CONTROLLABILITY OF SEMILINEAR SCHROEDINGER EQUATION VIA LOW-DIMENSIONAL SOURCE TERM

ANDREY SARYCHEV

ABSTRACT. We study controllability of 2D defocusing cubic Schroedinger equation under periodic boundary conditions and control applied via source term (additively). The source term is a linear combination of few complex exponentials (modes) with time-variant coefficients - controls. We manage to prove that controlling just 4 modes one can achieve controllability of this equation in any finite-dimensional projection of its evolution space $H^{1+\sigma}(\mathbb{T}^2)$, as well as approximate controllability in $H^{1+\sigma}(\mathbb{T}^2), \sigma > 0$. We also present negative result regarding exact controllability of cubic Schroedinger equation via a finite-dimensional source term.

Keywords: semilinear Schroedinger equation, approximate controllability, geometric control[1]

1. INTRODUCTION

Lie algebraic approach of geometric control theory to *nonlinear distributed systems* has been initiated recently. An example of its implementation is study of 2D Navier-Stokes/Euler equations of fluid motion controlled by *low-dimensional forcing* in [1, 2], where for the mentioned equations one arranged sufficient criteria for approximate controllability and for controllability in finite-dimensional projections of evolution space.

Here we wish to develop similar approach to another class of distributed system - cubic defocusing Schroedinger equation (cubic NLS):

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + F(t,x), \ u|_{t=0} = u^0, \tag{1}$$

controlled via source term $F(t,x)$.

We restrict ourselves to 2-dimensional periodic case: space variable x belongs to torus $\mathbb{T}^2$.

Our problem setting is distinguished by two features. First, control is introduced via source term, i.e. in additive form, on the contrast to bilinear form, characteristic for quantum control. More particular feature is *finite-dimensionality of the range of the controlled source term*:

$$F(t,x) = \sum_{k \in \hat{\mathcal{K}}} v_k(t) e^{ik \cdot x}, \ \hat{\mathcal{K}} \subset \mathbb{Z}^2 \text{ - finite}, \tag{2}$$

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which means that for each t the value $F(t)$ belongs to a *finite-dimensional subspace* $\mathcal{F}_{\hat{\mathcal{K}}} = \mathrm{Span}\{e^{ik\cdot x},\ k \in \hat{\mathcal{K}}\}$ of the evolution space for NLS.

The *control functions* $v_k(t),\ t \in [0,T],\ k \in \hat{\mathcal{K}}$, which enter the source term, can be chosen freely in $\mathbf{L}^\infty[0,T]$, or in any functional space, which is dense in $\mathbf{L}^1[0,T]$.

By this choice of 'small-dimensional' control our problem setting differs from the studies of controllability of NLS (see end of Section 3 for few references to alternative settings and approaches), in which controls have infinite-dimensional range. In some of the studies controls are supported on a subdomain and one is interested in tracing propagation of the controlled energy to other parts of domain. On the contrast, in our case controls affect few directions - modes - in functional evolution space for NLS and we are interested in the way this controlled action spreads to other (higher) modes.

One could opt for more general finitely generated control $\sum_{k\in\hat{\mathcal{K}}} v_k(t)F^k(x)$, but then representation of the NLS equation and in particular of its nonlinear term on $\mathbb{T}^2$ becomes much more intricate. Similar difficulties arise, when one studies NLS equation under general boundary conditions.

We will treat NLS equation (1) as an evolution equation in $H^{1+\sigma}(\mathbb{T}^2),\ \sigma > 0$. The 'high regularity' helps us to avoid certain analytic difficulties which are unrelated to the controllability issue.

Imposing the initial condition $u(0) = u_0 \in H^{1+\sigma}(\mathbb{T}^2)$, we set problems of:

(1) controllability in finite-dimensional projections, meaning that one can steer in time $T > 0$ the trajectory of the equation (1) from u_0 to a state $\hat{u} \in H^{1+\sigma}$ with any preassigned orthogonal projection $\Pi^{\mathcal{L}}\hat{u}$ onto any given finite-dimensional subspace $\mathcal{L} \subset H^{1+\sigma}$;
(2) approximate controllability meaning that attainable set of (1) from each u_0 is dense in $H^{1+\sigma}$;
(3) exact controllability in $H^{1+\sigma}$.

Definitions of some types of controllability and exact problem setting are provided in the next Section together with the main results. First of the results asserts that controllability in projection on each finite-dimensional subspace of $H^{1+\sigma}$ and approximate controllability in $H^{1+\sigma}$ can be achieved by (universal family of) 4-dimensional controls ($\sharp\hat{\mathcal{K}} = 4$). Corollary 6.5 describes a class of sets of controlled modes which suffice for achieving these types of controllability. The second main result asserts lack of exact controllability in $H^{1+\sigma}$ by controlling any finite number of modes.

2. Cubic Schroedinger equation on $\mathbb{T}^2$; problem setting and main results

2.1. **Controllability: definitions.**

2.1.1. *Global controllability.* As we said evolution space of NLS equation will be Sobolev space $H = H^{1+\sigma}(\mathbb{T}^2)$.

We say that control (2) *steers the system (1) from* $u_0 \in H$ *to* $\hat{u} \in H$ *in time* $T > 0$, if solution of (1) with initial condition $u|_{t=0} = u^0$ exists, is unique, belongs to $C([0,T], H)$ and satisfies $u(T) = \hat{u}$. The equation is *globally time-T (exactly) controllable* from u_0, if it can be steered in time T from u_0 to any point of H; it is globally (exactly) controllable from u_0, if for each $\hat{u}$ the equation can be steered from u_0 to $\hat{u}$ in some time $T > 0$.

2.1.2. *Controllability in finite-dimensional projections and in finite-dimensional component.* Let $\mathcal{L}$ be a closed linear subspace of H, $\Pi^{\mathcal{L}}$ be orthogonal projection of H onto $\mathcal{L}$.

Equation (1)-(2) is (time-T) globally *controllable from* u_0 *in projection onto* $\mathcal{L}$, if for each $\hat{q} \in \mathcal{L}$ the system can be steered (in time T) from u_0 to some point $\hat{u}$ with $\Pi^{\mathcal{L}}(\hat{u}) = \hat{q}$.

The NLS equation (1)-(2) is (time-T) globally *controllable from* u_0 *in finite-dimensional projections* if for each finite-dimensional subspace $\mathcal{L}$ it is (time-T) globally controllable from u_0 in projection onto $\mathcal{L}$; note that the set $\hat{\mathcal{K}}$ of controlled modes is assumed to be the same for all $\mathcal{L}$.

Whenever $\mathcal{L}$ is a 'coordinate subspace' $\mathcal{L} = \mathrm{span}\{e^{ik\cdot x} |\ k \in \mathcal{K}^o\}$, with $\mathcal{K}^o \subset \mathbb{Z}^2$ being a finite set of *observed modes*, then controllability in projection on $\mathcal{L}$ is called *controllability in observed* $\mathcal{K}^o$*-component*.

Remark 2.1. *It is convenient to characterize time-T controllability in terms of surjectiveness of the end-point map* $E_T : v(\cdot) \mapsto F(v(\cdot)) \mapsto u(T)$ *of the controlled NLS equation (1)-(2), which maps a control* $v(\cdot) = (v_k(t)), k \in \hat{\mathcal{K}}$, *into the 'final' point* $u(T)$ *of the trajectory* $u(t)$ *of this equation, driven by source term* $F = \sum_{k \in \hat{\mathcal{K}}} v_k(t) e^{ik\cdot x}$ *and starting at* $u(0) = u^0$. *Similarly controllability in projection on* $\mathcal{L}$ *means that the composition* $\Pi_{\mathcal{L}} \circ E_T$ *is onto (covers)* $\mathcal{L}$. $\square$

2.1.3. *Approximate controllability.* The NLS equation (1)-(2) is time-T approximately controllable from u_0 in H, if it can be steered from u_0 to each point of a dense subset of H. $\square$

2.1.4. *Solid controllability (cf.* [2]*).* On the contrast to previous definitions the word 'solid' does not refer to a new type of controllability but means property of stability of controllability with respect to certain class of perturbations.

Let $\Phi : \mathcal{M}^1 \mapsto \mathcal{M}^2$ be a continuous map between two metric spaces, and $S \subseteq \mathcal{M}^2$ be any subset. We say that Φ *covers* S *solidly*, if $S \subseteq \Phi(\mathcal{M}^1)$ and the inclusion is stable with respect to C^0-small perturbations of Φ, i.e. for some C^0-neighborhood Ω of Φ and for each map $\Psi \in \Omega$, there holds: $S \subseteq \Psi(\mathcal{M}^1)$.

Controllability in projection on finite-dimensional subspace $\mathcal{L}$ for the NLS equation (1)-(2) is *solid*, if for any bounded set $S \subseteq \mathcal{L}$ there exists a family of controls $V_S = \{v(t,b) |\ b \in B$ - compact in $\mathbb{R}^d\}$, such that projected end-point map $\left(\Pi^{\mathcal{L}} \circ E_T\right)|_{V_S}$ (see Remark 2.1) covers S solidly. We will say that S is solidly attained by the controlled NLS equation.

2.2. **Problem setting and main results.** Our first goal is establishing sufficient criteria for controllability of cubic defocusing NLS in all finite-dimensional projections and approximate controllability in $H^{1+\sigma}, \ \sigma > 0$. Common criterion is formulated in terms of a set of controlled modes $\hat{\mathcal{K}}$, which is fixed and the same for *all* projections and for approximate controllability.

Second objective is *negative* result regarding exact controllability of cubic NLS via finite-dimensional source term.

Main result 1 (criterion for controllability in finite-dimensional projections and approximate controllability). *Given 2D periodic defocusing cubic Schroedinger equation (1), controlled via source term (2), one can find a* 4*-element set* $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ *of controlled modes such that for any initial data* $u^0 \in H^{1+\sigma}(\mathbb{T}^2)$ *and any* $T > 0$*: i) for each finite-dimensional subspace* $\mathcal{L}$ *of* $H^{1+\sigma}(\mathbb{T}^2)$ *the equation (1)-(2) is time-T controllable from* u^0 *in projection on* $\mathcal{L}$*; ii) the equation is approximately controllable from* u^0 *in* $H^{1+\sigma}(\mathbb{T}^2)$. □

Remark 2.2. *An example of a set* $\hat{\mathcal{K}}$ *able to guarantee the controllability properties is* $\hat{\mathcal{K}} = \{(0,0),(1,0),(0,1),(1,1)\}$. *Corollary 6.5 introduces a class of sets* $\hat{\mathcal{K}}$ *of controlled modes, which suffice for the two types of controllability.* □

Main result 2 (negative result on exact controllability). *For 2D periodic defocusing cubic Schroedinger equation (1), controlled via source term (2) with arbitrary finite set* $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ *of controlled modes, for each* $T > 0$ *and each initial data* $u^0 \in H^{1+\sigma}(\mathbb{T}^2)$*, the time-T attainable set* $\mathcal{A}_{T,u^0}$ *of (1)-(2) from* u^0 *is contained in a countable union of compact subsets of* $H^{1+\sigma}(\mathbb{T}^2)$ *and therefore the complement* $H^{1+\sigma}(\mathbb{T}^2) \setminus \mathcal{A}_{T,u^0}$ *is dense in* $H^{1+\sigma}(\mathbb{T}^2)$. □

3. Outline of the approach: Lie extensions, fast-oscillating controls, resonances. Other approaches

Study of controllability of NLS equation is based (as well as our earlier joint work with A.Agrachev on Navier-Stokes/Euler equation) on method of iterated Lie extensions. *Lie extension* of control system $\dot{x} = f(x,u), \ u \in U$ is a way to add vector fields to the right-hand side of the system guaranteeing (almost) invariance of its controllability properties. The additional vector fields are expressed via Lie brackets of $f(\cdot,u)$ for various $u \in U$. If after a series of extensions one arrives to a controllable system, then the controllability of the original system will follow.

