal{I}} \widetilde{x}_i^{\setminus} - (1-\tau) \sum_{i \in \mathcal{I}} \omega_i^{\setminus} - ((1-\tau) + \tau\varepsilon) \sum_{\mathcal{H} \in \mathbf{H}_\nu} \eta^{\mathcal{H}\setminus} = 0 \quad (48.9)\\
& p^C - 1 = 0 \quad (48.10)
\end{cases} \tag{48}$$

and define the functions

$$H : \Xi \times [0,1] \to \mathbb{R}^{\dim \Xi}, \quad (\xi, \tau) \mapsto H(\xi, \tau) = \text{left hand side of the System (48)},$$

$$F : \Xi \to \mathbb{R}^{\dim \Xi}, \quad \xi \mapsto F(\xi) = H(\xi, 0),$$

$$G : \Xi \to \mathbb{R}^{\dim \Xi}, \quad \xi \mapsto G(\xi) = H(\xi, 1).$$

Note that F agrees with the left hand side of the System (39), while G with the left hand side of the following system

$$\begin{cases} \{i\} \in \mathbf{H}_\sigma, & Du_i(x_i) - \lambda_i p = 0 & (49.1) \\ \{i\} \in \mathbf{H}_\sigma, & -p(x_i - \widetilde{x}_i) = 0 & (49.2) \\ \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & Du_i(x_i) - \lambda_i p = 0 & (49.3) \\ \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p(x_i - \widetilde{x}_i - \varepsilon a_i \eta^{\mathcal{H}}) = 0 & (49.4) \\ \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & Du_i(\underline{x}_i) - \underline{\lambda}_i p = 0 & (49.5) \\ \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p(\underline{x}_i - \widetilde{x}_i) = 0 & (49.6) \\ \mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & \phi^{\mathcal{H}}(a_i) - \mu^{\mathcal{H}}\left(u_i(x_i) - u_i(\underline{x}_i)\right) = 0 & (49.7) \\ \mathcal{H} \in \mathbf{H}_\nu, & -\sum_{i \in \mathcal{H}} a_i + 1 = 0 & (49.8) \\ & \sum_{i \in \mathcal{I}} x_i^{\backslash} - \sum_{i \in \mathcal{I}} \widetilde{x}_i^{\backslash} - \varepsilon \sum_{\mathcal{H} \in \mathbf{H}_\nu} \eta^{\mathcal{H}\backslash} = 0 & (49.9) \\ & p^C - 1 = 0 & (49.10) \end{cases} \tag{49}$$

We have that F, G, H are continuous on their domains and G is C^1 on Ξ. As proved in the Appendix, we have

$$G^{-1}(0) = \{\xi^*\}, \tag{50}$$

$$D_\xi G(\xi^*) \in \mathbb{M}(\dim \Xi) \quad \text{is not singular,} \tag{51}$$

$$H^{-1}(0) \text{ is compact.} \tag{52}$$

Then (47) follows from Theorem 15. □

Proof of Statement 9.2 of Theorem 9. Since $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$, the result follows from Condition (14). □

Proof of Statement 9.3 of Theorem 9. Since $\mathbf{H} \in \mathfrak{H}^f(\mathcal{I})$ and $\chi \in \mathfrak{D}^e(\mathbb{R}^C_{++} \times \mathcal{E}(I, C))$, the result is a consequence of Proposition 6. □

Proof of Statement 9.4 of Theorem 9. Some preliminary remarks are needed. First of all, note that $\mathcal{F}$ is continuous on $\Xi \times \mathcal{E}(I, C)$ and the function

$$D_\xi \mathcal{F} : \Xi \times \mathcal{E}(I, C) \to \mathbb{M}(\dim \Xi)$$

is well defined and continuous on $\Xi \times \mathcal{E}(I, C)$, as well. Moreover, as proved in the Appendix,

$$\begin{gathered} \pi : \mathcal{F}^{-1}(0) \to \mathcal{E}(I, C), \quad (\xi, E) \mapsto E, \\ \text{is a proper function, that is, the pre-image of any compact set is compact.} \end{gathered} \tag{53}$$

In particular, as π is also continuous, π is closed, that is, the image of any closed set is closed. Given now the set

$$\mathcal{D} = \{E \in \mathcal{E}(I, C) : \mathcal{F}(\xi, E) = 0 \;\Rightarrow\; \det D_\xi \mathcal{F}(\xi, E) \neq 0\},$$

we have that $\mathcal{D}$ is open and dense in $\mathcal{E}(I, C)$. In fact, openness of $\mathcal{D}$ immediately follows from continuity of $\mathcal{F}$ and $\det D_\xi \mathcal{F}$ and (53). The proof of the density of $\mathcal{D}$ is instead more subtle. Fixed

$$E^* = \left((\omega_i^*, u_i^*)_{i \in \mathcal{I}}, (\eta^{\mathcal{H}*}, \theta^{\mathcal{H}*})_{\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})}\right) \in \mathcal{E}(I, C),$$

let us prove there exists a sequence $(E^{[n]})_{n=1}^{\infty}$ in $\mathcal{D}$ such that $E^{[n]} \to E^*$. To this end, define, for every $\gamma = (\gamma_i)_{i\in\mathcal{I}} \in \mathbb{R}^{CI}_{++}$, the element of $\mathcal{E}(I,C)$ given by

$$E^*[\gamma] = \Big((\omega_i^* + \gamma_i, u_i^*)_{i\in\mathcal{I}}, \big(\eta^{\mathcal{H}*}, \theta^{\mathcal{H}*}\big)_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\Big).$$

If we prove that there exists a full measure subset Ω^* of $\mathbb{R}^{CI}_{++}$ such that, for every $\gamma \in \Omega^*$, $E^*[\gamma] \in \mathcal{D}$, then the proof is complete. Indeed, if we consider any sequence $(\gamma^{[n]})_{n=1}^{\infty}$ in Ω^* such that $\gamma^{[n]} \to 0$, then $E^*[\gamma^{[n]}] \to E^*$. Consider now the function

$$\widetilde{\mathcal{F}} : \Xi \times \mathbb{R}^{CI}_{++} \to \mathbb{R}^{\dim \Xi}, \quad (\xi, \gamma) \mapsto \widetilde{\mathcal{F}}(\xi, \gamma) = \mathcal{F}(\xi, E^*[\gamma]).$$

A cumbersome computation allows to verify that 0 is a regular value for $\widetilde{\mathcal{F}}$ and then, from Theorem 16, we obtain the desired result.

We complete the proof defining $\mathcal{D}(I,C,\mathbf{H}) = \mathcal{D}$ and showing that every $E^* \in \mathcal{D}$ has all the properties required in Statement 9.2 of Theorem 9. Since 0 is a regular value of

$$\mathcal{F}(\cdot, E^*) : \Xi \to \mathbb{R}^{\dim \Xi}, \quad \xi \mapsto \mathcal{F}(\xi, E^*),$$

and since (53) holds true, we get

$$|\{\xi \in \Xi : \mathcal{F}(\xi, E^*) = 0\}| = K \quad \text{and} \quad \{\xi \in \Xi : \mathcal{F}(\xi, E^*) = 0\} = \{\xi^{k*}\}_{k=1}^{K}, \tag{54}$$

where K is a positive integer. From Theorem 14, there exist an open neighborhood $\mathcal{O}(E^*) \subseteq \mathcal{E}(I,C)$ of E^* and, for every $k \in \{1,\ldots,K\}$, an open neighborhood $\mathcal{O}(\xi^{k*}) \subseteq \Xi$ of ξ^{k*} and a function $\varphi_k : \mathcal{O}(E^*) \to \mathcal{O}(\xi^{k*})$ such that:

$$\varphi_k \in C^0, \varphi_k(E^*) = \xi^{k*}, \text{ and } \mathcal{O}(\xi^{k*}) \cap \mathcal{O}(\xi^{h*}) = \varnothing \text{ for } k \neq h, \tag{55}$$

$$\Big\{(\xi, E) \in \mathcal{O}(\xi^{k*}) \times \mathcal{O}(E^*) : \mathcal{F}(\xi, E) = 0\Big\} = \Big\{(\xi, E) \in \mathcal{O}(\xi^{k*}) \times \mathcal{O}(E^*) : \xi = \varphi_k(E)\Big\}. \tag{56}$$

Moreover, again from (53), we have that

$$\{(\xi, E) \in \Xi \times \mathcal{O}(E^*) : \mathcal{F}(\xi, E) = 0\} = \bigcup_{k=1}^{K} \{(\xi, E) \in \Xi \times \mathcal{O}(E^*) : \xi = \varphi_k(E)\}. \tag{57}$$

Of course, (54), (55), (56) and (57) imply (17), (18), (19) and (20), respectively. □

Proof of Statement 10.1 of Theorem 10. Consider the set

$$\mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I}) = \{\pi \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I}) : \mathcal{J} \subseteq \mathcal{I}, |\mathcal{J}| \geq 2, \forall j \in \mathcal{J}, \{j\} \in \pi \;\Rightarrow\; \mathcal{J} \notin \mathbf{H}\}.$$

