Financial tools for the abatement of traffic congestion : a dynamical analysis

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Abstract

In this article we propose a simple mechanism aimed at implementing and supporting environmental protection policies in urban areas based on innovative financial instruments issued by a policy maker, which can be buyed by two categories of involved agents, city users and agencies providing the city services.

According to this mechanism, virtuous service providers choosing to offer high quality services can obtain cost abatement. City users, reciprocally, have to pay for entering into the city, but can protect themselves against a city low quality of life by a self-insurance device.

The interaction of these two categories of economic agents is modelled by a two-population evolutionary game, where the population of city users strategically interacts with that of service providers. From the analysis of the model it emerges that such a dynamics may lead to a welfare-improving attracting Nash equilibrium at which all city users choose to use environmental-friendly means of transportation and all service providers choose to offer high quality services. However, the basin of attraction of that equilibrium may have a rather complex morphology. In particular more attractors and/or limit cycles can be present. In such a context we indicate sufficient conditions making the virtuous equilibrium a global attractor for all trajectories starting at a *mixed-strategy* point.

1 Introduction

Air pollution and the dangers for pedestrians and cyclists deriving from the widespread use of private cars in urban centres may incentive the use of private cars by city users. The choice of using the car in these areas instead of going by bicycle or on foot has a self-enforcing nature: the higher the urban

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air pollution and traffic congestion levels, the higher the incentive to go by car to reduce one's exposure to these problems. An increase in the use of private cars further increases air pollution and the dangers of urban traffic in its turn, thus reinforcing the decision to go by car. As showed by the literature on environmental self-protection choices (see, e.g., Hueting 1980, Antoci et al. 2008, Bonatti and Campiglio 2009), this mechanism may lead the urban community towards suboptimal Nash equilibria characterized by an excessive use of cars, unbearable levels of air pollution and traffic congestion.

Therefore, such a context calls for the definition of a system of sustainable mobility based on road pricing schemes (see e.g. Gwilliam 2008), according to which the policy maker increases the costs to enter into urban areas by car (parking taxes, congestion charges and so on) and collects revenues which can be used to finance private or public firms providing services aimed at the abatement of the negative effects of urban traffic and at the improvement of the quality of life in urban agglomerates (e.g. the management of public transport networks, of cycle and pedestrian lanes, of small-scale urban green areas etc.). Those services can reduce the relative convenience of entering into urban areas by private cars and consequently can stop the undesirable self-enforcing process described above.

Road pricing instruments have been used to control traffic in several urban centres (e.g. Bergen, London, Milan, Singapore, Shanghai, Stockholm). The effectiveness of such policy instruments highly depends on the amount of the revenues raised via road pricing and on the quality of services provided to defend individuals from the effects due to traffic congestion. Service providers that furnish high quality services bear extra-costs and policy makers have to give monetary incentives which reduce unitary costs. In this context, an increase in the proportion of individuals using private cars generates an increase in traffic congestion but at the same time leads to an increase in the raised funds, which can be used to incentive the provision of high quality services.

All the proposals set forth to reduce traffic congestion are based on fixed taxes that individuals have to pay when entering into urban areas by private cars. In our paper we propose a simple mechanism aimed at implementing and supporting environmental protection policies in urban areas based on innovative financial instruments issued by a policy maker (PM), which can be bought by two categories of involved agents, city users (CU) and agencies providing the city services (SP), which can be public or private. In particular, we consider the case of a city whose citizens and visitors (the city users) face the risk of a reduction in quality of life, caused by urban traffic, on one hand, and by a poor quality standard of urban services, on the other.

According to the proposed mechanism, each city user has to choose (ex-ante) whether to use a private car (choice A) or to use a more environment-preserving transport mode - foots, cycles, buses, trolleybuses, trams (choice B). The PM requires them to buy two different *tickets*, including *cash-or-nothing call options*, called A (at a price p_a) and B (at a price $p_b < p_a$), according to their choices. Each SP has to choose (ex-ante) whether to improve (option C) or not (option D) the services it furnishes. The PM requires them to subscribe two different contracts, similar to *cash-or-nothing put options*, called C and D, according to their choices. The ticket prices, p_a and p_b, are fixed by the PM.

The tickets A and B imply a cost for CU if the value of a properly defined index Q of the *city quality of life*, measured by an independent agency at the end of any fixed period, is above a fixed threshold value $Q \geq Q^*$, but offer a reimbursement in the case $Q < Q^*$. When $Q < Q^*$, the CU owning the *ticket* A receive a reimbursement equal to αp_a, while the CU owning the *ticket* B receive a reimbursement equal to βp_b, where α and β are two parameters satisfying the condition $0 < \alpha \leq \beta \leq 1$ ($\alpha = \beta = 1$ means that both amounts p_a and p_b are totally reimbursed). If $Q \geq Q^*$ the CU do not receive any refund.

On the side of the SP, those choosing option C, i.e. improving the services they furnish, will bear a unitary extra-cost but, if the quality target Q^* is achieved, will get a reward.

The contract C is very similar to the contract on which the "environmental policy bonds" regime, introduced by Horesh (2000a, 2000b, 2002a, 2002b), is based. Environmental policy bonds are auctioned by the Public Administration on the open market, but, unlike ordinary bonds, can be redeemed at the face value only if a specified environmental objective has been achieved. They do not bear any interest, and the yield investors can gain depends on the difference between the auctioned price and the face value in the case of redemption. Economic agents involved in the environmental objective, *either polluters or not*, once in possession of the bonds, have a strong interest to operate in such a way that the objective itself is quickly achieved, so to cash in the expected gains as soon as possible. Differently from the environmental policy bonds regime, the option C considered in our model can be only bought by SP which intend to provide high quality services (the quality of services is assumed to be observable).

The tickets A and B, bought by the CU, can be regarded as the joint implementation of a fixed environmental tax and a potential refund. The prospective of a refund, in case $Q < Q^*$, makes these policy instruments more acceptable to public opinion; in fact, they can be considered as self-insurance products whose purchase can offer protection (or mitigation) from some environmental risk. By relying on the citizens aversion to environmental risks, these self-insurance instruments can be a partial alternative to new taxes or forms of public indebtedness

According to the mechanism we propose, virtuous service providers choosing to offer high quality services can obtain cost abatement by subscribing the environmental contracts C. City users have to pay to enter into the city, but can protect themselves against a low city quality of life by a self-insurance device. The policy maker can achieve the goal of improving the city quality of life at a low cost, since the costs born by city users compensate, at least partially, the financial aids to virtuous services providers and so do not imply any worsening of the public budget. In such a context, an increase in the number of city users choosing to enter into the city by private cars has a negative effect on the value of the quality index Q; however, since they have to pay a higher ticket ($p_a > p_b$), they contribute to increase the funds used to incentive virtuous behavior by service providers, which, in turn, contributes to increase the value of Q. Therefore, as a consequence of the mechanism described above, a strong interdependency between city users and services providers' behavior occurs. The aim of this paper is to study the dynamics that may arise in such a context. To this purpose, the choice processes of city users and service providers are modelled via a two-population evolutionary game, where the population of city users strategically interacts with that of service providers. Specifically these processes are modelled by the so-called replicator dynamics (e.g., see Weibull 1995), according to which a given choice spreads among the population as long as its expected payoff is greater than the average one. As it emerges from the model, such a dynamics may lead to a welfare-improving attracting Nash equilibrium, in which all city users choose to use environmental-friendly means of transportation and all service providers choose to offer high quality services. The basin of attraction of this equilibrium expands as the reimbursement due to the visitors increases.