This approach can not be extended automatically onto infinite-dimensional setting due to the lack of adequate Lie algebraic tools. So far in the infinite-dimensional context Lie algebraic formulae are rather used as guiding tools, whose implementation has to be justified by analytic means. In the rest of this Section we provide geometric control sketch for the proof of main result.

When studying controllability we look at cubic NLS equation as at particular type of infinite-dimensional *control-affine system*:

$$-i\partial_t u = c(u,t) + \sum_{k\in\hat{\mathcal{K}}} e_k v_k(t), \ e_k = e^{ik\cdot x},$$

where $c(u,t)$ is cubic *drift* vector field, e_k are constant *controlled* vector field in $H^{1+\sigma}(\mathbb{T}^2)$ with values $e^{ik\cdot x} \in H^{1+\sigma}(\mathbb{T}^2)$.

Lie extensions, we use, are implemented iteratively. At each iteration they involve two controlled vector fields e_m, e_n and outcome is fourth-order Lie bracket $[e_n,[e_m,[e_m,c]]]$, which appears as *extending controlled vector field.* The vector field is constant (as far as the vector field c is cubic) and is seen as direction of action of an *extended control.*

Different type of Lie brackets which makes its appearance for each Lie extension is third-order Lie bracket $[e_m,[e_m,c]]$, which can be seen as *obstruction to controllability*, along the vector field 'unilateral drift' of the system takes place. This drift can not be locally compensated but for NLS equation one can nullify *average drift* by imposing integral (isoperimetric) relations onto the controls involved.

To design needed motion in the extending direction $[e_n,[e_m,[e_m,c]]]$ and to oppress motion in the directions, not needed, we employ *fast-oscillating controls.* Use of such controls is traditional for geometric control theory and although a 'general theory' is hardly available, the approach can be effectively applied in particular cases (see, for example treatment of 'single-bracket case' in [16]).

In our study we feed fast-oscillating controls

$$v_m(t)e^{ia_m t/\varepsilon}e^{im\cdot x}, v_n(t)e^{ia_n t/\varepsilon}e^{in\cdot x}$$

into the right-hand side of the NLS equation at looks at interaction of the two controls via the cubic term. The idea is to design needed resonance in the course of such interaction, that is to choose oscillation frequences and magnitudes in such a way that the interaction 'in average' influences dynamics of (few) certain modes. In our treatment we manage to limit the influence to unique basis mode $e^{i(2m-n)\cdot x}$; the resonance term is seen as additional (extending) control along this mode. The procedure is interpreted as *elementary extension* of the set of controlled modes: for any $m,n \in \hat{\mathcal{K}}$: $\hat{\mathcal{K}} \mapsto \hat{\mathcal{K}} \bigcup \{2m-n\}$.

Final controllability result is obtained by (finite) iteration of the elementary extensions. If one seeks controllability in observed $\mathcal{K}^o$-component with $\mathcal{K}^o \supset \hat{\mathcal{K}}$, then one should look (when possible) for a series of elementary extensions $\hat{\mathcal{K}} = \mathcal{K}^1 \subset \mathcal{K}^2 \subset \cdots \subset \mathcal{K}^N = \mathcal{K}^o$. Getting extended controls available for each observed mode $k \in \mathcal{K}^o$ we conclude controllability of the extended system in $\mathcal{K}^o$-component by an easy Lemma 5.2. On the contrast controllability of the original system in $\mathcal{K}^o$-component will follow by virtue of rather technical Approximative Lemma 5.1, which formalizes the resonance design.

From controllability for *each* finite-dimensional component one derives controllability in projection on each finite-dimensional subspace as well as approximate controllability; this is proved in Section 7.

Note that the analysis of interaction of different terms via cubic nonlinearity in the case of *periodic* NLS equation is substantially simplified by choice of special basis of exponential modes.

Besides the design of proper resonances there are two *analytic* problems to be fixed. First problem consists of studying NLS with fast-oscillating right-hand side and of establishing the continuity, approximating properties and the *limits* of corresponding trajectories, as the frequency of oscillation tends to $+\infty$. Second problem is to cope with the fact that at each iteration we are only able to *approximate* the desired motion, therefore the controllability criteria need to be stable with respect to the approximation errors.

The second problem is fixed with the help of the notion of *solid controllability* (see previous Section), which guarantees stability of controllability property with respect to approximation error.

The solution to the first problem in finite-dimensional setting is provided by *theory of relaxed controls.* For general nonlinear PDE such theory is unavailable; although for *semilinear infinite-dimensional control systems* relaxation results have been obtained in [9, 8]. We provide formulations and proofs needed for our analysis in Subsection 5.5.

What regards negative result on exact controllability stated in Main result 2, then the key point for its proof is continuity of input-trajectory map in some weaker topology of the (functional) space of inputs (controls) in which the space is countable union of compacts and as a consequence attainable sets are meager. This kind of argument has been used in [3] for establishing noncontrollability of some bilinear distributed systems. Finer method, based on estimates of Kolmogorov's entropy has been invoked in [15] for proving lack of exact controllability by finite-dimensional forcing for Euler equation of fluid motion.

At the end of the Section we wish to mention just few references to other approaches to controllability of linear and semilinear Schroedinger equation controlled via bilinear or additive control, this latter being "internal" or boundary.

First we address the readers to [18, 11] which provide nice surveys of the results on:

- exact controllability for linear Schroedinger equation with additive control in relation to observability of adjoint system and to geometric control condition ([13] and references in [18] on other results up to 2003);
- controllability of linear Schroedinger equation with control entering bilinearly; besides references in the above cited surveys there are notable results [4, 5] on local (exact) controllability in H^7 of 1-D equation; another interesting result is (obtained by geometric methods)

criterion [6] of approximate controllability for the case in which 'drift Hamiltonian' has discrete non-resonant spectrum (see bibliographic references in [4, 5, 6] to preceding work);

- exact controllability of semilinear Schroedinger equation by means of internal additive control; in addition to references in [18, 11] we mention more recent publications [7, 14] where the property has been established for 2D and 1D cases. The key tool in the study of the semilinear case is 'linearization principle', going back to [12]. In contrast our approach makes direct and exclusive use of the nonlinear term.

4. Preliminaries on existence, uniqueness and continuous dependence of trajectories

Notions of controllability, introduced above, involve trajectories of cubic NLS equation with source term. The trajectories are sought in the space $C([0,T];H)$, H being Hilbert space of functions $u(x)$ defined on $\mathbb{T}^2$. We opt for $H = H^{1+\sigma}(\mathbb{T}^2)$.

In this Section we collect results on existence/uniqueness and on continuity in the right-hand side for solutions of semilinear equations

$$(-i\partial_t + \Delta)\tilde{u} = G(t,\tilde{u}), \ \tilde{u}(0) = \tilde{u}^0 \tag{3}$$

and of its 'perturbation':

$$(-i\partial_t + \Delta)u = G(t,u) + \phi(t,u), u(0) = u^0. \tag{4}$$

Below we identify the equations (3),(4) with their integral forms (10),(11) obtained via applications of Duhamel formula.

We assume the nonlinear terms $G(t,\cdot), \phi(t,\cdot) : \ H \mapsto H$ to be continuous, and to satisfy the conditions

$$G(t,0) = 0, \tag{5}$$

$$\forall b > 0, \ \exists \beta_b(t) \in \mathbf{L}^1([0,T],\mathbb{R}_+), \ \text{such that } \forall t \in [0,T], \ \forall \|u\| \leq b,$$

$$\|G(t,u)\|_H \leq \beta_b(t), \ \|G(t,u') - G(t,u)\|_H \leq \beta_b(t)\|u' - u\|_H, \tag{6}$$

$$\|\phi(t,u)\|_H \leq \beta_b(t), \ \|\phi(t,u') - \phi(t,u)\|_H \leq \beta_b(t)\|u' - u\|_H. \tag{7}$$

Local existence of solutions under the assumptions could be established via fixed point argument for contracting map in $C([0,T];H)$.

Proposition 4.1 (local existence and uniqueness of solutions)**.** *Let G satisfy conditions (6). Then for each $B > 0$, $\exists T_B > 0$ such that for $\|\tilde{u}^0\|_H \leq B$ there exists unique strong solution $u(\cdot) \in C([0,T_B],H)$ of Cauchy problem (3).* □

We choose $H = H^{1+\sigma}(\mathbb{T}^2)$, so that the cubic term of the NLS equation (1) would satisfy conditions (6),(7). One can invoke the following technical result for verification.

Lemma 4.2 ('Product Lemma'; [17]). *For Sobolev spaces $H^s(\mathbb{T}^d)$ of functions on d-dimensional torus there holds:*
for $s \geq 0: \ \|fg\|_{H^s} \leq C(s,d)\left(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\right);$
for $s > d/2: \ \|fg\|_{H^s} \leq (C'(s,d)\|f\|_{H^s}\|g\|_{H^s}.$ □

This Lemma allows verification of the conditions (6),(7) for more general Nemytskii-type operators $u \mapsto G(t,u), \ u \mapsto \phi(t,u)$ of the form

$$u(t,x) \mapsto F_0(t,x) + \sum_{j=1}^{p} P_j(u(t,x), \bar{u}(t,x); t),$$

where $P_j : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are polynomials of degree j in $u, \bar{u}$ with coefficients $p_{j\alpha}(t) \in \mathbf{L}^1([0,T],\mathbb{C})$, while $F_0(t,x)$ belongs to $\mathbf{L}^1([0,T],\mathbb{C})$. Recall that the source term (2) is trigonometric polynomial in x and $F(t,x) \in \mathbf{L}^\infty\left([0,T], H^{1+\sigma}\right)$.

Global existence and uniqueness results for cubic *defocusing* NLS equation (1) are classical under assumptions we made; see, for example, [7] for respective global formulation for cubic defocusing NLS with source term.

