Market equilibrium with heterogeneous behavioral and classical investors' preferences

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Abstract

Starting from the theory of portfolio selection under Cumulative Prospect Theory (CPT) in a one period model, we firstly present some remarks connected with the violation of the so-called loss aversion in the case of power utility functions.

The main contribution of this paper comes from the analysis of two equilibrium models. In the first one, an Expected Utility (EU) maximizer, a CPT agent and an accommodating market maker are allowed to interact. We show that there can be equilibria with null, positive or total risky investment by the CPT trader. Our results are then compared to an analogous model with two EU maximizers. On the contrary, the second financial market is populated by a sufficiently large number of EU agents and CPT agents, each of them being *price maker* and endowed with possibly *heterogeneous* preferences, these two facts being new to the literature. This time EU traders fully invest in stocks whereas CPT traders stay out of the risky market. For both models, equilibrium existence and robustness is shown using analytical and numerical methods.

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1 Introduction

After the Nobel Prize winning work of Kahneman and Tversky, a lot of research has been done over the Cumulative Prospect Theory field (CPT in what follows). While the first steps primarily concerned laboratory experiments and empirical estimates with the aim of assessing the existence of loss-averse and myopic economic agents, in the past few years theoretical results became predominant, as undoubtedly real world economy is made up of *behavioral* agents too.

On one hand, the early works of Markowitz on portfolio choice theory and Sharpe about CAPM represent two corner-stones in one-period mathematical finance models, as long as we take for granted that EU investors are the only ones who can enter a financial market. On the other hand, if we restrict our attention solely to CPT agents, then the recent work of He and Zhou [12] provides a deep theoretical insight over portfolio choice theory. Obviously, that is not the first paper on the subject (for example, see the work by Barberis and Huang, [2]); however, it seems to us that it is the most complete and it retains the less restrictive set of assumptions. This is why we decided to take [12] as a starting point. Nonetheless, if the CPT agent is endowed with power utility function, optimization results can sometimes be controversial. In particular, it easily happens that a so called loss-averse investor violates her own loss aversion, in a sense to be made precise later. Moreover, He and Zhou where primarily interested on a general formulation of a portfolio selection problem, partially leaving aside a sensitivity analysis over the model's parameters.

Turning back to general equilibrium models, theoretical results are few and a full generality is far from being obtained. As a matter of fact, most the existing literature on this subject has restricted its attention to the case of an economy exclusively populated by homogeneous CPT agents. More specifically, De Giorgi, Hens and Rieger [10] shed light over the existence of market equilibria; however, their results are negative in the sense that CPT preferences are compatible with the absence of equilibria. Furthermore, they were able to show the existence of an equilibrium with non-negative constraints on

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final wealth and a continuum of agents in the market. We point out that in [10], the market can have only a finite number of states and all CPT agents share the same preference parameters.
Another positive result come from the work of Barberis and Huang, where in [2] they analytically proved under quite restrictive hypothesis the existence of an equilibrium which satisfies the well-known CAPM conditions. In particular, their assumptions concerned homogeneous investors' preferences with power utility functions and joint normally distributed stock returns. The interesting fact is that they don't have to rely on numerical analysis and the equilibrium is derived very similarly with respect to the *classical* case, where only EU maximizers are able to invest.
Furthermore, in [7] the authors analyzed a two agents economy and proved the existence of equilibrium based on efficient risk-sharing rules.
Other controversial results on this field come from the paper by De Giorgi, Hens and Levy [9], where it is stressed that in a multi-asset market with joint normally distributed returns and heterogeneous CPT preferences an equilibrium does not exist. However, in [12] a critical flaw in their argument has been highlighted and an equilibrium is shown to be attainable. Not only, in [1] non-existence of equilibrium is assessed in a variety of different market contexts, e.g. in a risk neutral monopoly, in an imperfect information monopoly and in a pure Bertrand competition setting.

The question we address now is: are there equilibria in an economy where both EU and CPT agent are present? This problem arises naturally as the real world financial markets are nothing but a merging of the decisions of different types of investors. Experimental evidence by List [18] confirms that experienced traders behave more likely as in the EU paradigm, whereas unexperienced traders are more subject to *behavioral* distortions when taking economic decisions; just as an example, they are much more affected by the so-called endowment effect and they are more reluctant to bargain. This is why we will look for equilibria where CPT agents do not invest in risky assets at all.
Another reason to reject the hypothesis of solely EU investors in financial markets comes from the recent survey conducted by Eriksen and Kvaløy [11] over a pool of financial advisors. As a surprising fact, they exhibited myopic loss aversion; nonetheless, when compared to a sample of students, those specialized workers exhibited a greater myopic loss aversion! Finally, the empirical work coming from Hwang and Satchell, [13], not only confirms a widespread loss aversion in UK and US financial markets but even finds higher loss aversion than previously expected and/or estimated. Besides this, "investors become far more loss averse during bull markets than during bear markets, indicating their more profound disutility for losses when others enjoy gains", [13].

Coming back to our question, our answer is fortunately positive. We were able to prove the existence (and the robustness) of several types of equilibria in different market models. We remark that we substantially use a game theoretical construction; for a survey on game theory issues with CPT participants, see [20].

The paper proceeds as follows. In Section 2 we recover the setting of the CPT portfolio choice by He and Zhou [12]; Section 3 explains their solution with a particular attention over the power utility case and the violation of loss aversion. Section 4 develops a model with one EU investor, one CPT investor and an accommodating market maker, whereas in Section 5 the market is made up of many EU and CPT agents. Finally, Section 6 concludes. Involved proofs are presented in the Appendix.

2 The uniperiodal model

Consider the problem of an investor who has to decide at time 0 how to allocate her wealth in a given financial market, supposing that the investment horizon is time T fixed once for all. Specifically, let's suppose that the agent owns an initial wealth W_0 and she is able to invest in a risk-free bond which yields r (i.e., one unit of currency eventually returns r units of currency at time T) and in a risky asset which yields a random return R. We further assume that the market is frictionless, short-selling is allowed and the investor has no bounds on the level of debts that she can bear.
A fundamental (random) variable for such an agent is the stock total excess return, namely $\tilde{R} := R - r$; we suppose that the investor is aware of its cumulative distribution function (c.d.f.) $F(\cdot)$. From the no-arbitrage rule (and in order to avoid a trivial model), we assume

$$0 < F(0) \equiv \mathbb{P}\{R \leq r\} < 1. \tag{2.1}$$

Our investor is supposed to be a *behavioral* one, in the sense that she follows CPT theory *à la* Kahneman and Tversky (see [24]) when distorting probabilities (through $w_\pm(\cdot)$) and when evaluating her final *utility* (through $u_\pm(\cdot)$), derived by the selected terminal wealth X (it will be defined in a moment).

Mathematically speaking, we will make the following assumptions, which will be in force throughout this paper.

Assumption 2.1. $u_\pm(\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *are continuous, strictly increasing, strictly concave and twice differentiable, with* $u_\pm(0)=0$.

Assumption 2.2. $w_\pm(\cdot):[0,1]\to[0,1]$ *are non-decreasing and differentiable, with* $w_\pm(0)=0$, $w_\pm(1)=1$.

Turning back to the model, we assume that the reference wealth of our agent is fixed at a given level $B\in\mathbb{R}$, hence a terminal position X greater than B has to be considered as a gain, while it will be a loss in the opposite case. To better understand the underlying meaning of this setting, we define $x_0:=rW_0-B$; that is, if the agent invest all her wealth in the risk-free bond, then she will obtain rW_0, therefore x_0 represents the spread of the reference point w.r.t. the riskless payoff. In this way, $x_0=0$ should be a parameter commonly observed among householders; on the other hand, $x_0\gg 0$ will be related to ambitious (or overconfident) investors.

If the agent decides to invest θ units of currency in the risky asset, then her final wealth will be the random variable

$$X(x_0,\theta)=x_0+B+\theta[R-r]. \tag{2.2}$$

According to the CPT theory, the final value attached to this random variable is defined as

$$V(X)=\int_B^{+\infty}u_+(x-B)d[-w_+(1-F_X(x))]-\int_{-\infty}^B u_-(B-x)d[w_-(F_X(x))], \tag{2.3}$$

where $F_X(\cdot)$ is the c.d.f of X and the integral is in the Lebesgue-Stieltjes sense. By straightforward computation, one can easily check that (2.3) is reduced to a value function $U(\cdot)$ which depends on θ; in particular, when $\theta=0$, we have

$$U(0)=\begin{cases}u_+(x_0), & \text{if } x_0\geq 0\\ -u_-(-x_0) & \text{if } x_0<0.\end{cases} \tag{2.4}$$

Moreover, by changing variables, we obtain for $\theta>0$

$$U(\theta)=\int_{-x_0/\theta}^{+\infty}u_+(\theta t+x_0)d[-w_+(1-F(t))]-\int_{-\infty}^{-x_0/\theta}u_-(-\theta t-x_0)d[w_-(F(t))], \tag{2.5}$$

while for $\theta<0$ we have

$$U(\theta)=\int_{-\infty}^{-x_0/\theta}u_+(\theta t+x_0)d[w_+(F(t))]-\int_{-x_0/\theta}^{+\infty}u_-(-\theta t-x_0)d[-w_-(1-F(t))]. \tag{2.6}$$

The problem of our agent will be:

$$\max_{\theta\in\mathbb{R}}U(\theta). \tag{P}$$

Before ending this section, we note that when there is no distortion, i.e. $w_\pm(\cdot)=id(\cdot)$, then the value function $U(\cdot)$ is nothing but the common S-shaped utility function, concave on gains and convex on losses.

3 The solution of the problem

In this section we simply report the results obtained in [12] in order to solve problem (P). In general it is not possible to obtain explicit expressions; however, some refinements are available in the power utility case and in the linear utility case.

As a first issue, well-posedness of the model needs to be proved. For this, we make the following technical assumption.

Assumption 3.1. $F(\cdot)$ *has a probability density function* $f(\cdot)$. *Moreover, there exists* $\epsilon_0>0$ *such that* $w'_\pm(1-F(x))f(x)=\mathcal{O}(|x|^{-2-\epsilon_0})$, $w'_\pm(F(x))f(x)=\mathcal{O}(|x|^{-2-\epsilon_0})$ *for* $|x|$ *sufficiently large and* $0<F(x)<1$.

Proposition 3.1 (Proposition 1, [12]). *Under Assumption 3.1, $U(\theta)$ has a finite value for any $\theta \in \mathbb{R}$, and $U(\cdot)$ is continuous on $\mathbb{R}$.*

We note that Assumption 3.1 is quite natural, albeit its technicality. Roughly speaking, it expresses a link between the probability distortions of the agent and the underlying market. To show that this hypothesis is generally unrestrictive, He and Zhou prove the following

Proposition 3.2 (Proposition 2, [12]). *If the stock return R follows a lognormal or normal distribution, and $w'_{\pm}(x) = \mathcal{O}(x^{-\alpha})$, $w'_{\pm}(1-x) = \mathcal{O}(x^{-\alpha})$ for sufficiently small $x > 0$ with some $\alpha < 1$, then Assumption 3.1 holds for any $\epsilon_0 > 0$.*

As already checked in [12], the empirical estimates about $w_{\pm}(\cdot)$ satisfy the assumption of the previous proposition for a wide range of parameters' values. On the other hand, the opposite case can easily occur for *ad hoc* choices of the same parameters.

The next step is trying to avoid the ill-posedness of the model; unfortunately, this is a quite hard issue and a full characterization of the well-posedness has not been obtained. However, a lot can be said about Problem (P) if we rely on the quantity

$$k := \lim_{x \to +\infty} \frac{u_-(x)}{u_+(x)} \geq 0, \tag{3.1}$$

assuming that the limit exists. This quantity k is also called *large-loss aversion degree* (LLAD for short) and it serves as a measure of the willingness of the agent to bear huge losses against huge gains. It is straightforward to see that $k = 0$ indicates a pleasure for a substantial gain greater than the pain for a loss of the same magnitude, whereas $k = +\infty$ stands for the opposite case. One of the main findings in [12] is

Theorem 3.1 (Theorem 1, [12]). *We have the following conclusions:*

(i) If $k = +\infty$, then $\lim_{|\theta| \to +\infty} U(\theta) = -\infty$, and consequently Problem (P) *is well-posed.*

(ii) If $k = 0$, then Problem (P) *is ill-posed.*

The remaining case, i.e. when $k \in (0, +\infty)$ is more intriguing and involving. Moreover, many of the commonly used utility functions $u_{\pm}(\cdot)$ satisfy this conditions, which amounts to say that the two utilities increase at the same speed. Therefore this case needs a special attention and new statistics need to be computed. In particular, we have

Lemma 3.1 (see Lemma 3, [12]). *Assume $\lim_{x \to +\infty} \frac{u_+(tx)}{u_+(x)} = g_+(t)\ \forall\, t \geq 0$ and $\lim_{x \to +\infty} \frac{u_-(tx)}{u_-(x)} = g_-(t)\ \forall\, t \geq 0$, then the following statistics*

$$a_1 := \int_0^{+\infty} g_+(t) d[-w_+(1 - F(t))], \tag{3.2}$$

$$a_2 := \int_{-\infty}^{0} g_-(-t) d[w_-(F(t))], \tag{3.3}$$

$$b_1 := \int_{-\infty}^{0} g_+(-t) d[w_+(F(t))], \tag{3.4}$$

$$b_2 := \int_0^{+\infty} g_-(t) d[-w_-(1 - F(t))], \tag{3.5}$$

are well-defined and strictly positive. If, in addition, $0 < k < +\infty$, then $g_+(t) \equiv g_-(t)$.

Now define the critical value

$$k_0 := \max\left(\frac{a_1}{a_2}, \frac{b_1}{b_2}\right). \tag{3.6}$$

Thanks to the previous Lemma, He and Zhou are able to prove the following well-posedness result.

Theorem 3.2. *Assume that $0 < k < +\infty$, $\lim_{x \to +\infty} u_+(x) = +\infty$, and $\lim_{x \to +\infty} \frac{u_+(tx)}{u_+(x)} = g(t)\ \forall\, t \geq 0$. We have the following conclusions:*

(i) If $k > k_0$, then $\lim_{|\theta| \to +\infty} U(\theta) = -\infty$, and consequently Problem (P) *is well-posed.*

(ii) If $k < k_0$, then either $\lim_{\theta\to+\infty} U(\theta) = +\infty$ *or* $\lim_{\theta\to-\infty} U(\theta) = +\infty$*, and consequently Problem* (P) *is ill-posed.*

We observe that the statistics a_1, a_2, b_1, b_2 are generalized *Choquet expectations* of some functional of the random variable R. Their economical interpretation is quite clear in the power utility case as well in the exponential utility case, in that they represent distorted preference criteria of the agent w.r.t. huge investment or short-selling in the risky asset. The comparison among the values of k and k_0 thus explains if the investor is relatively more attracted by the risk-free asset (and therefore she will choose null stock investment) or by the risky asset (which in turn leads to ill-posedness). The case $k = k_0$ is mathematically more subtle but at the same time it is economically quite unrealistic, so it is not investigated further in [12].

To conclude this section concerning the solution in the general case, we remark that a detailed analysis can be carried on the sensitivity of the value function $U(\cdot)$ near $\theta = 0$ and its asymptotic properties as $\theta \to \pm\infty$. In particular, one can show that the value function $U(\cdot)$ has a diminishing marginal value if the utility functions $u_\pm(\cdot)$ do. Moreover, if $\lim_{|\theta|\to+\infty} U(\theta) = -\infty$ and $\lim_{x\to+\infty} u'_\pm(x) = 0$, then $U(\cdot)$ is globally non-concave and non-convex. This results implies that duality theory or global optimization can not be exploited here.

3.1 The power utility case when the reference wealth coincides with the risk-free return

Let's suppose that our *behavioral* investor possesses power utility functions, i.e. $u_+(x) = x^\alpha$, $u_-(x) = k_- x^\beta$ with $k_- > 0$ and $0 < \alpha, \beta \le 1$. We recall that in [24] an empirical study showed $\alpha = \beta = 0.88$. Moreover we assume that the adjusted reference wealth x_0 is null; as already observed, this may be common among ordinary householders. In such a case, we can explicitly solve problem (P).