As proved in the Appendix,

$$\mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I}) \neq \varnothing. \tag{58}$$

Fix now $\pi^* \in \mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$ and define $\mathbf{H}^* = \pi^* \cup \mathbf{P}_\sigma(\mathcal{I})$. Note that

$$\pi^* \cap \mathbf{P}_\nu(\mathcal{I}) \neq \varnothing, \quad (\mathbf{H} \cap \mathbf{P}_\nu(\mathcal{I})) \setminus \pi^* \neq \varnothing, \quad \mathbf{H}^* \in \mathfrak{H}^f(\mathcal{I}), \quad \widehat{\pi}(\mathbf{H}^*) = \pi^*,$$

and remember the equalities $\mathbf{H}^*_\sigma = \mathbf{H}^* \setminus \mathbf{P}_\nu(\mathcal{I})$ and $\mathbf{H}^*_\nu = \mathbf{H}^* \cap \mathbf{P}_\nu(\mathcal{I})$. As shown in the proof of Statement 9.4 of Theorem 9, for every

$$\Big((u_i)_{i\in\mathcal{I}}, \big(\eta^{\mathcal{H}}, \theta^{\mathcal{H}}\big)_{\mathcal{H}\in\mathbf{H}^*_\nu}\Big) \in \mathcal{U}^I_C \times (\mathbb{R}^C_+ \setminus \{0\})^{|\mathbf{H}^*_\nu|} \times \left(\bigtimes_{\mathcal{H}\in\mathbf{H}^*_\nu} \Delta^{\mathcal{H}}\right),$$

there exists $(\omega_i)_{i\in\mathcal{I}} \in \mathbb{R}^C_{++}$ such that, for every

$$\left(\eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right)_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})\setminus\mathbf{H}^*_\nu} \in \bigtimes_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})\setminus\mathbf{H}^*_\nu} \left((\mathbb{R}^C_+ \setminus \{0\}) \times \Delta^{\mathcal{H}}\right), \tag{59}$$

we have

$$E = \left((\omega_i, u_i)_{i\in\mathcal{I}}, \left(\eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right)_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\right) \in \mathcal{D}(I, C, \mathbf{H}^*).$$

Let us build now $E^* \in \mathcal{D}(I, C, \mathbf{H}^*)$ as follows. Fix any $(u^*_i)_{i\in\mathcal{I}} \in \mathcal{U}^I_C$ and, for every $\mathcal{H} \in \mathbf{H}^*_\nu$, define[10]

$$\eta^{\mathcal{H}*} = 3 \cdot \mathbf{1}_C, \quad \theta^{\mathcal{H}*} = \left(\frac{1}{|\mathcal{H}|}, \dots, \frac{1}{|\mathcal{H}|}\right) \in \Delta^{\mathcal{H}}.$$

Consider then $(\omega^*_i)_{i\in\mathcal{I}}$ satisfying the above property and complete the definition of E^* choosing any vector in (59). Of course. from Theorem 9, we know there exists $(x^*, p^*) \in \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ such that $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}^*, \chi)$.

Let us observe now that if $E^{**} \in \mathcal{E}(I, C)$ is such that all its components but the ones in (59) agree with the ones of E^*, then $E^{**} \in \mathcal{D}(I, C, \mathbf{H}^*)$ and $(\pi^*, x^*, p^*) \in W_n(E^{**}, \mathbf{H}^*, \chi)$. On the basis of that observation, with a slight abuse of notation, we are going to properly choose the components of E^* in (59) in order to have $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}, \chi)$ and $(\pi^*, x^*) \in P_f(E, \mathbf{H})$.

Choose any $\left(\eta^{\mathcal{H}*}\right)_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})\setminus\mathbf{H}^*_\nu}$ such that

$$\sum_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})\setminus\mathbf{H}^*_\nu} \eta^{\mathcal{H}*} \ll \mathbf{1}_C.$$

Fix now $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu$. Since $\pi^* \in \mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$, we know there is $j^* \in \mathcal{H}$ such that $\{j^*\} \notin \pi^*$ and then $\mathcal{H}_{j^*}(\pi^*) \in \mathbf{P}_\nu(\mathcal{I})$, where $\mathcal{H}_{j^*}(\pi^*)$ is the unique element of π^* which j^* belongs to. From Proposition 18, we can find $\theta^{\mathcal{H}*} \in \Delta^{\mathcal{H}}$ such that

$$\begin{aligned} &\alpha^{\mathcal{H}}_{j^*}\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}*}, \theta^{\mathcal{H}*}\right) \\ &< \alpha^{\mathcal{H}_{j^*}(\pi^*)}_{j^*}\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}_{j^*}(\pi^*)}, \eta^{\mathcal{H}_{j^*}(\pi^*)}, \theta^{\mathcal{H}_{j^*}(\pi^*)}\right). \end{aligned} \tag{60}$$

Let us prove now that $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}, \chi)$. Indeed, Conditions (12) and (13) are trivially fulfilled, while Condition (14) follows since, for every $\mathcal{J} \in \mathbf{H}$, the relation

$$\forall i \in \mathcal{J},\ u_i\left(\chi^{\mathcal{J}}_i\left(p^*, E^{*\mathcal{J}}\right)\right) \geq u_i\left(\chi^{\mathcal{H}_i(\pi^*)}_i\left(p^*, E^{*\mathcal{H}_i(\pi^*)}\right)\right)$$

is never satisfied. Finally, let us prove that $(\pi^*, x^*) \in P_f(E, \mathbf{H})$. As proved in the Appendix,

$$\pi^{**} \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I}) \quad \Rightarrow \quad \sum_{\mathcal{H}\in\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} \leq \sum_{\mathcal{H}\in\pi^*\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}}. \tag{61}$$

If by contradiction there is $(\pi^{**}, x^{**}) \in A_f(E^*, \mathbf{H})$ such that $(u_i(x^{**}_i))_{i\in\mathcal{I}} > (u_i(x^*_i))_{i\in\mathcal{I}}$, then we have $(\pi^*, x^{**}) \in A_f(E^*, \mathbf{H})$ and we contradict Proposition 6.

Finally, as $E^* \in \mathcal{D}(I, C, \mathbf{H}^*)$, we can apply Theorem 9 and find an open neighborhood $\mathcal{O}(E^*) \subseteq \mathcal{D}(I, C, \mathbf{H}^*)$ of E^*, an open neighborhood $\mathcal{O}(x^*, p^*) \subseteq \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ of (x^*, p^*) and a continuous function $g : \mathcal{O}(E^*) \to \mathcal{O}(x^*, p^*)$ such that, for every $E \in \mathcal{O}(E^*)$, $(\pi^*, g(E)) \in W_n(E, \mathbf{H}^*, \chi)$. As, by Proposition 18, we know that, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, $\alpha^{\mathcal{H}}$ is continuous on its domain, we can find a

[10]In what follows, $\mathbf{1}_C = (1, \dots, 1) \in \mathbb{R}^C$.

(not renamed) smaller open neighborhood of E^* and a suitable restriction of g such that, for every $E \in \mathcal{O}(E^*)$,

- $\sum_{\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu} \eta^{\mathcal{H}} \ll 2 \cdot \mathbf{1}_C,$

- $\forall\, \mathcal{H} \in \mathbf{H}^*_\nu, \quad \eta^{\mathcal{H}} \gg 2 \cdot \mathbf{1}_C,$

- $\forall\, \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu, \quad \alpha^{\mathcal{H}}_{j^*}\left(g(E), (\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right) < \alpha^{\mathcal{H}_{j^*}(\pi^*)}_{j^*}\left(g(E), (\omega_i, u_i)_{i\in\mathcal{H}_{j^*}(\pi^*)}, \eta^{\mathcal{H}_{j^*}(\pi^*)}, \theta^{\mathcal{H}_{j^*}(\pi^*)}\right).$

Then, arguing as for E^*, we have that, for every $E \in \mathcal{O}(E^*)$, $(\pi^*, g(E)) \in W_n(E, \mathbf{H}, \chi)$ and the corresponding equilibrium allocation belongs to $P_f(E, \mathbf{H})$. The desired result is then proved defining $\mathcal{O}_1 = \mathcal{O}(E^*)$. □

Proof of Statement 10.2 of Theorem 10. As the present proof is very similar to the previous one, some details are omitted. Fix $\pi^* \in \mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$ and define $\mathbf{H}^* = \pi^* \cup \mathbf{P}_\sigma(\mathcal{I})$.