Proposition 4.3 (global existence and uniqueness). *Let time-variant source term $t \mapsto F(t,\cdot)$ belong to $\mathbf{L}^1([0,T],H^{1+\sigma})$. Then for each initial condition $u(0) = u^0 \in H^{1+\sigma}$ the Cauchy problem for the equation (1) has unique strong solution $u(\cdot) \in C([0,T],H^{1+\sigma})$.* □

No we provide few results on continuity of trajectories in the right-hand side of the NLS equation.

Proposition 4.4 (continuity in the right-hand side). *Let assumptions of the Proposition 4.1 hold and let $\tilde{u}(t) \in C([0,T],H)$ be solution of (3); assume* $\sup_{t\in[0,T]}\|u(t)\| < b$. *Then* $\exists\delta > 0, \ c > 0$ *such that whenever*

$$\|u^0 - \tilde{u}^0\| + \int_0^T \sup_{\|u\|\leq b} \|\phi(t,u)\|_H dt < \delta, \tag{8}$$

then solution $u(t)$ of the perturbed equation (4) exists on the interval $[0,T]$, is unique and admits an upper bound

$$\sup_{t\in[0,T]} \|u(t) - \tilde{u}(t)\| < c\left(\|u^0 - \tilde{u}^0\| + \int_0^T \sup_{\|u\|\leq b}\|\phi(t,u)\|_H dt\right). \ \square \tag{9}$$

Proof. As it is known the solution of equation (4) can be continued in time as long as $H^{1+\sigma}$-norm remains bounded. Therefore from the estimate (9) for sufficiently small $\delta > 0$ one gets extendibility of solution of (4) onto $[0,T]$.

To prove (9) we rewrite the differential equations (3),(4) in the integral form

$$\tilde{u}(t) = e^{it\Delta}\left(\tilde{u}^0 + i\int_0^t e^{-i\tau\Delta}G(\tau,\tilde{u}(\tau))d\tau\right), \tag{10}$$

$$u(t) = e^{it\Delta}\left(u^0 + i\int_0^t e^{-i\tau\Delta}\left(G(\tau,u(\tau)) + \phi(\tau,u(\tau))\right)d\tau\right). \tag{11}$$

Then

$$u(t)-\tilde{u}(t)=e^{it\Delta}\left((u^0-\tilde{u}^0)+i\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right)+$$
$$+e^{it\Delta}i\int_0^t e^{-i\tau\Delta}\left((G(\tau,u(\tau))-G(\tau,\tilde{u}(\tau)))+(\phi(\tau,u(\tau))-\phi(\tau,\tilde{u}(\tau)))\right)d\tau.$$

Given that $e^{it\Delta}$ is an isometry of $H^{1+\sigma}$, we get

$$\|u(t)-\tilde{u}(t)\|_H\leq\|u^0-\tilde{u}^0\|_H+\left\|\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H+ \tag{12}$$
$$+\int_0^t\|e^{-i\tau\Delta}\left(G(\tau,u(\tau))-G(\tau,\tilde{u}(\tau))+\phi(\tau,u(\tau))-\phi(\tau,\tilde{u}(\tau))\right)\|_H d\tau\leq$$
$$\leq\|u^0-\tilde{u}^0\|_H+\left\|\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H+2\int_0^t\beta_b(\tau)\|u(\tau)-\tilde{u}(\tau)\|_H d\tau.$$

By Gronwall inequality

$$\|u(t)-\tilde{u}(t)\|_H\leq \tag{13}$$
$$\leq\left(\|u^0-\tilde{u}^0\|_H+\left\|\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H\right)C'e^{C\int_0^t\beta_b(\tau)d\tau},$$

for some $C,C'>0$ and whenever (8) is satisfied, we get

$$\|u(t)-\tilde{u}(t)\|_H\leq C''\left(\|u^0-\tilde{u}^0\|+\int_0^t\|\phi(\tau,\tilde{u}(\tau))\|\,d\tau\right)\leq C''\delta. \tag{14}$$

□

Below we derive more general continuity result (Proposition 5.7) which incorporates perturbations $\phi(t,x)$, fast-oscillating in time, and *relaxation metric* for the right-hand sides.

Similarly to the previous Proposition one gets

Lemma 4.5. *Consider family of equations*

$$(-i\partial_t+\varepsilon\Delta)u^\varepsilon=\varepsilon G(t,u^\varepsilon)+\phi(t,u^\varepsilon),\ u^\varepsilon(0)=u^0,\ \varepsilon>0, \tag{15}$$

depending on parameter $\varepsilon>0$, with G,ϕ satisfying (6),(7). Consider 'limit equation' for $\varepsilon=0$:

$$-i\partial_t\tilde{u}=\phi(t,\tilde{u}),\ \tilde{u}|_{t=0}=u^0. \tag{16}$$

For solution $\tilde{u}(\cdot)\in C([0,T],H)$ of (16) there exists ε_0 (depending on T), such that for $\varepsilon\in[0,\varepsilon_0)$ solutions $u^\varepsilon(t)$ of (15) exist on $[0,T]$ and

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-\tilde{u}(t)\|_H=o(1),\ \textit{as } \varepsilon\to 0.\ \square$$

Proof. By Duhamel formula we get as in (12)

$$\|u^\varepsilon(t)-\tilde{u}(t)\|_H \leq \left\|e^{i\varepsilon t\Delta}u^0-u^0\right\| + \left\|\int_0^t e^{-i\varepsilon\tau\Delta}\varepsilon G(\tau,u(\tau))d\tau\right\| +$$

$$+\left\|\int_0^t e^{-i\varepsilon\tau\Delta}\phi(\tau,u^\varepsilon(\tau))-\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H \leq$$

$$\leq \varepsilon\left\|\int_0^t e^{-i\varepsilon\tau\Delta}G(\tau,u(\tau))d\tau\right\| + \left\|(e^{i\varepsilon t\Delta}-I)u^0\right\|_H +$$

$$+\left\|\int_0^t (e^{-i\varepsilon\tau\Delta}-I)\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H + \int_0^t \|\phi(\tau,u^\varepsilon(\tau))-\phi(\tau,\tilde{u}(\tau))\|_H\, d\tau.$$

The last addend at the right-hand side is bounded by $\int_0^t \beta(\tau)\|u^\varepsilon(\tau)-\tilde{u}(\tau)\|_H d\tau$. We will arrive to the needed conclusion by virtue of Gronwall inequality, when proving that the other three addends are $o(1)$ as $\varepsilon \to +0$.

We comment on the addend $\left\|\int_0^t(e^{-i\varepsilon\tau\Delta}-I)\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H$, the other two assertions being obvious. For each $\delta > 0$ one can approximate the function $\tau \mapsto \phi(\tau,\tilde{u}(\tau)),\ \tau \in [0,T]$ by a piecewise constant function $\psi^\delta(\tau)$: $\|\phi(\tau,\tilde{u}(\tau))-\psi^\delta(\tau)\|_{\mathbf{L}^1([0,T],H)} \leq \delta$. Then given that $\|e^{-i\varepsilon\tau\Delta}-I\| \leq 2$ one gets

$$\left\|\int_0^t (e^{-i\varepsilon\tau\Delta}-I)\phi(\tau,\tilde{u}(\tau))d\tau\right\|_H \leq \left\|\int_0^t (e^{-i\varepsilon\tau\Delta}-I)\psi^\delta(\tau)d\tau\right\|_H + 2\delta.$$

For a piecewise constant function ψ^δ the first addend tends to 0, as $\varepsilon \to 0$. □

5. Extension of control

Here we introduce our main tool - extension of control. The outcome of the Section, to be employed later, is Proposition 5.3 which establishes sufficient criterion for controllability in finite-dimensional component, wherefrom one will derive in Section 7 controllability in projections and approximate controllability (Main Result 1). Proposition 5.3 is in its turn derived from rather technical Approximative Lemma 5.1 for extensions, accompanied by elementary Lemma 5.2 on controllability by full-dimensional control.

In what follows the metrics $\mathbf{L}^1\left([t_0,t_1],H^{1+\sigma}\right), \mathbf{L}^1\left([t_0,t_1],\mathbb{C}^\kappa\right)$, $[t_0,t_1] \subset \mathbb{R}$ will be denoted both by $\mathbf{L}^1_t$ by abuse of notation.

5.1. Extensions: approximative lemma.

Consider NLS equation (1)-(2) with controls applied to the modes, indexed by a set $\hat{\mathcal{K}} \subset \mathbb{Z}^2$, or the same with the controlled source term $\sum_{k\in\hat{\mathcal{K}}} v_k(t)e^{ik\cdot x}$.

Pick two vectors r,s from the set $\hat{\mathcal{K}}$ and call $\mathcal{K} = \hat{\mathcal{K}}\bigcup\{2r-s\}$ an *elementary extension* of $\hat{\mathcal{K}}$. Call $\mathcal{K}$ *proper extension* of $\hat{\mathcal{K}}$ if there exits a finite sequence of sets $\hat{\mathcal{K}} = \mathcal{K}^1 \subset \mathcal{K}^2 \subset \cdots \subset \mathcal{K}^N = \mathcal{K}$, such that each $\mathcal{K}^j$ is elementary extension of $\mathcal{K}^{j-1},\ j \geq 2$.

The following Lemma states that controls (energy) fed into the modes, indexed by $\hat{\mathcal{K}}$, can be cascaded to and moreover can approximately control larger set $\mathcal{K}$ of modes, whenever $\mathcal{K}$ is proper extension of $\hat{\mathcal{K}}$.

Lemma 5.1 (approximative lemma). *Let $\mathcal{K}$ be a proper extension of $\hat{\mathcal{K}}$. Given a family of controls*

$$b \mapsto W(t;b) = \sum_{k \in \mathcal{K}} w_k(t,b) e^{ik\cdot x}, \ b \in B \text{ - compact in } \mathbb{R}^d, \tag{17}$$

parameterized by $b \in B$ continuously in L_1^t-metric, one can construct for each $\delta > 0$ another family of controls

$$b \mapsto V^{\delta}(\cdot, b) = \sum_{k \in \hat{\mathcal{K}}} v_k(t,b) e^{ik\cdot x}, \ b \in B, \tag{18}$$

continuous in $\mathbf{L}_t^1$-metric, such that for the respective end-point maps (see Remark 2.1) of the NLS equations,

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + W(t,b), \tag{19}$$

$$-i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x) + V^{\delta}(t,b), \tag{20}$$

controlled via source terms $F = W$ and $F = V^{\delta}$, there holds

$$\|E_T(V^{\delta}(b)) - E_T(W(b))\| \leq \delta, \ \forall b \in B. \ \Box \tag{21}$$

Remark 5.1. *Note that controls (18) take their values in 'low-dimensional' space $\mathcal{F}_{\hat{\mathcal{K}}}$ in comparison with the 'high-dimensional' space $\mathcal{F}_{\mathcal{K}}$ - the range of controls (17).* $\Box$

Remark 5.2. *It suffices to prove the Lemma for $\mathcal{K}$ being an* elementary extension *of $\hat{\mathcal{K}}$, the rest being accomplished by induction.* $\Box$

5.2. **Full-dimensional control.** Before proving that controllability can be achieved by means of low-dimensional controls we formulate general result for the case, where control is *full-dimensional.*

Lemma 5.2 (full-dimensional control lemma). *Controlled semi-linear equation*

$$-i\partial_t u(t,x) + \Delta u(t,x) = G(t,u) + \sum_{k \in \hat{\mathcal{K}} = \mathcal{K}^o} w_k(t) e^{ik\cdot x}, \ u(0) = u^0, \tag{22}$$

with coinciding sets of controlled and observed modes $\mathcal{K}^1 = \mathcal{K}^o$, is time-$T$ solidly controllable for each $T > 0$ in observed $\mathcal{K}^o$-component. $\Box$

Proof of Lemma 5.2. Without lack of generality assume the initial condition to be $u(0) = 0_H$. Take a ball B in $\mathcal{F}_{\mathcal{K}^o} = \text{span}\{e^{ik\cdot x} | \ k \in \mathcal{K}^o\}$. We will prove that B is solidly attainable for the controlled equation (22).