The model presented in this paper is a generalization of the model analyzed in Antoci et al. (2009a). In Antoci et al. (2007, 2009b, 2009c) a similar fund rising mechanism has been analyzed in a context in which firms have to decide about the adoption of an environment preserving tecnology.

The structure of the remainder of the paper is as follows. Section 2 develops the game theoretic model; in sections 3 and 4 the model is analyzed; section 5 concludes.

2 The model

We assume that, at each time t, city users (CU) and agencies providing the city services (SP) play a one shot population game (i.e. all CU and all SP play the game simultaneously). Each city user has to choose (ex-ante) whether to use a private car (choice A) or not (choice B). The policy-maker requires them to buy two different *tickets*, including *cash-or-nothing call options*, called A (at a price p_a) and B (at a price p_b), according to their choices. Each SP has to choose (ex-ante) between option C and option D, i.e. whether to improve or not the services it furnishes. The policy-maker requires them to subscribe two different contracts, similar to *cash-or-nothing put options*, called C and D, according to their choices.

We assume the two populations to be constant over the time and normalize to 1 the number of both CU and SP. Let the variable $x(t)$ denote the proportion of CU adopting choice A at time t ($0 \leq x(t) \leq 1$) and $1 - x(t)$ the proportion of CU adopting choice B. Analogously, let $y(t)$ denote the proportion of SP choosing option C at the time t ($0 \leq y(t) \leq 1$), and $1 - y(t)$ the proportion of SP choosing option D.

The ticket prices, $p_a > p_b$, are fixed by the Policy Maker (PM).

The index Q is a measure of the *city quality of life*, whose target is fixed by the PM at a sufficiently high value Q^*. We imagine that, at the end of any fixed period, an independent agency measures Q.

At the end of a period in which $Q < Q^*$, the CU owning the *ticket* A receive a reimbursement equal to αp_a, while the CU owning the *ticket* B receive a reimbursement equal to βp_b, where α and β are two parameters satisfying the condition $0 < \alpha \leq \beta \leq 1$ ($\alpha = \beta = 1$ means that both amounts, p_a and p_b, are totally reimbursed). If $Q \geq Q^*$ the CU do not receive any refund (this means that the value of the *call options* is zero). Hence the option values depend on the index Q, which is their *underlying.*

It is also reasonable to assume that each CU's payoff is affected not only by the price of the *tickets* and the amount of the reimbursements, but by the *city quality of life,* depending negatively on the use of private cars and positively on better urban services.

Consequently we assume the payoff of a CU choosing option A to be given by:

- $\pi_a^1 = \gamma_1 y - \delta x - \rho p_a$ if $Q \geq Q^*$
- $\pi_a^2 = \gamma_2 y - \delta x - \rho (1 - \alpha) p_a$ if $Q < Q^*$

while the payoff of a CU choosing option B will be given by:

- $\pi_b^1 = \varepsilon_1 y - \eta x - \rho p_b$ if $Q \geq Q^*$
- $\pi_b^2 = \varepsilon_2 y - \eta x - \rho (1 - \beta) p_b$ if $Q < Q^*$

All the parameters are positive. Moreover, $\gamma_1 > \gamma_2$, $\varepsilon_1 > \varepsilon_2$, $\varepsilon_i > \gamma_i$, $\varepsilon_1 - \varepsilon_2 > \gamma_1 - \gamma_2$, implying, in particular, that the citizens not using private cars derive more advantage from increases in service quality. All the citizen payoffs are negatively correlated to the number of citizens using private cars by the two parameters δ and η (we assume $\eta \geq \delta$). Notice that both p_a and p_b are multiplied by ρ. This parameter can be thought as a measure of the citizen willingness to pay for urban services. For the sake of simplicity, we assume that all the citizens have the same willingness.

On the side of the SP, those choosing option C, i.e. improving the services they furnish, will bear a unitary extra-cost θ, $\theta > 0$, but, if the quality target is achieved, will get a reward $\lambda + \mu x - \nu y$, where $\lambda - \nu > \theta$, $\mu, \nu > 0$. The parameter θ represents the cost of increasing the level of services, while λ is a fixed amount of money; $\mu > 0$ means that the reward is positively related to the number of CU (x) who decide to use a private car. The reason is obvious: if we fix $p_a > p_b$, the more are the citizens choosing option A, the larger are the financial resources available to pay the SP reward. On the other hand, an increase in y implies that more SP will be entitled to the financial aid, thus reducing the reimbursement available to each one. Therefore, the payoff of a SP choosing option C is given by:

- $\pi_c^1 = \lambda + \mu x - \nu y - \theta$ if $Q \geq Q^*$
- $\pi_c^2 = -\theta$ if $Q < Q^*$

Without loss of generality, we can normalize to zero the payoff of a SP choosing D (i.e. who decide not to provide high quality services).

Finally we assume:

$$P(Q \geq Q^*) = \sigma(1-x) + (1-\sigma)y,\ 0 < \sigma < 1$$

where P denotes a probability. This is equivalent to saying that, if all SP improve the services they provide and all CU renounce to private cars, then Q will be *almost surely* above the threshold level Q^* ($P(Q \geq Q^*) = 1$). The parameter σ represents the weight of CU renouncing to private cars on the probability that $Q \geq Q^*$.

