Ambiguity made easier

M. Del Vigna*

April 22, 2011

Abstract

In this paper we review some well-known simple models for portfolio selection under Knightian uncertainty, also known as ambiguity, and we compute a number of explicit optimal portfolio rules using elementary mathematical tools. In the case of a single period financial market, new results arise for an agent who is risk neutral and smoothly ambiguity averse, for a loss averse and smoothly ambiguity averse agent, for a Mean-Variance and α-Maxmin Expected Utility agent. In a continuous time setting, we are able to recover some existing results on optimal investment strategies employing trivial stochastic analysis and avoiding the complicated BSDE machinery.

Keywords: Knightian uncertainty, Maxmin Expected Utility, smooth ambiguity aversion, loss aversion.

AMS classification (2000): 91B06, 91G10.

JEL classification: D81, G11.

1 Introduction

Since the early works of F. Knight in the 1920s and the subsequent analysis of D. Ellsberg in 1961, [7], the role of ambiguity in economic modeling has become more and more important. While risk and attitude toward uncertainty[1] constitute two well-known and studied phenomena, both from a theoretical and an empirical point of view, ambiguity seems to be somewhat fleeting. In fact, everyone is able to recognize uncertainty in everyday economic decisions and any human being is naturally endowed with and/or develops a personal "strategy" to face risky problems. On the contrary, it is more subtle to think of a plurality of possible ways of risk disclosure, which amounts to to so-called Knightian uncertainty.

As a matter of fact, the introduction of ambiguity in financial modeling is in favour of a more realistic description of the actual market conditions. It is hard to think of a decision maker who is perfectly aware of the risks she is going to bear, as it was supposed in the classical Markowitz and Merton's portfolio problems and in their close successors. However, introducing a set of plausible measures governing risk also increases the computational complexity and it often leads to the lack of closed form solutions. Our work simply consists in removing this complexity as much as possible.

There exists a variety of different ways to model ambiguity and ambiguity aversion. A first axiomatic approach was proposed in 1989 by Gilboa and Schmeidler [12] and it is widely known

*Université Paris-Dauphine & Università degli Studi di Pisa, matteo.delvigna@for.unipi.it.

[1] In our language, risk and uncertainty are the same thing and they denote the randomness linked to a particular phenomenon; on the contrary, ambiguity and Knightian uncertainty are synonyms and they refer to the ignorance of the probability law which governs that phenomenon.

as Maxmin Expected Utility (MEU for short). This naming is easily explained because the decision maker's objective function (to be maximized) is the minimum of a set of expected utilities computed with respect to several priors (probability distributions). Loosely speaking, the investor chooses her optimal investment strategy only once the extreme pessimistic point of view has been taken. Not only, this approach does not allow a distinction between ambiguity and ambiguity attitude.

To circumvent this flaw, a slight modification of the MEU has been implemented in 2004 by Ghirardato, Maccheroni and Marinacci. Their α-MEU paradigm states that the objective function is a convex combination of the minimum (pessimistic) and the maximum (optimistic) of a set of expected utilities, with weights α and $1-\alpha$ respectively. In this way, ambiguity is modeled only by the specification of the set of priors, whereas the magnitude of the parameter α reflects the attitude towards ambiguity. The cases $\alpha = 1$ and $\alpha = 0$ reduce to extreme pessimism and extreme optimism respectively; the midway $\alpha = 1/2$ stands for ambiguity neutrality.

An even more sophisticated approach is the one developed by Klibanoff, Marinacci and Mukerji in 2005, [17], and it is introduced as a *smooth* model for ambiguity. This paradigm too allows a separation between Knightian uncertain and the attitude toward it. While the first one is still identified by a particular set of priors, the second one is specified by a sufficiently regular (smooth) distortion function Φ of the expected utilities. By considering a distribution q over the set of priors, we can summarize the goal of our decision maker as the maximization of

$$\mathbb{E}_q\left[\Phi\left(\mathbb{E}_{\mathbb{P}}[U(\tilde{W})]\right)\right],$$

where $\tilde{W}$ is the random terminal wealth and U is her utility function; see equation (2.2) for a more detailed explanation. In particular, it is not difficult to recognize a nested von Neumann-Morgestern expected utility in a mathematical *expected ambiguity*, where Φ acts similarly to U. Hence, it is natural to interpret a concave (convex) Φ as aversion (desire) toward ambiguity, where the linear case reduces to the standard expected utility paradigm. Not only, a concave Φ reflects aversion to mean-preserving spread in expected utility values.