Restrict (22) to an interval $[0, \varepsilon]$, where small $\varepsilon > 0$ will be specified later on. Proceed with time substitution $t = \varepsilon\tau, \ \tau \in [0,1]$ under which (22) takes form:

$$-i\partial_\tau u + \varepsilon\Delta u = \varepsilon G(t,u) + \varepsilon \sum_{k\in\mathcal{K}^o} w_k(t)e^{ik\cdot x},\ u(0)=0,\ \tau\in[0,1]. \tag{23}$$

Fix $\gamma > 1$. For each $b \in \gamma B,\ b = (b_1,\ldots,b_N)$ consider control $w(\cdot;b) = -i\varepsilon^{-1}\sum_{k\in\mathcal{K}^o} b_k e^{ik\cdot x}$. Substituting the control into (23) we get

$$-i\partial_\tau u + \varepsilon\Delta u = \varepsilon G(t,u) - i\sum_{k\in\mathcal{K}^o} b_k e^{ik\cdot x},\ u(0)=0,\ \xi\in[0,1].$$

For $\varepsilon = 0$ we get the 'limit equation'

$$\partial_\tau u = \sum_{k\in\mathcal{K}^o} b_k e^{ik\cdot x},\ u(0)=0,\ ,\ \tau\in[0,1]. \tag{24}$$

Let E_1^0 be the time-1 end-point map of (24). In the basis $e^{ik\cdot x}$ of $H^{1+\sigma}$ it has form $(b_1,\ldots,b_N)\mapsto \sum_{k\in\mathcal{K}^o} b_k e^{ik\cdot x}$.

Obviously the map $b\mapsto\Phi(b) = \Pi^o\circ E_1^0(w(t;b))$, where Π^o is orthogonal projection onto $\mathcal{L}^o$, coincides on $\gamma B\supset B$. with the identity map $\mathrm{Id}_{\gamma B}$ and $(I-\Pi^o)E_1^0(w(t;b)) = 0$.

According to Lemma 4.5 for the continuous maps $\Phi^\varepsilon : b\mapsto E_1^\varepsilon(w(\cdot,b))$, where E_1^ε are end-point maps of the control systems (23), there holds $\|\Phi^\varepsilon - \Phi^0\|_{C^0(B)}\to 0$ as $\varepsilon\to 0$.

By degree theory argument there exists ε_0 such that $\forall\varepsilon\le\varepsilon_0$ the image of $\left(\Pi^0\circ\Phi^\varepsilon\right)(\gamma B)$ covers B solidly. $\square$

Remark 5.3. *In fact we only established controllability for small times $T > 0$. Still controllability in any time can be concluded by a standard trick of guiding the system from u^0 to the origin of $H^{1+\sigma}$ in small time $\delta > 0$, maintaining it at the origin under zero control for time length $T - 2\delta$ and then guiding it to preassigned $\hat{u}$ in time $\delta > 0$.* $\square$

Remark 5.4. *From the proof of the Lemma it follows, that in addition to controllability one can arrange for each $\delta > 0$ a proper choice of controls, so that the estimate $\|(I-\Pi^o)\left(u(T)-u^0\right)\|\le\delta$ will hold for the projection $I-\Pi^o = \Pi^\perp$ onto orthogonal complement to $\mathcal{F}_{\mathcal{K}^o}$.* $\square$

Remark 5.5. *Without lack of generality we may assume, that $w(t,b)$ are smooth with respect to t and that any finite number of derivatives $\frac{\partial^j w}{\partial t^j}(\cdot,b)$ depend continuously in $\mathbf{L}_t^1$-metric on $b\in B$. Indeed smoothing $w(t,b)$ by convolution with a smooth ε-approximation $h_\varepsilon(t)$ of Dirac function $\delta(t)$, one gets a family of smooth controls $w^\varepsilon(t,b)$, which provides solid controllability, for small $\varepsilon > 0$. The continuous dependence in $\mathbf{L}_t^1$-metric of $\frac{\partial w^\varepsilon}{\partial t}(\cdot,b)$ on b is verified directly.* $\square$

5.3. **Controllability in finite-dimensional component via extensions.** The following result regarding controllability in observed component is a corollary of Lemmae 5.1,5.2.

Proposition 5.3. *If a set of observed modes $\mathcal{K}^o$ is proper extension of a set of controlled modes $\mathcal{K}^1$, then NLS equation*

$$-i\partial_t u + \Delta u = |u|^2 u + V(\cdot, b),$$

is solidly controllable in the observed $\mathcal{K}^o$-component. □

Proof. Let S be a compact subset of $\mathcal{F}_{\mathcal{K}^o} = \text{span}\{f_k | \ k \in \mathcal{K}^o\}$. According to Lemma 5.2 we can choose a family of $\mathcal{F}_{\mathcal{K}^o}$-valued controls $W(\cdot, b)$ which provides solid controllability. If $\delta > 0$ is small enough and family $V(t; b)$ satisfies conclusion of the Approximative Lemma 5.1, then $\Pi^o \circ E_T(V(t; b))$ covers S solidly. □

5.4. **Proof of Approximative Lemma 5.1.** According to the Remark 5.2 it suffices to treat the case where $\mathcal{K}$ is elementary extension of $\hat{\mathcal{K}}$:

$$\mathcal{K} = \hat{\mathcal{K}} \bigcup \{2r - s\}, \ r, s \in \hat{\mathcal{K}}.$$

It is convenient to proceed with time-variant change of basis in $H^{1+\sigma}$, passing from the exponentials $e^{ik\cdot x}$ to the exponentials

$$f_k = e^{i(k\cdot x + |k|^2 t)}, \ k = (k_1, k_2) \in \mathbb{Z}^2.$$

Therefore from now on we consider $\mathcal{F}_{\mathcal{K}}$-valued family of controls

$$b \mapsto W(t, b) = \sum_{k\in\mathcal{K}} w_k(t; b) f_k, \tag{25}$$

parameterized by $b \in B$ - compact in Euclidean space. We wish to construct family of controls $V(t; b) = \sum_{k\in\hat{\mathcal{K}}} v_k(t; b) f_k$, whose range has one dimension less and which satisfy (21).

5.4.1. *Substitution of variables.* We will seek the family $b \mapsto V(t, b)$ in the form

$$V(t, b) = \tilde{W}(t, b) + \partial_t v_r(t, b) f_r + \partial_t v_s(t, b) f_s, \tag{26}$$

where $\tilde{W}(t, b)$, whose range is $\mathcal{F}_{\mathcal{K}}$, and families of Lipschitzian functions $t \mapsto v_r(t, b), v_s(t, b)$ will be specified in the course of the proof. For some time we will omit dependence on b in notation.

Feeding the controls (26) into the right-hand side of equation (1) we get

$$(-i\partial_t + \Delta)u = |u|^2 u + \tilde{W}(t) + \dot{v}_r(t) f_r + \dot{v}_s(t) f_s. \tag{27}$$

This equation can be given form

$$(-i\partial_t + \Delta)\left(u - iV_{rs}(t)\right) = |u|^2 u + \tilde{W}(t), \tag{28}$$

where $V_{rs}(t) = v_r(t) f_r + v_s(t) f_s$. We used the fact that $(-i\partial_t + \Delta) f_k = 0, \ \forall k \in \mathbb{Z}^2$.

By time-variant substitution

$$u^* = u - iV_{rs}(t), \tag{29}$$

we transform(28) into equation:

$$(30)\qquad (-i\partial_t+\Delta)u^* = |u^*+iV_{rs}(t)|^2(u^*+iV_{rs}(t))+\tilde{W}(t) =$$
$$= |u^*|^2u^* - i(u^*)^2\bar{V_{rs}} + 2i|u^*|^2V_{rs} - V_{rs}^2\bar{u^*} + 2u^*|V_{rs}|^2 + i|V_{rs}|^2V_{rs} + \tilde{W}(t).$$

Imposing constraints

$$(31)\qquad v_r(0)=v_s(0)=0,\ v_r(T)=v_s(T)=0,$$

we keep end-points unchanged under the substitution (29): $u(0)=u^*(0)$, $u(T)=u^*(T)$. Hence the end-point maps E_T for the controlled equations (27) and (30) coincide for those Lipschitzian controls $v_r(t), v_s(t)$, which meet (31) .

5.4.2. *Fast oscillations and resonances.* Now we put into game *fast-oscillations,* by choosing $V_{rs}(t)$ in (29),(30) of the form

$$(32)\qquad V_{rs}(t)=v_r(t)f_r+v_s(t)f_s = e^{i(t/\varepsilon+\rho(t))}\check{v}_r(t)f_r+e^{i2t/\varepsilon}\check{v}_s(t)f_s,$$

where $\check{v}_r(t), \check{v}_s(t), \rho(t)$ are Lipschitzian *real-valued* functions, which together with small $\varepsilon>0$, will be specified in the course of the proof.

The terms at the right-hand side of (30), which contain $V_{rs}, \bar{V_{rs}}$, are to be classified as *non-resonant* and *resonant* with respect to the substitution (32). We call a term non-resonant if, after the substitution it results in a sum of fast-oscillating factors of the form $p(u,V_{rs},t)e^{i\beta t/\varepsilon},\ \beta\neq 0$, where $p(u,V_{rs},t)$ is polynomial in $u,\bar{u},V_{rs},\bar{V}_{rs}$, with coefficients Lipschitzian in t, independent of ε. Otherwise, when no factor $e^{i\beta t/\varepsilon}$ is present, the term is resonance. Crucial fact, which will be established below, is that *influence of non-resonant (fast-oscillating) terms to the end-point map can be made arbitrarily small*, when the frequency of the oscillating factor $e^{i\beta t/\varepsilon}$ is sufficiently large.