Let us build now $E^* \in \mathcal{D}(I, C, \mathbf{H}^*)$ as follows. Fix any $(u^*_i)_{i\in\mathcal{I}} \in \mathcal{U}^I_C$ and, for every $\mathcal{H} \in \mathbf{H}^*_\nu$, define

$$\eta^{\mathcal{H}*} = \mathbf{1}_C, \quad \theta^{\mathcal{H}*} = \left(\frac{1}{|\mathcal{H}|}, \dots, \frac{1}{|\mathcal{H}|}\right) \in \Delta^{\mathcal{H}}.$$

Choose then $(\omega^*_i)_{i\in\mathcal{I}}$ satisfying the same property used in the proof of Statement 10.1 of Theorem 10 and complete the definition of E^* choosing any vector in (59). From Theorem 9, we know there exists $(x^*, p^*) \in \mathbb{R}^{CI}_{++} \times \mathbb{R}^C_{++}$ such that $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}^*, \chi)$. As already noted, if $E^{**} \in \mathcal{E}(I, C)$ is such that all its components but the ones in (59) agree with the ones of E^*, then $E^{**} \in \mathcal{D}(I, C, \mathbf{H}^*)$ and $(\pi^*, x^*, p^*) \in W_n(E^{**}, \mathbf{H}^*, \chi)$. Then, with a slight abuse of notation, we are going to properly choose the components of E^* in (59) in order to have $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}, \chi)$ and $(\pi^*, x^*) \notin P_f(E, \mathbf{H})$.

Fixed $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu$, choose any $\eta^{\mathcal{H}*} \in \mathbb{R}^C_{++}$ such that

$$\eta^{\mathcal{H}*} \gg 3|\mathbf{H}^*_\nu| \cdot \mathbf{1}_C.$$

Moreover, we know there is $j^* \in \mathcal{H}$ such that $\{j^*\} \notin \pi^*$ and then $\mathcal{H}_{j^*}(\pi^*) \in \mathbf{P}_\nu(\mathcal{I})$. From Proposition 18, we can find $\theta^{\mathcal{H}*} \in \Delta^{\mathcal{J}}$ such that (60) holds true.

The economy E^* just built satisfies $(\pi^*, x^*, p^*) \in W_n(E^*, \mathbf{H}, \chi)$. Finally, let us prove that $(\pi^*, x^*) \notin P_f(E^*, \mathbf{H})$. Indeed, consider $\mathcal{K} \in (\mathbf{H} \cap \mathbf{P}_\nu(\mathcal{I})) \setminus \mathbf{H}^*_\nu$ and define

$$\pi^{**} = \{\mathcal{K}\} \cup \{\{i\} : i \in \mathcal{I} \setminus \mathcal{K}\} \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I}).$$

Of course

$$\sum_{\mathcal{H} \in \pi^{**} \cap \mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} = \eta^{\mathcal{K}} \gg \sum_{\mathcal{H} \in \pi^{*} \cap \mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}}$$

and we can immediately build $(\pi^{**}, x^{**}) \in A_f(E, \mathbf{H})$ such that such that $(u_i(x^{**}_i))_{i\in\mathcal{I}} > (u_i(x^*_i))_{i\in\mathcal{I}}$. Then we have $(\pi^*, x^*) \notin P_f(E, \mathbf{H})$.

As $E^* \in \mathcal{D}(I, C, \mathbf{H}^*)$, we can apply Theorem 9 and find an open neighborhood $\mathcal{O}(E^*) \subseteq \mathcal{D}(I, C, \mathbf{H}^*)$ of E^*, an open neighborhood $\mathcal{O}(x^*, p^*) \subseteq \mathbb{R}^C_{++}$ of (x^*, p^*) and a continuous function $g : \mathcal{O}(E^*) \to \mathcal{O}(x^*, p^*)$ such that, for every $E \in \mathcal{O}(E^*)$, $(\pi^*, g(E)) \in W_n(E, \mathbf{H}^*, \chi)$. As by Proposition 18 we know that, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, $\alpha^{\mathcal{H}}$ is continuous on its domain, we can find a (not renamed) smaller open neighborhood of E^* and a suitable restriction of g such that, for every $E \in \mathcal{O}(E^*)$,

- $\forall\, \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu, \quad \eta^{\mathcal{H}} \gg 2|\mathbf{H}^*_\nu| \cdot \mathbf{1}_C,$
- $\forall\, \mathcal{H} \in \mathbf{H}^*_\nu, \quad \eta^{\mathcal{H}} \ll 2 \cdot \mathbf{1}_C,$
- $\forall\, \mathcal{H} \in \mathbf{P}_\nu(\mathcal{I}) \setminus \mathbf{H}^*_\nu, \quad \alpha^{\mathcal{H}}_{j^*}\left(g(E), (\omega_i, u_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}}, \theta^{\mathcal{H}}\right)$
$$< \alpha^{\mathcal{H}_{j^*}(\pi^*)}_{j^*}\left(g(E), (\omega_i, u_i)_{i\in\mathcal{H}_{j^*}(\pi^*)}, \eta^{\mathcal{H}_{j^*}(\pi^*)}, \theta^{\mathcal{H}_{j^*}(\pi^*)}\right).$$

Then, arguing as for E^*, we have that, for every $E \in \mathcal{O}(E^*)$, $(\pi^*, g(E)) \in W_n(E, \mathbf{H}, \chi)$ and the corresponding equilibrium allocation does not belong to $P_f(E, \mathbf{H})$. The desired result is then proved defining $\mathcal{O}_2 = \mathcal{O}(E^*)$. □

Proof of Theorem 13.1. Our purpose is to build an economy

$$E^* = \left((\omega^*_i, u^*_i)_{i\in\mathcal{I}}, (\eta^{\mathcal{H}*}, \theta^{\mathcal{H}*})_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\right) \in \mathcal{E}(I, C, \mathbf{P}(\mathcal{I})),$$

and find a suitable price $p^* \in \mathbb{R}^C_{++}$ such that (23) and (24) are satisfied, provided x^* and x^{**} in the statement are obtained by Condition (12). Define first, for every $i \in \mathcal{I}$, $\omega^*_i = \mathbf{1}_C$, and

$$u^*_i : \mathbb{R}^C_{++} \to \mathbb{R}, \quad x_i \mapsto u^*_i(x_i) = \sum_{c\in\mathcal{C}} \ln(x^c_i).$$

Consider now $p^* = \mathbf{1}_C$. It is simple to verify that, for every $i \in \mathcal{I}$,

$$\chi^{\{i\}}_i(p^*, (\omega^*_i, u^*_i)) = \mathbf{1}_C, \quad u^*_i\left(\chi^{\{i\}}_i(p^*, (\omega^*_i, u^*_i))\right) = 0$$

and, in particular,

$$\sum_{i\in\mathcal{I}} \chi^{\{i\}}_i(p^*, (\omega^*_i, u^*_i)) = \sum_{i\in\mathcal{I}} \omega^*_i. \tag{62}$$

In order to complete the definition of E^*, we have to choose, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$, $(\eta^{\mathcal{H}*}, \theta^{\mathcal{H}*}) \in \mathbb{R}^C_{++} \times \Delta^{\mathcal{H}}$ in such a way that (23) and (24) hold true. Define then, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| \geq 3$,

$$\eta^{\mathcal{H}*} = |\mathcal{H}| \cdot \mathbf{1}_C, \quad \theta^{\mathcal{H}*} = \frac{1}{|\mathcal{H}|}(1, \ldots, 1) \in \Delta^{\mathcal{H}},$$

and, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| = 2$,

$$\eta^{\mathcal{H}*} = 4 \cdot \mathbf{1}_C, \quad \theta^{\mathcal{H}\diamond} = \frac{1}{2}(1, 1) \in \Delta^{\mathcal{H}}.$$

Then we are left with defining, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| = 2$, $\theta^{\mathcal{H}*} \in \Delta^{\mathcal{H}}$. Note that, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| \geq 3$, $i \in \mathcal{H}$,

$$\chi^{\mathcal{H}}_i\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}*}, \theta^{\mathcal{H}*}\right) = 2 \cdot \mathbf{1}_C, \quad u^*_i\left(\chi^{\mathcal{H}}_i\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}*}, \theta^{\mathcal{H}*}\right)\right) = C\ln(2),$$

and, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| = 2$, $i \in \mathcal{H}$,

$$\chi^{\mathcal{H}}_i\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}*}, \theta^{\mathcal{H}\diamond}\right) = 3 \cdot \mathbf{1}_C, \quad u^*_i\left(\chi^{\mathcal{H}}_i\left(p^*, (\omega^*_i, u^*_i)_{i\in\mathcal{H}}, \eta^{\mathcal{H}*}, \theta^{\mathcal{H}\diamond}\right)\right) = C\ln(3).$$