There is an extensive literature on ambiguity in financial market models and investment problems which employs empirical research and/or numerical methods to compute the optimal solution; see [1], [21], [10], [16] and [5] only to mention a few. Early research on a stylized portfolio selection problem is [6]; theoretical developments can be found in [9] for a discrete time framework and in [4] for a continuous time setting. Moreover, Taboga [19] provides a mathematically elegant version of portfolio optimization under the smooth ambiguity model and Gollier [13] investigates the comparative statics of ambiguity aversion under the same model. The MEU model with a finite set of priors is analyzed in [18]. In [8], Epstein and Miao study a dynamical equilibrium model involving heterogeneous agents in a complete and ambiguous market; finally, in the recent years ambiguity and asset pricing has also been study experimentally in a laboratory environment; see [3].

The paper is organized as follows. In Section 2 we study a single period investment problem under smooth ambiguity aversion. Section 3 is devoted to an analogous problem with Mean-Variance risk averse agent who follows the α-MEU paradigm. In Section 4 we analyze a continuous time market with a MEU investor. Finally, Section 5 concludes.

2 Smooth ambiguity in a single period market

We consider the classical portfolio selection model in a stylized, frictionless, single period financial market, where the investment opportunity set is made up by a risk-free asset (bond) which yields a constant exogenous return r_f and a risky asset (stock) with stochastic return described by the random variable[2] $\tilde{r}$. To be more precise, one can think of the risky asset as of an index-linked derivative or some other representative contingent claim.

Let $W_0 \geq 0$ be the initial wealth at disposal of our agent. Then, if she decides to invest w units of currency in the risky asset, she would get the random terminal wealth

$$\tilde{W} := W_0 r_f + w(\tilde{r} - r_f). \tag{2.1}$$

As is usual in decision making models under uncertainty, the preferences of the agent are represented by a utility function $U : \mathbb{R} \to \mathbb{R}$, which is supposed to be sufficiently regular and increasing. Applying the standard von Neumann-Morgenstern expected utility theory, if the investor believes that $\mathbb{P}$ is the prior that correctly describes the stock returns, then she evaluates her final position as $\mathbb{E}_{\mathbb{P}}[U(\tilde{W})]$, where $\mathbb{E}_{\mathbb{P}}$ denotes the expectation w.r.t. $\mathbb{P}$.

Now, we model the ambiguity concerning the financial market by allowing for a non-unique prior (probability measure) governing the risky asset's return. A mathematically sound but sophisticated approach for such a framework can be found in [19]; in our case, it is sufficient to assume the existence of a set $\mathcal{Q}$ of probability measures, each describing a particular distribution for $\tilde{r}$. Economically speaking, the investor does not know the effective underlying measure which describes the uncertainty linked to the risky asset. Therefore, she has to consider the whole set $\mathcal{Q}$ when selecting her optimal portfolio. Note that $\mathcal{Q}$ can be a finite, a countable or even an uncountable set. For instance, Gollier [13] assumes a continuum set of probability measures in $\mathcal{Q}$ which follow a Gaussian distribution; this *ad hoc* hypothesis allows to obtain an explicit solution to a single period investment problem.

On the other hand, we separately model the ambiguity attitude of our representative trader assuming that she behaves as predicted by the *smooth ambiguity* paradigm, as axiomatized in [17]. Mathematically speaking, we suppose that our agent is endowed with a sufficiently regular, increasing and strictly concave function $\Phi : \mathbb{R} \to \mathbb{R}$ which describes her ambiguity aversion by distorting expected utilities. Now, we assume a finite set of priors $\mathcal{Q} = \{\mathbb{Q}_j, j = 1, \ldots, J\}$ and a measure q over $\mathcal{Q}$ identified by a vector $(q_1, \ldots, q_J)$ of weights, where q_j represents the likelihood assigned to the j-th prior by our agent, $q_j > 0$, $q_1 + \ldots + q_J = 1$. If we denote by $\mathbb{E}_j$ the expectation w.r.t. the probability $\mathbb{Q}_j$, then the objective function of the investor is given by

$$\max_{w \in \mathbb{R}} F(w) \equiv \mathbb{E}_q[\Phi(\mathbb{E}[U(\tilde{W})])] \equiv \sum_{j=1}^{J} q_j \Phi\left(\mathbb{E}_j[U(\tilde{W})]\right). \tag{2.2}$$

Using equation (2.2) and assuming U, Φ being twice continuously differentiable, it is not difficult to derive necessary and sufficient optimality conditions on w. Due to the complexity of the first order conditions, the literature has concentrated on numerical simulations and there is still a lack of clear explicit results. In order to fill this gap, we are going to impose additional (and sometimes strong) hypotheses that allow us to obtain explicit formula for the optimal risky position, also getting simple and intuitive economic explanations.