Theorem 3.3 (Theorem 3, [12])**.** *Assume $x_0 = 0$ and that the utility functions are of the power type. We have the following conclusions:*

(i) If $\alpha > \beta$, or $\alpha = \beta$ and $k_- < k_0$, then (P) *is ill-posed.*

(ii) If $\alpha = \beta$ and $k_- > k_0$, then the only optimal solution to (P) *is $\theta^* = 0$.*

(iii) If $\alpha = \beta$ and $k_- = k_0 = a_1/a_2$, then any $\theta^* \ge 0$ is optimal to (P)*.*

(iv) If $\alpha = \beta$ and $k_- = k_0 = b_1/b_2$, then any $\theta^* \le 0$ is optimal to (P)*.*

(v) If $\alpha < \beta$, then the only optimal solution to (P) *is*

$$\theta^* = \begin{cases} \left[\frac{1}{k_-}\frac{\alpha}{\beta}\frac{a_1}{a_2}\right]^{\frac{1}{\beta-\alpha}} & \text{if } a_1^\beta/a_2^\alpha \ge b_1^\beta/b_2^\alpha, \\ -\left[\frac{1}{k_-}\frac{\alpha}{\beta}\frac{b_1}{b_2}\right]^{\frac{1}{\beta-\alpha}} & \text{if } a_1^\beta/a_2^\alpha < b_1^\beta/b_2^\alpha. \end{cases} \tag{3.7}$$

We end by noting that from the proof of the previous theorem it is immediate to show that in cases (ii), (iii) and (iv), the optimal value of problem (P) is $U(0) = 0$, whereas in case (v) it is given by

$$U(\theta^*) = \begin{cases} k_-^{-\frac{\alpha}{\beta-\alpha}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}\right]\left[\frac{a_1^\beta}{a_2^\alpha}\right]^{\frac{1}{\beta-\alpha}} & \text{if } a_1^\beta/a_2^\alpha \ge b_1^\beta/b_2^\alpha, \\ k_-^{-\frac{\alpha}{\beta-\alpha}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}\right]\left[\frac{b_1^\beta}{b_2^\alpha}\right]^{\frac{1}{\beta-\alpha}} & \text{if } a_1^\beta/a_2^\alpha < b_1^\beta/b_2^\alpha. \end{cases} \tag{3.8}$$

3.2 Loss averse investors violating loss aversion...

The parameter k_- which appears in Section 3.1 in the particular choice of $u_-(\cdot)$, the loss part of the S-shaped utility function of our CPT agent, is usually referred to as "loss aversion coefficient". The original idea which underlies this definition is that "losses loom larger than gains", [15]. Unfortunately, this concept is systematically violated *for every* choice of $k_- \ge 1$ whenever we assume an S-shaped utility function of the power type with $\alpha < \beta$. To see this, let x represent the deviation from the reference point of our investor. Now we can note that

$$V(x) = u_+(x) = x^\alpha, \quad V(-x) = u_-(x) = k_- x^\beta, \qquad \forall\, x \ge 0.$$

Thus, a loss is more unpleasant than a gain of the same magnitude if and only if $V(-x) \geq V(x)$ for $x > 0$. Now, assuming $\alpha < \beta$ and setting $\zeta := k_-^{\frac{1}{\alpha-\beta}}$, by reversing the previous inequality we find that

$$\text{Loss aversion violated} \iff 0 < x < \zeta. \tag{3.9}$$

Besides this, it is not clear at all which exact value can be represented by ζ. For example, if $k_- = 1$, then we have $\zeta = 1$ independently of α and β. But "1" can represent 1 unit of currency, or 1 billion units of currency! [1]

As a consequence, the naming "loss aversion coefficient" is by no doubt misleading. As Bernard and Ghossoub point out in [5], Section 2.1, loss aversion should be a *behavioral* concept in the sense that the comparison between the pain associated with a loss and the pleasure connected to a gain should take into account *behavioral* quantities, e.g. probability distortions. Therefore, the choice of a specific value, namely k_-, attached to the utility function for losses $u_-(\cdot)$ is unable to fully explain the loss aversion of an agent and hence it is clearly a simplifying assumption.
Secondly, we do not have to forget that a generic utility function $u(\cdot)$ (in our case both $u_+(\cdot)$ and $u_-(\cdot)$) has an ordinal nature and not a cardinal nature, as it reflects a specific preference ordering. As it is important to stress the independence of $u(\cdot)$ on the magnitude of the terminal wealth, the same should be true if we consider $u_+(\cdot)$ and the magnitude of a gain or $u_-(\cdot)$ and the magnitude of a loss. On the contrary, power utilities constrain loss aversion only over the interval $[\zeta, +\infty)$, thus leaving our agent the possibility to *cheat* on his own loss aversion. A confirm of this unpleasant fact comes from the analysis of the results in our Theorem 3.3, i.e. Theorem 3 in [12]:

(i) being (P) ill-posed, loss aversion is violated "by definition". More precisely, in the case $\alpha > \beta$ equation (3.9) says that violation occurs if our agent selects $\theta^* > \zeta$; but this is indeed true as $\lim_{\theta\to+\infty} U(\theta) = +\infty$. A similar phenomenon happens if $\alpha = \beta$ and $k_- < k_0$. Just as an example, if $w_+(\cdot) \equiv w_-(\cdot)$ and $F(\cdot)$ is the symmetric c.d.f. of a centered random variable, i.e. $F(t) = 1 - F(-t)\ \forall\, t \geq 0$, then we have $k_0 = 1$. Thus, $k_- < 1$ ensures violation;[2]

(iii) by the relation (3.9), we have violation whenever $\frac{a_1}{a_2} < 1$ and $\theta^* > 0$ is chosen; case *(iv)* is analogous;

(v) using (3.7) in the case $a_1^\beta / a_2^\alpha \geq b_1^\beta / b_2^\alpha$, and (3.9), we see that

$$\text{Loss aversion violated} \iff \theta^* < \zeta \iff \frac{a_1}{a_2} < \frac{\beta}{\alpha}. \tag{3.10}$$

It is important to stress that the previous condition does *not* depend on k_-, to so-called loss aversion coefficient! In particular, whenever $a_1 \leq a_2$ we have violation, as β is supposed to be greater than α. Not only, we can note that the greater is β/α, the more probably a violation happens. Even the following easy example shows how frequent a violation can be.

Example 3.1. Assume $0 < \alpha < \beta \leq 1$, a null risk-free rate of return, i.e. $r = 1$, and a stock excess return normally distributed, namely $R \sim \mathcal{N}(\mu, \sigma^2)$. It is easy to see that we have

$$F(t) = \mathcal{N}\left(\frac{t - \mu + 1}{\sigma}\right),$$

where $\mathcal{N}(\cdot)$ is the c.d.f. of a standard Gaussian random variable. Let's now fix the expected stock return, $\mu = 1.02$, and its standard deviation, $\sigma = 0.5$. Furthermore, we suppose that our *behavioral* investor possesses probability distortions of the Kahneman-Tversky type, i.e.

$$w_+(p) = \frac{p^\gamma}{[p^\gamma + (1-p)^\gamma]^{1/\gamma}}, \qquad w_-(p) = \frac{p^\delta}{[p^\delta + (1-p)^\delta]^{1/\delta}}, \tag{3.11}$$

with $\gamma = \delta = 0.65$ as obtained in laboratory experiments.

Thanks to the hypothesis $w_+(\cdot) \equiv w_-(\cdot)$ and the skewness of $F(\cdot)$, it is immediate to see that the condition $a_1^\beta / a_2^\alpha \geq b_1^\beta / b_2^\alpha$ is indeed fulfilled. At this point, it remains to numerically compute the statistics a_1 and a_2 in order to verify when (3.10) is assured. For a better graphical representation,

[1] We can even show that loss aversion is violated whenever $\alpha = \beta$ and $k_- < 1$ or $\alpha > \beta$ and $x > \zeta$.

[2] Things can get even worse if $F(\cdot)$ has a positive skewness; see Section 3.3 and equation (3.2) in particular.

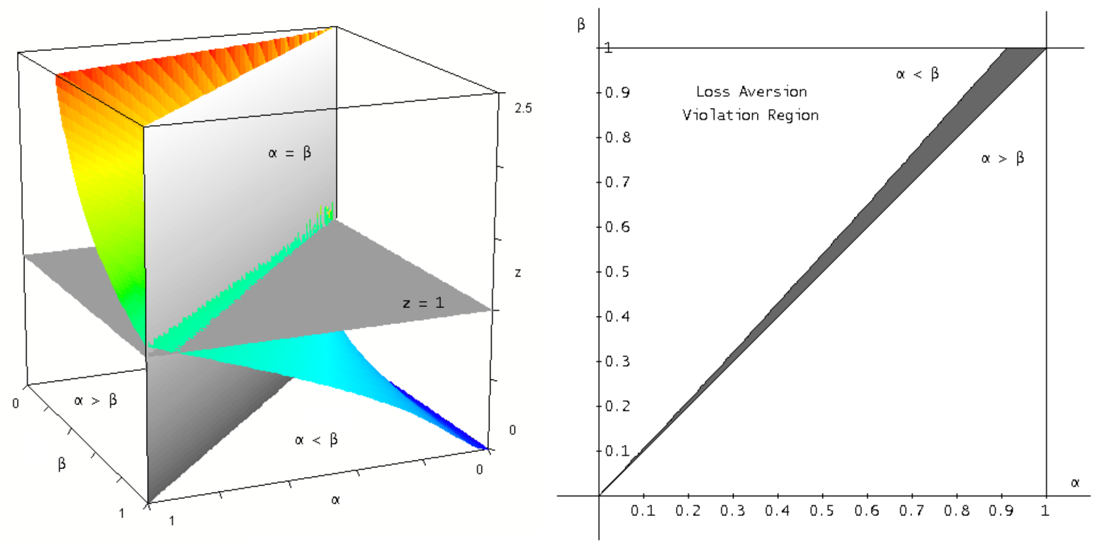

Figure 1: violation regions of loss aversion in the case $0 < \alpha < \beta \leq 1$.

we computed the quantity $z := \frac{\alpha}{\beta}\frac{a_1}{a_2}$; it is straightforward to check that loss aversion violation is now equivalent to $z < 1$. Results are shown in Figure 1. In particular, in the left plot we depicted a surface representing the value of z; for the sake of clarity, we also reported the vertical plane $\alpha = \beta$. Restricting our attention to the half part with $\alpha < \beta$, it is immediate to see that $z < 1$ is the most common situation, unless the discrepancy $\beta - \alpha$ is effectively small. More evidently, the right plot of Figure 1 shows the curve given by the intersection between the 3D-surface of z and the plane $z = 1$. Ignoring the half-plane with $\alpha > \beta$, we can see that only the shaded area corresponds to a CPT agent who does not "violate". Hence, our supposed loss averse agent can (and often do) violate her own so-called loss aversion in order to reach her maximal prospect value.

To conclude, if we wish to retain the power utility assumption, we inevitably have to abandon the belief of being modeling a loss averse agent. Alternatively, we can modify our concept of loss aversion using a different measure for this phenomenon which undoubtedly affects real economic decisions; see e.g. [16]. As we are not discussing the best way to model loss aversion, neither if power utility is able to capture actual investors' behavior, we continue to use this particular paradigm, paying particular attention when selecting the parameters α, β and k_-, which we will continue to denote as the "loss aversion coefficient".

3.3 CPT preferences and stochastic dominance

As Kahneman and Tversky pointed out in [24], one of the main advantages of *Cumulative* Prospect Theory w.r.t. to the earlier Prospect Theory is the fact that this preference structure is compatible with First Order Stochastic Dominance. In other words, if a portfolio/financial position (strictly) first-order stochastically dominates another portfolio, then the CPT value of the first one is (strictly) greater than that of the second one. Using our notation, this fact can be easily seen from equation (2.3) using integration by parts. Nonetheless, within their restricted settings, a similar result is also proved in [2], Proposition A1, and in [5], Proposition 4.4.
Concerning Second Order Stochastic Dominance, we recall that loosely speaking, an investor whose preferences "satisfy" second order stochastic dominance always selects the lowest variance portfolio among a class of admissible ones, as long as their mean value remains the same[3]. A fundamental result which links CPT and second order stochastic dominance is Proposition A2 in [2]. We now frame it adapting their notation to ours.

Proposition 3.3 (Proposition A2, [2]). *Let the final wealths X_1 and X_2 be given and assume the following:*

- *the utility functions are of the power type with $\alpha = \beta$, $\alpha \in (0, 1]$, $k_- > 1$ and $x_0 = 0$;*

[3]It is well-known that this is only a consequence (and not a characterization) of second order stochastic dominance; see Proposition 4.1.2 in [3].

- *the probability weightings are of the Kahneman-Tversky type (as in equation* (3.11)*) with* $\gamma = \delta$, $\delta \in (0.28, 1)$ *and* $\alpha < 2\delta$*;*

- $\mathbb{E}[X_1] = \mathbb{E}[X_2] \geq 0$*;*

- X_1 *and* X_2 *are symmetrically distributed;*

- X_1 *and* X_2 *satisfy the single-crossing property, so that if* P_1 *and* P_2 *are their respective CDF, there exists* z *such that* $P_1(x) \leq P_2(x)$ *for* $x < z$ *and* $P_1(x) \geq P_2(x)$ *for* $x > z$*.*

Then $V(X_1) \geq V(X_2)$ *and the inequality holds strictly whenever the single-crossing property is strictly satisfied for some* x*.*

Briefly, the proof of this result strongly depends on most of the hypothesis and it is based on the fact that the value of a generic prospect X can be written as

$$V(X) = c(a_1 - k_- a_2),$$

for a suitable constant c, which in turn can be shown to be negatively dependent w.r.t. a mean-preserving spread. In particular, if we remove the hypothesis $\alpha = \beta$ then the second order stochastic dominance is definitely lost, as Bernard and Ghossoub note in their Remark 4.2, [5]. Finally, we recall that the bound $\delta \in (0.28, 1)$ is necessary to have strictly increasing probability distortions over $[0, 1]$.

4 An equilibrium model with one EU and one CPT agent

In this section we are going to build, in a game theoretical fashion, a one period equilibrium model of a single risky asset market where one *classical* (or EU) agent and one *behavioral* (or CPT) agent interact. More precisely, we have in mind an hypothetical situation where a non-strategic market maker is willing to buy (or sell) any desired amount of shares from these two investors; at the same time, the demand level for the stock will affect its equilibrium yield which in turn will influence the equilibrium price. We denote with θ_B and θ_C the proportions of wealth invested in shares by the *behavioral* and the *classical* investors respectively; moreover, we suppose that the initial endowments of these agents are normalized with $W_0^B = W_0^C = 1$, therefore θ_B and θ_C also represent absolute levels of wealth. As it will be clear by the subsequent analysis, this can be done without loss of generality for the *classical* agent in that her maximization problem which has to be solved later is independent of her initial wealth. On the contrary, the *behavioral* investor's policy can be seriously affected by her initial endowment, as her strategy depends on the adjusted reference wealth level x_0; however our opinion is that it seems reasonable to suppose a common starting wealth equal to 1 (see also the following assumptions and the subsequent observations).

It is now important to stress that, if not otherwise stated, we impose leverage as well as short-selling constraints on strategies, i.e. $\theta_B \in [0, 1]$ and $\theta_C \in [0, 1]$. On one hand, this will allow us to make easier computations; on the other hand, in some cases the equilibrium itself will avoid such a kind of behavior. To be more precise, we will return on this issue later, when analyzing the strategies of our investors.

Now let's specify the market structure and the preferences of our agents.

Assumption 4.1 (Market structure)**.**

- *There is a risk-free asset (bond) with null return, i.e.* $r = 1$*;*
- *There is single risky asset (stock) with a normally distributed return* $R \sim \mathcal{N}(\hat{\mu}, \hat{\sigma}^2)$*, where*

$$\hat{\mu} = \mu(\theta_B, \theta_C) = \mu + \epsilon(\theta_B + \theta_C), \tag{4.1}$$

$$\hat{\sigma}^2 = \sigma^2(\theta_B, \theta_C) = \sigma^2 + \eta(|\theta_B| + |\theta_C|). \tag{4.2}$$

The parameters' ranges are as follows:

$$\mu \in [1, +\infty), \quad \epsilon \in [0, +\infty), \quad \sigma \in (0, +\infty), \quad \eta \in [0, +\infty). \tag{4.3}$$

We used the symbol μ (σ) to denote both the drift (volatility) impact function and the constant which appears in that expression; however, we hope that this will cause no notation problems in what follows.