Using Proposition 18 and because of the special structure of the objects we have introduced, it is simple to verify that there exists $\beta \in \mathbb{R}$ and, for every $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ such that $|\mathcal{H}| = 2$, $\theta^{\mathcal{H}*} \in \Delta^{\mathcal{H}}$ such that if $\mathcal{H} = \{i_1, i_2\}$ with $i_1 < i_2$, $(i_1, i_2) \neq (1, I)$, then

$$C \ln\left(\tfrac{11}{4}\right) < u_{i_1}^* \left(\chi_{i_1}^{\{i_1,i_2\}} \left(p^*, (\omega_i^*, u_i^*)_{i\in\{i_1,i_2\}}, \eta^{\{i_1,i_2\}*}, \theta^{\{i_1,i_2\}*}\right)\right)$$

$$< \beta < u_{i_2}^* \left(\chi_{i_2}^{\{i_1,i_2\}} \left(p^*, (\omega_i^*, u_i^*)_{i\in\{i_1,i_2\}}, \eta^{\{i_1,i_2\}*}, \theta^{\{i_1,i_2\}*}\right)\right),$$

while if $\mathcal{H} = \{1, I\}$, then

$$C \ln\left(\tfrac{11}{4}\right) < u_I^* \left(\chi_I^{\{1,I\}} \left(p^*, (\omega_i^*, u_i^*)_{i\in\{1,I\}}, \eta^{\{1,I\}*}, \theta^{\{1,I\}*}\right)\right)$$

$$< \beta < u_1^* \left(\chi_1^{\{1,I\}} \left(p^*, (\omega_i^*, u_i^*)_{i\in\{1,I\}}, \eta^{\{1,I\}*}, \theta^{\{1,I\}*}\right)\right).$$

Let us verify now that the economy E^* just built satisfies (23) and (24). Let us prove at first that

$$(\mathbf{P}_\sigma(\mathcal{I}), x^*, p^*) \in W^{MC}(E^*, \mathbf{P}(\mathcal{I}), \chi), \tag{63}$$

where, for every $i \in \mathcal{I}$, $x_i^* = \chi_i^{\{i\}}(p^*, E^*)$. Of course, Conditions 12.1 and 12.2 in Definition 12 follow by definition and by (62), respectively. In order to verify that Condition 12.3 in Definition 12 holds true, consider any $\mathcal{H} \in \mathbf{P}_\nu(\mathcal{I})$ and find a suitable $\mathcal{K} \in \mathbf{P}_\nu(\mathcal{I})$ such that the required inequalities hold true. Using (2), we have that if $|\mathcal{H}| \geq 3$, then we can choose any $\mathcal{K} \subseteq \mathcal{H}$ such that $|\mathcal{K}| = 2$, while if $\mathcal{H} = \{i_1, i_2\}$, $i_1 < i_2$, then we can choose

$$\begin{array}{lll} \mathcal{K} = \{i_1 - 1, i_1\} & \text{if} & 1 < i_1, \\ \mathcal{K} = \{i_1, I\} & \text{if} & i_1 = 1, i_2 < I, \\ \mathcal{K} = \{I - 1, I\} & \text{if} & i_1 = 1, i_2 = I. \end{array}$$

We stress that the above argument requires $I \geq 3$. Defined now, for every $i \in \mathcal{I}$, $x_i^{**} = \chi_i^{\{I\}}(p^*, E^*)$, it is immediate to prove both $(\{\mathcal{I}\}, x^{**}, p^*) \in W^{MC}(E^*, \mathbf{H}, \chi)$ and (24). □

A Appendix

Proof of (35). In order to show that $\Phi^{\mathcal{H}}$ is continuous, it suffices to prove that each sequence $(\zeta^{\mathcal{H}[n]}, p^{[n]}, e^{\mathcal{H}[n]})_{n=1}^\infty$ in $(\mathcal{G}^{\mathcal{H}})^{-1}(0)$, such that $(p^{[n]}, e^{\mathcal{H}[n]})_{n=1}^\infty$ converges in $\mathbb{R}_{++}^C \times \mathcal{E}^{\mathcal{H}}(I, C)$, admits a converging subsequence in $(\mathcal{G}^{\mathcal{H}})^{-1}(0)$. More precisely, assume that

$$(p^{[n]}, e^{\mathcal{H}[n]}) = (p^{[n]}, (\omega_i^{[n]}, u_i^{[n]})_{i\in\mathcal{H}}, \eta^{\mathcal{H}[n]}, \theta^{\mathcal{H}[n]}) \to$$

$$(p^\diamond, e^{\mathcal{H}\diamond}) = (p^\diamond, (\omega_i^\diamond, u_i^\diamond)_{i\in\mathcal{H}}, \eta^{\mathcal{H}\diamond}, \theta^{\mathcal{H}\diamond}) \in \mathbb{R}_{++}^C \times \mathcal{E}^{\mathcal{H}}(I, C).$$

Then it suffices to show that, up to a subsequence, $(\zeta^{\mathcal{H}[n]})_{n=1}^\infty$ converges to a certain $\zeta^{\mathcal{H}\diamond} \in Z^{\mathcal{H}}$ as the condition $\mathcal{G}^{\mathcal{H}}(\zeta^{\mathcal{H}\diamond}, p^\diamond, e^{\mathcal{H}\diamond}) = 0$ follows by the continuity of $\mathcal{G}^{\mathcal{H}}$. We are going to use a diagonal argument and then, in particular, when we speak about a converging sequence, in fact we only mean it has a converging subsequence. Of course, for every $i \in \mathcal{H}$, we have $a_i^{[n]} \to a_i^\diamond \in [0, 1]$.

Fix $i \in \mathcal{H}$ and prove first that $(x_i^{[n]})_{n=1}^\infty$ converges to an element of $\mathbb{R}_{++}^C$. Note that $\omega_i^{[n]} \to \omega_i^\diamond$. Then, for every $n \in \mathbb{N}^*$,

$$u_i^{[n]}(x_i^{[n]}) \geq u_i^{[n]}(\omega_i^{[n]}) \geq \min_{x_i \in S_i} u_i^{[n]}(x_i) \geq \min_{x_i \in S_i} u_i^\diamond(x_i) - \varepsilon_n,$$

where $S_i = \{\omega_i^{[n]}\}_{n=1}^{\infty} \cup \{\omega_i^{\diamond}\}$ is a compact subset of $\mathbb{R}^C_{++}$ and

$$\varepsilon_n = \max_{x_i \in S_i} \left| u_i^{[n]}(x_i) - u_i^{\diamond}(x_i) \right|.$$

Of course, by the definition of the open-compact topology on $C^2(\mathbb{R}^C_{++})$, we have $\varepsilon_n \to 0$ as $n \to \infty$. Let $x_i^* \in S_i$ be such that

$$\min_{x_i \in S_i} u_i^{\diamond}(x_i) = u_i^{\diamond}(x_i^*),$$

and let $\delta > 0$ be small enough such that $x_i^* - 2\delta \mathbf{1}_C \in \mathbb{R}^C_{++}$. Since by (2), $u_i^{\diamond}(x_i^*) > u_i^{\diamond}(x_i^* - \delta \mathbf{1}_C)$, there exists $n_1 \in \mathbb{N}^*$ such that, for every $n \geq n_1$,

$$u_i^{[n]}(x_i^{[n]}) \geq u_i^{\diamond}(x_i^*) - \varepsilon_n > u_i^{\diamond}(x_i^* - \delta \mathbf{1}_C). \tag{64}$$

Moreover, for every $n \in \mathbb{N}^*$, we have that

$$p^{[n]} x_i^{[n]} \leq p^{[n]} \omega_i^{[n]} + a_i^{[n]} p^{[n]} \eta^{\mathcal{H}[n]},$$

and then, since $p^{[n]} \to p^{\diamond}$, $\omega_i^{[n]} \to \omega_i^{\diamond}$, $\eta^{\mathcal{H}[n]} \to \eta^{\mathcal{H}\diamond}$ and $a_i^{[n]} \in [0,1]$, it is immediate to prove that there exists a constant $k > 0$ such that, for every $n \in \mathbb{N}^*$, $x_i^{[n]} \in [0, k\mathbf{1}_C]$.

Let us consider now $x_i^{\diamond} \in [0, k\mathbf{1}_C]$ a cluster point of the set $\{x_i^{[n]}\}_{n=1}^{\infty}$ and assume $x_i^{[n]} \to x_i^{\diamond}$. Consider any $\widetilde{x}_i \in \mathbb{R}^C_{++}$ such that $u_i^{\diamond}(\widetilde{x}_i) = u_i^{\diamond}(x_i^* - 2\delta \mathbf{1}_C)$. For $n \in \mathbb{N}^*$ large enough, by (64) we get $u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\widetilde{x}_i) \geq 0$ and then

$$\begin{aligned} 0 &\leq u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\widetilde{x}_i) \leq Du_i^{[n]}(\widetilde{x}_i)(x_i^{[n]} - \widetilde{x}_i) \\ &= \left(Du_i^{[n]}(\widetilde{x}_i) - Du_i^{\diamond}(\widetilde{x}_i) \right)(x_i^{[n]} - \widetilde{x}_i) + Du_i^{\diamond}(\widetilde{x}_i)(x_i^{[n]} - \widetilde{x}_i). \end{aligned}$$

Taking now the limit as n goes to infinity in the previous inequality, we obtain $Du_i^{\diamond}(\widetilde{x}_i)(x_i^{\diamond} - \widetilde{x}_i) \geq 0$. More precisely, the following relation holds true

$$x_i^{\diamond} \in \bigcap_{\widetilde{x}_i \in \left\{ z_i \in \mathbb{R}^C_{++} : u_i^{\diamond}(z_i) = u_i^{\diamond}(x_i^* - 2\delta \mathbf{1}_C) \right\}} \left\{ y_i \in \mathbb{R}^C : Du_i^{\diamond}(\widetilde{x}_i)(y_i - \widetilde{x}_i) \geq 0 \right\}. \tag{65}$$

Moreover, Assumptions (3) and (4) and a well-known result from convex analysis assure that the right hand side of (65) is equal to the set

$$\{y \in \mathbb{R}^C_{++} : u_i^{\diamond}(y) \geq u_i^{\diamond}(x_i^* - 2\delta \mathbf{1}_C)\}.$$

Then $x_i^{\diamond} \in \mathbb{R}^C_{++}$ and the proof is complete. A very similar argument allows to prove that, for every $i \in \mathcal{H}$, the sequence $(\underline{x}_i^{[n]})_{n=1}^{\infty}$ converges to $\underline{x}_i^{\diamond} \in \mathbb{R}^C_{++}$. Moreover, from the above results, we immediately have that, for every $i \in \mathcal{H}$, $(\lambda_i^{[n]})_{n=1}^{\infty}$ converges to $\lambda_i^{\diamond} \in \mathbb{R}_{++}$ and $(\underline{\lambda}_i^{[n]})_{n=1}^{\infty}$ converges to $\underline{\lambda}_i^{\diamond} \in \mathbb{R}_{++}$.