To begin, we set $U = id$ unless otherwise stated; this will reduce our agent to a risk neutral one and will highlight the implications of different ambiguity attitudes. Obviously, such an investor only cares about the expected return of the risky asset, thus ignoring any other relevant feature about the shape of the distribution function of $\tilde{r}$ under $\mathbb{Q}_j$. We set $\mu = (\mu_1, \ldots, \mu_J)$, where we define $\mu_j := \mathbb{E}_j[\tilde{r}] - r_f$; moreover, without loss of generality we assume $\mu_1 < \ldots < \mu_J$; that is, μ_j is the expected excess return if $\mathbb{Q}_j$ is the relevant prior.

Proposition 2.1. *A necessary condition to have a unique optimal solution for* (2.2) *is* $\mu_j < 0 < \mu_{j+1}$ *for some* j*; otherwise, Problem* (2.2) *is ill-posed.*

[2] We do not specify here the underlying probability space as we want to keep our presentation as simple as possible; for our purposes, the only relevant feature about $\tilde{r}$ is its distribution function.

Proof. Using (2.1), it is immediate to see that the optimization problem reduces to

$$\max_{w\in\mathbb{R}} F(w) = \sum_{j=1}^{J} q_j \Phi(W_0 r_f + w\mu_j).$$

First and second order optimality conditions for $w = w^*$ become

$$F'(w^*) = \sum_{j=1}^{J} q_j \Phi'(W_0 r_f + w^*\mu_j)\mu_j = 0,$$
$$F''(w^*) = \sum_{j=1}^{J} q_j \Phi''(W_0 r_f + w^*\mu_j)\mu_j^2 < 0.$$

Thanks to the concavity of Φ, F is a strictly concave function on $\mathbb{R}$, hence it possesses at most a strict unique maximum. However, if $\mu_j < 0 \ \forall\ j = 1,\dots,J$, then $F'(w) < 0 \ \forall\ w \in \mathbb{R}$ and the optimal strategy would be an infinite short selling. A similar analysis goes for $\mu_j > 0 \ \forall\ j$, leading to an infinite risky investment. □

Example 2.1 (Quadratic smooth ambiguity)**.** Let $\Phi(x) = x - \frac{b}{2}x^2$, $b > 0$; obviously, we have to restrict our attention to the subset $0 \le x \le \frac{1}{b}$. In such a case, it is easy to compute

$$w^* = \frac{1 - bW_0 r_f}{b} \frac{\mathbb{E}_q[\mu]}{\mathbb{E}_q[\mu^2]}, \tag{2.3}$$

where we set $\mu^2 = (\mu_1^2,\dots,\mu_J^2)$ with an abuse of notation. From (2.3), it is immediate to recognize a kind of Sharpe ratio which depends on the distribution over the set of priors $\mathcal{Q}$. This similarity led Chen and Epstein [4] to the naming *market price of ambiguity*[3] for the ratio $\mathbb{E}_q[\mu]/\mathbb{E}_q[\mu^2]$; for more details, see Section 4. We also remark that this example can be easily generalized to the case of a continuum set of priors.

Example 2.2 (Constant Absolute Ambiguity Aversion)**.** Let $\Phi(x) = -e^{-\gamma x}$, $\gamma > 0$; similarly to the Constant Absolute Risk Aversion (CARA) utility function, Gollier [13] defines such a smooth ambiguity function as a CAAA function. Unfortunately, explicit computations are available only with additional assumptions over the set of priors $\mathcal{Q}$. In particular, we suppose that our investor is only aware of two different possible measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ with positive probabilities q_1 and $1-q_1$ respectively. An economic interpretation for this setting is to think about a decision maker who receives investment advices from two distinct financial experts; these two experts' suggestions are clearly different and our agent assigns them particular likelihoods. Moreover, we recall that we must have $\mu_1 < 0 < \mu_2$ to obtain a sensible solution; this in turn can be interpreted as a bearish (bullish) market forecast by the first (second) advisor. First order conditions lead to

$$w^* = \frac{1}{\gamma(\mu_2 - \mu_1)} \log\left(-\frac{1-q_1}{q_1}\frac{\mu_2}{\mu_1}\right), \tag{2.4}$$

from which we deduce $w^* \gtreqless 0$ if and only if $\mathbb{E}_q[\mu] \gtreqless 0$, i.e. it only depends on the sign of the market price of ambiguity.