Finally we obtain the expected payoffs:

- $E\pi_a = (\gamma_1 y - \delta x - \rho p_a) P(Q \geq Q^*) + [\gamma_2 y - \delta x - \rho(1-\alpha) p_a] P(Q < Q^*)$
- $E\pi_b = (\varepsilon_1 y - \eta x - \rho p_b) P(Q \geq Q^*) + [\varepsilon_2 y - \eta x - \rho(1-\alpha) p_b] P(Q < Q^*)$
- $E\pi_c = (\lambda + \mu x - \upsilon y - \theta) P(Q \geq Q^*) - \theta P(Q < Q^*)$
- $E\pi_d = 0$

The process of adopting strategies is modelled by the so called replicator dynamics (see, e.g., Weibull, 1995), according to which the strategies whose expected payoffs are greater than the average payoff spread within the populations at the expense of the others. In our case:

$$\begin{aligned} \dot{x} &= x\left(E\pi_a - \overline{E\pi}_{CU}\right) \\ \dot{y} &= y\left(E\pi_c - \overline{E\pi}_{SP}\right) \end{aligned} \tag{1}$$

where

$$\begin{aligned} \overline{E\pi}_{CU} &= x \cdot E\pi_a - (1-x) \cdot E\pi_b \\ \overline{E\pi}_{SP} &= y \cdot E\pi_c - (1-y) \cdot E\pi_d \end{aligned}$$

are the average payoffs, respectively, of the two populations of CU and SP.

We assume that, in our context, replicator dynamics is generated by the following "expectation forming" mechanism. At the end of each period t (whose length, in a continuous time framework, is reduced to zero), the values of x and y become common knowledge to the agents (e.g. one can immagine that these values are frequently reported and updated on the webpage of the Public Administration and on the local media). On the basis of such values, agents form their expectations about the relative performance of the available strategies in the next period (in other words, the current values of x an y are used as a proxy for the values of these variables in the close future).

The replicator system (1) in the square $[0,1]^2$ can be written as:

$$\begin{aligned} \dot{x} &= x(1-x)(E\pi_a - E\pi_b) = x(1-x)F(x,y) \\ \dot{y} &= y(1-y)(E\pi_c - E\pi_d) = y(1-y)G(x,y) \end{aligned} \tag{2}$$

Setting

$$\begin{array}{rcl} \varphi & = & (\varepsilon_1 - \gamma_1) - (\varepsilon_2 - \gamma_2) \\ \psi & = & \varepsilon_2 - \gamma_2 \\ \zeta & = & \eta - \delta \\ \tau & = & \rho\,(\beta p_b - \alpha p_a) \\ \omega & = & \rho\,[(1-\alpha)\,p_a - (1-\beta)\,p_b] \end{array}$$

we obtain:

$$\begin{array}{rcl} F\,(x,y) & = & (-\varphi y + \tau)\,[\sigma\,(1-x) + (1-\sigma)\,y] + \zeta x - \psi y - \omega \\ G\,(x,y) & = & (\lambda + \mu x - \nu y)\,[\sigma\,(1-x) + (1-\sigma)\,y] - \theta \end{array}$$

where:

$$\varphi, \psi > 0,\ \zeta \geq 0,\ \omega > \max(0,\tau);\ 0 < \sigma < 1;\ \lambda, \mu, \nu, \theta > 0,\ \lambda - \nu > \theta \qquad (3)$$

3 Fixed points and stability

The analysis of fixed points of (2) and their stability is summarized by the following Propositions.

Proposition 1 *The boundary fixed points are at least four (the vertices of* $[0,1]^2$*) and at most nine (the maximum can be attained). At least one and at most four of them are attractors: in particular, the vertex* $(0,1)$ *is always attracting and the four vertices are all attracting if and only if the boundary fixed points are eight.*

Proof. The conditions (3) imply, as it is easily checked, that the vertex $(0,1)$ is attracting. Moreover the hyperbola $F(x,y) = 0$, having asymptotes $y = \frac{\sigma\tau - \zeta}{\sigma\varphi}$ and $-\sigma x + (1-\sigma)\,y = c$, for a suitable c, has at most one intersection with each horizontal edge of the square $[0,1]^2$. On the other hand $G(x,y) = 0$ has one intersection with $x = 0$ at some point $(0, \overline{y})$ with $\overline{y} > 1$. It follows that $G(x,y) = 0$ has at most one intersection with the vertical edge $x = 0$ of the square and at most two intersections with the vertical edge $x = 1$. Hence the boundary fixed points are at most nine. Moreover, as in $(0,0)$ and $(0,1)$ $\dot{x} < 0$, the possible fixed points on the open horizontal edges cannot be attracting. Analogously, since $\dot{y} > 0$ in $(0,1)$ and $\dot{y} < 0$ in $(1,1)$, the possible fixed point on the open vertical edge $x = 0$ and one of the possible fixed points on the open vertical edge $x = 1$ cannot be attracting either. In fact, it follows that, when the boundary fixed points are eight, the four vertices are all attracting. In case of nine boundary fixed points, instead, the vertices $(0,0)$, $(0,1)$ and $(1,0)$ are still attracting, while $(1,1)$ is a saddle and there exists a fourth attractor on the open edge $x = 1$. In all the other cases the attractors are less than four. ■

Figure 1 illustrates the case when the boundary fixed points are eight and thus all the four vertices are attractors.

Proposition 2 *The minimum number of interior fixed points (i.e. belonging to $(0,1)^2$) is zero and the maximum (which can be attained) is three. At most one of them is an attractor and at most one of them is a saddle. When the interior fixed points are three, two of them are repellors and one is a saddle.*

Proof. The two hyperbolas $F(x,y)=0$ and $G(x,y)=0$ have a common asymptotic direction (that of the line $-\sigma x+(1-\sigma)\,y=0$). Hence their intersections in the euclidean plane are at most three. It is easily seen that only one branch of $G(x,y)=0$ intersects $(0,1)^2$. On the other hand it can be checked that $F(x,y)=0$ does not intersect the vertical edge $x=0$ of the square and, in case $\sigma\tau-\zeta\geq 0$ (i.e. $\frac{\partial F}{\partial x}<0$ in $(0,1)^2$), it does not intersect the horizontal edge $y=0$ either. It follows that the possible intersection of $F(x,y)=0$ with $(0,1)^2$ is the graph of a function $x=f(y)$ defined in an interval $(y',y'')\subseteq(0,1)$. Therefore the intersections between the two hyperbolas in $(0,1)^2$ are the solutions of the equation $G(f(y),y)=0$ with $y\in(y',y'')$, hence the possible zeros in (y',y'') of a third degree polynomial $P(y)$. By possibly exchanging $P(y)$ with $-P(y)$, it is easily observed that, if $(\overline{x},\overline{y})$ is a fixed point of the system in $(0,1)^2$, then $sign\det J\,(\overline{x},\overline{y})=signP'(\overline{y})$, where J denotes the Jacobian matrix. As a consequence, if two consecutive simple zeros of $P(y)$ correspond to interior fixed points of system (2), one of them is a saddle and the other one is generically either an attractor or a repellor. Vice-versa a double zero of $P(y)$, giving rise to an intersection between $F=0$ and $G=0$ in $(0,1)^2$, corresponds to a saddle-node of the system, while, as we will see, a triple zero of $P(y)$, if it corresponds to an interior fixed point of the system, gives rise to an *improper repellor*. Let us consider now the case of three interior fixed points (see Figure 2). From the previous considerations, through a careful investigation, one can check that in this case $\frac{\partial F}{\partial x}>0$ and $\frac{\partial F}{\partial y}<0$ along $\{F=0\}\cap(0,1)^2$ while $\frac{\partial G}{\partial y}>0$ along $\{G=0\}\cap(0,1)^2$. Hence the function $x=f\,(y)$, above defined, is increasing, with $0<y<\overline{y}\leq 1$, and no one of the interior fixed points, say $P_i=(x_i,y_i)$, $i=1,2,3$, is attracting (since $traceJ\,(P_i)>0$). Next we want to show that P_3, the one with the highest coordinates, is a repellor. In fact, it is easily seen that the arc $x=f\,(y)$, $y_3<y<\overline{y}$, lies in the region $\{G>0\}\cap(0,1)^2$. Consider then a *curvilinear triangle* having two *edges* on $F=0$ and $G=0$ and vertices $P_3=(x_3,y_3)$, $Q'=(f\,(y^*)\,,y^*)$, $Q''=(g\,(y^*)\,,y^*)$, where $y_3<y^*<\overline{y}$, $G\,(g\,(y^*)\,,y^*)=0$ and $(g\,(y^*)\,,y^*)\in(0,1)^2$. Then the backward (i.e. negative) trajectory of Q' cannot leave the *triangle* and thus converges to P_3. It follows that P_3, being a non-degenerate fixed point with a parabolic repelling sector (see footnote 1), is a repellor. From what precedes, we conclude that P_1 is also a repellor and P_2 is a saddle. ■