We are left with proving that $a_i^{\diamond} \in (0,1)$ and $(\mu^{\mathcal{H}[n]})_{n=1}^{\infty}$ converges in $\mathbb{R}$. Assume by contradiction there exists $i^* \in \mathcal{H}$ such that $a_{i^*}^{[n]} \to a_{i^*}^{\diamond} = 0$. Of course, we have

$$\omega_{i^*}^{[n]} + a_{i^*}^{[n]} \eta^{\mathcal{H}[n]} \to \omega_{i^*}^{\diamond} \quad \text{and} \quad \omega_{i^*}^{[n]} \to \omega_{i^*}^{\diamond},$$

and then [11]

$$\lim_{n\to\infty}\left(u_{i^*}^{[n]}(x_{i^*}^{[n]}) - u_{i^*}^{[n]}(\underline{x}_{i^*}^{[n]})\right) = 0.$$

From (29.5), we deduce the equality

$$\mu^{\mathcal{H}[n]} = \frac{\theta_i^{\mathcal{H}[n]}\lambda_i^{[n]}p^{[n]}\eta^{\mathcal{H}[n]}}{u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\underline{x}_i^{[n]})}, \tag{66}$$

where $i \in \mathcal{H}$ is arbitrarily chosen. Using (66) with $i = i^*$, we obtain

$$\lim_{n\to\infty}\mu^{\mathcal{H}[n]} = +\infty$$

for the denominator goes to zero as n goes to infinity and it is always positive, and the numerator converges to a positive number. From (29.6) we obtain that there exists $i_* \in \mathcal{H}$ such that $a_{i_*}^{[n]} \to a_{i_*}^{\diamond} > 0$ and then

$$\lim_{n\to\infty}\left(u_{i_*}^{[n]}(x_{i_*}^{[n]}) - u_{i_*}^{[n]}(\underline{x}_{i_*}^{[n]})\right) = u_{i_*}^{\diamond}(x_{i_*}^{\diamond}) - u_{i_*}^{\diamond}(\underline{x}_{i_*}^{\diamond}) > 0.$$

Using now (66) with $i = i_*$, we obtain that $(\mu^{\mathcal{H}[n]})_{n=1}^{\infty}$ is uniformly bounded from above and the contradiction is found. Then we have that, for every $i \in \mathcal{H}$, $a_i^{\diamond} > 0$ and by (29.6) we get that, for every $i \in \mathcal{H}$, $a_i^{\diamond} \in (0,1)$. Finally, the convergence of $(\mu^{\mathcal{H}[n]})_{n=1}^{\infty}$ follows from (66) and the previous considerations. □

Proof of (50). Let us consider the vector

$$\xi^* = \left((x_i^*, \lambda_i^*)_{\{i\}\in\mathbf{H}_\sigma}, \left((x_i^*, \lambda_i^*, \underline{x}_i^*, \underline{\lambda}_i^*, a_i^*)_{i\in\mathcal{H}}, \mu^{\mathcal{H}*}\right)_{\mathcal{H}\in\mathbf{H}_\nu}, p^*\right) \in \Xi$$

defined as follows:

$$\begin{array}{ll}
\forall\,\{i\} \in \mathbf{H}_\sigma, & x_i^* = \widetilde{x}_i, \\
\forall\,\{i\} \in \mathbf{H}_\sigma, & \lambda_i^* = D_{x_i^C} u_i(\widetilde{x}_i), \\
\forall\,\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & x_i^* = \widetilde{x}_i + \frac{\varepsilon}{|\mathcal{H}|}\eta^{\mathcal{H}}, \\
\forall\,\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & \lambda_i^* = D_{x_i^C} u_i\left(\widetilde{x}_i + \frac{\varepsilon}{|\mathcal{H}|}\eta^{\mathcal{H}}\right), \\
\forall\,\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & a_i^* = \frac{1}{|\mathcal{H}|}, \\
\forall\,\mathcal{H} \in \mathbf{H}_\nu, & \mu^{\mathcal{H}*} = 0, \\
 & p^* = \widetilde{p}, \\
\forall\,\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & (\underline{x}_i^*, \underline{\lambda}_i^*) \text{ is the unique solution to the system} \left\{\begin{array}{l} Du_i(\underline{x}_i) - \underline{\lambda}_i\widetilde{p} = 0 \\ -\widetilde{p}(\underline{x}_i - \widetilde{x}_i) = 0 \end{array}\right.
\end{array} \tag{67}$$

It is simple to verify that $G(\xi^*) = 0$. In order to prove that ξ^* is indeed the unique solution to $G(\xi) = 0$, let us consider $\xi^{\diamond}$ such that $G(\xi^{\diamond}) = 0$ and show that $\xi^* = \xi^{\diamond}$. First of all, note that, for every $\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}$,

$$p^{\diamond}x_i^{\diamond} = p^{\diamond}\left(\widetilde{x}_i + \varepsilon a_i^*\eta^{\mathcal{H}}\right) > p^{\diamond}\widetilde{x}_i = p^{\diamond}\underline{x}_i^{\diamond}.$$

[11] Note that the solution (x_i, λ_i) to the problem

$$\left\{\begin{array}{l} Du_i(x_i) - \lambda_i p = 0 \\ -p(x_i - \omega_i) = 0 \end{array}\right.$$

continuously depends on $(p, \omega_i, u_i) \in \mathbb{R}_{++}^C \times \mathbb{R}_{++}^C \times \mathcal{U}_C$.

Then $u_i(x_i^\diamond) > u_i(\underline{x}_i^\diamond)$. Consider now $\mathcal{H} \in \mathbf{H}_\nu$ and prove that $\mu^{\mathcal{H}\diamond} = 0$. If by contradiction $\mu^{\mathcal{H}\diamond} > 0$, then, for every $i \in \mathcal{H}$, $\phi^{\mathcal{H}}(a_i^\diamond) > 0$ and then $a_i^\diamond < \frac{1}{|\mathcal{H}|}$. Then it follows

$$\sum_{i\in\mathcal{H}} a_i^\diamond < 1$$

and the contradiction is found. Analogously, if by contradiction $\mu^{\mathcal{H}\diamond} < 0$, then, for every $i \in \mathcal{H}$, $\phi^{\mathcal{H}}(a_i^\diamond) < 0$ and then $a_i^\diamond > \frac{1}{|\mathcal{H}|}$. Then it follows

$$\sum_{i\in\mathcal{H}} a_i^\diamond > 1$$

and the contradiction is found again. Then we obtain $\mu^{\mathcal{H}\diamond} = 0$ and, for every $i \in \mathcal{H}$, $a_i^\diamond = \frac{1}{|\mathcal{H}|}$.