Example 2.3 (Constant Relative Ambiguity Aversion)**.** Let $\Phi(x) = \frac{x^\gamma}{\gamma}$, $0 < \gamma < 1$; analogously to the previous terminology, we now have a CRAA function. Note that Φ is correctly defined

[3]In the language of Chen and Epstein [4], uncertainty is a synonym to ambiguity.

only over $[0,+\infty)$; hence, we arbitrarily set $\Phi(x) := -\infty$ for $x < 0$. Once again we assume the existence of two priors with positive probabilities q_1 and $1-q_1$ respectively. Performing standard computations, we obtain

$$w^* = W_0 r_f \frac{1 - \left(-\frac{1-q_1}{q_1}\frac{\mu_2}{\mu_1}\right)^{1/(\gamma-1)}}{\mu_2\left(-\frac{1-q_1}{q_1}\frac{\mu_2}{\mu_1}\right)^{1/(\gamma-1)} - \mu_1}, \tag{2.5}$$

and we see that $w^* \gtreqless 0$ if and only if $\mathbb{E}_q[\mu] \gtreqless 0$. Not only, one can easily check that we obtained a non negative terminal wealth $\tilde{W}$ under both priors.

Example 2.4 (Ambiguity and loss aversion)**.** We now remove the assumption of a risk neutral investor and we suppose to model a loss averse decision maker. There is strong empirical evidence that supports loss aversion among financial investors; see for example [15]. Loosely speaking, our agent reacts differently towards gains and losses which are defined w.r.t. a reference wealth level $B \in \mathbb{R}$. In particular, once such a benchmark B is fixed, any final reward greater than B is considered as a gain by our agent; on the contrary, getting less than B would mean a loss. Not only, to her a loss is more painful than a gain with the same magnitude. An easy way to mathematically frame loss aversion is to consider the piecewise linear utility (or value) function

$$U(x) = \begin{cases} x - B & \text{if } x \geq B, \\ \lambda(x-B) & \text{if } x < B, \end{cases} \tag{2.6}$$

with $\lambda > 1$. After that, a critical step consists in choosing the proper reference level B. In our simple scenario, $B = W_0 r_f$ seems to be a correct choice; in fact, this would correspond to a situation where a full safe investment represents the benchmark. At this point, it is easy to see that the objective function of our trader becomes

$$F(w) = \sum_{j=1}^{J} q_j \Phi\left(w[\mu_j^+ + \lambda\mu_j^-]\right), \tag{2.7}$$

where μ_j^+ and μ_j^- are the upper and the lower partial first moments of the random variable $\tilde{r} - r_f$ respectively, that is to say

$$\mu_j^+ := \mathbb{E}_j[(\tilde{r} - r_f) I_{\tilde{r}-r_f \geq 0}], \qquad \mu_j^- := \mathbb{E}_j[(\tilde{r} - r_f) I_{\tilde{r}-r_f < 0}].$$

Setting $\nu_j := \mu_j^+ + \lambda\mu_j^-$, it is straightforward to obtain the first order condition

$$\sum_{j=1}^{J} q_j \Phi'\left(w^*\nu_j\right)\nu_j = 0, \tag{2.8}$$

from which we see the close analogy with the previously models and we deduce an optimality result similar to Proposition 2.1. However, it is important to stress that now ν_j also depends on the preferences of the investor via the loss aversion coefficient λ; hence, the ill posedness can have an additional source.

Before concluding this section, we note that adding constraints that model the risk aversion to our optimization problem does not influence the attitude toward ambiguity. Just as an example, employing the standard Roy's Safety First Principle as in [14] or a VAR constraint as in [2] only restricts the set of admissible portfolios and keeps our analysis unaffected.

3 α-Maxmin Expected Utility in a single period market

We now change the way to model the ambiguity of our financial market and the ambiguity aversion of our representative investor. We take the α-MEU model as starting point; this paradigm has been first axiomatized in [11] and it is a generalization of the classical MEU model by Gilboa and Schmeidler [12]. Assuming the existence of a set of priors $\mathcal{Q}$, the MEU paradigm represents ambiguity aversion via a pessimistic choice principle, that is to say that our agent solves the problem

$$\max_{w\in\mathbb{R}} \min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[U(\tilde{W})]. \tag{3.1}$$

On the contrary, the α-MEU model assumes that the decision maker weights with a constant $\alpha \in [0,1]$ a pessimistic and an optimistic scenario; in other words, her goal becomes

$$\max_{w\in\mathbb{R}} \left\{ \alpha \min_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[U(\tilde{W})] + (1-\alpha) \max_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[U(\tilde{W})] \right\}, \tag{3.2}$$

where $\alpha = 1$ reduces the previous equation to the MEU case. It is also easy to see that if $\alpha = 0$, then the investor is taking into consideration only an optimistic point of view, whereas for $\alpha = 1/2$ we recover an ambiguity neutral case. Finally, for $\alpha > 1/2$ we observe ambiguity aversion, while $\alpha < 1/2$ stands for ambiguity loving.