4 Limit cycles and bifurcations

The main result of this Section is the proof that system (2), defined in the square $[0,1]^2$, can possess two limit cycles, an attracting one surrounded by a repelling one. This number may not be the maximum. However the result already shows the complexity of configurations that our apparently simple system can exhibit. Moreover we show how the two limit cycles can reduce to one, either repelling or attracting, through, respectively, a Hopf or a saddle-connection bifurcation, or can disappear through a *collision* (then the bifurcating cycle is interiorly attracting and externally repelling). In the latter case the vertex $(0,1)$ becomes the only *attracting set* of the system. Finally we provide necessary and sufficient conditions for $(0,1)$ to attract all the trajectories lying in $(0,1)^2$.

We start with the following

Proposition 3 *If system* (2) *exhibits some limit cycles in* $(0,1)^2$*, they all surround the same interior fixed point.*

Proof. It is well-known that the sum of the indexes of the fixed points surrounded by a limit cycle must be $+1$[1]. As we have shown, if there are two non-degenerate interior fixed points, one of them is a saddle. Vice-versa, in the case the interior fixed points are three, accounting for multiplicity, the proof of Proposition 2 shows that the one of them with the highest coordinates cannot lie in the interior of a cycle. This proves the present Proposition. ■

Theorem 4 *System* (2) *exhibits, for suitable values of the parameters, two limit cycles in* $(0,1)^2$*, an attracting one surrounded by a repelling one.*

Proof. Let us first consider a system Σ_0 satisfying, besides (3), the following conditions:

1. $G_0(0,0) > 0 > G_0(1,1)$
2. $F_0(1,1) > 0 > F_0(1,0)$
3. There exists a fixed point $P_0 = (x^*, 1)$, $0 < x^* < 1$, such that $F_0(x^*, 1) = 0 < G_0(x^*, 1)$
4. There exists a fixed point $Q_0 = (1, y^*)$, $0 < y^* < 1$, such that $F_0(1, y^*) = G_0(1, y^*) = \frac{\partial G_0}{\partial y}(1, y^*) = 0$

Conditions (1)-(4), in addition to the previous ones, are easily seen to imply:

- $\frac{\partial F_0}{\partial x} > 0$ for $y \in [y^*, 1]$
- $\frac{\partial G_0}{\partial x} < 0$ along $\{G_0(x,y) = 0\} \cap [0,1]^2$
- $\frac{\partial G_0}{\partial y} > 0$ along $\{G_0(x,y) = 0\} \cap [0,1] \times [0, y^*)$
- $\frac{\partial G_0}{\partial y} < 0$ along $\{G_0(x,y) = 0\} \cap [0,1] \times (y^*, 1]$

[1] For a non-degenerate fixed point the index is $+1$ if it is not a saddle, -1 if it is a saddle, while a saddle-node has index 0 etc. (see, e.g., Lefschetz 1977)

It follows that Σ_0 has exactly six fixed points in $[0,1]^2$, all lying on the boundary: the attractor $(0,1)$, the saddles $(0,0)$, $(1,0)$, $(1,1)$, $(x^*,1)$ and the degenerate fixed point $Q_0=(1,y^*)$, whose Jacobian matrix has the form $J=\begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix}$ with $c>0$. Actually it can be checked that Q_0 possesses an elliptic sector and a parabolic sector lying in $(0,1)^2$, plus a hyperbolic sector lying outside the square[2]. More precisely, the elliptic sector is bounded by a curve (a *polycycle*) constituted by the stable manifold, in $(0,1)^2$, of $P_0=(x^*,1)$ and by the segments $\{x^*\leq x\leq 1,\, y=1\}$ and $\{x=1,\, y^*\leq y\leq 1\}$; while the parabolic sector is constituted by trajectories having α-limit in Q_0 and ω-limit in $(0,1)$.

Let us now perturb Σ_0 into Σ, by replacing F_0 and G_0 with:

$$\begin{aligned} F(x,y) &= F_0(x,y)+\alpha\varepsilon^2 \\ G(x,y) &= G_0(x,y)-\varepsilon \end{aligned} \tag{4}$$

where $\varepsilon>0$ is arbitrarily small and α is suitably chosen. As a consequence, the fixed points of Σ are the four vertices of $[0,1]^2$ and a point P on $x=1$ near P_0 with the above characteristics, plus, in the place of Q_0, a point Q in $(0,1)^2$, which, generically, is either an attractor or a repellor.

In order to better understand the situation, let us consider the coordinates $u=1-x$ and $z=y-y^*$. In such coordinates F and G can be written as:

$$\begin{aligned} F(u,z) &= -u(a+bz)+cz-dz^2+\alpha\varepsilon^2 \\ G(u,z) &= u(k-lz)-mu^2-nz^2-\varepsilon \end{aligned} \tag{5}$$

where $a,b,c,d,k,l,m,n>0$, while the coordinates of $Q=(\widetilde{u},\widetilde{z})$ satisfy: $\widetilde{u}=\frac{\varepsilon}{k}+o(\varepsilon)$, $\widetilde{z}=\frac{a\varepsilon}{ck}+o(\varepsilon)$. It follows that, by multiplying F by a suitable $h>0$, we can set:

$$a=\left(l+2n\frac{a}{c}\right)y^*(1-y^*)$$

which implies

$$traceJ(Q)=\beta\varepsilon^2+o\left(\varepsilon^2\right) \tag{6}$$

Suppose, now, Q is an attractor for Σ. Hence, since Σ has no repelling fixed point in $[0,1]^2$, it follows from Poincaré-Bendixson Theorem (see Hirsch and Smale 1974) that there must exist a repelling limit cycle surrounding Q.