From (49.1), (49.2), (49.3) and (49.4) we have that, for every $i \in \mathcal{I}$, $u_i(x_i^\diamond) \geq u_i(x_i^*)$ and from (49.2), (49.4), (49.9) we have

$$\sum_{i\in\mathcal{I}} x_i^* = \sum_{i\in\mathcal{I}} \widetilde{x}_i + \varepsilon \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}},$$

and

$$\sum_{i\in\mathcal{I}} x_i^\diamond = \sum_{i\in\mathcal{I}} \widetilde{x}_i + \varepsilon \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}}.$$

If by contradiction $x^* \neq x^\diamond$, then we can define, for every $i \in \mathcal{I}$, $x_i^\circ = \frac{1}{2}(x_i^* + x_i^\diamond)$. Thus we have that $(x_i^\circ)_{i\in\mathcal{I}}$ satisfies

$$\sum_{i\in\mathcal{I}} x_i^\circ = \sum_{i\in\mathcal{I}} \widetilde{x}_i + \varepsilon \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}},$$

as well and, because of (3), for every $i \in \mathcal{I}$, $u_i(x_i^\circ) \geq u_i(x_i^*)$ with at least a strict inequality. Then, by the definition of $(x_i^*)_{i\in\mathcal{I}}$ and (41), a contradiction is found. As $x^\diamond = x^*$, we immediately have that, for every $i \in \mathcal{I}$, $\lambda_i^\diamond = \lambda_i^*$, $\underline{x}_i^\diamond = \underline{x}_i^*$, $\underline{\lambda}_i^\diamond = \underline{\lambda}_i^*$ and $p^\diamond = p^*$. □

Proof of (51). In order to prove (51), we have to show that if

$$\Delta\xi = \left((\Delta x_i, \Delta\lambda_i)_{\{i\}\in\mathbf{H}_\sigma}, \left((\Delta x_i, \Delta\lambda_i, \Delta\underline{x}_i, \Delta\underline{\lambda}_i, \Delta a_i)_{i\in\mathcal{H}}, \Delta\mu^{\mathcal{H}} \right)_{\mathcal{H}\in\mathbf{H}_\nu}, \Delta p \right) \in \mathbb{R}^{\dim \Xi} \tag{68}$$

is such that $DG(\xi^*)\Delta\xi = 0$, then $\Delta\xi = 0$. Using in particular the fact that, for every $\mathcal{H} \in \mathbf{H}_\nu$, $\mu^{\mathcal{H}*} = 0$, the linear system $DG(\xi^*)\Delta\xi = 0$ can be written as

$$\begin{cases}
\{i\} \in \mathbf{H}_\sigma, & D^2u_i(x_i^*)\Delta x_i - \Delta\lambda_i p^* - \lambda_i^* \Delta p = 0 \quad (69.1)\\
\{i\} \in \mathbf{H}_\sigma, & -p^*\Delta x_i = 0 \quad (69.2)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & D^2u_i(x_i^*)\Delta x_i - \Delta\lambda_i p^* - \lambda_i^* \Delta p = 0 \quad (69.3)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p^*\Delta x_i + \varepsilon p^* \eta^{\mathcal{H}} \Delta a_i = 0 \quad (69.4)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & D^2u_i(\underline{x}_i^*)\Delta\underline{x}_i - \Delta\underline{\lambda}_i p^* - \underline{\lambda}_i^* \Delta p = 0 \quad (69.5)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & -p^*\Delta\underline{x}_i - \Delta p(\underline{x}_i^* - \widetilde{x}_i) = 0 \quad (69.6)\\
\mathcal{H} \in \mathbf{H}_\nu, i \in \mathcal{H}, & D\phi^{\mathcal{H}}(a_i^*)\Delta a_i - \Delta\mu^{\mathcal{H}}\left(u_i(x_i^*) - u_i(\underline{x}_i^*)\right) = 0 \quad (69.7)\\
\mathcal{H} \in \mathbf{H}_\nu, & \sum_{i\in\mathcal{H}} \Delta a_i = 0 \quad (69.8)\\
 & \sum_{i\in\mathcal{I}} \Delta x_i^\backslash = 0 \quad (69.9)\\
 & \Delta p^C = 0 \quad (69.10)
\end{cases} \tag{69}$$

First of all, note that, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, we have $D\phi^{\mathcal{H}}(a_i^*) < 0$, $u_i(x_i^*) - u_i(\underline{x}_i^*) > 0$ and then from (69.7),

$$\Delta a_i = \frac{\Delta \mu^{\mathcal{H}}}{D\phi^{\mathcal{H}}(a_i^*)} \left(u_i(x_i^*) - u_i(\underline{x}_i^*) \right).$$

Summing up for $i \in \mathcal{H}$, we obtain

$$0 = \sum_{i \in \mathcal{H}} \Delta a_i = \frac{\Delta \mu^{\mathcal{H}}}{D\phi^{\mathcal{H}}(a_i^*)} \sum_{i \in \mathcal{H}} \left(u_i(x_i^*) - u_i(\underline{x}_i^*) \right).$$

That implies $\Delta \mu^{\mathcal{H}} = 0$ and then, for every $i \in \mathcal{H}$, $\Delta a_i = 0$.

From (69.1) we get, for every $\{i\} \in \mathbf{H}_\sigma$,

$$\Delta x_i D^2 u_i(x_i^*) \Delta x_i - \Delta \lambda_i p^* \Delta x_i - \lambda_i^* \Delta p \Delta x_i = 0$$

and from (69.2) we obtain

$$\frac{\Delta x_i D^2 u_i(x_i^*) \Delta x_i}{\lambda_i^*} = \Delta p \Delta x_i.$$

Analogously, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, using (69.3), (69.4) and the fact that $\Delta a_i = 0$, we obtain that

$$\frac{\Delta x_i D^2 u_i(x_i^*) \Delta x_i}{\lambda_i^*} = \Delta p \Delta x_i.$$

Then, from the above relations and (69.9) and (69.10),

$$\sum_{i \in \mathcal{I}} \frac{\Delta x_i D^2 u_i(x_i^*) \Delta x_i}{\lambda_i^*} = \Delta p^{\setminus} \left(\sum_{i \in \mathcal{I}} \Delta x_i^{\setminus} \right) + \Delta p^C \left(\sum_{i \in \mathcal{I}} \Delta x_i^C \right) = 0.$$

Then, for every $i \in \mathcal{I}$, we have $\Delta x_i = 0$. From (69.1), (69.3) and (69.10), it immediately follows that, for every $i \in \mathcal{I}$, $\Delta \lambda_i = 0$ and $\Delta p = 0$. Then from (69.5), (69.6) we finally have that, for every $i \in \mathcal{I}$, $\Delta \underline{x}_i = 0$ and $\Delta \underline{\lambda}_i = 0$ and the proof is complete. □

Proof of (52). Let us consider a sequence $(\xi^{[n]}, \tau^{[n]})_{n=1}^\infty$ in $\Xi \times [0,1]$ such that, for every $n \in \mathbb{N}^*$, $H(\xi^{[n]}, \tau^{[n]}) = 0$ and prove that, up to a subsequence, $(\xi^{[n]}, \tau^{[n]}) \to (\xi^\diamond, \tau^\diamond) \in \Xi \times [0,1]$. As in the proof of (35) we are going to use a diagonal argument and again, when we speak about a converging sequence, in fact we only mean it has a converging subsequence. We surely have $\tau^{[n]} \to \tau^\diamond \in [0,1]$ and, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, $a_i^{[n]} \to a_i^\diamond \in [0,1]$.

Fix now $\{i\} \in \mathbf{H}_\sigma$ and prove $(x_i^{[n]})_{n=1}^\infty$ converges to an element of $\mathbb{R}^C_{++}$. Note that

$$(1 - \tau^{[n]})\omega_i + \tau^{[n]} \widetilde{x}_i \to (1 - \tau^\diamond)\omega_i + \tau^\diamond \widetilde{x}_i.$$

Then, for every $n \in \mathbb{N}^*$,

$$u_i(x_i^{[n]}) \geq u_i\left((1 - \tau^{[n]})\omega_i + \tau^{[n]} \widetilde{x}_i \right) \geq$$

$$\min \left\{ u_i(x_i) : x_i \in \left\{ (1 - \tau^{[n]})\omega_i + \tau^{[n]} \widetilde{x}_i : n \in \mathbb{N}^* \right\} \cup \left\{ (1 - \tau^\diamond)\omega_i + \tau^\diamond \widetilde{x}_i \right\} \right\}.$$

As the set

$$S_i = \left\{ (1 - \tau^{[n]})\omega_i + \tau^{[n]} \widetilde{x}_i : n \in \mathbb{N}^* \right\} \cup \left\{ (1 - \tau^\diamond)\omega_i + \tau^\diamond \widetilde{x}_i \right\} \subseteq \mathbb{R}^C_{++}$$

is compact, there is an element $v_i \in S_i$ such that, for every $n \in \mathbb{N}^*$, $u_i(x_i^{[n]}) \geq u_i(v_i)$. Note also that, arguing as in the proof of Proposition 19, we have

$$\sum_{i\in\mathcal{I}} x_i^{[n]} - (1-\tau^{[n]})\sum_{i\in\mathcal{I}}\omega_i - \tau^{[n]}\sum_{i\in\mathcal{I}}\widetilde{x}_i - \Big((1-\tau^{[n]}) + \tau^{[n]}\varepsilon\Big)\sum_{\mathcal{H}\in\mathbf{H}_\nu}\eta^{\mathcal{H}} = 0. \tag{70}$$

It follows there exists $w_i \in \mathbb{R}^C_{++}$ such that, for every $n \in \mathbb{N}^*$, $x_i^{[n]} \leq w_i$. Then, for every $n \in \mathbb{N}^*$,

$$x_i^{[n]} \in C_i^1 \cap C_i^2$$

where

$$C_i^1 = \left\{x_i \in \mathbb{R}^C_{++} : u_i(x_i) \geq u_i(v_i)\right\}, \quad C_i^2 = \left\{x_i \in \mathbb{R}^C_{++} : 0 \leq x_i \leq w_i\right\}.$$

As by (4), $C_i^1 \cap C_i^2$ is a compact subset of $\mathbb{R}^C_{++}$, the sequence $(x_i^{[n]})_{n=1}^\infty$ has a subsequence converging to an element of $\mathbb{R}^C_{++}$. Fixed $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, the proof that $(x_i^{[n]})_{n=1}^\infty$ and $(\underline{x}_i^{[n]})_{n=1}^\infty$ converge to an element of $\mathbb{R}^C_{++}$ is completely analogous to the previous case. Moreover, from the above results, we immediately have that, for every $i \in \mathcal{I}$, $(\lambda_i^{[n]})_{n=1}^\infty$ converges to $\lambda_i^\diamond \in \mathbb{R}_{++}$, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, $(\underline{\lambda}_i^{[n]})_{n=1}^\infty$ converges to $\underline{\lambda}_i^\diamond \in \mathbb{R}_{++}$, and $(p^{[n]})_{n=1}^\infty$ converges to $p^\diamond \in \mathbb{R}^C_{++}$.