Some remarks about the previous assumptions are now in order. First of all, $r = 1$ can be imposed without loss of generality as the only important variable is the excess of return of the stock, namely $\hat{\mu}-r$. As $\mu \geq 1$ and $\epsilon \geq 0$, it turns out by (4.1) that this return spread is always positive as long as we admit only non negative demand levels. Note that this fact is also empirically observed in long-term analysis, as the equity premium puzzle confirms; see for example [19] or [4]. Secondly, a Gaussian distribution of the stock return implicitly makes some simplification when solving maximization problems (just recall the close connection between normally distributed asset returns and the mean-variance portfolio selection criterion).

Obviously, the absolute values which appear in the volatility impact function (4.2) are not necessary if short-selling is not allowed. Moreover, the parameters ϵ and η should not be too "big" w.r.t. μ and σ^2 respectively. In fact, a higher value of ϵ would naturally lead to a higher investment level in the stock caused by the FOSD property shared by both agents; without imposing short-selling, this could lead to an ill-posed model in the sense that traders have convenience to invest more and more in the risky asset to self-boost its expected return. On the other hand, a higher risky investment also raises volatility which negatively affects the demand levels for risk-averse agents. In conclusion, the interaction of these impact functions should generate a natural trade-off between risky and non-risky investment.

We stress the fact that the choice of such impact functions, which are obviously unrealistic, is for computational convenience. However, a linear impact function on the drift has also been used in an insider trading influenced market in [17] and in an optimal liquidation problem in [22], only to mention a few. From a theoretical point of view, volatility impact has sometimes been avoided, as it causes severe problems when analyzing equilibrium models or optimal trading/liquidation strategies with the presence of large traders; see for example [8] or [17]. On the contrary, some authors allow for an endogenous volatility ([6] and [21]) and the excess volatility puzzle documented by Shiller in [23] induced us to add the impact function (4.2), where the exogenous constant σ^2 represents the volatility induced by noise trading. Summarizing, the two agents are *partially price taker*, as $\hat{\mu}$ and $\hat{\sigma}^2$ have an exogenous and an endogenous component too. This feature should give to our model the flavor of a financial market where the interaction of many small traders is summed up by the presence of a market maker who provides exogenous parameters, plus two large traders who are able to influence the terminal return or the terminal price of the shares.

Assumption 4.2 (CARA classical agent). *The* classical *agent's utility function is*

$$u_C(x) = 1 - \exp(-x), \tag{4.4}$$

where the constant absolute risk aversion coefficient is 1.

Note that we could even choose a more general form like $u_C(x) = 1 - \exp(-\lambda x)$, where λ is the CARA coefficient. In this case, a higher λ implies a more risk-averse agent; for the moment we shall set $\lambda = 1$ as a normalization assumption.

Assumption 4.3 (CRRA behavioral agent). *The* behavioral *agent has the following utility functions:*

$$u_+(x) = x^\alpha, \quad u_-(x) = k_- x^\beta, \tag{4.5}$$

with $\alpha, \beta \in (0, 1]$ *and* $k_- > 0$. *Moreover, her adjusted reference wealth level is* $x_0 = 0$ *and her probability weighting functions are of the Kahneman-Tversy type (see equation* (3.11)*) with* $\gamma, \delta \in (0.28, 1)$.

We do not make further assumptions on the values of $\alpha, \beta, k_-, \gamma$ and δ. Their values will be specified later, depending on the type of equilibrium we wish to select. The previous hypothesis allows us to use the results of He and Zhou [12] recalled in Section 3.1 concerning the optimal policy of a *behavioral* agent. Furthermore, it is easily seen that thanks to our assumptions, the terminal wealths of our agents can be expressed as

$$X_i = \theta_i + (W_0^i - \theta_i)r = \theta_i R + (1 - \theta_i) = 1 + \theta_i(R - 1), \tag{4.6}$$

where $i \in \{B, C\}$ and we recall that the distribution of the random variable R depends on both θ_B and θ_C. These last two variables are those which need to be endogenously determined by the maximizing behavior of our agents; their values will in turn give us the equilibrium stock return mean and variance as long as we consider the market parameters μ, ϵ, σ^2 and η be exogenously given.

4.1 Equilibria with no *behavioral* agent's demand

As a first (and simpler) case, we would like to find some equilibria in which the optimal policy for the CPT agent is not to invest in the risky stock. Such a type of anomalous situation is interesting in that it should be a signal of extremely high loss aversion (if not violated ...) and/or risk aversion, which in turn would induce our *behavioral* agent to a completely safe investment in bonds; also recall that in [13] it was estimated a loss aversion coefficient, namely k_-, much greater than usually expected. This could lead some investors to exit the stock market during specific periods. A null risky investment could also be the optimal strategy for a CPT agent who experiences a predominant endowment effect, leading her to avoid any potential loss.

In order to reach this kind of equilibria, we are going to exploit Theorem 3.3; in particular, cases *(ii)*, *(iii)* and *(iv)* are those which admit such an optimal strategy. We immediately see that a necessary condition is to set $\alpha = \beta$; moreover, we will suppose that our *behavioral* investor's parameters are those empirically obtained in [24], i.e. $\alpha = \beta = 0.88$, $k_- = 2.25$, and the weighting functions are of the Kahneman and Tversky type (see (3.11)), which are assumed to be identical with a common exponent $\gamma = 0.65$. We remark that our choice of k_- is just to fix ideas, as all the subsequent analytic results of this section will be proved for every $k_- > 0$. Let's also note that these values fulfill the conditions imposed by Assumption 3.1, therefore we can use all the results of Section 3.

In what follows, the equilibrium values will be denoted with a *; thus our goal is to find equilibria with $\theta_B^* = 0$. It is easily seen that in equilibrium, the impact functions (4.1) and (4.2) are only influenced by the *classical* agent's policy and they will be

$$\hat{\mu}^* = \mu + \epsilon\theta_C^*, \qquad \hat{\sigma}^{2*} = \sigma^2 + \eta\theta_C^*. \tag{4.7}$$

As our model is a game theoretical one, we remark the fact that, with our assumptions, an equilibrium with $\theta_B^* = 0$ exists if and only if we are able to find an optimal strategy θ_C^* which maximizes the expected utility for the *classical* agent given $\theta_B^* = 0$ and at the same time the policy $\theta_B = 0$ is the optimal one for the *behavioral* agent. But this is indeed the case if for every $\theta_B \in [0,1]$ we have $k_- > k_0 \equiv k_0(\theta_B, \theta_C^*)$ (see Theorem 3.3, *(ii)*) or $k_- = k_0(\theta_B, \theta_C^*) = \frac{a_1(\theta_B,\theta_C^*)}{a_2(\theta_B,\theta_C^*)}$ and we select $\theta_B^* = 0$ (see Theorem 3.3, *(iii)* and similarly for case *(iv)*, replacing a_1 and a_2 with b_1 and b_2 respectively). Unfortunately, this is not an easy task as the critical statistic k_0 has not an explicit representation; moreover, it depends on the market parameters μ, ϵ, σ^2, η and on the optimal strategy θ_C^* too (which in turn depends on the market parameters ...). Therefore, we are going to follow these steps:

Step 1 Solve the maximization problem for the *classical* agent with $\theta_B = 0$, namely

$$\max_{\theta_C\in[0,1]} U_C(\theta_C) \equiv \max_{\theta_C\in[0,1]} \mathbb{E}[u_C(X_C)] \tag{4.8}$$

in order to find the candidate optimal strategies (if one exists) $\theta_C^*(\mu, \epsilon, \sigma^2, \eta)$.

Step 2 Fix some or all parameters' values (possibly within reasonable ranges) and check the optimality conditions of Theorem 3.3 to have $\theta_B^* = 0$.

If we succeed in solving the two steps, then we can try to enlarge as much as possible the previous ranges in order to retain the selected equilibrium $(\theta_B^*, \theta_C^*) = (0, \theta_C^*)$. Now, the first step can be easily implemented; we start by writing equation (4.8) explicitly, substituting for U_C and X_C and exploiting the normality of the return R:

$$\max_{\theta_C\in[0,1]} \int_{\mathbb{R}} \left(1 - \exp\left(-(\theta_C z + 1 - \theta_C)\right)\right) \frac{1}{\sqrt{2\pi}\hat{\sigma}} \exp\left\{-\frac{(z-\hat{\mu})^2}{2\hat{\sigma}^2}\right\} dz.$$

Now, replacing $\hat{\mu} = \mu + \epsilon\theta_C$ and $\hat{\sigma}^2 = \sigma^2 + \eta\theta_C$ and performing the Lebesgue integration we obtain[4]

$$\max_{\theta_C\in[0,1]} U_C(\theta_C) = \max_{\theta_C\in[0,1]} 1 - \exp\left\{\frac{\eta}{2}\theta_C^3 + \left(\frac{\sigma^2}{2} - \epsilon\right)\theta_C^2 + (1-\mu)\theta_C - 1\right\}. \tag{4.9}$$

Now, the structure of solution strongly depends on the choice $\eta = 0$ or not. The first case correspond to a null volatility impact, i.e. $\hat{\sigma}^2 \equiv \sigma^2$ and computations are obviously easier.

[4] It can be easily shown that similar formula can be obtained if we allow W_C^0 and λ to take values different from 1; however, the initial wealth adds only a multiplicative factor and this has no influence over the extreme points of (4.9), whereas $\lambda \neq 1$ involves the calculations as it affects the risk-aversion of the *classical* agent.

4.1.1 The case with null volatility impact

Let's start by considering the case $\eta = 0$. We recall that the constant part of the drift is $\mu \geq 1$ and we suppose a positive drift impact, i.e. $\epsilon > 0$ in order to avoid a trivial model. Now, equation (4.9) is obviously simplified and the first order conditions lead to

$$\frac{dU_C}{d\theta_C} = 0 \iff \theta_C = \frac{\mu - 1}{\sigma^2 - 2\epsilon}.$$

Obviously, we have to suppose $\sigma^2 > 2\epsilon$; moreover, this solution is admissible if and only if $\theta_C \in [0, 1]$. By the fact $\mu \geq 1$, we have to assume the further condition $\sigma^2 \geq 2\epsilon + (\mu - 1)$. This inequality has a clear economical explanation, in that the exogenous volatility parameter on the left-hand-side should be greater than an adjusted excess drift effect on the right-hand-side in order to discourage heavy risky investment. Moreover, in the special case $\mu = 1$ we have a null risk premium, which in turn leads to a null risky investment for the EU agent. We also note that $\epsilon = 0$ reduces the model to a standard portfolio selection and the optimal θ_C is nothing but the Sharpe ratio of the risky asset divided by its standard deviation; recalling the classical Merton portfolio choice problem, $\frac{\mu-1}{\sigma^2}$ represent an index of the performance of the risky investment.
With these hypothesis on the market parameters, we can set $\theta_C^* = \frac{\mu-1}{\sigma^2-2\epsilon}$ and this is indeed a maximum as implied by the second order conditions. Note that θ_C^* is increasing in both μ and ϵ whereas it is decreasing in σ^2, as expected.

Now it's turn to finish Step 2, trying to show that $\theta_B = 0$ can be optimal for the *behavioral* agent. Fortunately, we obtained the following positive result.

Proposition 4.1. *Assume $\eta = 0$. If $(\mu^*, \epsilon^*, \sigma^{2*})$ are such that*

$$\begin{cases} \sigma^{2*} > 2\epsilon^* \ , \\ \sigma^{2*} \geq 2\epsilon^* + (\mu^* - 1) \ , \\ k_0 \left(1, \frac{\mu^*-1}{\sigma^{2*}-2\epsilon^*}\right) \leq k_- \ , \end{cases} \tag{4.10}$$

then there exists an equilibrium for every choice of $(\mu, \epsilon, \sigma^2) \in [1, \mu^*] \times (0, \epsilon^*] \times [\sigma^{2*}, +\infty)$.

Proof. See the Appendix. □

The intuition behind this result is the following: if we are able to find a particular triple $(\mu^*, \epsilon^*, \sigma^{2*})$ such that it fulfills all the conditions of (4.10), then we can even obtain a non-zero Lebesgue measure set of parameters' values for which the equilibrium exists, and this (mathematical) robustness property is very pleasant from the point of view of an economist. In particular, we see that the first two conditions in (4.10) are necessary for the admissibility of the optimal θ_C, whereas the third inequality is the key to exploit Theorem 3.3.

If the last inequality holds strictly, then we are selecting an equilibrium which is confirmed by Theorem 3.3, *(ii)*; otherwise, if we have an equality, then we can apply case *(iii)* of Theorem 3.3 imposing to the *behavioral* agent the choice $\theta_B = 0$. However, we remark that the second type of equilibrium is in fact an unrealistic one, as a very slight change in market parameters would destroy it. Mathematically speaking, they are unstable equilibria; in particular, for them there is lack of robustness as we are lying on the boundary of a subset of the 3-dimensional space (μ,ϵ,σ^2). In both cases, we can compute the equilibrium stock return R^*; this will be a random variable with Gaussian distribution $\mathcal{N}(\hat{\mu}^*, \hat{\sigma}^{2*})$, where

$$\hat{\mu}^* = \mu^* + \epsilon^* \left(\frac{\mu^* - 1}{\sigma^{2*} - 2\epsilon^*}\right), \qquad \hat{\sigma}^{2*} = \sigma^{2*},$$

and it is easily seen that $\hat{\mu}^*$ is increasing in both μ^* and ϵ^* and decreasing in σ^{2*}.

We have now to rely on numerical analysis in order to find our starting equilibrium triple $(\mu^*,\epsilon^*,\sigma^{2*})$. We can fix some of these three parameters and see what happens when letting the other(s) vary. Before starting, we note that a value of $\epsilon = 0.01$ implies that if both agents invest the totality of their respective wealths in the risky asset, then the equilibrium expected return (if the equilibrium exists) increases by 2%; moreover, $\mu = 1.10$ means that investing in stocks provides an expected additional yield by 10% w.r.t. the risk-free bond. Therefore, we choose to sensibly fix the values $\mu^* = 1.10$ and $\epsilon^* = 0.01$; from the no-leverage condition for the EU agent, namely the second inequality of (4.10), we obtain the

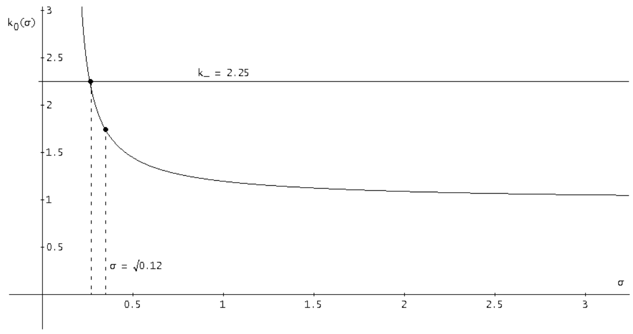

Figure 2: existence and σ-stability of the equilibrium with $\mu = 1.10$, $\epsilon = 0.01$, $\eta = 0$.

constraint $\sigma^{2*} \geq 0.12$. A standard quadrature formula[5] gave us $k_0(1,1) = 1.7506 < 2.25$. Therefore, Proposition 4.1 suggests the following ranges:

$$\mu \in [1, 1.10], \qquad \epsilon \in (0, 0.01], \qquad \sigma^2 \in [0.12, +\infty).$$

It is interesting to compute k_0 as a function of σ [6]; results are shown in Figure 2. First of all we see that the right-most point corresponds to $\sigma^2 = 0.12$; moreover $k_0(\sigma) \geq 1$ and it is decreasing in the exogenous volatility, as we already saw in the proof of Proposition 4.1. Obviously, $\sigma \geq \sqrt{0.12}$ guarantees the existence of the equilibrium; not only, we see that an equilibrium would exist even for lower values ($\sigma^2 \geq 0.068$, which is identified by the left-most point in the plot)[7]. An explanation of this fact is that allowing the *classical* agent to exploit leverage, thus raising the expected return of the stock, is not sufficient to induce a non-zero risky investment for the *behavioral* agent. [8]

4.1.2 The case with volatility impact

In this section we are going to see what are the main changes on the results previously obtained if we allow for a positive impact on the volatility to be endogenously determined, i.e. this time we have $\eta > 0$. The analysis for the *classical* agent in Step 1 can be carried quite similarly, whereas difficulties arise in Step 2. First order conditions deduced from equation (4.9) imply

$$\frac{dU_C}{d\theta_C} = 0 \iff \theta_C = \frac{-(\sigma^2 - 2\epsilon) \pm \sqrt{(\sigma^2 - 2\epsilon)^2 + 6\eta(\mu - 1)}}{3\eta},$$

where we implicitly suppose $\sigma^2 \geq 2\epsilon$. As we do not allow short-selling, we have to immediately discard the negative solution, thus the only candidate remains

$$\theta_C^* = \frac{-(\sigma^2 - 2\epsilon) + \sqrt{(\sigma^2 - 2\epsilon)^2 + 6\eta(\mu - 1)}}{3\eta}. \tag{4.11}$$

Second order conditions confirm once again that (4.11) is a maximum point. Note that $\mu = 1$ again produces a null risk premium, which implies an admissible $\theta_C^* = 0$; otherwise, we have to impose further assumptions on the market parameters in order to have $\theta_C^* \leq 1$. Straightforward calculations show that the correct condition is

$$3\eta + 2\left(\sigma^2 - 2\epsilon - (\mu - 1)\right) \geq 0,$$

which is equivalent to impose upper bounds on the drift parameters μ and ϵ or lower bounds on the volatility parameters σ^2 and η; this confirms once again economic intuition, as the *classical* agent will avoid leverage when the market behaves "normally". It is important to stress the fact that we do

[5] If not otherwise stated, we used the Simpson quadrature for the numerical approximation of our integrals, as it revealed faster convergence w.r.t. adaptive method, such as the Lobatto quadrature method.