[2] Let P be an isolated fixed point of a smooth planar system. Consider an open neighborhood D of P (e.g. a disc). A region $U\subseteq D-P$ is said (see,e.g., Andronov 1973):

- an elliptic sector of P if it is constituted by trajectories having P both as α and ω limit-set;
- a parabolic sector of P if it is constituted by trajectories having P either as α or as ω limit-set;
- a hyperbolic sector of P if it is constituted by trajectories having P neither as α nor as ω limit-set.

We want to show that this cycle is not originated by a Hopf bifurcation. In fact, in such a case, its diameter should be, because of (6), of the order ε^2. But we can show the existence of a *winding inward* trajectory starting at a point whose distance from Q is $\rho\varepsilon$, with $\rho > 0$ independent of ε. Hence this trajectory is contained in the region bounded by the repelling cycle and, consequently, the latter cannot be originated by a Hopf bifurcation. Therefore the Hopf bifurcation at Q, which generically takes place when α crosses a suitable value $\overline{\alpha}$, is supercritical, i.e. an attracting limit cycle arises around the fixed point Q when it becomes a repellor. Thus, in this case, two (at least) limit cycles exist, an attracting one surrounded by a repelling one.

So, we have to show the existence of a *winding inward* trajectory starting at a distance from Q of the order ε. We will utilize the coordinates (u, z), but referring to the system $\Sigma = \left(\dot{x}, \dot{y}\right)$. Let us first consider the line $r) \frac{\partial G}{\partial z} = \frac{\partial G}{\partial y} = 0$. Then $r)$ $lu + 2nz = 0$ intersects $G = 0$ at a point $(\widehat{u}, \widehat{z})$, where $\widehat{u} = \frac{\varepsilon}{k} + o(\varepsilon)$, $\widehat{z} = -\frac{l\widehat{u}}{2n}$. By straightforward computations one can check that the *backward* (i.e. negative) trajectory starting at $H_0 = (u_0, z_0)$, with $z_0 = -\frac{l\varepsilon}{4nk}$, $G(u_0, z_0) = 0$, intersects $F(u, z) = 0$ at a point $H_1 = (u_1, z_1)$ without intersecting r. Moreover one can see that $G(u_1, z_1) = o(\varepsilon)$.[3] Consider now the backward trajectory of H_1 and its intersection $H_2 = (u_0, z_2)$ with $u = u_0$. Let $\overline{z}$ satisfy $F(u_0, \overline{z}) = 0$. Then, from the above considerations and recalling $y^* > \frac{1}{2}$, one can show that $z_2 - \overline{z} - (\overline{z} - z_0) = z_2 + z_0 - 2\overline{z} > \gamma\varepsilon$, where $\gamma > 0$ is independent of ε for ε sufficiently small. Take now the first intersections with $F = 0$ of the *forward* (i.e. positive) trajectory of H_0 and the *backward* trajectory of H_2, and denote them, respectively, as $H_3 = (u_3, z_3)$ and $H_4 = (u_4, z_4)$. If were $u_3 \geq u_4$ (and thus $x_3 \leq x_4$), then there should be a value $\overline{u}$, $u_0 < \overline{u} < u_4$, where, denoting by $(\overline{u}, z')$, $(\overline{u}, \overline{z})$, $(\overline{u}, z'')$ the intersections of $u = \overline{u}$ with, respectively, the arc (H_0, H_3), the curve $F = 0$ and the arc (H_2, H_4), $z'' - \overline{z} = \overline{z} - z' = \delta > 0$. Moreover one can show that the above intersections should lie in the region $G > 0$. However, since $\delta^2 << \delta$ (δ being of order at most ε), it is easily seen that, in absolute value, the *speed* at $(\overline{u}, z')$ would be higher than the *speed* at $(\overline{u}, z'')$, i.e.

$$\frac{(z' + y^*)(1 - z' - y^*)G(\overline{u}, z')}{\overline{u}(1 - \overline{u})|F(\overline{u}, z')|} > \frac{(z'' + y^*)(1 - z'' - y^*)G(\overline{u}, z'')}{\overline{u}(1 - \overline{u})F(\overline{u}, z'')} \tag{7}$$

which implies $u_3 < u_4$, leading to a contradiction. Therefore it is $u_3 < u_4$, i.e. $x_3 < x_4$, which means that the trajectory through H_0 winds inward. This concludes the proof. ■

Figure 3 shows the case illustrated in the Theorem: there is only one interior fixed point, a repellor, surrounded by two limit cycles, respectively attracting

[3] Suppose, by contradiction, $|G(u_1, z_1)| \geq p\varepsilon$, with p independent of ε. Hence there would exist $(u', z') \in (H_1, H_0)$ such that $|G(u', z')| = \frac{p\varepsilon}{2}$. But then, as it is easily calculated, $\frac{p\varepsilon}{2} \leq \int_{z'}^{z_1} \frac{d|G(u(z), z)|}{dz} dz = \int_{z'}^{z_1} \left[\frac{(k - lz - 2mu)u(1-u)|F(u,z)|}{y(1-y)|G(u,z)|} + lu + 2nz\right] dz < q\varepsilon^2$ for a suitable q independent of ε, which produces a contradiction when ε is sufficiently small.

and repelling.

Now we want to show how the phase picture of system (2) can evolve, through bifurcations, starting from the one described in the above Theorem: that is, from two limit cycles to one (repelling or attracting) to no one. In the latter case we will see that $(0,1)$ becomes an attractor for all the trajectories lying in $(0,1)^2$ (except the one at the repelling fixed point), which can be considered a desirable outcome of the model.

Let us summarize the situation described in the Theorem, when two limit cycles appear. The following conditions hold:

- $\frac{\partial F}{\partial x} > 0$ along $\{F = 0\} \cap [0,1]^2$ and $\frac{\partial G}{\partial x} < 0$ along $\{G = 0\} \cap [0,1]^2$.
- $F = 0$ intersects the boundary of $[0,1]^2$ at the two (open) edges $x = 1$ and $y = 1$, while $G = 0$ intersects the boundary of $[0,1]^2$ at the two (open) edges $y = 0$ and $y = 1$.
- At the point $P = (\overline{x}, 1)$, where $F(\overline{x}, 1) = 0$, $G(\overline{x}, 1) > 0$.
- Consequently there is only one interior fixed point $Q = (\widetilde{x}, \widetilde{y})$ and we assume $\frac{\partial G}{\partial y}(\widetilde{x}, \widetilde{y}) < 0$.