We are left with proving that $a_i^\diamond \in (0,1)$ and $(\mu^{\mathcal{H}[n]})_{n=1}^\infty$ converges in $\mathbb{R}$. Fix $\mathcal{H} \in \mathbf{H}_\nu$ and assume by contradiction there exists $i^* \in \mathcal{H}$ such that $a_{i^*}^{[n]} \to a_{i^*}^\diamond = 0$. Of course, we have

$$(1-\tau^{[n]})\omega_{i^*} + \tau^{[n]}\widetilde{x}_{i^*} + \Big((1-\tau^{[n]}) + \tau^{[n]}\varepsilon\Big)\, a_{i^*}^{[n]}\eta^{\mathcal{H}} \to (1-\tau^\diamond)\omega_{i^*} + \tau^\diamond\widetilde{x}_{i^*},$$

and

$$(1-\tau^{[n]})\omega_{i^*} + \tau^{[n]}\widetilde{x}_{i^*} \to (1-\tau^\diamond)\omega_{i^*} + \tau^\diamond\widetilde{x}_{i^*},$$

and then

$$\lim_{n\to\infty}\Big(u_{i^*}(x_{i^*}^{[n]}) - u_{i^*}(\underline{x}_{i^*}^{[n]})\Big) = 0.$$

From (48.7) we deduce the equality

$$\mu^{\mathcal{H}[n]} = \frac{(1-\tau^{[n]})\theta_i^{\mathcal{H}}\lambda_i^{[n]}p^{[n]}\eta^{\mathcal{H}} + \tau^{[n]}\phi^{\mathcal{H}}(a_i^{[n]})}{u_i(x_i^{[n]}) - u_i(\underline{x}_i^{[n]})}, \tag{71}$$

where $i \in \mathcal{H}$ is arbitrarily chosen. Using now (71) with $i = i^*$, we obtain

$$\lim_{n\to\infty}\mu^{\mathcal{H}[n]} = +\infty$$

for the denominator goes to zero as n goes to infinity and it is always positive, and there exists $c > 0$ such that, for n large enough, the numerator is greater than c. Indeed, as long as $a_{i^*}^{[n]} < \frac{1}{|\mathcal{H}|}$, the numerator is positive. Moreover if $\tau^\diamond < 1$, then

$$\liminf_{n\to\infty}(1-\tau^{[n]})\theta_{i^*}^{\mathcal{H}}\lambda_{i^*}^{[n]}p^{[n]}\eta^{\mathcal{H}} > 0$$

while if $\tau^\diamond = 1$, then

$$\lim_{n\to\infty}\tau^{[n]}\phi^{\mathcal{H}}(a_{i^*}^{[n]}) = +\infty.$$

From (48.8) we obtain there exists $i_* \in \mathcal{H}$ such that $a_{i_*}^{[n]} \to a_{i_*}^\diamond > 0$ and then

$$\lim_{n\to\infty}\Big(u_{i_*}(x_{i_*}^{[n]}) - u_{i_*}(\underline{x}_{i_*}^{[n]})\Big) = u_{i_*}(x_{i_*}^\diamond) - u_{i_*}(\underline{x}_{i_*}^\diamond) > 0.$$

Using now (71) with $i = i_*$, we have that $(\mu^{\mathcal{H}[n]})_{n=1}^{\infty}$ is uniformly bounded from above and the contradiction is found. Then we have that, for every $i \in \mathcal{H}$, $a_i^{\diamond} > 0$ and by (48.8) we get that, for every $i \in \mathcal{H}$, $a_i^{\diamond} \in (0,1)$. Finally, it is immediate to prove that thanks to (48.8) we can assume there exists $i^{**} \in \mathcal{H}$ such that, for every $n \in \mathbb{N}^*$, $a_{i^{**}}^{[n]} \leq \frac{1}{|\mathcal{H}|}$. Then, the convergence of $(\mu^{\mathcal{H}[n]})_{n=1}^{\infty}$ follows from (71) with $i = i^{**}$ and the previous considerations. □

Proof of (53). In order to show that π is proper, we have to prove that each sequence $(\xi^{[n]}, E^{[n]})_{n=1}^{\infty}$ in $\mathcal{F}^{-1}(0)$, such that $(E^{[n]})_{n=1}^{\infty}$ converges in $\mathcal{E}(I, C)$, admits a converging subsequence in $\mathcal{F}^{-1}(0)$. Assume that

$$(E^{[n]})_{n=1}^{\infty} = \left((\omega_i^{[n]}, u_i^{[n]})_{i\in\mathcal{I}}, (\eta^{\mathcal{H}[n]}, \theta^{\mathcal{H}[n]})_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\right) \to$$

$$E^{\diamond} = \left((\omega_i^{\diamond}, u_i^{\diamond})_{i\in\mathcal{I}}, \left(\eta^{\mathcal{H}\diamond}, \theta^{\mathcal{H}\diamond}\right)_{\mathcal{H}\in\mathbf{P}_\nu(\mathcal{I})}\right) \in \mathcal{E}(I, C).$$

Then it suffices to show that, up to a subsequence, $(\xi^{[n]})_{n=1}^{\infty}$ converges to a certain $\xi^{\diamond} \in \Xi$ as the condition $\mathcal{F}(\xi^{\diamond}, E^{\diamond}) = 0$ follows by the continuity of $\mathcal{F}$. As in the proof of (35) we are going to use a diagonal argument and again, when we speak about a converging sequence, in fact we only mean it has a converging subsequence. We surely have, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, $a_i^{[n]} \to a_i^{\diamond} \in [0,1]$.

Fix $\{i\} \in \mathbf{H}_\sigma$ and prove $(x_i^{[n]})_{n\in\mathbb{N}}$ converges to an element of $\mathbb{R}^C_{++}$. Note that $\omega_i^{[n]} \to \omega_i^{\diamond}$. Then, for every $n \in \mathbb{N}^*$,

$$u_i^{[n]}(x_i^{[n]}) \geq u_i^{[n]}(\omega_i^{[n]}) \geq \min_{x_i\in S_i} u_i^{[n]}(x_i) \geq \min_{x_i\in S_i} u_i^{\diamond}(x_i) - \varepsilon_n,$$

where $S_i = \{\omega_i^{[n]}\}_{n=1}^{\infty} \cup \{\omega_i^{\diamond}\}$ is a compact subset of $\mathbb{R}^C_{++}$ and

$$\varepsilon_n = \max_{x_i\in S_i} \left| u_i^{[n]}(x_i) - u_i^{\diamond}(x_i) \right|.$$

By the definition of the open-compact topology on $C^2(\mathbb{R}^C_{++})$, we have that $\varepsilon \to 0$ as $n \to \infty$. Let $x_i^* \in S_i$ be such that

$$\min_{x_i\in S_i} u_i^{\diamond}(x_i) = u_i^{\diamond}(x_i^*),$$

and let $\delta > 0$ be small enough such that $x_i^* - 2\delta\mathbf{1}_C \in \mathbb{R}^C_{++}$. Since by (2), $u_i^{\diamond}(x_i^*) > u_i^{\diamond}(x_i^* - \delta\mathbf{1}_C)$, there exists $n_1 \in \mathbb{N}^*$ such that, for every $n \geq n_1$,

$$u_i^{[n]}(x_i^{[n]}) \geq u_i^{\diamond}(x_i^*) - \varepsilon_n > u_i^{\diamond}(x_i^* - \delta\mathbf{1}_C). \tag{72}$$

Of course, we can find $n_2 \geq n_1$ such that, for every $n \geq n_2$,

$$0 \ll x_i^{[n]} \leq \sum_{i\in\mathcal{I}} \omega_i^{[n]} + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}[n]} \leq \sum_{i\in\mathcal{I}} \omega_i^{\diamond} + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}\diamond} + \mathbf{1}_C.$$