[6] We use σ and not σ^2 in our graphics; therefore, its lower bound becomes $\sqrt{0.12} \approx 0.346$.

[7] Low values of σ^2 seem to be more realistic; just as an example, with $\mu = 1.05$, $\sigma^2 = 0.09$ implies a 20% probability of observing a maximal $\pm 7\%$ excess return and a 90% probability of a maximal $\pm 50\%$. On the contrary, $\sigma^2 = 1$ implies a 20% probability of a maximal $\pm 25\%$ excess return and a 90% probability of a maximal $\pm 165\%$!

[8] If we fix just one parameter and let the other two vary, graphical analysis shows existence and stability of the equilibrium for non negligible ranges. 3D-plots are available on request

not have to set any *a priori* condition over the parameters; the previous inequality simply specifies a "coordination" that our market must have to possibly admit an equilibrium without leverage by the EU investor. We also note that this condition is just a generalized version of the second inequality in (4.10); this time the additional parameter η is taken into account. Finally, from (4.11) one can compute the derivatives of θ_C^* w.r.t. market parameters and it is easily seen that θ_C^* is still increasing in both μ and ϵ, whereas it is decreasing in both σ^2 and η.

It remains to show that $\theta_B = 0$ can be optimal for the *behavioral* agent. Proceeding similarly to the previous case, we state the following result in the particular case of null drift impact.

Proposition 4.2. *Assume $\eta > 0$ and $\epsilon = 0$. If $(\mu^*, \sigma^{2*}, \eta^*)$ are such that*

$$\begin{cases} 3\eta^* + 2\left(\sigma^{2*} - (\mu^* - 1)\right) \geq 0\,, \\ k_0\left(0, \frac{-\sigma^{2*} + \sqrt{(\sigma^{2*})^2 + 6\eta^*(\mu^* - 1)}}{3\eta^*}\right) \leq k_-\,, \end{cases} \tag{4.12}$$

then there exists an equilibrium for $\mu = \mu^*$ and every choice of $(\sigma^2, \eta) \in (\sigma^{2*}, +\infty) \times [\eta^*, +\infty)$.

Proof. See the Appendix □

Apart the presence of $\eta > 0$, which obviously makes computations more difficult, the interpretation of Proposition 4.2 is similar to that of case $\eta = 0$. There are two main differences which worth a deeper explanation. Firstly, the second inequality in (4.12) contains $k_0\left(0, \theta_C^*\right)$ instead of $k_0\left(1, \theta_C^*\right)$; the reason is that now low investment values by the CPT investor can destroy the existence condition $k_0 \leq k_-$. Secondly, once a particular triple $(\mu^*, \sigma^{2*}, \eta^*)$ is found, then μ can not be arbitrarily lowered, as a variation of this parameter increases the drift as well as the volatility(and this is also why we need $\epsilon = 0$).

Finally, the way to select the equilibrium is the same as explained in the case with null volatility impact and the equilibrium stock return can be computed similarly. In particular, we have

$$\hat{\mu}^* = \mu^*, \qquad \hat{\sigma}^{2*} = \frac{2}{3}\sigma^{2*} + \frac{1}{3}\sqrt{(\sigma^{2*})^2 + 6\eta^*(\mu^* - 1)},$$

and it is easily seen that the equilibrium volatility is increasing in μ^*, σ^{2*} and η^*.

Graphical analysis available on request shows that, while keeping $\mu = 1.2$ fixed, the equilibrium existence condition (4.12) is fulfilled unless volatility parameters σ and/or η are sufficiently close to 0. Note that in this case the value of μ is exceptionally high; even if we were not able to prove that k_0 is increasing in μ, numerical simulations suggests that with lower μ there would still be an equilibrium, even for lower σ and/or η. Similar surfaces can be obtained if we fix one of these two parameters; moreover, non-zero Lebesgue measure sets of parameters which support the equilibrium can be easily computed.

At least, we turn back to the general case with $\epsilon > 0$ and $\eta > 0$. The technical problem that arises now is the fact that there is no more a monotonic dependence of k_0 in θ_B; to see this, note that an increase in the *behavioral* agent's demand produces at the same time greater endogenous drift and volatility. The resulting combined effect is hard to estimate with analytical techniques; therefore, we will limit ourselves to numerical and graphical analysis. First of all, let's set the market values as

$$\mu = 1.05, \quad \epsilon = 0.01, \quad \sigma^2 = 0.09, \quad \eta = 0.3.$$

In such a case it is easy to represent k_0 as a function of θ_B (recall that we are looking for equilibria with $\theta_B^* = 0$; therefore, using game-theory jargon, the *classical* agent's optimal strategy must be based on this conjecture). In particular we obtain Figure 3, from which we see that the *behavioral* investor will indeed select $\theta_B^* = 0$ as long as she can not get an approximate leverage of 740 times her initial wealth! Finally, a magnification of Figure 3 with $\theta_B \in [0, 1]$ shows that k_0 is decreasing, thus confirming the absence of global monotonicity. With the previous parameters, it is easy to compute the equilibrium values as

$$\theta_B^* = 0, \quad \theta_C^* \approx 0.265, \quad \hat{\mu}^* \approx 1.05265, \quad \hat{\sigma}^{2*} \approx 0.1695. \tag{4.13}$$

The *classical* agent will thus invest about a quarter of her wealth in stocks, whereas the remaining amount is used to buy risk-less bonds.

At this point one can wonder if the parameters we selected are *ad hoc*; fortunately, the answer seems to be: No. A possible way to show this is to fix from time to time three of the four parameters to

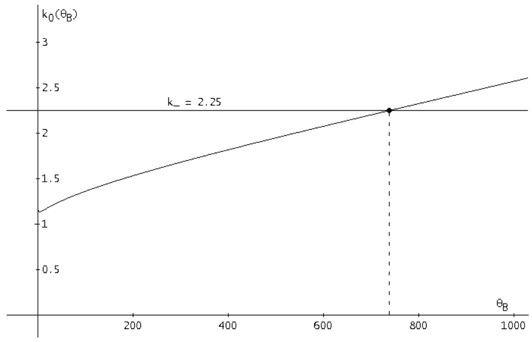

Figure 3: existence and stability of the equilibrium depending on the *behavioral* agent's demand.

the values which appear in (4.13) and see how k_0 behaves depending on the remaining free parameter and θ_B. Numerical analysis (3D plots available on request) show that the existence regions are still "wide" in the sense that the equilibrium can be sustained for not-negligible ranges. At this point we should make a step back. From the beginning of Section 4, we assumed as given the preferences of our investors, letting the market "adjust" in order to have equilibria. A natural question now is: what happens if those preferences change? Or, which is equivalent, what if we wrongly estimated utility function parameters or probability distortions? Just to give an idea, we performed calculations slightly changing some preference parameters and it is not surprising at all that results are qualitatively the same and they quantitatively only changed a few. To convince yourself, it would be sufficient to see what happens to the critical statistics k_0 if we make some perturbations.

First of all, changing α (or β, they must be equal) only affects the magnitude of $a_1(\theta_B, \theta_C^*)$, $a_2(\theta_B, \theta_C^*)$ and their ratio k_0, not its shape. Secondly, we modified probability distortions by choosing $\gamma = 0.61$ and $\delta = 0.69$ in (3.11), as estimated in [24]. This time we had an additional difficulty in that $a_1(\theta_B, \theta_C^*) \neq b_2(\theta_B, \theta_C^*)$, $a_2(\theta_B, \theta_C^*) \neq b_1(\theta_B, \theta_C^*)$ and $a_1(\theta_B, \theta_C^*) \geq a_2(\theta_B, \theta_C^*)$ does not always hold. However, this was not very problematic as the values of γ and δ were quite similar and they are not sufficient to heavily distort our analysis.

Finally, we recall that one assumption was $\lambda = 1$ for the CARA utility function of the *classical* agent; that was necessary in order to simplify calculations and comparisons. As this coefficient is a measure of risk-aversion, we expect it to have a role similar to those of σ^2 or η; therefore, we now let it vary over $(0, +\infty)$. At first, we have to solve Step 1 in a more general version and this time the optimal strategy will depend on λ too. Choosing $\epsilon, \eta > 0$ and discarding the negative solution, we find

$$\theta_C^*(\lambda, \mu, \epsilon, \sigma^2, \eta) = \frac{-(\lambda\sigma^2 - 2\epsilon) + \sqrt{(\lambda\sigma^2 - 2\epsilon)^2 + 6\lambda\eta(\mu - 1)}}{3\lambda\eta}.$$

Comparing the previous equation with (4.11), it is immediate to see that

$$\theta_C^*(\lambda, \mu, \epsilon, \sigma^2, \eta) = \theta_C^*(1, \mu, \epsilon, \lambda\sigma^2, \lambda\eta), \quad \forall\, \lambda \in (0, +\infty).$$

Therefore, adding the parameter λ amounts to perform a positive homogeneous transformation over the volatility parameters. Obviously, second order conditions are still fulfilled and the no-leverage constraint becomes

$$3\lambda\eta + 2\left(\lambda\sigma^2 - 2\epsilon - (\mu - 1)\right) \geq 0,$$

thus incorporating the same transformation. Fianlly, straightforward calculations show that θ_C^* is decreasing in λ as expected.

Remark 4.1. Our analysis has been conducted under a no-leverage and a no-short selling constraint. Removing the first one, i.e. allowing $\theta_B^*, \theta_C^* > 1$, has no dramatic consequences. In fact, by the side of the EU agent we just have to properly modify the first inequality in (4.12), whereas her optimal strategy remains unchanged. Therefore, in the case $\epsilon = 0$ the argument that prove Proposition 4.2 is still valid; on the other hand, in the case $\epsilon > 0$ we can rely on Figure 3 which shows that the *behavioral* agent should be able to borrow huge amounts of money[9] in order to deviate from $\theta_B^* = 0$. Furthermore,

[9] She would gamble using the wealth of someone else trying to manipulate the market ...

some graphics we provided also contain parameters' values that do not satisfy the no-leverage condition for the *classical* agent and we observed that an equilibrium is still attainable.

We also note that the no-short selling constraint is substantially unbinding as long as our investors are of the mean-variance type; loosely speaking, $\theta_B < 0$ or $\theta_C < 0$ would decrease the drift and increase the volatility at the same time thanks to Assumption 4.1. More specifically, a rapid look at equation (4.9) shows that allowing for negative θ_C would simply add a modulus in the third degree term. By the second order conditions, this implies a unique maximum point $\theta_C^* > 0$ for suitable choices of the market parameters satisfying the first inequality of (4.12) (or a different inequality if we allow taking leverage). Given this, the CPT agent would behave in the same way for $\theta_B \geq 0$; furthermore, she will never choose a moderate $\theta_B < 0$ under normal market conditions as she is substantially a mean-variance maximizer; in fact, $\theta_B < 0$ would produce lower expected return and higher volatility. Finally, graphical analysis shows that for negative θ_B the shape of k_0 is qualitatively specular to that of Figure 3, thus she would short sell (if possible) only huge amounts of the risky asset in order to manipulate the market.

4.2 Other equilibria

The remaining equilibria which are sustainable within our framework are those with $\theta_B^* \neq 0$. With Assumption 4.3 and recalling Theorem 3.3, for the existence of such equilibria it is necessary that the *behavioral* agent has preferences fulfilling cases *(iii)*, *(iv)* or *(v)*[10]. However, we already observed that the first two cases possess undesirable features; in particular, they are unstable and not realistic at all. Moreover, if we are looking for a non-zero demand of the *behavioral* agent, there is an additional "internal" problem concerning this kind of equilibrium. In fact, $\theta_B \neq 0$ implies that θ_C depends on θ_B too; therefore, the statistics a_1, a_2, b_1 and b_2 are difficult to control in the sense that a little variation of one's demand will surely destroy the equilibrium as $k_- = k_0(\theta_B^*, \theta_C^*)$ is a necessary condition in cases *(iii)* and *(iv)*.

The same problem arises even in case *(v)*, but it has different consequences. Apart from being computationally more difficult, we see that if we assume $\alpha < \beta$, then there can be another source of ill-posedness; the explanation relies on the fact that we can not use (3.7) to compute θ_B^*. Actually, equation (3.7) can be exploited only if we have a_1 and a_2 which do not depend on θ_B! In our case, those statistics are indeed influenced by the demand level of the *behavioral* investor, therefore we have to use a different argument in order to find the optimal θ_B. Moreover, we will show later that in many circumstances the CPT agent has an incentive to invest as much as possible in the risky asset, i.e. θ_B^* tends to explode, thus leading to an ill-posed problem if we do not impose some restrictions on her leverage.

After this introductory discussion, let's specify well the hypothesis under which we will work. We retain Assumption 4.1 with the same parameters' ranges if not otherwise stated and Assumption 4.2 with $\lambda = 1$ as a normalization. Concerning our *behavioral* agent, there are some important issues which deserve an explanation. We choose for her a typical S-shaped power utility function, this time with $0 < \alpha < \beta \leq 1$, a loss aversion coefficient $k_- = 1$ and probability distortion $w_+(\cdot)$ and $w_-(\cdot)$ of the Kahneman-Tversky type in (3.11), with $\gamma = \delta = 0.65$ as empirically estimated in [24].

Remark 4.2. As is known, in [24] the laboratory observations gave $\alpha = \beta = 0.88$; however, we must distort them in order to fulfill our assumptions. We also note that those values were obtained analyzing a sample of students and not a pool of professional financial investors. Nonetheless, this values are prone to errors, as every estimation is; thus it is plausible to observe different CRRA coefficients for gains and losses, expecially in real world financial markets. A confirm comes from the paper by Hwang and Satchell [13], where the difference $\beta - \alpha$ is estimated using US and UK market data. In particular, they find the values 0.2 and 0.25 respectively; moreover, they suggest $\alpha = 0.7$, $\beta = 0.9$ for the US investors and $\alpha = 0.7$, $\beta = 0.95$ for the UK investors.[11] To begin, we will arbitrarily set $\alpha = 0.80$ and $\beta = 0.95$, as we expect them to be quite close each other and not so different w.r.t. the original estimates. Obviously, in what follows we shall also analyze the effects that different values of those parameters have on the market equilibria.

[10] Imposing the no-short selling constraint, case *(iv)* is excluded and case *(v)* is allowed only with $\frac{a_1^\beta}{a_2^\alpha} \geq \frac{b_1^\beta}{b_2^\alpha}$; see equation (3.7). However, we will show that with our assumptions on the market structure, this inequality is automatically satisfied.

[11] In the light of Figure 1, we can not conclude that in real financial markets we have a violation of loss aversion. This is because we do not know what are the real probability distortions $w_+(\cdot)$ and $w_-(\cdot)$ and, more importantly, stock excess return is not normally distributed.