It follows that the fixed points of the system in $[0,1]^2$ are six: the attractor $(0,1)$, the saddles $(0,0)$, $(1,0)$, $(1,1)$, $(\overline{x}, 1)$ and the interior fixed point $Q = (\widetilde{x}, \widetilde{y})$ which can be either an attractor or a repellor (observe that $\overline{x} < \widetilde{x}$). In fact, multiplying F by a suitable $h > 0$, we can make $traceJ(Q) > 0$, so that Q is a repellor. In such a case there may exist, as we have seen, two limit cycles surrounding Q. Suppose we have precisely this phase picture. Now, if we multiply F by a positive $h < 1$, the attracting limit cycle shrinks, until it disappears when, at $h = \overline{h}$, $traceJ(Q) = 0$. For smaller values of h the system exhibits only one repelling limit cycle, surrounding Q (see Figure 4). This is the effect of a supercritical Hopf bifurcation. Let us see, instead, how only the attracting limit cycle can remain. By the usual notations, let us modify the coefficients $\zeta$and ω of F, leaving all the others unvaried. Precisely, we replace ζ and ω by $\zeta' = \zeta + \rho$ and $\omega' = \omega + \rho\widetilde{x}$, $\rho > 0$. At the same time we multiply G by some $k > 1$ in such a way that $traceJ(Q)$ remains positive but does not increase (or suitably decreases). Then, when ρ, starting from 0, increases, the fixed point $P = (\overline{x}, 1)$ moves rightward, while the repelling limit cycle expands vertically, until, for a suitable pair $(\overline{\rho}, \overline{k})$, $F(\overline{x}, 1) = G(\overline{x}, 1) = 0$. At this stage P is a *saddle-node* and the repelling limit cycle has become a loop through P (i.e. a *saddle-connection*). For higher values of ρ there is only one attracting limit cycle surrounding Q, while P has become a repellor and there exists a new interior fixed point, a saddle $S = (\widehat{x}, \widehat{y})$, with $\widehat{x} < \widetilde{x}$ and $\widehat{y} > \widetilde{y}$. The basin of the attracting cycle is bounded, precisely, by the stable manifold of S, whose *arcs* originate in P (see Figure 5). We observe, however, that, by just letting the curve $F = 0$ approach the curve $G = 0$ on the edge $y = 1$ of the square, without adjusting at the same time $traceJ(Q)$, the resulting phase picture might have been different: that is, when the boundary fixed point, say $\widetilde{P}$, becomes a repellor and an interior saddle S appears, the two limit cycles may persist (see Figure 6).

Finally we show that both limit cycles can disappear through a *collision*. In fact, let us start again from the two limit cycles *configuration*. Then, by multiplying F by some $h > 1$, $traceJ(Q)$ increases, while the attracting limit cycle around Q expands and the repelling one shrinks, until, at a suitable value $\overline{\overline{h}}$, they coincide (i.e. *collide*). The resulting *compound* limit cycle is interiorly attracting and externally repelling (see Figure 7). Then, for $h > \overline{\overline{h}}$, no cycle exists and all the trajectories starting in $(0,1)^2$ (except, of course, the one at Q) converge to the *virtuous* point $(0,1)$.

Actually we can give necessary and sufficient conditions for the *global attractiveness* of $(0,1)$ relatively to the open square $(0,1)^2$, namely:

Proposition 5 *The vertex* $(0,1)$ *attracts all the trajectories lying in the open square* $(0,1)^2$ *if and only if the following conditions hold:*

1. $\sigma\lambda \geq \theta$;
2. $\omega \geq \zeta$;
3. *there exists* $h > 0$ *such that* $hF(1,1) + G1,1) \leq 0$;
4. $F(x,y) < 0$ *along* $\{G(x,y) = 0\} \cap (0,1)^2$

We omit the proof of the above Proposition, which can be drawn through straightforward steps.

Let us observe that the first three conditions guarantee that the vertices, in the order, $(0,0)$, $(0,1)$ and $(1,1)$ are not attractors, while the fourth condition, together with the other three, guarantees that no interior fixed point exists and no other boundary fixed point, except $(0,1)$, attracts trajectories lying in $(0,1)^2$.

We can also give a sufficient condition, which allows an easier economical interpretation, for the vertex $(0,1)$ to attract all the trajectories lying in $(0,1)^2$. This condition is obtained observing that $\dot{x} < 0$ holds for every values of x and y in the open square $(0,1)^2$ if:

$$\min\{(1-\alpha)\,p_a - (1-\beta)\,p_b,\; p_a - p_b + \sigma\,(\beta p_b - \alpha p_a)\} \geq \frac{\eta - \delta}{\rho} \tag{8}$$

Under condition (8) all the trajectories in $(0,1)^2$ approach the edge $x = 0$ of the square. If both (8) and $\sigma\lambda \geq \theta$ are satisfied, then the vertex $(0,1)$ attracts all the trajectories lying in the open square $(0,1)^2$.

5 Conclusions

In this paper we have analyzed the effects on economic agents' behavior generated by the introduction of a simple mechanism aimed at implementing and supporting environmental protection policies in urban areas. According to the proposed mechanism, service providers choosing to offer high quality services can obtain cost abatement; while city users have to pay for entering into urban areas, but can protect themselves against a low quality of life in these areas by a self-insurance device. The policy maker can achieve the goal of improving the city users' quality of life at a low cost, since the costs born by city users compensate, at least partially, the financial aids to virtuous services providers and so do not imply any worsening of the public budget. The dynamics arising from the interaction between these categories of economic agents are modelled by a two-population evolutionary game according to which the population of city users strategically interacts with that of service providers.

From the analysis of the model it emerges that the welfare-improving equilibrium $(x, y) = (0, 1)$, where all city users choose an environment-preserving transport mode and all services agencies provide high quality services, is always a local attractor under evolutionary dynamics. However, the basin of attraction of such an equilibrium may have a rather complex morphology. In fact, in sections 3 and 4 we show that other attracting fixed points can coexist with $(0, 1)$ and, furthermore, that two limit cycles may arise, an attracting one surrounded by a repelling one. In our construction the latter separates the basin of attraction of the former from the basin of attraction of the *virtuous vertex* $(0, 1)$. Therefore, evolutionary dynamics is strongly path-dependent. This feature of dynamics may be viewed as a shortcoming of the proposed mechanism. Nevertheless we have seen a sufficient condition assuring the global attractiveness of the equilibrium $(0, 1)$ relatively to the open square $(0, 1)^2$:

$$\sigma\lambda \geq \theta \text{ and } \min\left\{(1-\alpha)\,p_a - (1-\beta)\,p_b,\, p_a - p_b + \sigma\left(\beta p_b - \alpha p_a\right)\right\} \geq \frac{\eta - \delta}{\rho} \tag{9}$$

which requires (coeteris paribus):

(1) a sufficiently large difference $p_a - p_b$ between the prices of the *tickets* A and B bought by the city users;

(2) a sufficiently large difference $\beta - \alpha$ between the reimbursement rates of the *tickets* B and A, respectively;

(3) a sufficiently large fixed amount of money λ (i.e. the component of the reward $\lambda + \mu x - \nu y$ not depending on the shares x and y) received, when $Q \geq Q^*$, by each service provider deciding to provide high quality services;

(4) a sufficiently high negative impact (measured by σ) generated by the use of private cars on the probability that $Q \geq Q^*$.