Let us consider now

$$x_i^{\diamond} \in \left[0, \sum_{i\in\mathcal{I}} \omega_i^{\diamond} + \sum_{\mathcal{H}\in\mathbf{H}_\nu} \eta^{\mathcal{H}\diamond} + \mathbf{1}_C\right]$$

a cluster point of the set $\{x_i^{[n]}\}_{n=1}^{\infty}$ and assume $x_i^{[n]} \to x_i^{\diamond}$. Consider any $\widetilde{x}_i \in \mathbb{R}^C_{++}$ such that $u_i^{\diamond}(\widetilde{x}_i) = u_i^{\diamond}(x_i^* - 2\delta\mathbf{1}_C)$. For $n \in \mathbb{N}^*$ large enough, by (72) we get $u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\widetilde{x}_i) \geq 0$ and then

$$0 \leq u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\widetilde{x}_i) \leq Du_i^{[n]}(\widetilde{x}_i)(x_i^{[n]} - \widetilde{x}_i)$$

$$= \left(Du_i^{[n]}(\widetilde{x}_i) - Du_i^{\diamond}(\widetilde{x}_i)\right)(x_i^{[n]} - \widetilde{x}_i) + Du_i^{\diamond}(\widetilde{x}_i)(x_i^{[n]} - \widetilde{x}_i).$$

Taking now the limit as n goes to infinity in the previous inequality, we obtain $Du_i^\diamond(\widetilde{x}_i)(x_i^\diamond - \widetilde{x}_i) \geq 0$. More precisely, the following relation holds true

$$x_i^\diamond \in \bigcap_{\widetilde{x}_i \in \left\{z_i \in \mathbb{R}^C_{++} : u_i^\diamond(z_i) = u_i^\diamond(x_i^* - 2\delta \mathbf{1}_C)\right\}} \left\{y_i \in \mathbb{R}^C : Du_i^\diamond(\widetilde{x}_i)(y_i - \widetilde{x}_i) \geq 0\right\}. \tag{73}$$

Moreover, Assumptions (3) and (4) and a well-known result from convex analysis assure that the right hand side of (73) is equal to the set

$$\{y \in \mathbb{R}^C_{++} : u_i^\diamond(y) \geq u_i^\diamond(x_i^* - 2\delta \mathbf{1}_C)\}.$$

Then $x_i^\diamond \in \mathbb{R}^C_{++}$ and the proof is complete. A very similar argument allows to prove that, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, the sequences $(x_i^{[n]})_{n=1}^\infty$ and $(\underline{x}_i^{[n]})_{n=1}^\infty$ converge in $\mathbb{R}^C_{++}$. Moreover, from the above results, we immediately have that, for every $i \in \mathcal{I}$, $(\lambda_i^{[n]})_{n=1}^\infty$ converges to $\lambda_i^\diamond \in \mathbb{R}_{++}$, for every $\mathcal{H} \in \mathbf{H}_\nu$, $i \in \mathcal{H}$, $(\underline{\lambda}_i^{[n]})_{n=1}^\infty$ converges to $\underline{\lambda}_i^\diamond \in \mathbb{R}_{++}$, and $(p^{[n]})_{n=1}^\infty$ converges to $p^\diamond \in \mathbb{R}^C_{++}$.

We are left with proving that $a_i^\diamond \in (0,1)$ and $(\mu^{\mathcal{H}[n]})_{n=1}^\infty$ converges in $\mathbb{R}$. Fix $\mathcal{H} \in \mathbf{H}_\nu$ and assume by contradiction there exists $i^* \in \mathcal{H}$ such that $a_{i^*}^{[n]} \to 0$. Of course, we have

$$\omega_{i^*}^{[n]} + a_{i^*}^{[n]} \eta^{\mathcal{H}} \to \omega_{i^*}^\diamond \quad \text{and} \quad \omega_{i^*}^{[n]} \to \omega_{i^*}^\diamond,$$

and then

$$\lim_{n\to\infty} \left(u_{i^*}^{[n]}(x_{i^*}^{[n]}) - u_{i^*}^{[n]}(\underline{x}_{i^*}^{[n]}) \right) = 0.$$

From (39.7), we obtain the equality

$$\mu^{\mathcal{H}[n]} = \frac{\theta_i^{\mathcal{H}[n]} \lambda_i^{[n]} p^{[n]} \eta^{\mathcal{H}[n]}}{u_i^{[n]}(x_i^{[n]}) - u_i^{[n]}(\underline{x}_i^{[n]})}, \tag{74}$$

where $i \in \mathcal{H}$ is arbitrarily chosen. Using (74) with $i = i^*$, we have that

$$\lim_{n\to\infty} \mu^{\mathcal{H}[n]} = +\infty$$

for the denominator goes to zero as n goes to infinity and it is always positive, and the numerator converges to a positive number. From (39.8), there exists $i_* \in \mathcal{H}$ such that $a_{i_*}^{[n]} \to a_{i_*}^\diamond > 0$ and then

$$\lim_{n\to\infty} \left(u_{i_*}^{[n]}(x_{i_*}^{[n]}) - u_{i_*}^{[n]}(\underline{x}_{i_*}^{[n]}) \right) = u_{i_*}^\diamond(x_{i_*}^\diamond) - u_{i_*}^\diamond(\underline{x}_{i_*}^\diamond) > 0.$$

Using now (74) with $i = i_*$, we have that $(\mu^{\mathcal{H}[n]})_{n=1}^\infty$ is uniformly bounded from above and the contradiction is found. Then we have that, for every $i \in \mathcal{H}$, $a_i^\diamond > 0$ and using (39.8) we get that, for every $i \in \mathcal{H}$, $a_i^\diamond \in (0,1)$. Finally, the convergence of $(\mu^{\mathcal{H}[n]})_{n=1}^\infty$ follows from (74) and the previous considerations. □

Proof of (58). Assume by contradiction that $\mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I}) = \varnothing$. Fixed $\pi_1 \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I})$, there exists $\mathcal{J}_1 \in \mathbf{P}_\nu(\mathcal{I}) \cap \mathbf{H}$ such that, for every $j \in \mathcal{J}_1$, $\{j\} \in \pi_1$. Then we can define

$$\pi_2 = (\pi_1 \setminus \{\{j\} : j \in \mathcal{J}_1\}) \cup \{\mathcal{J}_1\} \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I}).$$

Note that $|\pi_1| > |\pi_2|$. Arguing as before, we can build a sequence $(\pi_n)_{n=1}^\infty$ in $\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$ such that, for every $n \in \mathbb{N}^*$, $|\pi_n| > |\pi_{n+1}|$. In particular, for every $n, m \in \mathbb{N}^*$, $n \neq m$, we have $\pi_n \neq \pi_m$. Then we have shown that $\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$ is infinite and the contradiction is found. □

Proof of (61). Consider any $\pi^{**} \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I})$ and prove that

$$\sum_{\mathcal{H}\in\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} \leq \sum_{\mathcal{H}\in\pi^{*}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}}. \tag{75}$$

Assume there exists $\mathcal{H}^* \in (\pi^* \cap \mathbf{P}_\nu(\mathcal{I})) \setminus (\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I}))$. Then (75) immediately follows as

$$\sum_{\mathcal{H}\in\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} = \left(\sum_{\mathcal{H}\in\pi^{*}\cap\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} \right) + \left(\sum_{\mathcal{H}\in(\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I}))\setminus\pi^*} \eta^{\mathcal{H}} \right)$$

$$\leq \left(\sum_{\mathcal{H}\in\pi^{*}\cap\pi^{**}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}} \right) + \eta^{\mathcal{H}^*} \leq \sum_{\mathcal{H}\in\pi^{*}\cap\mathbf{P}_\nu(\mathcal{I})} \eta^{\mathcal{H}}.$$

Assume now that $(\pi^* \cap \mathbf{P}_\nu(\mathcal{I})) \subseteq (\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I}))$. We show (75) proving that $\pi^{**} = \pi^*$. First of all, let us prove that $(\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I})) \subseteq (\pi^* \cap \mathbf{P}_\nu(\mathcal{I}))$. Suppose by contradiction there exists $\mathcal{H}^* \in (\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I})) \setminus (\pi^* \cap \mathbf{P}_\nu(\mathcal{I}))$. Then, for every $\mathcal{H} \in \pi^* \cap \mathbf{P}_\nu(\mathcal{I})$, $\mathcal{H} \neq \mathcal{H}^*$. Moreover, since $(\pi^* \cap \mathbf{P}_\nu(\mathcal{I})) \subseteq (\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I}))$ and $\pi^{**} \in \mathfrak{P}_{\mathbf{H}}(\mathcal{I})$, we have that, for every $\mathcal{H} \in \pi^* \cap \mathbf{P}_\nu(\mathcal{I})$, $\mathcal{H} \cap \mathcal{H}^* = \varnothing$. Then, for every $i \in \mathcal{H}^*$, $\{i^*\} \in \pi^*$ and the contradiction is found as $\pi^* \in \mathfrak{S}\mathfrak{P}_{\mathbf{H}}(\mathcal{I})$.

Then we have $(\pi^* \cap \mathbf{P}_\nu(\mathcal{I})) = (\pi^{**} \cap \mathbf{P}_\nu(\mathcal{I}))$. It is immediate to prove that the previous equality implies $\pi^{**} = \pi^*$. □

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