Remark 4.3. Our arbitrary choice of $k_- = 1$ serves as a normalization and it can be convincingly motivated recalling our analysis contained in Section 3.2. In fact, with $\alpha < \beta$ we saw that loss aversion violation can be a feature of this kind of models. Specifically, this time we have $\zeta = 1$, so that the violation range is given by $(0, 1)$ for any choice of $0 < \alpha < \beta \leq 1$. Within our setting, we note that a fixed amount of money x represents a deviation from the reference point which in our model was set as the risk-free return on the initial wealth, namely $W_0^B r = 1$. Therefore, any optimal terminal wealth $X_B \in (1, 2)$ is the result of a violation happened during the maximization procedure of our CPT agent. As expected, this condition will be verified with high probability and this phenomenon is retained even if we change the value of k_-. To explain this fact, let's choose a $k_- > 1$ to fix ideas; consequently, ζ is reduced and the violation range shrinks but at the same time the optimal demand level θ_B^* is drastically reduced, as k_- negatively affects the prospect value $V(\cdot)$ (see equation (2.3)). Hence, the final wealth X_B will be lower too and the probability of a violation will still be elevated. In conclusion, letting k_- vary only has quantitative and not qualitative effects on our equilibrium model. Nonetheless, analytical results of this section have been proved for every choice of $k_- > 0$, thus ensuring mathematical correctness.

Another important issue concerns the constraints that we are going to impose on the strategies of our agents. We will always admit only $\theta_B > 0$ as we want to avoid market manipulation with heavy short selling which could be exploited in order to reach equilibrium return and volatility favorable to the CPT agent. At the same time, the no-leverage constraint $\theta_B \leq 1$ is not imposed because one of our goals is to see what happens if she is able to (moderately) borrow money from the bond market. However, an upper bound will be fixed to make the model sensible and it is sometimes necessary to have an equilibrium; see Lemma 4.2 in the next section. On the contrary, we are not imposing any restriction to the *classical* agent, as they will be specified in time to time.

The equilibrium values (if they exist) will be denoted with a *, as usual. In particular, if the investors select the pair (θ_B^*, θ_C^*), then we will have

$$\hat{\mu}^* = \mu + \epsilon\theta_B^* + \epsilon\theta_C^*, \qquad \hat{\sigma}^{2*} = \sigma^2 + \eta\theta_B^* + \eta|\theta_C^*|. \tag{4.14}$$

Recalling the game theoretical nature of our model, to discover equilibria we shall implement the following procedure:

Step 1 Solve the maximization problem for the *classical* agent given any strategy θ_B of the *behavioral* investor, namely

$$\max_{\theta_C \in D} U_C(\theta_B, \theta_C), \tag{4.15}$$

where $D \subseteq \mathbb{R}$ is a suitable set of admissible strategies.
Let $\theta_C^*(\theta_B) := \theta_C^*(\theta_B, \mu, \epsilon, \sigma^2, \eta)$ be a maximizer of (4.15), if one exists.

Step 2 Fix the market parameters $(\mu, \epsilon, \sigma^2, \eta)$ and for any $\theta_C^*(\theta_B)$ previously obtained, solve the optimization problem for the *behavioral* agent, namely

$$\sup_{\theta_B > 0} V\left(X_B\left(\theta_B, \theta_C^*(\theta_B)\right)\right), \tag{4.16}$$

where the value function $V(\cdot)$ is given by (2.5) and the terminal wealth X_B is given by (4.6).

Step 3 If (4.16) actually is a maximum, then compute $\theta_C^*(\theta_B^*)$, $\hat{\mu}^*$ and $\hat{\sigma}^{2*}$ in order to obtain the equilibrium strategies and the equilibrium market parameters; else restrict the set of admissible strategies for the CPT agent imposing a sensible upper bound and solve (4.16) once more. If the sup in (4.16) is attained for $\theta_B \downarrow 0$, then there are no equilibria.

Step 1 is not difficult to implement as it is a generalization of its counterpart in the case $\theta_B^* = 0$. Performing similar calculations, we see that (4.15) is equivalent to

$$\max_{\theta_C \in D} 1 - \exp\left\{\frac{\eta}{2}|\theta_C|^3 + \left(\frac{\sigma^2}{2} - \epsilon + \frac{\eta}{2}\theta_B\right)\theta_C^2 + (1 - \mu - \epsilon\theta_B)\theta_C - 1\right\}. \tag{4.17}$$

Therefore, we have the modulus in the third degree term as θ_C is not necessarily positive; moreover, there are two additional terms which depends on θ_B. As our goal is to minimize the exponential argument, we see that the quadratic term acts negatively and it is more influent as η and θ_C^2 become greater. On the contrary, the linear term positively together with ϵ and θ_C. Thus, their overall effect has to be analyzed and we shall again distinguish between the cases of totally exogenous or endogenous volatility.

4.2.1 The case with null volatility impact

In the simpler case with given exogenous volatility $\sigma^2 > 0$, we see that equation (4.17) reduces to

$$\max_{\theta_C \in D} \; 1 - \exp\left\{\left(\frac{\sigma^2}{2} - \epsilon\right)\theta_C^2 + (1 - \mu - \epsilon\theta_B)\theta_C - 1\right\}.$$

Therefore it remains just one additional term depending on θ_B and there is no more distinction between a positive or negative θ_C. First order conditions give

$$\frac{\partial U_C}{\partial \theta_C}(\theta_B, \theta_C) = 0 \iff \theta_C = \frac{(\mu - 1) + \epsilon\theta_B}{\sigma^2 - 2\epsilon}. \tag{4.18}$$

With the usual parameters assumptions $\sigma^2 - 2\epsilon > 0$, $\epsilon > 0$ and having imposed $\theta_B > 0$, we see that $\theta_C^*(\theta_B) = \frac{(\mu-1)+\epsilon\theta_B}{\sigma^2-2\epsilon}$ is the unique strictly positive global maximizer as the second order conditions confirm. As a consequence, we can even choose $D = \mathbb{R}$ as the set of admissible strategies. Clearly, since $\epsilon > 0$, θ_C^* is increasing in θ_B; this positive dependence is no surprising if one thinks to the positive impact of someone's demand on the equilibrium drift $\hat{\mu}^*$ and the null impact over the volatility. However, for the moment we can not state how θ_C^* varies depending on the market parameters, as we do not know how θ_B^* behaves. We will analyze these facts later.

Before fixing market parameters, we state some general facts about the maximization problem of the *behavioral* agent in Step 2. The following two lemmas are indeed valid for every choice of the preference parameters $0 < \alpha < \beta \leq 1$ and $\lambda > 0$ [12]; the proofs can be adjusted in a straightforward way.

Lemma 4.1. *Assume $\eta = 0$ and $\sigma^2 > 2\epsilon > 0$. Then we have $\frac{a_1(\theta_B)^\beta}{a_2(\theta_B)^\alpha} \geq \frac{b_1(\theta_B)^\beta}{b_2(\theta_B)^\alpha}$ for every $\theta_B > 0$.*

Proof. Using integration by parts formula, it is immediate to see that we have

$$a_1(\theta_B) = \int_0^{+\infty} \alpha t^{\alpha-1} w(1 - F(t))dt \geq \int_{-\infty}^0 \alpha(-t)^{\alpha-1} w(F(t))dt = b_1(\theta_B),$$

$$a_2(\theta_B) = \int_{-\infty}^0 \beta(-t)^{\beta-1} w(F(t))dt \leq \int_0^{+\infty} \beta t^{\beta-1} w(1 - F(t))dt = b_2(\theta_B),$$

which holds for every $\theta_B > 0$ thanks to the positive skewness of the distribution of the risky asset return. The fact $\alpha < \beta$ concludes. □

Remark 4.4. We note that even with a positive η, as long as $\mu \geq 1$, $\epsilon > 0$ and the agent's demands are positive, the c.d.f. $F(\cdot)$ which appears in the previous proof maintains its asymmetry.

As we focus on equilibria with $\theta_B > 0$, by Lemma 4.1 we see that we recover case *(v)* of Theorem 3.3. However, we can not use (3.7) to find the optimal strategy of the *behavioral* agent, as observed at the beginning of this section.

Lemma 4.2. *With the same assumptions of Lemma 4.1, the optimization problem* (4.16) *of the behavioral agent is equivalent to*

$$\sup_{\theta_B > 0} \; \theta_B^\alpha a_1(\theta_B) - k_- \theta_B^\beta a_2(\theta_B). \tag{4.19}$$

Moreover, problem (4.19) *is ill-posed and we have*

$$\lim_{\theta_B \downarrow 0} \; \theta_B^\alpha a_1(\theta_B) - k_- \theta_B^\beta a_2(\theta_B) = 0. \tag{4.20}$$

Proof. See the Appendix. □

We immediately observe that the equivalence between the two problems and equation (4.20) holds true even with $\eta > 0$ but the arguments must be changed (for more details, see the proof in the Appendix). The previous lemma will now be exploited to numerically implement Step 2.

Due to the ill-posedness result, we see that there can not be an equilibrium unless we restrict θ_B over a rightward closed interval, e.g. $\theta_B \in (0, L]$. In such a case, we can modify (4.19), obtaining

$$\sup_{\theta_B \in (0,L]} \; \theta_B^\alpha a_1(\theta_B) - k_- \theta_B^\beta a_2(\theta_B). \tag{4.21}$$

Note that we can have two distinct types of equilibria:

[12] Using a $\lambda \neq 1$ amounts to replacing σ^2 with $\lambda\sigma^2$ in (4.18). Thus, the qualitative effects on the equilibria can be simply analyzed.

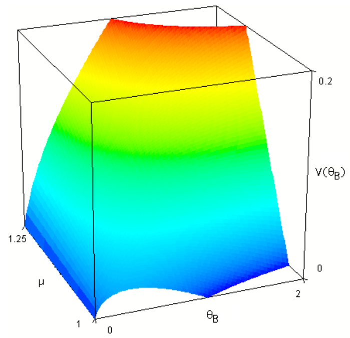

Figure 4: prospect value depending on the exogenous constant drift μ.

- *(internal equilibrium)* in this case, an equilibrium exists if there is at least one local maximizer of (4.21); obviously, θ_B^* will be the "best" maximizer of (4.21) only if the respective prospect value for the *behavioral* investor is greater than the value obtained when selecting $\theta_B = L$.
- *(boundary equilibrium)* if there are no local maximizers of (4.21), we can choose $\theta_B^* = L$.

We observe that this last case will always produce an equilibrium unless the supremum of (4.21) is zero. Our opinion is that it seems to be somewhat unrealistic that the *behavioral* agent invest all her wealth in the risky asset (or she get as much leverage as she can). Typically, householders prefer to invest all their wealth in bonds instead of risking everything in the stock market. However, if one thinks to a *behavioral* fund manager who is trying to beat the benchmark through massive stock investment, then it would be more plausible to observe such an extreme optimal policy.

For the sake of clarity, let's arbitrarily fix $\theta_B \in (0, 2]$, thus the CPT investor is allowed to borrow as much as her initial wealth. At this point, (4.21) results in a function depending on the market parameters (μ,ϵ,σ^2) with the constraints $\mu \geq 1$, $\sigma^2 > 2\epsilon > 0$. Therefore, we need to fix two of these three values in order to obtain 3D-pictures and see if there exists a maximum point over θ_B as long as the free parameter varies; in such cases, we are also able to depict 2D-graphs representing the "implicit maximum curve", i.e. $\theta_B^*(\cdot)$ where the dot represents one market parameter. Then, we will plot those curves for scattered values of one remaining parameter.

We discuss the main results, depending on the free parameter. With an abuse of notation, we will denote with $V(\theta_B)$ the prospect value $V(X_B(\theta_B, \theta_C^*(\theta_B)))$, dropping the dependence on (μ,ϵ,σ^2). Graphics are provided only when μ varies; for the other cases, they are available on request.

- **Figure 4**: $\mu \in [1, 1.25]$, $\epsilon = 0.005$, $\sigma^2 = 0.49$. From the 3D-surface it is immediate to see that internal equilibria exist only if μ is sufficiently low; otherwise, we will have boundary equilibria as $V(\cdot)$ is strictly increasing for every $\theta_B \in [0, 2]$.
- **Figure 5**, *left plot*: $\mu \in [1, 1.25]$, $\epsilon = 0.005$. Graphical analysis suggests

$$\frac{\partial \theta_B^*}{\partial \mu} > 0, \qquad \frac{\partial \theta_B^*}{\partial \sigma^2} < 0.$$

  We also note that σ^2 has to be sufficiently high in order to have an internal equilibrium and if we wish no leverage for the *behavioral* agent, i.e. $\theta_B^* \leq 1$, then we must choose an even higher variance. The explanation of these facts is obvious if one thinks that our *behavioral* investor is willing to invest more as the volatility decreases or as the expected return increases.

- **Figure 5**, *right plot*: $\mu \in [1, 1.2]$, $\sigma^2 = 0.49$. This time we have

$$\frac{\partial \theta_B^*}{\partial \mu} > 0, \qquad \frac{\partial \theta_B^*}{\partial \epsilon} > 0.$$

  It is interesting to observe that for sufficiently high values of ϵ we are able to find two distinct levels of demand which are stationary points, where the lowest one, θ_{B1}, is a local maximum and the higher one, θ_{B2} is a local minimum (in the plot they are denoted with dashed lines). Thus, the CPT investor has to compare $V(2)$ with $V(\theta_{B1})$, i.e. she evaluates if it is more convenient to totally exploit the leverage and set $\theta_B^* = 2$ or to choose the local maximizer and select $\theta_B^* = \theta_{B1}$.

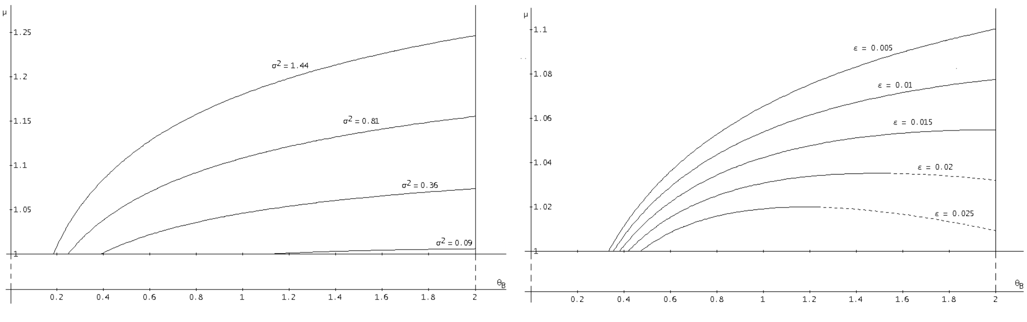

Figure 5: implicit maximum curves depending on the exogenous constant drift μ.

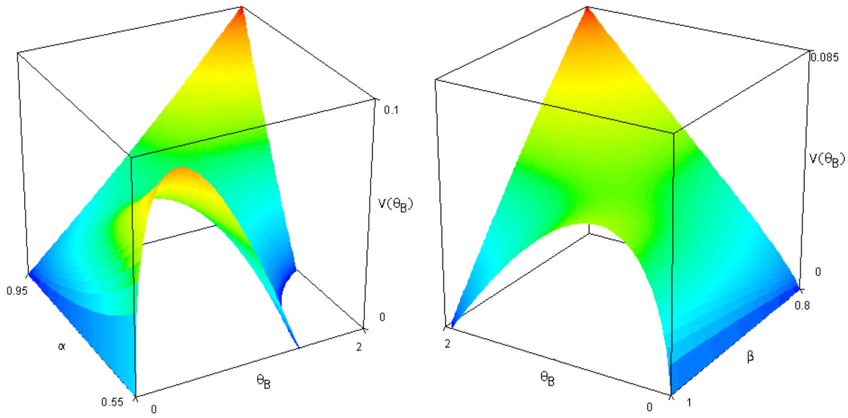

Figure 6: prospect value depending on the preference parameters of the *behavioral* agent.

We note that this comparison is unnecessary for lower values of ϵ as we only find one stationary point θ_{B1} which is a local maximum; hence the prospect value $V(\cdot)$ is strictly decreasing over $(\theta_{B1}, 2]$ and the optimal strategy is $\theta_B^* = \theta_{B1}$.

For the sake of completeness, when ϵ is the free parameter, we observe that while keeping μ fixed, θ_B^* is increasing in ϵ and decreasing in σ^2, whereas if σ^2 is fixed, then θ_B^* is increasing in both μ and ϵ. Finally, when σ^2 is the free parameter, we observe θ_B^* to be increasing in μ and decreasing in σ^2 as long as ϵ is kept fixed, whereas θ_B^* is increasing in ϵ and decreasing in σ^2 as long as mu is held fixed.

Before concluding Step 3, it remains to see what happens if we change the preference parameters of our *behavioral* investor; the motivation comes from [13], where we recall that $\beta - \alpha$ was highlighted as an important quantity in real world financial markets. Hence, having fixed $\mu = 1.05$, $\epsilon = 0.005$, $\sigma^2 = 0.49$, we performed a graphical and numerical analysis whose main results are shown in Figure 6.