We now assume the presence of a mean-variance expected utility maximizer in a single period portfolio optimization problem. Moreover, we suppose that the investor receives advices from two experts as in Example 2.2. In particular, expert j suggests an expected excess return μ_j and a standard deviation $\sigma_j > 0$ attached to the risky asset return. To fix ideas, we set $0 < \mu_1 < \mu_2$ and $0 < \sigma_1 < \sigma_2$. Note that a reversion in the standard volatility parameters implies mean-variance dominance and makes our problem a trivial one. Therefore, the α-MEU optimization problem is

$$\max_{w\in\mathbb{R}} f(w) \equiv \left\{ \alpha(f_1 \wedge f_2)(w) + (1-\alpha)(f_1 \vee f_2)(w) \right\}, \tag{3.3}$$

where we define

$$f_1(w) := \mu_1 w - \frac{\gamma}{2}\sigma_1^2 w^2, \qquad f_2(w) := \mu_2 w - \frac{\gamma}{2}\sigma_2^2 w^2 \tag{3.4}$$

for $\gamma > 0$. Note that Problem (3.3) has already been solved in [18] in the case of J different multiple priors but with $\alpha = 1$, i.e. in the MEU paradigm. Nonetheless, Problem (3.3) has a trivial solution for $\alpha = 0$ with J priors, as it is sufficient to select the w^* which attains the maximum among the vertex of J distinct parabolas. Finally, for $\alpha = 1/2$ and two priors it is immediate to see that we have $f \equiv \frac{1}{2}(f_1 + f_2)$ and we just have to maximize a concave parabola. Specifically, one would obtain $w^* = \frac{1}{\gamma}\frac{\mu_1+\mu_2}{\sigma_1^2+\sigma_2^2}$ as expected. Hence, it remains to analyze the open set $\alpha \in (0,1)$, $\alpha \neq 1/2$.

To begin we observe that $f_1(0) = f_2(0) = 0$ and $0 < f_1'(0) < f_2'(0)$. Hence, choosing a $w < 0$ will never be optimal for our decision maker. Moreover, f_1 and f_2 cross twice in 0 and $w_C := \frac{2}{\gamma}\frac{\mu_2-\mu_1}{\sigma_2^2-\sigma_1^2} > 0$ respectively. Thus, the maximum of f can be found analyzing it over the open intervals $(0, w_C)$ and $(w_C, +\infty)$ and finally comparing the respective maximal values with $f(w_C)$. In particular, we see that

- on $(0, w_C)$ we have $f_2 > f_1$ and the maximum of f is attained at $w_L := \frac{1}{\gamma}\frac{\alpha\mu_1+(1-\alpha)\mu_2}{\alpha\sigma_1^2+(1-\alpha)\sigma_2^2}$ if and only if $\alpha < \alpha_L := \frac{\mu_2(\sigma_1^2+\sigma_2^2)-2\mu_1\sigma_2^2}{(\mu_2-\mu_1)(\sigma_2^2-\sigma_1^2)}$; moreover, in such a case we have $f(w_L) > f(w_C)$;
- on $(w_C, +\infty)$ we have $f_1 > f_2$ and the maximum of f is attained at $w_R := \frac{1}{\gamma}\frac{\alpha\mu_2+(1-\alpha)\mu_1}{\alpha\sigma_2^2+(1-\alpha)\sigma_1^2}$ if and only if $\alpha < \alpha_R := \frac{\mu_1(\sigma_1^2+\sigma_2^2)-2\mu_2\sigma_1^2}{(\mu_2-\mu_1)(\sigma_2^2-\sigma_1^2)}$; moreover, this time we have $f(w_R) > f(w_C)$.

We summarize the overall analysis in the following result.

Proposition 3.1. *In the α-MEU model, with the previous assumptions and using the previous notations, we have*

- *if $\alpha \geq (\alpha_L \vee \alpha_R)$, then $w^* = w_C$;*
- *if $\alpha \in (\alpha_L, \alpha_R)$, then $w^* = w_R$;*
- *if $\alpha \in (\alpha_R, \alpha_L)$, then $w^* = w_L$;*
- *if $\alpha \leq (\alpha_L \wedge \alpha_R)$, then w^* is obtained from the comparison of $f(w_L)$ and $f(w_R)$.*

Before concluding, we note that $\alpha_L + \alpha_R \equiv 1$ and they can assume both positive and negative values; thus there are five possible distinct cases depending on the sign and the magnitude of α_L and α_R. Quite interestingly, we see that if $\alpha_L < 0$ ($\alpha_R < 0$), then w_L (w_R) is always the optimal portfolio, independently on α. On the contrary, if $\alpha_L \in (0,1)$, then w_C is the solution of our problem for sufficiently high ambiguity aversion and in the extreme case $\alpha_L = \alpha_R = 1/2$ we have that w_C is optimal for all levels of ambiguity aversion. Obviously, the role of γ is confined to a scale parameter.