Direct verification shows that the terms

$$i(u^*)^2\bar{V_{rs}},\ 2i|u^*|^2V_{rs},\ V_{rs}^2\bar{u^*}$$

at the right-hand side of (30) are all non-resonant with respect to (32).

5.4.3. *Resonance monomials in the quadratic term $2u^*|V_{rs}|^2$: an obstruction.* Consider the quadratic term $2u^*|V_{rs}|^2$, which after the substitution (32) takes form

$$2u^*|v_{rs}|^2 = 2u^*\left(|\check{v}_r(t)|^2+|\check{v}_s(t)|^2\right)+4u^*\check{v}_r(t)\check{v}_s(t)\mathrm{Re}\left(e^{-it/\varepsilon}e^{i\rho(t)}f_r\bar{f}_s\right).$$

The last addend in the parenthesis is non-resonant, while the resonant term $2u^*(|\check{v}_r(t)|^2+|\check{v}_s(t)|^2)$ is an example of so-called *obstruction to controllability* in terminology of geometric control.

We can not annihilate or compensate this term but, as far as the group $e^{it\Delta}$ corresponding to linear Schroedinger equation is quasiperiodic, one can impose conditions on controls in such a way, that for a chosen $T>0$ the influence of the obstructing term onto time-T end-point map E_T will be nullified.

Indeed, proceeding with time-variant substitution:

$$u^\star = u^* e^{-2i\Upsilon(t)}, \ \Upsilon(t) = \int_0^t (|\check{v}_r(t)|^2 + |\check{v}_s(t)|^2) d\tau, \tag{33}$$

one gets for $u^\star$ the equality:

$$(-i\partial_t + \Delta)u^\star e^{2i\Upsilon(t)} = (-i\partial_t + \Delta)u^* - 2u^*(|\check{v}_r(t)|^2 + |\check{v}_s(t)|^2).$$

The equation (30) rewritten for $u^\star$ gets form

$$\begin{aligned}(-i\partial_t + \Delta)u^\star = |u^\star|^2 u^\star - i(u^\star)^2 \bar{V_{rs}} e^{2i\Upsilon(t)} + 2i|u^\star|^2 V_{rs} e^{-2i\Upsilon(t)} - \\ -V_{rs}^2 \bar{u^\star} e^{-4i\Upsilon(t)} + 4u^* 2\mathrm{Re}\left(e^{i(t/\varepsilon + \rho(t))} v_r(t)\bar{v}_s(t)\right) e^{-2i\Upsilon(t)} + \\ +\tilde{W}(t)e^{-2i\Upsilon(t)} + i|V_{rs}|^2 V_{rs} e^{-2i\Upsilon(t)}.\end{aligned} \tag{34}$$

For the sake of maintaining (for a given $T > 0$) the time-T end-point map E_T unchanged, additional *isoperimetric conditions* on $\check{v}_r(t), \check{v}_s(t)$

$$\int_0^T (|\check{v}_r(t)|^2 + |\check{v}_s(t)|^2) dt = \Upsilon(T) = \pi N, \ N \in \mathbb{Z}, \tag{35}$$

could be imposed. The equality would imply $u^\star(0) = u^*(0), \ u^\star(T) = u^*(T)$.

Remark 5.6. *Although right-hand side of (34) gained 'oscillating factors' of the form $e^{-bi\Upsilon(t)}$, the notion of resonant and resonant terms will not suffer changes, as long as $e^{-2i\Upsilon(t)}$ is not 'fast oscillating'; in further construction $\Upsilon(t)$ will be chosen bounded uniformly in t and b with bounds independent of $\varepsilon > 0$.* □

We introduce the notation $\tilde{\mathcal{N}}^\varepsilon(u,t)$ for the sum of non-resonant terms at the right-hand side of (34) getting

$$(-i\partial_t + \Delta)u^\star = |u^\star|^2 u^\star + \tilde{W}(t)e^{-2i\Upsilon(t)} + i|V_{rs}|^2 V_{rs} e^{-2i\Upsilon(t)} + \tilde{\mathcal{N}}^\varepsilon(u,t). \tag{36}$$

5.4.4. *Extending control via cubic resonance monomial.* Now we work with the cubic term

$$i|V_{rs}|^2 V_{rs} e^{-2i\Upsilon(t)} = i(v_r(t)f_r + v_s(t)f_s)^2(\bar{v}_r(t)\bar{f}_r + \bar{v}_s(t)\bar{f}_s)e^{-2i\Upsilon(t)}, \tag{37}$$

where $v_r(t), v_s(t), \Upsilon(t)$ are defined by (32).

Rewriting (37) as polynomial in $\check{v}_r(t), \check{v}_s(t)$ with time-variant coefficients we extract the only resonant monomial

$$e^{2i(\rho(t)-\Upsilon(t))}\check{v}_r^2(t)\check{v}_s(t)f_r^2\bar{f}_s, \tag{38}$$

and join all the non-resonant monomials to the term $\tilde{\mathcal{N}}^\varepsilon(u,t)$ in (36).

Recalling that $f_m = e^{i(m\cdot x + |m|^2)t}, \ m \in \mathbb{Z}^2$, we compute

$$f_r^2\bar{f}_s = e^{i((2r-s)\cdot x + (2|r|^2 - |s|^2)t)} = f_{2r-s}e^{i((2|r|^2 - |s|^2 - |2r-s|^2)t)} = f_{2r-s}e^{-2i|r-s|^2 t},$$

and rewrite (38) in the form

$$\check{v}_r^2(t)\bar{\check{v}}_s(t)e^{2i(\rho(t) - |r-s|^2 t - \Upsilon(t))}f_{2r-s},$$

which we will see as *extending control* for the mode f_{2r-s}.

The equation (36) can be represented as

$$(-i\partial_t + \Delta)u^\star = |u^\star|^2 u^\star + \tilde{W}(t)e^{-2i\Upsilon(t)} +$$
$$+\check{v}_r^2(t)\bar{\check{v}}_s(t)e^{2i(\rho(t)-|r-s|^2t-\Upsilon(t))}f_{2r-s} + \tilde{\mathcal{N}}^\varepsilon(u,t). \tag{39}$$

In Subsection 5.5 we will show that the influence of the fast-oscillating term $\tilde{\mathcal{N}}^\varepsilon(u,t)$ onto the end-point map can be made arbitrarily small by choice of small $\varepsilon > 0$. By now we will take care of other addends at the right-hand side of (39). We wish to choose families of functions $\tilde{W}(t;b), \check{v}_r(t;b), \check{v}_s(t;b)$ in such a way that

$$\tilde{W}(t)e^{-2i\Upsilon(t)} + \check{v}_r^2(t)\bar{\check{v}}_s(t)e^{2i(\rho(t)-|r-s|^2t-\Upsilon(t))}f_{2r-s}$$

approximates $W(t;b)$ in $\mathbf{L}^1_t$-metric uniformly in $b \in B$.

Get family of controls $\hat{W}(t;b) = \sum_{k\in\hat{\mathcal{K}}} w_k(t;b)f_k$, by truncating the summand $w_{2r-s}f_{2r-s}$ from $W(t;b)$ (see (25)). We put $\tilde{W}(t;b) = \hat{W}(t;b)e^{2i\Upsilon(t;b)}$.

The controls $\check{v}_r(t;b), \check{v}_s(t;b)$ will be constructed according to the

Lemma 5.4. *For a continuous in L^t_1-metric family of controls $b \mapsto w(t;b) \in \mathbf{L}^\infty[0,T]$, and each $\varepsilon' > 0$ one can construct continuous in L^t_1-metric families of real-valued functions*

$$b \mapsto \check{v}_r(t;b,\varepsilon'), b \mapsto \check{v}_s(t;b,\varepsilon'), \tag{40}$$

such that: i) they are Lipschitzian in t; ii) their partial derivatives in t depend on b continuously in L^t_1-metric; iii) for each b, ε' the conditions (31),(35) hold for them; iv) their L^t_2-norms are equibounded for all $\varepsilon' > 0, b \in B$; and v)

$$\|D^{\varepsilon'}_{rs}\|_{L^t_1} = \int_0^T \left|\check{v}_r^2(t;b,\varepsilon')\check{v}_s(t;b,\varepsilon') - |w_{2r-s}(t,b)|\right| dt \leq \varepsilon'. \tag{41}$$

uniformly in $b \in B$. □

The Lemma is proved in Appendix. Now we formulate a corollary, which defines the family $b \mapsto \rho(t;b,)$.

Corollary 5.5. *Given family (40), constructed in the Lemma, there exists a continuous in L^1_t-metric family of Lipschitzian functions $b \mapsto \rho(\cdot;b)$ for which*

$$\int_0^T \left|\check{v}_r^2(t;b,\varepsilon')\bar{\check{v}}_s(t;b,\varepsilon')e^{2i(\rho(t)-|r-s|^2t-\Upsilon(t))} - w_{2r-s}(t,b)\right| dt \leq \varepsilon'. \ \square \tag{42}$$

Recall that $\Upsilon(t)$ is defined by (33).

To prove the Corollary we choose

$$\rho(t;b) = \frac{1}{2}Arg\left(w_{2r-s}(t,b)\right) + |r-s|^2t + \Upsilon(t;b). \tag{43}$$

According to Remark 5.5 we may think that $w_{2r-s}(t,b)$ are smooth in t and hence $\rho(t;b)$ is Lipschitzian in t. Its dependence on b is continuous in $\mathbf{L}^1_t$-metric. By (41),(43) we conclude (42). □

Taking $\varepsilon' = \varepsilon$ and substituting the constructed controls $v_r, v_s, \tilde{W}$ into (36) we get the equation

$$(-i\partial_t + \Delta)u^\star = |u^\star|^2 u^\star + W(t) + D^{\varepsilon}_{rs}(t) + \tilde{\mathcal{N}}^{\varepsilon}(u^\star, t, b). \tag{44}$$

By construction the end-point maps $\tilde{E}_T$ and E_T of the systems (44) and (27) coincide on the set of controls, satisfying (31),(32),(35).

Lemma 5.6. *The end-point map $E^{\varepsilon}_T(b)$ of the system (44) calculated for the family of controls, defined by Proposition 5.4, tends to the end-point map E^{lim}_T of the 'limit system' (19) uniformly in b as $\varepsilon \to 0$.* $\square$

Would the term $\tilde{\mathcal{N}}^{\varepsilon}(u^\star, t, b)$ be missing in (44) we could derive Lemma 5.6 from Proposition 4.4. The passage to limit, as $\varepsilon \to 0$, in the presence of fast-oscillating $\tilde{\mathcal{N}}^{\varepsilon}(t, u)$ tends to 0, will be established in Proposition 5.7.

The proof of Approximative Lemma 5.1 is complete modulo proof of Lemmae 5.4,5.6 .