- **Figure 6**, *left plot*: $\alpha \in [0.55, 0.95)$, $\beta = 0.95$. The 3D-surface represents the prospect value for the CPT agent. It is immediate to see that such an investor has substantially different reactions depending on the difference between β and α. Recalling equations (4.20) and (4.21), we observe that for $\alpha \approx 0.95$ (i.e. when the difference tends to be null) we do not have a local maximum, thus it is optimal $\theta_B^* = 2$. On the other hand, a greater discrepancy has the notable effect to produce stationary points; in particular, we obtain local maxima for lower θ_{B1} and local minima for higher θ_{B2}.These last type of stationary points has not been represented because with our choice of the market parameters we have $\theta_{B2} > 2$. Furthermore, if we select a different β and let vary $\alpha < \beta$, then we get qualitatively identical plots.

- **Figure 6**, *right plot*: $\alpha = 0.8$, $\beta \in (0.8, 1]$. A somewhat similar result can be observed; once again the important quantity is the difference $\beta - \alpha$, which can produce the local maxima or not. Finally, with a lower α, the surfaces we obtained for $\beta \in (\alpha, 1]$ are very similar to the one we depicted.

To conclude Step 3, we propose a particular equilibrium. Note that if we fix a triple (μ,ϵ,σ^2), then it is easy to numerically compute θ_B^* and then replacing its value in the explicit expression of $\theta_C^*(\cdot)$.

After that, one can find the equilibrium drift and volatility simply using (4.14). Just as an example, choosing the triple (1.06, 0.005, 0.64) and approximating with order 10^{-5} we obtain

$$(\theta_B^*, \theta_C^*) = (0.67110, 0.10056), \quad \hat{\mu}^* = 1.0638, \quad \hat{\sigma}^{2*} = 0.64; \tag{4.22}$$

hence, the *behavioral* agent will risky invest about two thirds of her wealth, whereas the *classical* agent will choose only an approximate 10% of stock buying. Obviously, these optimal strategies does not reflect the typical policies observed in the real world. As a matter of fact, we were able to show that it is easy to provide existence of an equilibrium despite its undesirable properties.

4.2.2 The case with volatility impact

Turning back to equation (4.17) with $\eta \neq 0$, the optimal strategy for the *classical* investor can be computed distinguishing the cases $\theta_C \geq 0$ or $\theta_C < 0$. Using first and second order optimality conditions, given any positive *behavioral* agent's demand level $\theta_B > 0$, it is straightforward to obtain

$$\theta_C^*(\theta_B) = \frac{-(\sigma^2 - 2\epsilon + \eta\theta_B) + \sqrt{(\sigma^2 - 2\epsilon + \eta\theta_B)^2 + 6\eta(\mu - 1 + \epsilon\theta_B)}}{3\eta}. \tag{4.23}$$

Recalling Assumption 4.1 on the ranges of our market parameters and the no-short selling constraint that binds the CPT investor, we see that $\theta_C^* \geq 0$ and the equality holds if and only if $\mu = 1$ and $\epsilon = 0$, i.e. when the stock expected excess return is null and there is no drift impact. Hence, we can choose once again $D = \mathbb{R}$ as there is only one global maximizer for (4.17).

Just like we did in the case $\eta = 0$, we now analyze the dependence of $\theta_C^*(\cdot)$ on θ_B. It is interesting to observe that for every $\theta_B > 0$ we have

$$\frac{\partial \theta_C^*}{\partial \theta_B}(\theta_B) \gtreqless 0 \iff \epsilon^2 - 2\epsilon\sigma^2 + 2\eta(\mu - 1) \lesseqgtr 0. \tag{4.24}$$

Therefore, we can have positive, negative or null derivative and this will only depend on an exogenous condition over market parameters. Moreover, as ϵ reasonably assumes small values, we can ignore the term ϵ^2 and (4.24) reduces to

$$\frac{\partial \theta_C^*}{\partial \theta_B}(\theta_B) \gtreqless 0 \iff \frac{(\mu - 1)}{\epsilon} \lesseqgtr \frac{\sigma^2}{\eta},$$

where we can see a comparison between the ratios of the "constant" parts of the market drift and volatility, and the impact parameters. In conclusion, a positive dependence takes place whenever the relative impact effect on the variance is greater than the relative impact effect on the drift and vice versa. An economical interpretation is immediate if one thinks to the consequences of a greater volatility impact coefficient η; in fact, the more η is elevated, the more risky investment "hurts" and it is more probable to have a negative derivative. On the other hand, a greater drift impact coefficient ϵ implies higher expected utility, thus probably leading to a positive dependence. These results are quite obvious if we think to the FOSD and the SOSD which stands behind this model; however, we can not conclude anything about what may happen *in equilibrium* because for the moment we do not know how θ_B^* changes when we vary market parameters. We refer the reader to the end of this section for a more detailed analysis, when we will depict some equilibrium curves just like we did in the case with null volatility impact.

Now we shift our attention to Step 2, namely the *behavioral* investor's problem. We recall that in the previous case with $\eta = 0$, we were able to prove Lemma 4.1 and Lemma 4.2. In particular, we remarked that even if we assumed $\eta > 0$ all their conclusions were still valid, but the ill-posedness of (4.19) could not be proved in the same way. Analyzing the proof of Lemma 4.2, we see that a crucial role was played by the relation $\frac{\partial \theta_C^*}{\partial \theta_B}(\theta_B) > 0$ for every $\theta_B > 0$. Unfortunately, we have just seen that this fact does not hold true anymore! However, we can prove the following result which strongly separates the cases $\epsilon = 0$ and $\epsilon > 0$, i.e. when there is or not a positive drift impact.

Lemma 4.3. *Under Assumption 4.1 with $\eta > 0$,*

- *if $\epsilon = 0$, then for every $\theta_B > 0$ we have*

$$\frac{\partial \hat{\mu}}{\partial \theta_B}(\theta_B, \theta_C^*(\theta_B)) = 0, \qquad \frac{\partial \hat{\sigma}^2}{\partial \theta_B}(\theta_B, \theta_C^*(\theta_B)) > 0. \tag{4.25}$$

Moreover,

$$\lim_{\theta_B \to +\infty} V(\theta_B) = -\infty, \tag{4.26}$$

therefore Problem (4.19) *is well-posed;*

- *if* $\epsilon > 0$*, then for every* $\theta_B > 0$ *we have*

$$\frac{\partial \hat{\mu}}{\partial \theta_B}(\theta_B, \theta_C^*(\theta_B)) > 0, \qquad \frac{\partial \hat{\sigma}^2}{\partial \theta_B}(\theta_B, \theta_C^*(\theta_B)) > 0. \tag{4.27}$$

Moreover,

$$\lim_{\theta_B \to +\infty} V(\theta_B) = +\infty, \tag{4.28}$$

therefore Problem (4.19) *is ill-posed.*

The proof is relegated to the Appendix, as it is quite technical and cumbersome; now we give an interpretation of the previous results and their consequences. First of all, recall that $\hat{\mu}$ and $\hat{\sigma}^2$ are the overall market drift and volatility; in the case $\epsilon = 0$ we have no drift impact, which obviously implies the first equality of (4.25) and even produces well-posedness thanks to the SOSD which affects the preferences of our *behavioral* agent. In fact, we have already seen that a higher θ_B only produces an increase in the overall volatility and its main consequence is the disadvantage caused by huge investment in the stock.
The case $\epsilon > 0$ is sharply different and equation (4.27) means that a higher demand of the *behavioral* agent will lead to an increase in the global drift and volatility of the market, independently on the dependence of $\theta_C^*(\cdot)$ on θ_B. This was quite obvious in the case $\eta = 0$ but it was not in the present situation, as a growth of θ_B leads to greater drift and variance, which partly encourages and partly discourages risky investment. Fortunately, this combined effect can be shown to be favorable to our *behavioral* investor, as the ill-posedness of (4.19) states. This is equivalent to say that $\lim_{\theta_B \to +\infty} V(\theta_B) = +\infty$; on the other hand, numerical simulations show that even for very high levels of θ_B we can have $V(\theta_B) < 0$, thus massive risky investment can be necessary before $V(\cdot)$ undertakes a monotone increasing phase. These results motivate our choice of allowing $\theta_B \in (0, 2]$ as we did before and we will subsequently look for *internal* and *boundary* equilibria, which always exist unless $\sup_{\theta_B \in (0,2]} V(\theta_B) = 0$.

At this point nothing more can be said from an analytical point of view and we have to rely on numerical simulations. As a first step, we provide 3D-plots which represent the prospect value $V(\cdot)$ of our *behavioral* agent, depending on $\theta_B \in (0, 2]$ and one market parameter, where the other three are kept fixed. As our aim is to show the influence of $\eta \neq 0$ on the existence and on the properties of equilibria that we could obtain, we decided to maintain the same drift parameters as in Section 4.1, namely $\mu = 1.05$ and $\epsilon = 0.005$; on the other hand, we chose $\sigma^2 = 0.09$ and $\eta = 0.2$. Note that in the previous analysis we selected $\sigma^2 = 0.49$, i.e. a higher exogenous volatility arising from noise trading, simply because the equilibrium results were more interesting. Nonetheless, in this new setting a lower σ^2 is sufficient, as the effect of a positive volatility impact is to raise the overall variance thanks to the fact that $\theta_C^*(\theta_B) \geq 0$ and $\theta_B^* > 0$[13]. Moreover, recall that lower values of σ correspond to more realistic scenarios, as we noted in Section 4.1.1.

- **Figure 7**, *top-left plot*: $\mu \in [1, 1.25]$, $\epsilon = 0.005$, $\sigma^2 = 0.09$, $\eta = 0.2$. At a first glance we note the similarity with the 3D-surface in Figure 4, hence the additional parameter $\eta > 0$ seems to have no qualitative effect on the exogenous drift μ.

- **Figure 7**, *top-right plot*: $\mu = 1.05$, $\epsilon \in [0, 0.02]$, $\sigma^2 = 0.09$, $\eta = 0.2$. We note that we have an internal equilibrium for the represented values of ϵ. On the contrary, increasing ϵ would produces boundary equilibria.

- **Figure 7**, *down-left plot*: $\mu = 1.05$, $\epsilon = 0.005$, $\sigma^2 \in [0, 1.44]$, $\eta = 0.2$. If we let σ vary with $\eta > 0$, even for $\sigma \approx 0$ we can observe a positive (and sufficiently elevated) global volatility; this in turn produces the "tunnel" shape of the down-left surface of Figure 7.
  From a theoretical point of view, we could even suppose $\sigma = 0$; economically speaking, this would mean absence of noise trading, or equivalently no other agent is influencing the market price except for our *classical* and *behavioral* investors. This setting is by no doubt interesting as it could represent an hypothetical competitive bargaining model between our two agents; we note

[13] Recall that $\theta_C^* = 0$ if and only if $\mu = 1$ and $\epsilon = 0$; see equation (4.23).

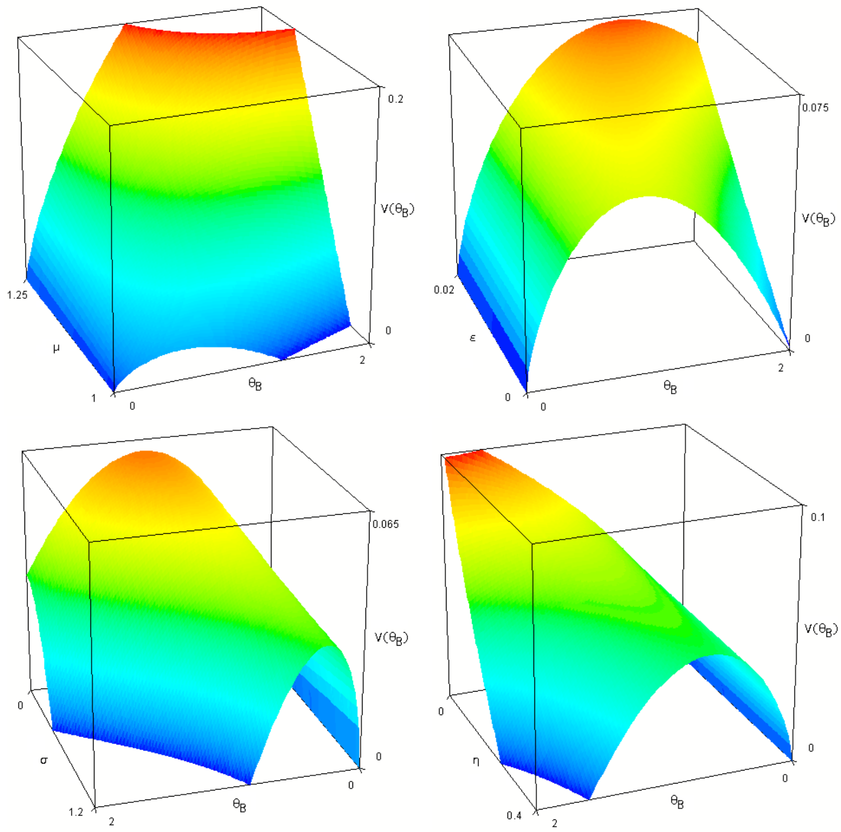

Figure 7: prospect value depending on θ_B and one market parameter.

that the exogenous drift constant μ is not forced to assume a particular value, as it arises from the fundamentals of the firm whose stocks are priced (or the fundamentals of the underlying economy, if our risky asset replicates some financial index). From (4.24) we see that $\frac{\partial \theta_C^*}{\partial \theta_B} \geq 0$, i.e. we always have a higher θ_C^* whenever the *behavioral* agent invests more.

- **Figure 7**, *down-right plot*: $\mu = 1.05$, $\epsilon = 0.005$, $\sigma^2 = 0.09$, $\eta \in (0, 0.4]$. The right-down plot of Figure 7 shows that for low values of η we find once again a monotone growth of the prospect value, whereas sufficiently high η provides a unique local maximum as expected. Therefore, a parameter which is important in order to determine the optimal strategy of our *behavioral* agent and to produce internal or boundary equilibria is the summed effect of σ and η, as they both influence the equilibrium volatility.

Performing graphical analysis, we observed a strong similarity of the implicit maximum curves which can be obtained in the case $\eta > 0$ with those in the case of null volatility impact. Moreover, scattering η with a fixed exogenous volatility is quite the same thing that scattering σ in the case $\eta = 0$.

To conclude, an interesting analysis can be made by depicting on the plane (θ_C^*, θ_B^*) some equilibrium curves, i.e. we can fix two market parameters and compute the equilibrium pair (θ_C^*, θ_B^*) which depends on the remaining free parameter. We show some equilibrium curves in Figure 8; in particular, the big rounded dot in the plots represent the equilibrium obtained with the following parameters' values: $\mu = 1.02$, $\epsilon = 0.005$, $\sigma = 0.6$ and $\eta = 0.1$. Hence, approximating with order 10^{-5}, we have

$$(\theta_B^*, \theta_C^*) = (0.50097, 0.05511), \quad \hat{\mu}^* = 1.02278, \quad \hat{\sigma}^{2*} = 0.41561.$$

On each plot we reported the lower and the upper bound of the range of the free parameter, which is scattered with a constant mesh size. As expected, Figure 8 shows that greater drift parameters imply a growth in the optimal demand levels of both investors, whereas an increase in the volatility parameters has an opposed effect. Quite interestingly, from the down-left plot we see that as σ varies it seems that the equilibrium curve is convex. Furthermore, the down-right plot shows that with this choice of market parameters, the effect of η on θ_C^* is very weak.

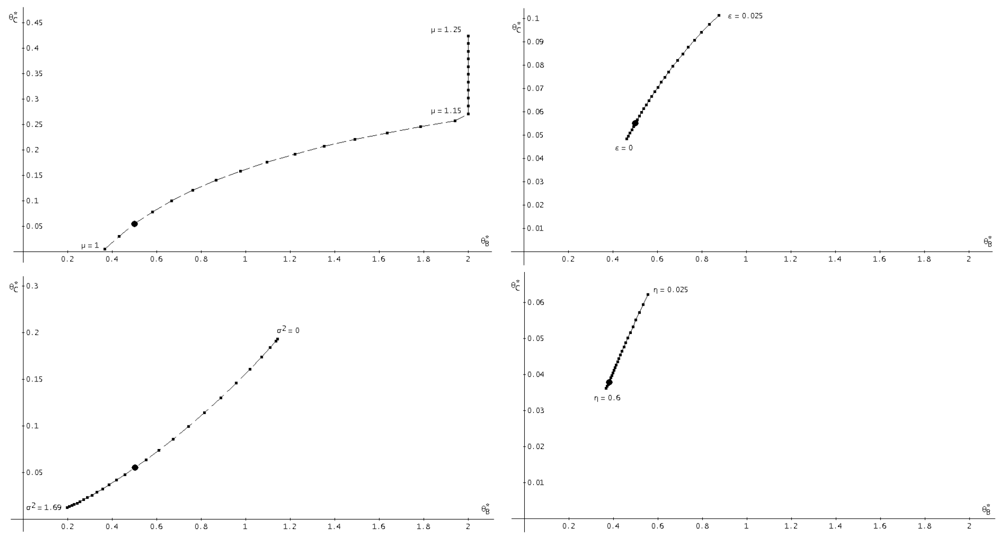

Figure 8: equilibrium curves depending on one market parameter with positive volatility impact.