4 Maxmin Expected Utility in a continuous time market

In this section, we consider a continuous time portfolio/consumption optimization problem for a risk averse and ambiguity averse decision maker. In particular, her attitude toward Knightian uncertainty is described by the MEU model, as previously introduced in equation (3.1). This problem has been widely analyzed by Chen and Epstein in [4] using quite involved BSDE methods. On one hand, they were able to provide existence results of a recursively defined utility function; on the other hand, explicit representation formula for the optimal portfolio/consumption process could only be obtained in very special and simple cases. Our aim is to derive their closed form results using easier computation techniques.

To begin, we fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a terminal time $T > 0$ and a standard scalar Brownian motion $W^{\mathbb{P}}$ over $[0,T]$ defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Moreover, the Brownian filtration is $\mathbb{F} := \{\mathscr{F}_t\}_{t\in[0,T]}$, where $\mathscr{F}_t$ is generated by $\sigma(W_s, s \leq t)$ and the $\mathbb{P}$-null sets of $\mathscr{F}$. Note that for the sake of clarity we are working with a scalar Brownian motion but this framework can be generalized to a multi-dimensional setting.

Next, we model the ambiguity surrounding the investment problem. To fix ideas, one can think of $\mathbb{P}$ as the historical probability governing market prices. At the same time, our agent is not completely sure of $\mathbb{P}$ being the "right" measure; hence, she considers a non trivial set of alternative priors. Following [4], we define a *density generator* as a scalar process θ over $[0,T]$ such that

$$z_t^\theta := \exp\left\{-\frac{1}{2}\int_0^t \theta_s^2 \,\mathrm{d}s - \int_0^t \theta_s \,\mathrm{d}W_s^{\mathbb{P}}\right\}, \tag{4.1}$$

is an $(\mathbb{F}, \mathbb{P})$-martingale over $[0,T]$. A sufficient condition for this to happen is the classical Novikov estimate

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2 \,\mathrm{d}s\right)\right] < +\infty.$$

By the Girsanov theorem, it follows that θ induces a measure $\mathbb{Q}^\theta$ on $(\Omega, \mathscr{F}_T)$, $\mathbb{Q}^\theta \sim \mathbb{P}$, defined by

$$\frac{\mathrm{d}\mathbb{Q}^\theta}{\mathrm{d}\mathbb{P}} = z_T^\theta, \tag{4.2}$$

and the process

$$W_t^\theta := W_t^{\mathbb{P}} + \int_0^t \theta_s \,\mathrm{d}s, \tag{4.3}$$

is an $(\mathbb{F}, \mathbb{Q}^\theta)$-Brownian motion over $[0,T]$. Henceforth, to form a set of priors one can describe a set Θ of density generators and the corresponding set of induced probabilities will be

$$\mathcal{Q}^\Theta = \left\{\mathbb{Q}^\theta : \theta \in \Theta \text{ and } \mathbb{Q}^\theta \text{ is defined by (4.2)}\right\}. \tag{4.4}$$

We now suppose that the financial market contains a risk-free asset (bond) and a risky asset (stock) whose prices at time t are denoted by B_t and S_t respectively. Their dynamics are given by

$$\begin{aligned} dB_t &= B_t r_t dt, \quad B_0 = 1; \\ dS_t &= S_t(b_t dt + \sigma_t dW_t^{\mathbb{P}}), \quad S_0 = s_0 > 0, \end{aligned}$$

where r, b, σ are deterministic functions that satisfy the usual integrability assumptions with $\inf_{t\in[0,T]} \sigma_t > 0$. Hence, we can compute $\eta_t := \frac{b_t - r_t}{\sigma_t}$, which Chen and Epstein in [4] call the *market price of uncertainty* as it reflects both risk and ambiguity. To see this, note that by using equation (4.3) it is immediate to compute the $\mathbb{Q}^\theta$-dynamics of the process S. Hence, ambiguity works through the drift coefficient of the stock prices, whereas the volatility is unaffected. The strong assumptions on the market coefficient to be deterministic is necessary to obtain closed form solutions for our portfolio choice problem. Nonetheless, in [4] r and η are supposed to be deterministic constants in order to obtain explicit formula; moreover, their assumption implies b being an affine transformation of σ.