5.5. **On continuity of solutions in the right-hand side with respect to relaxation metric.** The results, we are going to present briefly in this Section, regard continuous dependence of solutions of NLS equation on the perturbations of its right-hand side, which are small in so-called *relaxation norm*. This norm is suitable for treating fast oscillating terms. In finite-dimensional context the continuity results are part of *theory of relaxed controls*. A number of relaxation results for semilinear systems in Banach spaces can be found in [8, 9]. Below we provide version adapted for our goal - proof of Lemma 5.6.

Consider semilinear equation (3) and its perturbation (4).

We assume the perturbations $\phi : [0, T] \times H \to H$ to belong to a family Φ. Elements of Φ are continuous; the family Φ is equibounded and equi-Lipschitzian meaning that each $\phi \in \Phi$ together with $G : [0, T] \times H \to H$ satisfy properties (5),(6), (7) with the same function $\beta_b(t)$.

Besides we admit *complete boundedness assumption*, which would guarantee the complete boundedness (*precompactness*) in H of the set $\{\phi(t, u(t)) |\ t \in [0, T], \phi \in \Phi\}$ for each choice of $u(\cdot) \in C([0, T], H)$. To get the property it suffices, for example, to assume *complete boundedness of the sets* $\Phi(t, u) = \{\phi(t, u) |\ \phi \in \Phi\}$ for each *fixed* couple (t, u) together with *upper semicontinuity of the set valued map* $(t, u) \mapsto \Phi(t, u)$.

We introduce *relaxation seminorm* $\|\cdot\|^{\mathrm{rx}}_b$ for the elements of Φ by the formula:

$$\|\phi\|^{\mathrm{rx}}_b = \sup_{t,t' \in [0,T], \|u\| \le b} \left\| \int_t^{t'} \phi(\tau, u) d\tau \right\|_H .$$

The seminorm is well adapted to the functions oscillating in time. The relaxation seminorms of fast-oscillating functions are small. For example $\|f(t)e^{it/\varepsilon}\|^{rx} \to 0$, as $\varepsilon \to 0$ for each function $f \in \mathbf{L}^1[0, T]$ (Lebesgue-Riemann lemma).

Now we formulate needed continuity result from which Lemma 5.6 will follow.

Proposition 5.7. *Let solution $\tilde{u}(t)$ of the NLS equation (3) exist on $[0,T]$, belong to $C([0,T],H)$ and satisfy $\sup_{t\in[0,T]}\|u(t)\|_H < b$. Let family Φ of perturbations satisfy the continuity, equiboundedness, equi-Lipschitzianness and complete boundedness assumption, introduced above. Then $\forall\varepsilon > 0 \exists\delta > 0$ such that whenever $\phi \in \Phi$, $\|\phi\|_b^{\mathrm{rx}} + \|u^0 - \tilde{u}^0\|_H < \delta$, then the solution $u(t)$ of the perturbed equation (4) exists on the interval $[0,T]$, is unique and satisfies the bound $\sup_{t\in[0,T]}\|u(t) - \tilde{u}(t)\|_H < \varepsilon$.* □

Sketch of the proof. Under the assumptions of the Proposition solutions of the equations (3),(4) exist locally and are unique (see Proposition 4.1). Global existence will follow from the bound on the $H^{1+\sigma}$-norm of the solution on $[0,T]$.

We start with the estimate (13) obtained in the course of the proof of Proposition 4.4:

$$\|u(t) - \tilde{u}(t)\| \leq \left(\|u^0 - \tilde{u}^0\| + \left\|\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right\|\right) C' e^{C\int_0^t \beta_b(\tau)d\tau}.$$

The conclusion of Proposition 5.7 will follow from

Lemma 5.8. *Let family Φ satisfy assumptions of the Proposition 5.7, and let $\tilde{u}(t)$ be solution of (3). Then $\forall\varepsilon > 0$, $\exists\delta > 0$ such that $\forall\phi \in \Phi$:*

$$\|\phi\|^{rx} < \delta \Rightarrow \left\|\int_0^t e^{-i\tau\Delta}\phi(\tau,\tilde{u}(\tau))d\tau\right\| < \varepsilon. \ \square$$

Proof of this Lemma can be found in Appendix. We finish by remark on validity of conditions of Proposition 5.7 for NLS.

Remark 5.7. *The nonlinear terms $\mathcal{N}^\varepsilon(u,t)$ at the right-hand side of (44) is Nemytskii-type operator of the form*

$$\mathcal{N}^\varepsilon(u,t) = W^0(t,x) + uW^{11}(t,x) + \bar{u}W^{12}(t,x) + u^2W^{21}(t,x) + |u|^2W^{22}(t,x),$$

where $W^{ij}(t,x)$ have form $w(t)e^{ik\cdot x}e^{i\rho(t)}e^{iat/\varepsilon}$, where $w(t),\rho(t)$ are Lipschitzian, $a > 0$. The Lipschitzian and boundedness properties are concluded by application of 'Product Lemma' cited in Section 4. Substituting the factors $e^{iat/\varepsilon}$ by $e^{i\theta}$ we see that for any continuous $\tilde{u}(t)$ the range of the function $\mathcal{N}^\varepsilon(u,t)$ is contained for all $\varepsilon > 0$ in the compact range of a continuous function of the variables $t \in [0,T]$, $\theta \in \mathbb{T}^1$.

6. Saturating sets of controlled modes and controllability

Starting with a set $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ and appealing to definition of elementary extension we define sequence of sets $\mathcal{K}^j \subset \mathbb{Z}^2$, $\mathcal{K}^1 = \hat{\mathcal{K}}$:

$$\mathcal{K}^j = \left\{2m - n|\ m,n \in \mathcal{K}^{j-1},\right\}; j = 2,\ldots,\ \mathcal{K}^\infty = \bigcup_{j=1}^{\infty}\mathcal{K}^j. \tag{45}$$

Taking $m = n$ in (45) we conclude that $\mathcal{K}^1 \subseteq \cdots \subseteq \mathcal{K}^j \subseteq \cdots \mathcal{K}^\infty$.

Definition 6.1. *A finite set $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ of modes is called saturating if $\mathcal{K}^\infty = \mathbb{Z}^2$.* $\square$

From Proposition 5.3 we conclude

Proposition 6.2. *Let set $\hat{\mathcal{K}}$ of controlled modes, involved in the source term (2), be saturating. Then for each $T > 0$ the controlled NLS equation (1)-(2) on $\mathbb{T}^2$ is time-T solidly controllable in each finite-dimensional component.* $\square$

As we will see in the next section controllability in each finite-dimensional component (in projection on each coordinate subspace) implies controllability in projection on each finite-dimensional subspace and also approximate controllability.

Corollary 6.3. *Let the set $\hat{\mathcal{K}}$ of controlled modes be saturating. Then for any $T > 0$ the controlled defocusing NLS equation (1)-(2) on $\mathbb{T}^2$ is time-T solidly controllable in each finite-dimensional projection and $H^{1+\sigma}$-approximately controllable.* $\square$

Now we introduce a class of saturating sets.

Proposition 6.4. *Let vectors $k, \ell \in \mathbb{Z}^2$ be such that $k \wedge \ell = \pm 1$. Then the set $\{\mathbf{0}, k, \ell, k+\ell\} \subset \mathbb{Z}^2$ is saturating.* $\square$

Proof. i) First note that if $z \in \mathcal{K}^\infty$, then $-z = 2 \cdot \mathbf{0} - z \in \mathcal{K}^\infty$.

We prove that $\mathcal{K}^\infty$ coincides with the set of all integer combinations $\mathcal{C} = \{\alpha k + \beta \ell |\ \alpha, \beta \in \mathbb{Z}\}$.

ii) The set $\mathcal{C}$ is obviously invariant with respect to the operation $(v, w) \mapsto v - 2w$. We will prove that $\mathcal{K}^\infty \supset \mathcal{C}$.

If $\pm z \in \mathcal{K}^\infty$, then by induction $z + 2\alpha k + 2\beta \ell \in \mathcal{K}^\infty$, $\forall \alpha, \beta \in \mathbb{Z}$. In particular

$$2\alpha k + 2\beta \ell \in \mathcal{K}^\infty, \ \forall \alpha, \beta \in \mathbb{Z} \text{ and } k + 2\alpha k + 2\beta \ell, \ell + 2\alpha k + 2\beta \ell \in \mathcal{K}^\infty.$$

Thus $\mathcal{K}^\infty$ contains all the combinations $mv + nw$ with at least one of the coefficients m, n even. Note that the set of such combinations is invariant with respect to the operation $(x, y) \mapsto 2x - y$ involved in (45) and $0, k, \ell$ all are "combinations" of this type.

iii) "Invoking" $k + \ell \in \hat{\mathcal{K}}$ we conclude by ii) that $\forall \alpha, \beta \in \mathbb{Z}$:

$$(2\alpha + 1)k + (2\beta + 1)\ell = (k + \ell) + 2\alpha k + 2\beta \ell \in \mathcal{K}^\infty.$$

2) Now we prove that whenever $k \wedge \ell = \pm 1$, then the set $\{\alpha k + \beta \ell |\ \alpha, \beta \in \mathbb{Z}\}$ coincides with $\mathbb{Z}^2$.

Assume $k \wedge \ell = 1$. Take any vector $y \in \mathbb{Z}^2$. Set $\alpha = y \wedge \ell, \ -\beta = y \wedge k$; obviously α, β are integer. We claim that $\alpha k + \beta \ell = y$.

By direct computation

$$(\alpha k + \beta \ell) \wedge \ell = \alpha (k \wedge \ell) = \alpha, \ (\alpha k + \beta \ell) \wedge k = \beta (\ell \wedge k) = -\beta.$$

Then $(y - (\alpha k + \beta \ell)) \wedge \ell = 0, \ (y - (\alpha k + \beta \ell)) \wedge k = 0$. As far as k, ℓ are linearly independent, we conclude $y - (\alpha k + \beta \ell) = 0$. $\square$

Corollary 6.5. *Let vectors $k, \ell \in \mathbb{Z}^2$ be such that $k \wedge \ell = \pm 1$ and the controlled source term (2) of the NLS equation (1) be of the form*

$$v_0(t) + v_k(t)e^{ik\cdot x} + v_\ell(t)e^{i\ell\cdot x} + v_{k+\ell}(t)e^{i(k+\ell)\cdot x}$$

Then for any $T > 0$ the NLS equation (1) is time-T controllable in each finite-dimensional projection and $H^{1+\sigma}$-approximately controllable. □

The space of controlled modes, introduced in Remark 2.2, satisfies hypothesi of the Corollary for $k = (1,0), \ell = (0,1)$.

7. Controllability proofs (Main result 1)

7.1. **Approximate controllability.** We have established that whenever set of controlled modes is saturating, then NLS is solidly controllable in projection on any finite-dimensional *coordinate* subspace. Using this fact we will now prove $H^{1+\sigma}$-approximate controllability and controllability in each finite-dimensional projection.

Let us fix $\tilde{\varphi}, \hat{\varphi} \in H_2$ and $\varepsilon > 0$ and assume that we want to steer the NLS equation from $\tilde{\varphi}$ to the ε-neighborhood of $\hat{\varphi}$ in H_2-metric.