Notice that in (8) the threshold value $\frac{\eta-\delta}{\rho}$ increases (coeteris paribus) if the difference between the negative impacts of city users choosing option A, respectively, on those choosing option B (measured by η) and on those choosing option A (measured by δ) increases. In fact, in a context characterized by a high value of $\eta - \delta$, the decision to use a private car has a strong self-enforcing nature that favours the emergence of undesirable outcomes characterized by a widespread use of private cars.

Furthermore, the numerical simulation showed in Figure 8 suggests the possibility to expand the basin of attraction of $(0,1)$ by increasing the reimbursement rate β due to city users renouncing to private cars (option B). This implies that if the policy maker levies a simple entrance ticket on the city users with no chance of being reimbursed, this could minimize the basin of attraction of the virtuous equilibrium $(0,1)$, increasing the initial critical mass of $1-x$ and y that are needed to approach it. Increasing the reimbursement share, therefore, might paradoxically lower the costs of the financial mechanism for the policy maker: if the system converges to $(0,1)$, no reimbursement will be paid by the policy maker to city users and the *extra* entries obtained from the *call option* component of the *tickets* can be used by the policy maker to finance service providers for their virtuous behavior.

In our opinion, the present analysis could be extended in several directions. In particular, in an optimal control framework in which the policy maker aims at maximizing its own objective function by choosing the values of the control variables p_a, p_b, α and β, it would be interesting to compare the costs for the policy maker of the two alternative regimes described above (with and without reimbursement), taking the budget constraint explicitly into account.

6 References

Andronov A.A. et al., 1973. *Qualitative theory of second-order dynamic systems,* John Wiley, Jerusalem

Antoci A., Marletto G., Russu P., Sanna S., 2009a. I social policy bonds 'modificati' come strumento di regolazione del trasporto urbano: una simulazione concettuale, in *Trasporti, Ambiente e Territorio. La Ricerca di un Nuovo Equilibrio*, Marletto

and Musso E. (eds.), Franco Angeli, Milan.

Antoci A., Dei R., Galeotti M., 2009b. Financing the adoption of environment preserving technologies via innovative financial instruments: an evolutionary game approach, *Nonlinear Analysis* 71, 952-959.

Antoci A., Borghesi S., Galeotti M., 2009c. Environmental options and technological innovation: an evolutionary game model, Nota di lavoro FEEM n. 2009.090, Milan.

Antoci A., Borghesi S., Galeotti M., 2008. Should we replace the environment? Limits of economic growth in the presence of self-protective choices, *International Journal of Social Economics* 35, 283-297.

Antoci A., Galeotti M., Geronazzo L., 2007. Visitor and firm taxes versus environmental options in a dynamical context", *Journal of Applied Mathematics*, vol. 2007, Article ID 97540, 1-15.

Bonatti L., Campiglio E., 2009. Mobility systems and economic growth: a theoretical analysis of the long-term effects of alternative transportation policies, Discussion Papers 7-2009, Department of Economics, University of Trento, Trento.

Costanza R., Perrings C., 1990. A flexible assurance bonding system for improved environmental management, *Ecological Economics* 2, 57-
75.

Hirsch M. W., Smale S., 1974. *Differential equations, dynamical systems, and linear algebra,* Academic Press, Orlando.

Horesh R., 2000a. *Injecting incentives into the solution of social and environmental problems: social policy bonds*, University Press, New York.

Horesh R., 2000b. Injecting incentives into the solution of social problems. Social policy bonds, *Economic Affairs* 20, 39-42.

Horesh R., 2002a. Better than Kyoto: climate stability bonds, *Economic Affairs* 22, 48-52.

Horesh R., 2002b. Environmental policy bonds: Injecting market incentives into the achievement of society's environmental goals, OECD Paper,
Paris.

Hueting, R., 1980. *New Scarcity and Economic Growth. More Welfare Through Less Production?*, North Holland , Amsterdam.

Gwilliam K., 2008. A review of issues in transit economics, *Research in Transportation Economics* 23, 4-22.

Lefschetz S., 1977. *Differential equations: geometric theory*, Dover Publications, New York.

Perrings C., 1989. Environmental bonds and environmental research in innovative activities, *Ecological Economics* 1, 95-110.

Torsello L., Vercelli A., 1998. Environmental bonds: a critical assessment. In: Chichilnisky G., Heal G. and Vercelli A. (eds.), *Sustainability:*
dynamics and uncertainty, Martin Kluwer, Amsterdam, 243-256.

Weibull J.W., 1995. *Evolutionary game theory*, MIT Press, Cambridge.

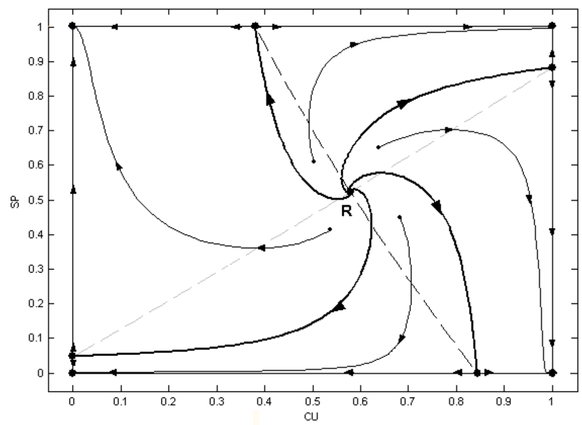

Figure 1: All the four vertices of the square are attractors. Parameters: $\lambda = 460$, $\mu = 120$, $\nu = 40$, $\psi = 48$, $\varphi = 25$, $\omega = 320$, $\theta = 240$, $\tau = 315$, $\sigma = 0.5$, $\zeta = 350$.