Then, suppose that our decision maker is endowed with an initial wealth $x_0 > 0$ and her risk attitude is represented by a strictly increasing, strictly concave utility function U. If we denote by X the wealth process and by ψ a self-financing portfolio[4], assuming a continuous time frictionless trading with no consumption we can compute the wealth dynamics as

$$dX_t = [r_t + \psi_t(b_t - r_t)]X_t dt + \psi_t \sigma_t X_t dW_t^{\mathbb{P}}, \quad t \in [0,T]; \quad X_0 = x_0. \tag{4.5}$$

In this setting, admissible portfolios ψ are supposed to be deterministic functions of time t such that

$$\int_0^T |b_t \psi_t| \,\mathrm{d}t < +\infty, \qquad \int_0^T (\sigma_t \psi_t)^2 \,\mathrm{d}t < +\infty.$$

The reason for ψ being deterministic is easily explained if one thinks to the ambiguity concerning our problem. In fact, if the investor is not sure about the measure $\mathbb{P}$ being the historical one, it seems by no doubt economically inappropriate to select a stochastic portfolio which is somehow linked to $\mathbb{P}$ or any other measure in $\mathcal{Q}^\Theta$ as this will engender a wealth process which does not reflect the true underlying probability.

Let A be the set of admissible portfolios. In such a case, the wealth process X is geometric and the optimization problem for a MEU, risk averse investor who cares about her terminal wealth X_T can be summarized as

$$\max_{\psi \in A} \min_{\theta \in \Theta} \mathbb{E}_\theta \left[U(X_T)\right], \tag{4.6}$$

where $\mathbb{E}_\theta$ denotes the expectation w.r.t. $\mathbb{Q}^\theta$.

[4]In the scalar case, ψ is the proportion of wealth invested in the risky asset; in the multi-dimensional case, ψ will be a vector of portfolio weights with $1 - \sum_i \psi_i$ the proportion invested in the risk-less asset.

Before proceeding further, it remains to specify a suitable set Θ to describe Knightian uncertainty. An *ad hoc* assumption which simplifies computations is to suppose κ*-Ignorance* (see Section 3.3, [4]). Loosely speaking, ambiguity is symmetrical around $\mathbb{P}$ in the sense that

$$\Theta = \left\{ \theta : \sup_{t\in[0,T]} \theta_t \leq \kappa \right\}, \tag{4.7}$$

for some $\kappa > 0$. Moreover, we will assume $\kappa < \sup_{t\in[0,T]} |\eta_t|$ as in [4], equation (5.16), i.e. ambiguity aversion is "small"[5].

Now, let's consider the logarithmic utility case, $U(x) = \log x$; in [4] the power utility case is considered, $U(x) = \alpha^{-1}x^{\alpha}$, $\alpha \in (0, 1)$; the logarithm can be retrieved as the limit for $\alpha \to 0^+$ and computations are very similar. Thus, using the $\mathbb{Q}^{\theta}$-dynamics of the wealth process X, equation (4.6) becomes

$$\begin{aligned}
&\max_{\psi\in A} \min_{\theta\in\Theta} \mathbb{E}_{\theta} \left[\log(X_T)\right] \\
&= \max_{\psi\in A} \min_{\theta\in\Theta} \mathbb{E}_{\theta} \left[\log x_0 + \int_0^T [r_t + \psi_t(b_t - r_t) - \frac{1}{2}\psi_t^2\sigma_t^2]\, \mathrm{d}t + \int_0^T \psi_t\sigma_t \, \mathrm{d}W_t^{\mathbb{P}} \right] \\
&= \log x_0 + \int_0^T r_t \, \mathrm{d}t + \max_{\psi\in A} \left\{ \int_0^T [\psi_t(b_t - r_t) - \frac{1}{2}\psi_t^2\sigma_t^2]\, \mathrm{d}t + \min_{\theta\in\Theta} \mathbb{E}_{\theta} \left[\int_0^T -\psi_t\sigma_t\theta_t \, \mathrm{d}t \right] \right\}. \qquad (4.8)
\end{aligned}$$

It is now easy to deduce the solution of the inner minimization problem in (4.8). In fact, thanks to the κ*-Ignorance* assumption, we have

$$\theta_t^* = \begin{cases} \kappa & \text{if } \psi_t \geq 0, \\ -\kappa & \text{if } \psi_t < 0. \end{cases} \tag{4.9}$$

Therefore, our problem reduces to

$$\max_{\psi\in A} \int_0^T [\psi_t(b_t - r_t - \kappa\sigma_t \operatorname{sgn}(\psi_t)) - \frac{1}{2}\psi_t^2\sigma_t^2]\, \mathrm{d}t,$$

where sgn denotes the sign function. By the definition of the market price of uncertainty η and the assumption of "small" ambiguity, we obtain

$$\psi_t^* = \frac{1}{\sigma_t}(\eta_t - \kappa \operatorname{sgn}(\eta_t)), \tag{4.10}$$

which is exactly the same as in [4], Section 5.3, in the scalar case with $\alpha \to 0^+$.[6] We finally remark that even without the restriction on deterministic portfolios, in [4] it is found a deterministic ψ^*, thus confirming the preceding economic intuition.