In order to avoid misunderstanding of these results, we recall that equation (4.24) specifies the way θ_C^* reacts on θ_B (but not the reaction on θ_B^* when we vary one or more parameters!). Therefore, we can observe positive correlation among the optimal demand levels even if $\frac{\partial \theta_C^*}{\partial \theta_B} < 0$. Just as an example, we have $\frac{\partial \theta_C^*}{\partial \theta_B} < 0$ if and only if $\sigma < 0.6344$ but at the same time a decrease in σ "pushes up" $\theta_C^*(\cdot)$ thanks to the SOSD. In our case, it turns out that the overall effect of a decrease in σ is to raise θ_B^* and $\theta_C^*(\theta_B^*)$ also for $\sigma < 0.6344$.

4.3 The model with two EU agents

We now perform a detailed analysis of a model analogous to the previous one, but this time we consider the case of two interacting *classical* agents, both characterized by CARA utility functions. The aim of this section is to highlight similarities and differences w.r.t. to the preceding scenario, also clarifying some aspects of our setting thanks to the greater availability of explicit formulas. However, we will see that in the general case graphical analysis still remains the best tool; in fact the complexity due to the presence of many parameters makes it difficult to obtain friendly expressions.

To begin, we keep the same hypothesis on the market structure, i.e. Assumption 4.1 will be in force throughout this section. Concerning our traders, we will identify them with the subscripts $_1$ and $_2$; we suppose a common initial endowment $W_{0,1} = W_{0,2} = 1$ and their strategies will be denoted as θ_1 and θ_2. Moreover, regarding their preferences we have

Assumption 4.4 (CARA classical agents with no-short selling). *The classical agents' utility functions are*

$$u_1(x) = 1 - \exp(-\lambda_1 x), \tag{4.29}$$

$$u_2(x) = 1 - \exp(-\lambda_2 x), \tag{4.30}$$

where $\lambda_1, \lambda_2 > 0$ are the constant absolute risk aversion coefficients. Moreover, we assume

$$\min\{\lambda_1, \lambda_2\} > \frac{2\epsilon}{\sigma^2}. \tag{4.31}$$

Finally, short-selling is not allowed, i.e. $\theta_1, \theta_2 \geq 0$.

Assumptions 4.1 and 4.4 are imposed to retain a framework as much as possible similar to the previous one; in particular, we will partially recover analogous results. We observe that the constraint (4.31) is nothing but a second order condition which ensures the well-posedness of our model; in particular, it is imposed to have a maximum for the objective functions of our investors, similarly to what we did in the case of one EU and one CPT trader. Nonetheless, the presence of two possibly distinct CARA coefficients makes the analysis more involved (and interesting). Finally, w.l.o.g. we restrict our attention to equilibria with $(\theta_1, \theta_2) \in \mathbb{R}_+^2$. Intuitively, this can be done as our investors are substantially of the mean-variance type, thus short-selling would induce a lower mean and a higher variance. In fact, the reason is similar to that explained at the beginning of Section 4.2.2: under reasonable market conditions, if one agents selects $\theta_i^* \geq 0$, then for the other trader it is never optimal to choose a negative risky investment level. However, we are not able to exclude *a priori* the existence of an equilibrium with both negative θ_i; of course, this can be considered a pathological situation as Assumptions 4.1 imposes a non negative exogenous risk premium; hence it will not receive our attention.

We now proceed as usual; at first, we write down the objective functions of the two investors. Then, we look for pure strategy equilibria and this will be done by using first and second order optimality conditions, also providing explicitly the equilibrium strategies in some particular cases. We remark the importance of the symmetry that underlies our model; this fact allow us to develop the analysis only for one trader, the other being almost identical as it is sufficient to interchange subscripts. The main result concerning this model is the following Proposition, whose proof can be found in the Appendix together with those of the subsequent corollaries.

Proposition 4.3. *Under Assumptions 4.1 and 4.4, for every choice of the parameters* $(\mu, \epsilon, \sigma^2, \eta)$ *and* (λ_1, λ_2) *the model is well-posed and there exist an equilibrium. Moreover, there are only two types of equilibria, namely:*

- boundary equilibria *with* $\theta_1^* = \theta_2^* = 0$*;*
- internal equilibria *with* $(\theta_1^*, \theta_2^*) \in \mathbb{R}^2_{++} := (0, +\infty)^2$.

To better understand the conclusions of this model, it is necessary to split the cases with $(\eta > 0)$ or without $(\eta = 0)$ volatility impact. We begin with the easier one.

Corollary 4.1. *Assume* $\eta = 0$*. Then the equilibrium strategies are*

$$\theta_1^* = \frac{(\mu-1)(\lambda_2\sigma^2 - \epsilon)}{(\lambda_1\sigma^2 - 2\epsilon)(\lambda_2\sigma^2 - 2\epsilon) - \epsilon^2}, \qquad \theta_2^* = \frac{(\mu-1)(\lambda_1\sigma^2 - \epsilon)}{(\lambda_1\sigma^2 - 2\epsilon)(\lambda_2\sigma^2 - 2\epsilon) - \epsilon^2}. \tag{4.32}$$

Consequently, we have the following characterizations:

- $\theta_1^* = \theta_2^* = 0 \iff \mu = 1$*;*
- *if* $\mu > 1$*, then* $\theta_1^* \gtreqless \theta_2^* \iff \lambda_1 \lesseqgtr \lambda_2$.

Before passing to the case with positive volatility impact, there are some interesting facts to highlight. First of all, the case $\epsilon = 0$ usually reduces to the classic results. Secondly, these equilibria can be represented as straight lines on the plane (θ_1, θ_2). In fact, it is immediate to see that if $\mu \geq 1$, then we have

$$\theta_1^* = \left(\frac{\lambda_2\sigma^2 - \epsilon}{\lambda_1\sigma^2 - \epsilon}\right)\theta_2^*,$$

which obviously implies the preceding characterization. Now, a straightforward sensitivity analysis gives these results:

$$\frac{\partial\theta_1^*}{\partial\mu} > 0, \qquad \frac{\partial\theta_1^*}{\partial\epsilon} > 0, \qquad \frac{\partial\theta_1^*}{\partial\sigma^2} < 0, \qquad \frac{\partial\theta_1^*}{\partial\lambda_1} < 0, \qquad \frac{\partial\theta_1^*}{\partial\lambda_2} < 0.$$

In particular, the last inequality can be interpreted in this way: the more agent 2 is risk-averse, the less she will invest in stock; as a consequence, agent 1 will lower her risky exposure too.
Note that we can also explicitly compute a no-leverage condition; specifically, agent 1 will not borrow money if and only if

$$\lambda_1 \geq \lambda_1^{NL} := \frac{(\mu-1)(\lambda_2\sigma^2 - \epsilon) + \epsilon(2\lambda_2\sigma^2 - 3\epsilon)}{\sigma^2(\lambda_2\sigma^2 - 2\epsilon)},$$

and a similar expression can be obtained for agent 2. Standard computations imply

$$\frac{\partial\lambda_1^{NL}}{\partial\mu} > 0, \qquad \frac{\partial\lambda_1^{NL}}{\partial\epsilon} > 0, \qquad \frac{\partial\lambda_1^{NL}}{\partial\sigma^2} > 0, \qquad \frac{\partial\lambda_1^{NL}}{\partial\lambda_2} < 0.$$

Therefore, if the return parameters μ and ϵ increase, then our investor is more favorable to exploit leverage. Hence, she behaves in this way because she tries to reach a higher terminal wealth by gambling with the money of someone else. Note that this policy is adopted even if the exogenous volatility grows; thus, risky investment is more attractive when someone else bears negative results!

To conclude, we make a comparison to the case with one CPT trader and no volatility impact. We stress that with the presence of a CPT investor, the existence of the equilibrium was not always

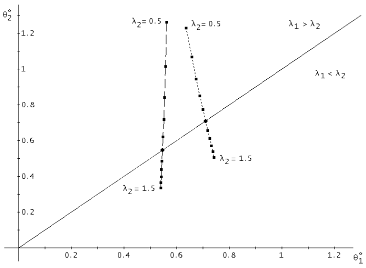

Figure 9: different joint behavior of equilibrium strategies depending on λ_2.

ensured and it also depended on μ, whereas in this case (4.31) does not and well-posedness always confirms existence. This observation applies to the case $\eta > 0$ as well, because Proposition 4.3 imposes no restrictions on η.

Finally, we note that equilibria with null risky investment by both EU and CPT agents were available when $\mu = 1$, as it happens now. Nonetheless, we had to impose the additional necessary condition $\alpha = \beta$, which can now be removed. In fact, in this model zero-demand levels can be simply characterized by a market parameter condition, thus ignoring the preferences of our investors. Passing to the endogenous volatility case, we obtained the subsequent result.

Corollary 4.2. *Assume $\eta > 0$. We have the following facts:*

(i) if $\lambda_1 = \lambda_2 = \lambda$, then

$$\theta_1^* = \theta_2^* = \frac{-(\lambda\sigma^2 - 3\epsilon) + \sqrt{(\lambda\sigma^2 - 3\epsilon)^2 + 10\eta\lambda(\mu - 1)}}{5\eta\lambda}; \tag{4.33}$$

in particular, $\theta_1^* = \theta_2^* = 0 \iff \mu = 1$;

(ii) if $\lambda_1 \neq \lambda_2$, then $\theta_1^* = \theta_2^* = 0 \Longrightarrow \mu = 1$ but not viceversa;

(iii) if $\mu > 1$, then $\theta_1^* \gtreqless \theta_2^* \iff \lambda_1 \lesseqgtr \lambda_2$.

In the case $\lambda_1 = \lambda_2$, the equilibria always lie on the bisector of the plane (θ_1, θ_2). Thanks to the explicit expression (4.33), we are able to perform a sensitivity analysis over θ_1^*, obtaining the same results as before. On the contrary, it is interesting to compute the updated no-leverage condition, which becomes

$$\lambda_1 \geq \lambda_1^{NL} := \frac{2(3\epsilon + \mu - 1)}{5\eta + 2\sigma^2};$$

in turn, this gives a negative dependence on σ^2.

Remarkably, in the case $\mu > 1$ the characterization of the demand levels is still valid and it only depends on the comparison between CARA coefficients. This property is exactly the same which can be found in a standard portfolio selection problem, i.e. when $\epsilon = \eta = 0$, so the model reduces to a non-impact version. In case *(ii)* of Corollary 4.2, we do not have the reversed implication; this is simply due to the possible presence of such an impact, which may induce non-zero risky investment.

A final observation concerns the joint behavior of the optimal policies. It is not difficult to find two numerical examples[14] which show that they can reveal different trends. In Figure 9 we can see that the most-left curve exhibit both decreasing investment levels as λ_2 grows, whereas in the right-most curve we have an increasing θ_1^*.

5 An equilibrium model with many heterogeneous investors

In the previous section we analyzed an hypothetical financial market where only two active agents were allowed to place orders, and those investors could be thought as large traders, in that they affected the equilibrium stock return. Implicitly, we hid a number of small traders behind the action of a single market maker who was able to absorb every demand or offer level. Now, we would like to give an interpretation of what happens "behind the curtains "; in other words, we are going to build a model

[14]In the left-most curve we selected $\mu = 1.01$, $\epsilon = 0.03$, $\sigma = 0.2$, $\eta = 0.05$ and $\lambda_1 = 1$, whereas in the right-most it is $\mu = 1.07$, $\epsilon = 0.01$, $\sigma = 0.02$, $\eta = 0.05$ and $\lambda_1 = 1$.

where many small EU agent and many small CPT agents have access to a financial market. Then, we will try to merge the economic decisions of all these investors and this should engender an equilibrium with endogenous prices or returns. An interesting feature of our model is that it resembles a pure competition scenario but at the same time every agent is supposed to be price maker. However, we will see that in equilibrium only the *classical* agents play a role in determining the equilibrium quantities, whereas the *behavioral* traders stay out of the risky asset market. Consequently, when the number of EU investors grows we expect to see something similar to a pure competition economy with price taking agents, and this is indeed the case (see Remark 5.1).

Now, we introduce the main hypothesis regarding the traders and the financial market.

Assumption 5.1 (CARA classical agents). *There are I* classical *agents, each of them endowed with an initial wealth $W^i > 0$, $i = 1, \dots, I$, and an utility function*

$$u_C^i(x) = 1 - \exp(-\lambda^i x), \qquad i = 1, \dots, I, \tag{5.1}$$

where $\lambda^i > 0$ is the constant absolute risk aversion coefficient of the i-th agent.

Assumption 5.2 (Risk-neutral loss-averse behavioral agents). *There are H* behavioral *agents, each of them endowed with an initial wealth $W^h > 0$, $h = 1, \dots, H$ and utility functions*

$$u_+^h(x) = x, \quad u_-^h(x) = k_-^h x, \quad h = 1, \dots, H, \tag{5.2}$$

with $k_-^h \geq 1$ for $h = 1, \dots H$. Moreover, the adjusted reference wealth is $x_0 = 0$ for every behavioral agent and the probability weighting functions $w_+^h(\cdot)$ and $w_-^h(\cdot)$ satisfy Assumption 2.2 and the hypothesis of Proposition 3.2 for $h = 1, \dots, H$.

Note that we are going to use absolute wealth and investment levels instead of percentage levels. Moreover, we suppose risk-neutrality for the CPT agents in order to avoid loss-aversion violation and, at the same time, to retain some analytic tractability. In fact, as long as $k_-^h \geq 1$, loss aversion is now a property of these preferences and the unpleasant consequences of CRRA utility functions that we saw in Section 3.2 are ruled out[15]. Besides this, we do not assume Kahneman-Tversky type probability distortions, as we will prove our main results under more general hypothesis. Finally, when maximizing their respective objective functions, our agents take into consideration that a lot of small traders are active in their market. To the best of our knowledge, this is the first time that many EU and many CPT investors, both with *heterogeneous* preferences, are considered in the literature.

Assumption 5.3 (Market structure).

- *There is a risk-free asset (bond) in perfectly elastic supply with unit price set equal to* 1 *and a deterministic return $r > 0$;*
- *There is a risky asset (stock) with $n > 0$ outstanding shares and a per-share dividend normally distributed, namely $D \sim \mathcal{N}(\mu, \sigma^2)$, with $\mu > 0$, $\sigma > 0$.*

It is clear that normal market conditions require $r \geq 1$, i.e. a non negative interest rate. However, we do not impose any other restriction over these parameters. Moreover, we will allow nor short-selling neither taking leverage by our agents and we suppose that there are not trading frictions or other constraints. Importantly, this is a game-theoretical model too. To better understand it, we propose a specific pure strategy equilibrium and we will check that this equilibrium can be sustained by a system of conjectures where every agent considers as given the strategy of every other agent. Obviously, all the traders must share the same beliefs about the return of the risky asset in order to avoid as much as possible computational difficulties. On one hand, we are able to provide sufficient conditions and an analytic proof for the existence of such equilibrium. Not only, we will see that it is indeed robust, in the sense that removing some constraints or varying some parameters is substantially innocuous, as well as a *coalition proofness* analysis reveals that it is resistant to multiple deviations. On the other hand, we were not able to prove that our suggested equilibrium is unique and we still rely on numerical and graphical analysis when doing some comparative statics.

Now, we need some further notations. Firstly, θ_C^i and θ_B^h denote the risky investment level of i-th EU agent and h-th CPT agent respectively. Hence, the no short-selling and no leverage constraints can be easily written as

$$\theta_C^i \in [0, W^i], \;\; \theta_B^h \in [0, W^h], \qquad i = 1, \dots, I, \;\; h = 1, \dots, H. \tag{5.3}$$

[15] Obviously, we lose the overall S-shaped form of the utility function.