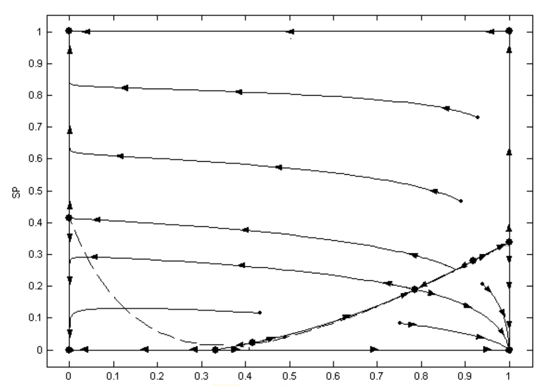

Figure 2: Three interior fixed points. Parameters: $\lambda = 11.45$, $\mu = 36$, $\nu = 0.25$, $\psi = 0.25$, $\varphi = 210$, $\omega = 10.0833$, $\theta = 8$, $\tau = 10$, $\sigma = 0.5$, $\zeta = 20.50$.

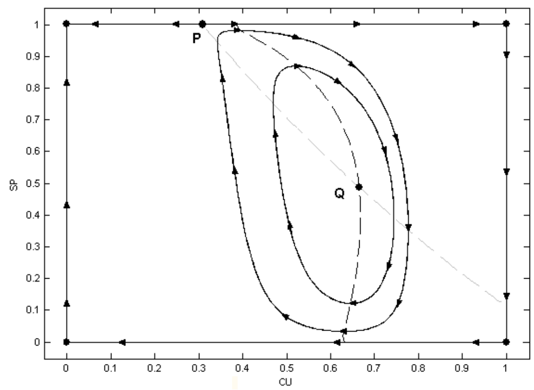

Figure 3: Two limit cycles surrounding the only interior (repelling) fixed point Q. Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 96$, $\theta = 140$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 93$.

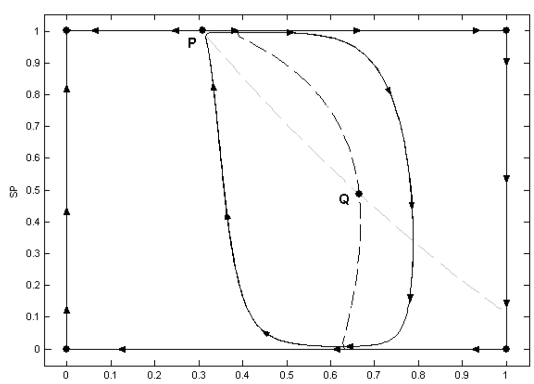

Figure 4: One repelling limit cycle. Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.663$, $\varphi = 5.5252$, $\omega = 70.7232$, $\theta = 140$, $\tau = 69.6182$, $\sigma = 0.7$, $\zeta = 68.5131$.

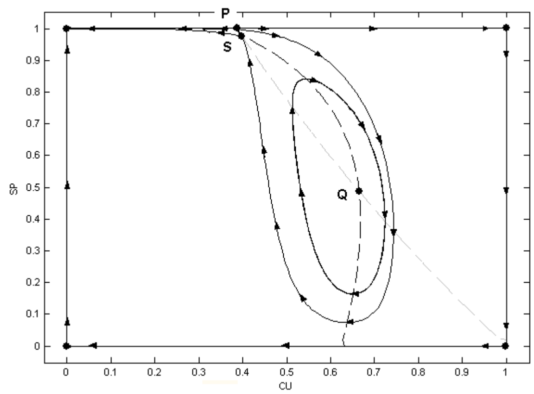

Figure 5: One attracting limit cycle. Parameters: $\lambda = 676$, $\mu = 26$, $\nu = 442$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 101.9827$, $\theta = 182$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 102$.

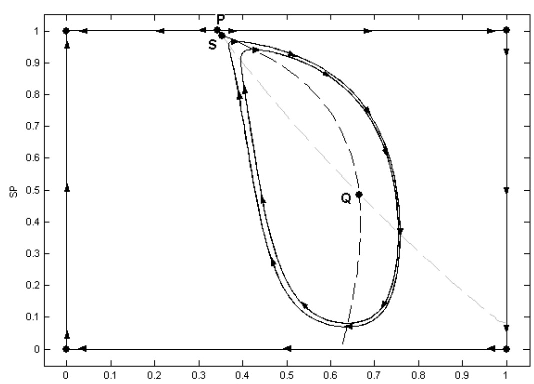

Figure 6: Two limit cycles with two interior fixed points. Parameters: $\lambda = 508.0347$, $\mu = 38$, $\nu = 340$, $\psi = 0.9$, $\varphi = 7.5$, $\omega = 98.3266$, $\theta = 140$, $\tau = 94.5$, $\sigma = 0.7$, $\zeta = 96.5$.

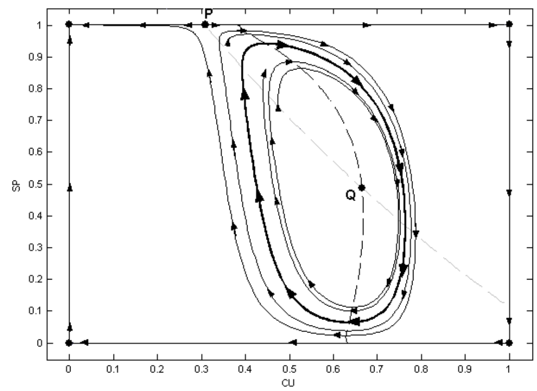

Figure 7: One *compound* limit cycle (internally attracting and externally repelling). Parameters: $\lambda = 520$, $\mu = 20$, $\nu = 340$, $\psi = 0.9483$, $\varphi = 7.9024$, $\omega = 101.1504$, $\theta = 140$, $\tau = 99.5699$, $\sigma = 0.7$, $\zeta = 97.9894$.

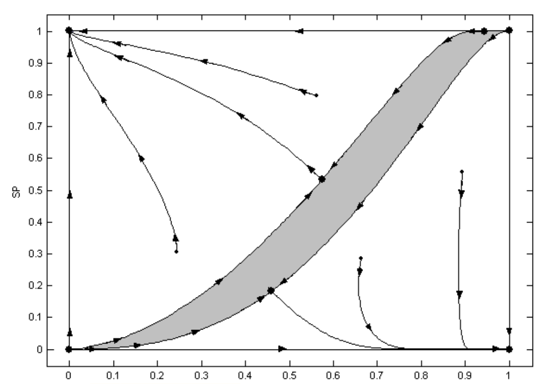

Figure 8: Attractors $(0,1)$ and $(1,0)$. Increasing the reimbursement rate β (hence increasing ω and τ) the basin of $(0,1)$ expands. Parameters: $\lambda = 8$, $\mu = 6$, $\nu = 0.5$, $\psi = 1$, $\varphi = 4$, $\omega = 0.2\,(\omega' = 1.8)$, $\theta = 5$, $\tau = 1.2\,(\tau' = 3)$, $\sigma = 0.8$, $\zeta = 2$.