Consider the Fourier expansions for $\tilde{\varphi}, \hat{\varphi}$ with respect to $e^{ik\cdot x},\ k \in \mathbb{Z}^2$. Denote by Π_N the projection of $\varphi \in H^{1+\sigma}$ onto the space of modes $e^{ik\cdot x},\ |k| \leq N$. Obviously $\Pi_N(\tilde{\varphi}) \to \tilde{\varphi}, \Pi_N(\hat{\varphi}) \to \hat{\varphi}$ in H_0 as $N \to \infty$.

Choose such N that the $H^{1+\sigma}$-norms of $\Pi_N^\perp(\tilde{\varphi}) = -\Pi_N(\tilde{\varphi}) + \tilde{\varphi}, \Pi_N^\perp(\hat{\varphi}) = -\Pi_N(\hat{\varphi}) + \hat{\varphi}$ are $\leq \varepsilon/4$.

By Lemma 5.2 there exists family of controls $W(b) = \sum_{\|k\| \leq N} w_k(t;b) f_k$ such that $\Pi_N(W(b))$ covers $\Pi_N(\hat{\varphi})$ solidly and besides $\|\Pi_N^\perp E_T(W(b)) - \Pi_N^\perp(\tilde{\varphi})\| \leq \varepsilon/4$. Then $\|\Pi_N^\perp E_T(W(b))\| \leq \varepsilon/2$.

If a set $\hat{\mathcal{K}}$ of controlled modes is saturating, then $\{k|\ |k| \leq N\}$ is proper extension of $\hat{\mathcal{K}}$. By Approximative Lemma 5.1 there exists family of controls $V(b) = \sum_{k \in \hat{\mathcal{K}}} v_k(t;b) f_k$ such that

$$\|E_T(V(b)) - E_T(W(b))\| \leq \varepsilon/4,\ \forall b \in B,$$

and $\Pi_N E_T(V(b))$ covers the point $\Pi_N(\hat{\varphi})$. Then $\forall b:\ \|\Pi_N^\perp E_T(V(b))\| \leq 3\varepsilon/4$ and for some $\hat{b}$: $\Pi_N E_T(V(\hat{b})) = \Pi_N \hat{\varphi}$. Then $\|E_T(V(\hat{b})) - \hat{\varphi}\| \leq \varepsilon$. □

7.2. **Controllability in finite-dimensional projections.** Let $\mathcal{L}$ be ℓ-dimensional subspace of $H^{1+\sigma}$ and $\Pi^{\mathcal{L}}$ be orthogonal projection of $H^{1+\sigma}$ onto $\mathcal{L}$.

First we construct a finite-dimensional *coordinate* subspace which is projected by $\Pi^{\mathcal{L}}$ onto $\mathcal{L}$. Moreover for each $\varepsilon > 0$ one can find a finite-dimensional *coordinate subspace* $\mathcal{L}^{\mathcal{S}}$ with its ℓ-dimensional (non-coordinate) subsubspace $\mathcal{L}_\varepsilon$, which is ε-close to $\mathcal{L}$. The latter means that not only $\Pi^{\mathcal{L}}\mathcal{L}_\varepsilon = \mathcal{L}$ but also the isomorphism $\Pi_\varepsilon = \Pi^{\mathcal{L}}|_{\mathcal{L}_\varepsilon}$ is ε-close to the identity operator. It is an easy linear-algebraic computation; which can be found in [1, Section 7].

Without lack of generality we may assume that $\|\Pi_S(\tilde{\varphi}) - \tilde{\varphi}\|_0 \leq \varepsilon$.

As far as the set $\hat{\mathcal{K}}$ of controlled modes is saturating, $\mathcal{S}$ is proper extension of $\hat{\mathcal{K}}$ and the system is solidly controllable in the observed component $q^{\mathcal{S}}$.

Let B be a ball in $\mathcal{L}$. Consider $B^\varepsilon = (\Pi_\varepsilon)^{-1}B$; obviously $B^\varepsilon \subset \mathcal{L}^\varepsilon \subset \mathcal{L}^{\mathcal{S}}$. We take a ball $B_{\mathcal{S}}$ in $\mathcal{L}^{\mathcal{S}}$, which contains B^ε and hence $\Pi_{\mathcal{L}}(B_{\mathcal{S}}) \supset B$.

Reasoning as in the previous Subsection one establishes existence of a family of controls $V(b) = \sum_{k\in\hat{\mathcal{K}}} v_k(t;b)f_k$ such that $\Pi_{\mathcal{S}}E_T(V(b))$ covers $B_{\mathcal{S}}$ solidly and $\forall b:\ \|\Pi_{\mathcal{S}}^{\perp}E_T(V(b))\| \leq 2\varepsilon$.

Then choosing $\varepsilon > 0$ sufficiently small we achieve that

$$\Pi^{\mathcal{L}}E_T(V(b)) = \Pi^{\mathcal{L}}\left(\Pi_{\mathcal{S}} + \Pi_{\mathcal{S}}^{\perp}\right)E_T(\hat{\mathcal{V}}(b))$$

covers B.

8. Lack of exact controllability proof (Main result 2)

Let us write cubic defocusing NLS equation (1)-(2) in the form

$$(-i\partial_t + \Delta)u = |u|^2u + \sum_{k\in\hat{\mathcal{K}}} \dot{w}_k(t)f_k,\ u|_{t=0} = u^0 \in H^{1+\sigma}, \tag{46}$$

where $f_k = e^{i(k\cdot x+|k|^2t)}$, $\hat{\mathcal{K}} \subset \mathbb{Z}^2$ is a finite set, $\sharp\hat{\mathcal{K}} = \kappa$. Controls $\dot{w}_k(t)$ are taken from $\mathbf{L}^1([0,T],\mathbb{C})$ and therefore are derivatives of absolutely continuous functions $w_k(t),\ w_k(0) = 0$. In this Section $\mathbf{W}^{1,1}([0,T],\mathbb{C}^\kappa)$ stays for the space of $\mathbb{C}^\kappa$-valued absolutely continuous functions, vanishing at $t = 0$.

Global existence and uniqueness results for solution of this equation in $C([0,T],H^{1+\sigma})$ is classical (Section 4).

Consider the end-point map $E_T : (\dot{w}_k(t)) \mapsto u|_{t=T}$ which maps the space of inputs $(\dot{w}_k(t)) \in \mathbf{L}^1([0,T],\mathbb{C}^\kappa)$ into the state space $H^{1+\sigma}$. The image of E_T is time-T attainable set of the controlled equation (46). We wish to prove that this set is contained in a countable union of compacts and in particular has a dense complement in $H^{1+\sigma}$.

Introducing $W(t,x) = \sum_{k\in\hat{\mathcal{K}}} w_k(t)f_k,$ we rewrite (see Subsection 5.4) the equation (46) as $(-i\partial_t + \Delta)(u - iW(t,x)) = |u|^2u$, and after time-variant substitution $u - iW(t,x) = u^*(t)$ in the form

$$(-i\partial_t + \Delta)u^* = |u^* + iW(t,x)|^2(u^* + iW(t,x)),\ u|_{t=0} = u^0, \tag{47}$$

which we look at as semilinear control system with the *input* $W(t)$. Obviously for each absolutely continuous $W(t) = (w_k(t)),\ k \in \hat{\mathcal{K}}$ solution of (47) exists and is unique on $[0,T]$.

Introduce input-trajectory map $E^* : W(\cdot) \mapsto u^*(\cdot)$ of (47). The following result is essentially a corollary of Proposition 4.4.

Lemma 8.1. *Input-trajectory map E^* is Lipschitzian on any ball $B_R = \{W(\cdot) \in \mathbf{W}^{1,1}([0,T],\mathbb{C}^\kappa)|\ \|W(\cdot)\|_{\mathbf{W}^{1,1}} \leq R\}$, endowed with $\mathbf{L}^1([0,T],\mathbb{C}^\kappa)$-metric, while the space of trajectories $u^*(\cdot)$ is endowed with $C([0,T],H^{1+\sigma})$-metric. In other words*

$$\exists L_R > 0:\ \|u_2^*(t) - u_1^*(t)\|_H \leq L_R\int_0^T \|W_2(t) - W_1(t)\|_{\mathbb{C}^\kappa}dt,\ \forall t \in [0,T],$$

$\forall W_1(\cdot), W_2(\cdot) \in B_R$ *and corresponding trajectories* $u_1^*(t), u_2^*(t)$ *of (47).* □

From Lemma 8.1, proved in Appendix, Main Result 2 can be deduced easily.

Consider composition of maps

$$(\dot{w}_k)_{k\in\hat{\mathcal{K}}} \mapsto W(\cdot) = (w_k)_{k\in\hat{\mathcal{K}}} \mapsto E_T^*(W) = E^*(W)|_{t=T};$$

E_T^* is the end-point map $W(\cdot) \mapsto u|_{t=T}$ for the equation (47).

The relation between the end-point maps of the controlled equations (46) and (47) results $E_T((\dot{w}) = E_T^*(W) + iW(T,x)$ and therefore the image of E_T (the attainable set) is contained in the image of the map

$$\Theta : (W(\cdot), \vartheta) \mapsto E_T^*(W(\cdot)) + \vartheta, \ (W(\cdot), \vartheta) \in \mathbf{W}^{1,1}([0,T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa.$$

Represent $\mathbf{L}^1([0,T], \mathbb{C}^\kappa)$ as a union of balls $\bigcup_{n\geq 1} B_n$ of radii $n \in \mathbb{N}$. The image of each B_n under the map $\mathcal{I} : \dot{w}(\cdot) \mapsto (w(\cdot), w(T))$ is bounded in $\mathbf{W}^{1,1}([0,T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa$. If one endows $\mathbf{W}^{1,1}([0,T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa$ with the metric of $\mathbf{L}^1([0,T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa$ then $\mathcal{I}(B_n)$ is pre-compact (and completely bounded) in this metric.

By Lemma 8.1 the map E_T^* is Lipschitzian in the metric of $\mathbf{L}^1([0,T], \mathbb{C}^\kappa)$; hence Θ is also Lipschitzian in the metric of $\mathbf{L}^1([0,T], \mathbb{C}^\kappa) \times \mathbb{C}^\kappa$ and therefore $E_T(B_n)$ is contained in completely bounded image $\Theta(\mathcal{I}(B_n)) \subset H^{1+\sigma}$. Hence the attainable set of (46) is contained in a countable union of pre-compacts $\bigcup_{n\geq 1} \Theta(\mathcal{I}(B_n))$ and by Baire category theorem has a dense complement in $H^{1+\sigma}$. □

9. Appendix: proofs of Lemmae 5.4, 5.8, 8.1

9.1. *Proof of Lemma 5.4.* First we choose $\check{v}_r^2(t)$ coinciding with real-valued nonnegative continuous piecewise-linear function, which vanishes at $\{0, T\}$, is constant and equal $\pi(T - \varepsilon^2)^{-1}$ on $[\varepsilon^2, T - \varepsilon^2]$ and is linear on $[0, \varepsilon^2]$ and $[T - \varepsilon^2, T]$. Evidently $\int_0^T \check{v}_r^2(t)dt = \pi$.