5 Conclusions

We have studied portfolio selection problems in an ambiguous setting under different market frameworks and a variety of attitudes toward Knightian uncertainty. We focused on the research

[5] A different way to model ambiguity would be uncertainty about the true market coefficients with pre-specified confidence intervals. For a treatment in a single-period market, see [10].

[6] In the power utility case, the only extra computation concerns the variance of a centered lognormal random variable.

of closed form solutions for the optimal portfolio in order to get useful insights on the economic meaning of our models. We have seen that explicit formula are easy to obtain if we impose *ad hoc* but still sensible assumptions; at the same time, the required mathematical machinery is extremely reduced.

In a number of examples, we provided the solution for a smooth ambiguity averse agent as axiomatized in [17]. In a single period market model, the aversion of our investor was supposed to be quadratic, exponential (Constant Absolute Ambiguity Aversion) or of the power law (Constant Relative Ambiguity Aversion) in the case of risk neutrality. Otherwise, one can even suppose linear loss aversion to generalize the previous results.

In the case of a Mean-Variance risk averse investor, we were able to find an explicit solution if ambiguity aversion is modeled through the α-MEU paradigm, as presented in [11]. While this problem was already solved for the classical MEU version in a single period setting in [18], our analysis is new to the literature.

Finally, in a continuous time financial market, we recover the optimal solution proposed in [4] using a much more simple scheme.

References

[1] A. B. Abel. An exploration of the effects of pessimism and doubt on asset returns. *Journal of Economic Dynamics and Control*, 26:1075-1092, 2002.

[2] G.J. Alexander and A.M. Baptista. Active portfolio management with benchmarking: Adding a value-at-risk constraint. *Journal of Economic Dynamics and Control*, 32:779-820, 2008.

[3] P. Bossaerts, P. Ghirardato, S. Guarnaschelli and W. R. Zame. Ambiguity in Asset Markets: Theory and Experiment. *The Review of Financial Studies*, 23:1325-1359, 2010.

[4] Z. Chen and L. G. Epstein. Ambiguity, Risk, and Asset Returns in Continuous Time. *Econometrica*, 70(4):1403-1443, 2002.

[5] Z. Dong, Q. Gu and X. Han. Ambiguity aversion and rational herd behaviour. *Applied Financial Economics*, 20:331-343, 2010.

[6] J. Dow and S. R. Werlang. Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio. *Econometrica*, 60(1):197-204, 1992.

[7] D. Ellsberg. Risk, Ambiguity and the Savage Axioms. *Quarterly Journal of Economics*, 75:643-669, 1961.

[8] L. G. Epstein and J. Miao. A two-person dynamic equilibrium under ambiguity. *Journal of Economic Dynamics and Control*, 27:1253-1288, 2003.

[9] L. G. Epstein and T. Wang. Intertemporal Asset Pricing under Knightian Uncertainty. *Econometrica*, 62(2):283-322, 1994.

[10] L. Garlappi, R. Uppal and T. Wang. Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach. *The Review of Financial Studies*, 20:41-81, 2006.

[11] P. Ghirardato, F. Maccheroni and M. Marinacci. Differentiating Ambiguity and Ambiguity Attitude. *Journal of Economic Theory*, 118:133-173, 2004.

[12] I. Gilboa and D. Schmeidler. Maxmin Expected Utility with a Non-Unique Prior. *Journal of Mathematical Economics*, 18:141-153, 1989.

[13] C. Gollier. Portfolio choices and asset prices: The comparative statics of ambiguity aversion. *Working Paper*, available at *http://idei.fr/doc/wp/2009/ambiguity.pdf*.

[14] M.R. Haley and M.K. McGee. Tilting safety first and the Sharpe portfolio. *Finance Research Letters*, 3:173-180, 2006.

[15] S. Hwang and S.E. Satchell. How loss averse are investors in financial markets? *Journal of Banking and Finance*, 34:2425-2438, 2010.

[16] R. Kan and G. Zhou. Optimal Portfolio Choice with Parameter Uncertainty. *Journal of Financial and Quantitative Analysis*, 42(3):621-656, 2007.

[17] P. Klibanoff, M. Marinacci and S. Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849-1892, 2005.

[18] F. Lutgens and P. C. Schotman. Robust Portfolio Optimization with Multiple Experts. *Review of Finance*, 14(2):343-383, 2010.

[19] M. Taboga. Portfolio selection with two-stage preferences. *Finance Research Letters*, 2:152-164, 2005.

[20] A. Tversky and D. Kahneman. Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty*, 5:297-323, 1992.

[21] R. Uppal and T. Wang. Model Misspecification and Underdiversification. *The Journal of Finance*, 58(6):2465-2486, 2003.