According to Remark 5.5 we may assume $w_{2r-s}(t,b), \partial_t w_{2r-s}(t,b)$ to be smooth in t and depend on b continuously in $\mathbf{L}_t^1$-metric. This implies that $\|w_{2r-s}(t,b)\|_{\mathbf{L}^\infty}$ are equibounded by $C_w > 0$.

Denote $\mathcal{I}_\varepsilon = [0, \varepsilon^2] \bigcup [T - \varepsilon^2, T]$ and $w^\varepsilon(t,b)$ the restrictions of $w_{2r-s}(t,b)$ onto the interval $[0,T] \setminus \mathcal{I}_\varepsilon$. Let

$$\int_{\varepsilon^2}^{T-\varepsilon^2} |w^\varepsilon(t,b)|^2 dt = A(b), \ A = \max_{b\in B} A(b);$$

the maximum is achieved. Put $N = [A/\pi] + 1$ and extend $|w^\varepsilon(t,b)|$ to a Lipschitzian function $\check{v}_s(t,b)$ on $[0,T]$ in such a way that $\check{v}_s(0,b) = \check{v}_s(T,b) =$

0 and $\int_0^T |\check{v}_s(t,b)|^2 dt = \pi N$.[2] Then

$$\int_0^T |\check{v}_r(t)|^2 + |\check{v}_s(t,b)|^2 dt = \pi(N+1).$$

Obviously $\check{v}_r^2(t;b)\check{v}_s(t;b) = |w_{2r-s}(t,b)|$ on $[\varepsilon^2, T-\varepsilon^2]$. Also

$$\int_{\mathcal{I}_\varepsilon} |\check{v}_s(t)|^2 dt \leq \pi N,$$

and by Cauchy-Schwarz inequality

$$\int_{\mathcal{I}_\varepsilon} |\check{v}_s(t;b)| dt \leq \varepsilon\sqrt{2\pi N}.$$

Then

$$\int_0^T \left|\check{v}_r^2(t)\check{v}_s(t,b) - |w_{2r-s}(t)|\right| dt =$$

$$= \int_{\mathcal{I}_\varepsilon} \left(\left|\check{v}_r^2(t)\check{v}_s(t,b)\right| + |w_{2r-s}(t;b)|\right) dt \leq \|\check{v}_r^2(t)\|_{\mathbf{L}^\infty}\varepsilon\sqrt{2\pi N} + 2C_w\varepsilon^2. \ \square$$

9.2. **Proof of Lemma 5.8.** . Given that $\tilde{u}(t)$ is continuous and ϕ possesses Lipschitzian property, we can conclude that $\forall\delta > 0\ \exists\delta' > 0$ such that $\forall\phi \in \Phi$:

$$\sup_{t,t'\in[0,T],\|u\|\leq b} \left\| \int_t^{t'} \phi(\tau,u)d\tau \right\| < \delta' \Rightarrow \sup_{t,t'\in[0,T]} \left\| \int_t^{t'} \phi(\tau,\tilde{u}(\tau))d\tau \right\| < \delta. \tag{48}$$

Indeed (compare with [10, Chap.4]) if $\omega(\tau)$ is modulus of continuity for $\tilde{u}(t)$ and $\sup_{t\in[0,T]} \|\tilde{u}(t)\| \leq b$, then

$$\sup_{t,t'\in[0,T]} \left\| \int_t^{t'} \phi(\tau,\tilde{u}(\tau))d\tau \right\| = \left\| \int_{\underline{t}}^{\underline{t}'} \phi(\tau,\tilde{u}(\tau))d\tau \right\| \leq \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \phi(\tau,\tilde{u}(\tau))d\tau \right\|,$$

where $\underline{t} = t_0 < t_1 < \cdots < t_N = \underline{t}'$ is a partition of $[\underline{t}, \underline{t}'] \subset [0,T]$ into $N \leq T/\eta$ subintervals of length η. Then

$$\sup_{t,t'\in[0,T]} \left\| \int_t^{t'} \phi(\tau,\tilde{u}(\tau))d\tau \right\| \leq \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \left(\phi(\tau,\tilde{u}(\tau)) - \phi(\tau,\tilde{u}(t_j))\right) d\tau \right\| +$$

$$+ \sum_{j=0}^{N-1} \left\| \int_{t_j}^{t_{j+1}} \phi(\tau,\tilde{u}(t_j))d\tau \right\| \leq \sum_{j=0}^{N-1} \int_{t_j}^{t_{j+1}} \beta_b(\tau)\|\tilde{u}(\tau)) - \tilde{u}(t_j)\|d\tau + N\|\phi\|_{rx} \leq$$

$$\leq C\omega(\eta) + \frac{T}{\eta}\|\phi\|_{rx}.$$

---

[2] One can take for example $\check{v}_s(t,b) = \varepsilon^{-2}\sqrt{a_1(b)t + a_2(b)\varepsilon^{-2}t^2}$ on $[0,\varepsilon^2]$. Parameters $a_1(b), a_2(b)$ can be chosen continuously depending on b. Similar construction can be arranged for the interval $[T-\varepsilon^2, T]$.

Choosing $\eta = \|\phi\|_{rx}^{1/2}$ we get

$$\sup_{t,t' \in [0,T]} \left\| \int\limits_t^{t'} \phi(\tau, \tilde{u}(\tau)) d\tau \right\| \leq C\omega(\|\phi\|_{rx}^{1/2}) + T\|\phi\|_{rx}^{1/2}$$

and conclude (48).

Introduce

$$\tilde{\Phi} = \{\phi(\tau, \tilde{u}(\tau)), \ \phi \in \Phi\}.$$

According to the aforesaid it suffices to prove the assertion

$$\varphi \in \tilde{\Phi} \bigwedge \|\varphi\|^{rx} < \delta \Rightarrow \left\| \int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau \right\| < \varepsilon. \tag{49}$$

The set $R = \{\varphi(\tau) | \ \tau \in [0,T], \varphi \in \tilde{\Phi}\}$ is completely bounded by assumption.

Taking an orthonormal basis $h_1, h_2, \ldots, h_n, \ldots$ in H and denoting by Π_n the orthogonal projection onto $\mathrm{Span}\{h_1, \ldots, h_n\}$, we conclude by standard compactness criterion that $\sup_{x \in P} \|x - \Pi_n x\| \to 0$, as $n \to \infty$.

Take a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ of the interval $[0,T]$ into subintervals of lengths $\eta = T/N$. We represent the integral in (49) as a sum

$$\int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau = \int_0^t e^{-i\tau\Delta} \left(\varphi(\tau) - \Pi_n \varphi(\tau)\right) d\tau +$$

$$+ \sum_{j=1}^{\omega} e^{-i\tau_j\Delta} \int_{\tau_{j-1}}^{\tau_j} \Pi_n \varphi(\tau) d\tau + \sum_{j=0}^{N-1} \int_{\tau_{j-1}}^{\tau_j} e^{-i\tau_j\Delta} \left(e^{-i(\tau-\tau_j)\Delta} - I\right) \Pi_n \varphi(\tau) d\tau.$$

Recalling that:

- **i:** $e^{-i\tau\Delta}$ is an isometry of H;
- **ii:** $\left\| \int_{\tau_{j-1}}^{\tau_j} \Pi_n \varphi(\tau) d\tau \right\| \leq \|\varphi\|^{rx}$;
- **iii:** $\| \left(\varphi(\tau) - \Pi_n \varphi(\tau)\right) \| \leq \rho_n, \ \rho_n \stackrel{n\to\infty}{\longrightarrow} 0$ uniformly for $\varphi \in \tilde{\Phi}, \ \tau \in [0,T]$;
- **iv:** $\sup_{0 \leq \xi \leq \tau} \| \left(e^{-i\xi\Delta} - I\right) \circ \Pi_n \| = \gamma_n(\tau), \ \forall n : \ \lim_{\tau \to 0} \gamma_n(\tau) = 0$,

we conclude

$$\left\| \int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau \right\| \leq T\rho_n + T\eta^{-1} \|\varphi\|^{rx} + \gamma_n(\eta) \int_0^T \|\varphi(\tau)\| d\tau. \tag{50}$$

Recall that $\int_0^T \|\varphi(\tau)\| d\tau$ are bounded by a constant c_1 for all $\varphi \in \tilde{\Phi}$.

Taking n large enough so that $T\rho_n < \varepsilon/3$, we then choose $\eta > 0$ small enough so that $c_1\gamma_n(\eta) < \varepsilon/3$. If we impose $\|\varphi\|^{rx} < \varepsilon\eta/3T$, then (50) will imply $\left\| \int_0^t e^{-i\tau\Delta} \varphi(\tau) d\tau \right\| < \varepsilon$. $\Box$

9.3. **Proof of Lemma 8.1.** By the inequalities (13)-(14) we get

$$\|u_2^*(t)-u_1^*(t)\|_H \leq C\int_0^T \|\Phi_{12}(\tau,u_1^*(\tau))\|_H d\tau e^{C'\int_0^T \beta_b(\tau)d\tau},$$

where $\Phi_{12}(\tau,u)$ and $\beta_b(t)$ are defined by

$$\Phi_{12}(\tau,u)=|u+iW_1(t,x)|^2(u+iW_1(t,x))-|u+iW_2(t,x)|^2(u+iW_2(t,x)),$$

$$\||u'+iW(t,x)|^2(u'+iW(t,x))-|u+iW(t,x)|^2(u+iW(t,x))\|_H \leq \tag{51}$$
$$\leq \beta_b(t)\|u'-u\|_H, \ \forall W(\cdot)\in B_R, \ \|u\|_H \leq b.$$

What regards $\beta_b(t)$, then by Product Lemma 4.2 the left-hand side of (51) is bounded from above by $C(1+b^2+\|W(t)\|_H^2)$. Hence β_b can be chosen constant, equal to $C'(1+b^2+R^2)$, as far as $W(t,x)$ are trigonometric polynomials in x with t-variant coefficients equibounded in $\mathbf{W}^{1,1}[0,T]$.

Similarly

$$\|\Phi_{12}(\tau,u_1^*(\tau))\|_H \leq C_1(1+b^2+R^2)\|W_2(\tau)-W_1(\tau)\|_{\mathbb{C}^\kappa}.$$

Then for $L_R = CC_1(1+b^2+R^2)e^{C'(1+b^2+R^2)T}$:

$$\|u_2^*(t)-u_1^*(t)\|_H \leq L_R\int_0^T \|W_2(\tau)-W_1(\tau)\|d\tau. \ \square$$

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DiMaD, University of Florence, Italy
E-mail address: asarychev@unifi.it