Other quantities that will result useful are:

$$W^C := \sum_{i=1}^{I} W^i, \qquad W^{C-i} := W^C - W^i; \tag{5.4}$$

in other words, W^C is the total wealth at disposal of the pool of *classical* agents, whereas from the point of view of the i-th agent, W^{C-i} is the total wealth of the other EU investors joined together. Given this, market clearing condition imply that the unitary price p of the risky asset is determined by the following condition:

$$\sum_{i=1}^{I} \theta_C^i + \sum_{h=1}^{H} \theta_B^h = np, \tag{5.5}$$

i.e. the overall demand must equal the total supply. Thus, once every optimal strategy is known, it is straightforward to compute the equilibrium price. Whatever the price is, we are able to find the risky asset return R, namely

$$R := \frac{D}{p}, \qquad R \sim \mathcal{N}\left(\frac{\mu}{p}, \frac{\sigma^2}{p^2}\right). \tag{5.6}$$

Let's specify the suggested equilibrium and the sufficient conditions that ensure its sustainability. As usual, equilibrium quantity will be denoted with a *.

Equilibrium conjecture. *There exists an equilibrium with trading strategies*

$$\theta_C^{i*} = W^i, \qquad i = 1, \dots, I, \tag{5.7}$$

$$\theta_B^{h*} = 0, \qquad h = 1, \dots, H. \tag{5.8}$$

A sensible economic interpretation of these strategies is that the representative small CPT trader behaves like a typical householder who assigns all her savings to bonds, whereas the small EU agent makes a totally risky investment. [16]

Now, if our conjecture leads to an effective equilibrium, then we would have

$$p^* = \frac{W^C}{n}, \qquad R^* \sim \mathcal{N}\left(\frac{\mu}{p^*}, \frac{\sigma^2}{p^{*2}}\right). \tag{5.9}$$

We will prove its existence under the following assumption.

Assumption 5.4 (Sufficient condition for the existence of the equilibrium)**.**

- *the equilibrium risk premium is strictly positive, i.e.*

$$\frac{\mu}{p^*} > r; \tag{5.10}$$

Intuitively, condition (5.10) assures a positive risky investment in equilibrium by the representative EU agent; next, we look for ranges of the initial wealths of the EU traders and for the loss aversion coefficients for which the conjecture reveals true. We can prove the following result.

Proposition 5.1. *Under Assumptions 5.1 - 5.4, there exists a threshold*

$$\overline{W^i} = \begin{cases} \frac{2\sqrt{3}}{3}\sqrt{\frac{nW^{C-i}(\mu-\lambda^i n\sigma^2)}{r}} \sin\left(\frac{\pi}{3} + \frac{1}{3}\mathrm{asin}\left(\frac{3\sqrt{3}\lambda^i\sigma^2}{2}\sqrt{\frac{rnW^{C-i}}{(\mu-\lambda^i n\sigma^2)^3}}\right)\right) - W^{C-i} & \text{if } \lambda^i < \frac{\mu}{n\sigma^2}, \\ \sqrt[3]{\frac{\mu n W^{C-i2}}{r}} - W^{C-i} & \text{if } \lambda^i = \frac{\mu}{n\sigma^2}, \\ -\frac{2\sqrt{3}}{3}\sqrt{\frac{nW^{C-i}(\mu-\lambda^i n\sigma^2)}{r}} \sin\left(\frac{1}{3}\mathrm{asin}\left(\frac{3\sqrt{3}\lambda^i\sigma^2}{2}\sqrt{\frac{rnW^{C-i}}{(\mu-\lambda^i n\sigma^2)^3}}\right)\right) - W^{C-i} & \text{if } \lambda^i > \frac{\mu}{n\sigma^2}, \end{cases} \tag{5.11}$$

where asin$(\cdot)$ denotes the principal arcsine, such that $\forall\ W^i \in (0, \overline{W^i}]$ the optimal strategy for the i-th classical *agent is the conjectured one, namely $\theta_C^{i*} = W^i$, $i = 1, \dots, I$.*

[16] We note that this policy is indeed consistent with the classical theory of portfolio selection if we think to our two assets as of two mutual funds (e.g. our risky asset can represent some index-linked derivative) and EU agents seeking an expected return equal to that of the tangency portfolio in an efficient frontier framework. In fact, this would imply null investment in the risk-free asset and a mean-variance efficient frontier portfolio selection, as expected by the EU nature of those traders.

Proof. See the Appendix. □

The meaning of the bound imposed by $\overline{W^i}$ is that if W^i is small enough not to excessively distort the risky asset return, then it will be optimal for the i-th agent to select $\theta_C^{i*} = W^i$. Moreover, $\overline{W^i}$ usually represent a non-negligible fraction of the overall wealth of the economy, thus the representative EU investor is not necessarily too *small*. In the particular case of uniformly distributed endowments, we are able to provide further details.

Remark 5.1. If we further assume the same initial endowment W^i for every EU agent, then it is straightforward to see that $W^C = IW^i$ and $W^{C-i} = (I-1)W^i$. In this case, we obtain the upper bound

$$\overline{W^i} = \frac{n(I-1)(I\mu - \lambda^i n\sigma^2)}{rI^3}, \quad i = 1, \ldots, I. \tag{5.12}$$

To see this, simply replace W^{C-i} with $(I-1)W^i$ in (A.17) and solve the polynomial equation $P(W^i) = 0$. For $\overline{W^i}$ to be strictly positive, we have to impose an additional condition over the preferences of our *classical* agent; more specifically, we have

$$\overline{W^i} > 0 \quad \forall\, i = 1, \ldots, I \iff \max_{i=1,\ldots I} \lambda^i < \frac{I\mu}{n\sigma^2}. \tag{5.13}$$

Given this, we can select any positive initial wealth level $W^i \in (0, \min_{i=1,\ldots,I} \overline{W^i}]$ and then compute the equilibrium price p^* and the corresponding return R^* as indicated by (5.9).

In particular, we see that $\overline{W^i}$ positively depends on μ, whereas it depends negatively on r, σ^2 and λ^i. Moreover, as the number of EU agents becomes larger and approaches infinity, condition (5.13) is automatically satisfied. At the same time, we have $\lim_{I\to+\infty} \overline{W^i} = 0$; hence, we recover a setting very similar to a perfect competition framework, where every trader has a negligible impact over the equilibrium price due to her vanishing influence over the market.

An even more interesting analysis can be made if we restrict our attention to the case with homogeneous preferences, i.e. $\lambda^i = \lambda$ for $i = 1, \ldots, I$, and we choose the maximum initial endowment, that is to say

$$W^i = \overline{W} := \frac{n(I-1)(I\mu - \lambda n\sigma^2)}{rI^3}.$$

Hence, we are able to explicitly compute

$$p^* \equiv \frac{I\overline{W}}{n} = \frac{(I-1)(I\mu - \lambda n\sigma^2)}{rI^2};$$

by straightforward computations, we can see that

$$\frac{\partial p^*}{\partial I} = \frac{(I\mu - \lambda n\sigma^2) + (I-1)\lambda n\sigma^2}{rI^3} > 0.$$

Besides this, we have

$$\lim_{I\to+\infty} p^* = \frac{\mu}{r},$$
$$\lim_{I\to+\infty} \mathbb{E}\left[R^*\right] = r.$$

Otherwise stated, the equilibrium risk premium of the stock asymptotically tends to vanish in a monotone decreasing way, as the number of EU agents increases. This is just another confirm of the perfect competition scenario that appears in the limit, where expected profits are reduced to zero and each agent simply becomes price taker. Finally, we note that p^* is increasing in μ and decreasing in σ^2, n, r and λ, as economic intuition suggests.

Turning back to the existence problem, we now have to show that our Equilibrium Conjecture indeed suggests the best strategies for the CPT agents too. In other words, we must check that every *behavioral* investor optimally chooses $\theta_B^h = 0$, given that $\theta_C^i = W^i$, $i = 1, \ldots, I$, and $\theta_B^k = 0$, $k = 1, \ldots, h-1, h+1, \ldots, H$. Formally, we state the following result.

Proposition 5.2. *Under Assumptions 5.1 - 5.4, there exists a threshold*

$$\overline{k_-^h} := \frac{\displaystyle\int_0^{+\infty} w_+^h \left(1 - \mathcal{N}\left(\frac{(x+r)p^* - \mu}{\sigma}\right)\right) dx}{\displaystyle\int_0^r w_-^h \left(\mathcal{N}\left(\frac{(-x+r)p^* - \mu}{\sigma}\right)\right) dx}. \tag{5.14}$$

such that $\forall\, k_-^h \in [\overline{k_-^h}, +\infty)$ *the optimal strategy for the h-th* behavioral *agent is the conjectured one, namely* $\theta_B^{h*} = 0$, $h = 1, \ldots, H$.

Proof. See the Appendix. □

We note the range imposed in (5.14) ensures $\theta_B^{h*} = 0$ for the h-th CPT investor. Now, it is clear that if these bounds are too restrictive, then the existence result will be weak in the sense that reality hardly matches our requirements. However, we will see later that $\overline{k_-^h}$ is near to 1, hence we do not require strong loss aversion, which intuitively guarantees a null risky investment. Furthermore, the analytically derived bound $\overline{k_-^h}$ reveals very accurate; that is to say that numerical evidence shows that even for k_-^h really close but lower than $\overline{k_-^h}$, we no more have the equilibrium. The subsequent graphical analysis will highlight this fact.

Finally, combining Proposition 5.1 and Proposition 5.2, we can analytically state the existence of our proposed equilibrium. Once again, we observe that all the sufficient hypotheses are nothing but Assumption 5.4 and the two bounds given by (5.11) and (5.14).

Remark 5.2. We note that in the Appendix we also prove with equation (A.20) that the stock price p does not influence the policy of the *behavioral* investor, who is only interested in the terminal return. This fact completely agrees with economic intuition and it seems a quite natural requirement. However, we also note that in the CRRA case this is no longer true unless $\alpha = \beta$ (and this is one reason why we discarded power utility functions).

Furthermore, Property (iii) in the proof of Proposition 5.2 shows that the value function is strictly increasing in the dividend μ and strictly decreasing in the risk-free return r. Consequently, if we are able to find a particular set of values which support the equilibrium, then we still have an equilibrium *ceteris paribus* but decreasing μ (or increasing r) as long as (5.10) is fulfilled. Once again, this reflects economic intuition, as a higher r makes risk-free investment more favorable for a FOSD agent, hence making our equilibrium with $\theta_B^{h*} = 0$ resist. Numerical evidence suggests that this is also true for higher levels of the volatility σ^2; however, we were not able to provide an analytic proof.
Finally, the argument for the existence of the equilibrium admits any positive wealth level W^h for the h-th *behavioral* trader; this is because we made our proof taking the supremum over $\theta_B^h \in (0, +\infty)$. Therefore, the no-leverage constraint for the CPT agents can even be removed as it is not binding at all.

Now it's time to make some graphical analysis in order to see what happens when we change preference and/or market parameters. We start by choosing the usual Kahneman-Tversky probability distortions as in (3.11) and by arbitrarily fixing a particular set of values; then we provide 3D plots where all but one parameter are kept fixed and the selected one is allowed to vary together with θ_B^h. Results are shown in Figure 10. We represent on the z-axis the prospect value $U^h(\cdot)$ of the h-th CPT investor and we can verify the existence of our equilibrium by depicting the plane $U^h = 0$: if for a parameter's value the surface lies below that plane for all $\theta_B^h > 0$, then the equilibrium exists; otherwise, our agent has an incentive to deviate by choosing the maximizing θ_B^h. In particular, market parameters are

$$n = 10^3, \qquad W^C = 10^3, \qquad r = 1.02, \qquad \mu = 1.06, \qquad \sigma = 0.2,$$

whereas those of the h-th *behavioral* trader are

$$k_-^h = 1.5, \qquad \gamma^h = 0.65, \qquad \delta^h = 0.65.$$

- **Figure 10**, *top-left plot*: $r \in [1, 1.06)$. We sensibly choose $r \geq 1$ to have a non negative interest rate and $r < 1.06$ to have a positive risk premium in equilibrium. As the prospect value $U^h(\cdot)$ is strictly decreasing in r for every fixed $\theta_B^h > 0$, we can see that for $r > 1.012$ the equilibrium can be substained.

- **Figure 10**, *top-right plot*: $\mu \in [1.02, 1.16]$. The lower bound for μ follows from equation (5.10), as $p^* = 1$ in this case. As noted before, lower levels of μ are acceptable, whereas a higher payoff of the risky asset attracts investment in the stock. The range of the parameter for which we have the equilibrium is approximately $\mu < 1.0676$;

- **Figure 10**, *center-left plot*: $\sigma \in [0.02, 0.5]$. Economic intuition is confirmed by the fact that a higher volatility makes risk-free investment more desirable, thus forcing the surface below the horizontal plane $U^h = 0$. We computed a standard deviation $\sigma = 0.1688$ as the lowest value which ensures equilibrium;

- **Figure 10**, *center-right plot*: $k_-^h \in [1, 2]$. We clearly see that for a sufficiently high loss-averse investor, equilibrium existence can be retained; this perfectly agrees with our analytic result that a higher k_-^h are still selectable. In this particular case, we computed $\overline{k_-^h} = 1.400699015$, whereas graphical experiments show that $k_-^h = 1.40069$ is a sufficient lower bound with a 10^{-30} error. Hence, our estimation for $\overline{k_-^h}$ reveals very accurate;

- **Figure 10**, *down-left plot*: $\gamma^h \in (0.28, 1]$. Quite surprisingly, the shape of probability weighting of the Kahneman-Tversky type for the gains that we choose does not affect the existence result, as long as we maintain the same $w_-^h(\cdot)$ for the losses. We recall that $\gamma^h > 0.28$ is necessary in order to have $w_+^h(\cdot)$ strictly increasing over $[0, 1]$; moreover, for $\gamma^h = 1$ we can not perform the integration by parts used in the proof of Proposition 5.2. This is because $w_+^h(\cdot)$ does not fulfill the hypothesis of Proposition 3.2. However, the proof can be slightly changed and numerical computations can still be performed, clearly leading to existence.

- **Figure 10**, *down-right plot*: $\delta^h \in [0.28, 1]$. On the contrary, changing δ^h has remarkable consequences, in that only for $\delta^h < 0.7264$ the equilibrium is retained. This is due to the fact that a higher δ^h amounts to smaller distortions for extreme probabilities, which translates in a more objective perception of losses. This in turn leads our agent to increase her risky exposure, as she *becomes* more risk-neutral.

A final observation concerns the so-called *coalition proofness* of our equilibrium. Loosely speaking, what happens if two or more agent cooperate in order to deviate from their respective optimal strategies? We already know that the no-leverage constraint for the *behavioral* agents can be removed without affecting our results. Thus, even if two or more CPT investors agree to merge their initial endowments, they will not able to find a better policy as long as the market parameters remain the same. Things become far more complicated if we allow cooperation among EU traders. In fact, once a specific pool of those agents is selected, we could repeat from the beginning the argument which discovers the optimal strategy $\theta_C^{i*} = W^i$. The problem now is that the price that each agent of this pool has to consider depends on every demand level of themselves; moreover, the conjectured invested wealth becomes W^C minus their total initial endowments. Consequently, the analysis become extremely involved and an easy solution can hardly be found. Similar conclusions are valid if we allow cooperation among the two types of investor. On a heuristic basis, we can say that if the total endowments of the cooperating agents is small w.r.t. the conjectured invested wealth, then they are not able to distort the equilibrium price; as a consequence, every member of this coalition will not have the incentive to deviate.

6 Conclusions

This paper is primarily concerned with the problem of assessing the existence of equilibria in simple financial markets where multiple and possibly *heterogeneous* agents are allowed to interact. Our main results give a positive answer to this question; firstly, in the case of a market-maker driven scenario, if we model investment decisions by one large *classical* and one large *behavioral* agent, then several types of equilibria are shown to be sustainable. This analysis has been mostly performed through numerical computations and graphical evidence. Secondly, when many small EU and many small CPT traders enter the market, then an equilibrium is still attainable, even if the preferences inside each pool are not the same. To the best of our knowledge, this approach to the problem is completely innovative and this is the first time that the existence of such an equilibrium has been shown. Notably, in our second model, each agent is supposed to be *price maker* but in equilibrium she substantially becomes *price taker*, thus accurately mimicking actual financial markets. Furthermore, economic intuition is confirmed in both models and numerical experiments provide robustness results.

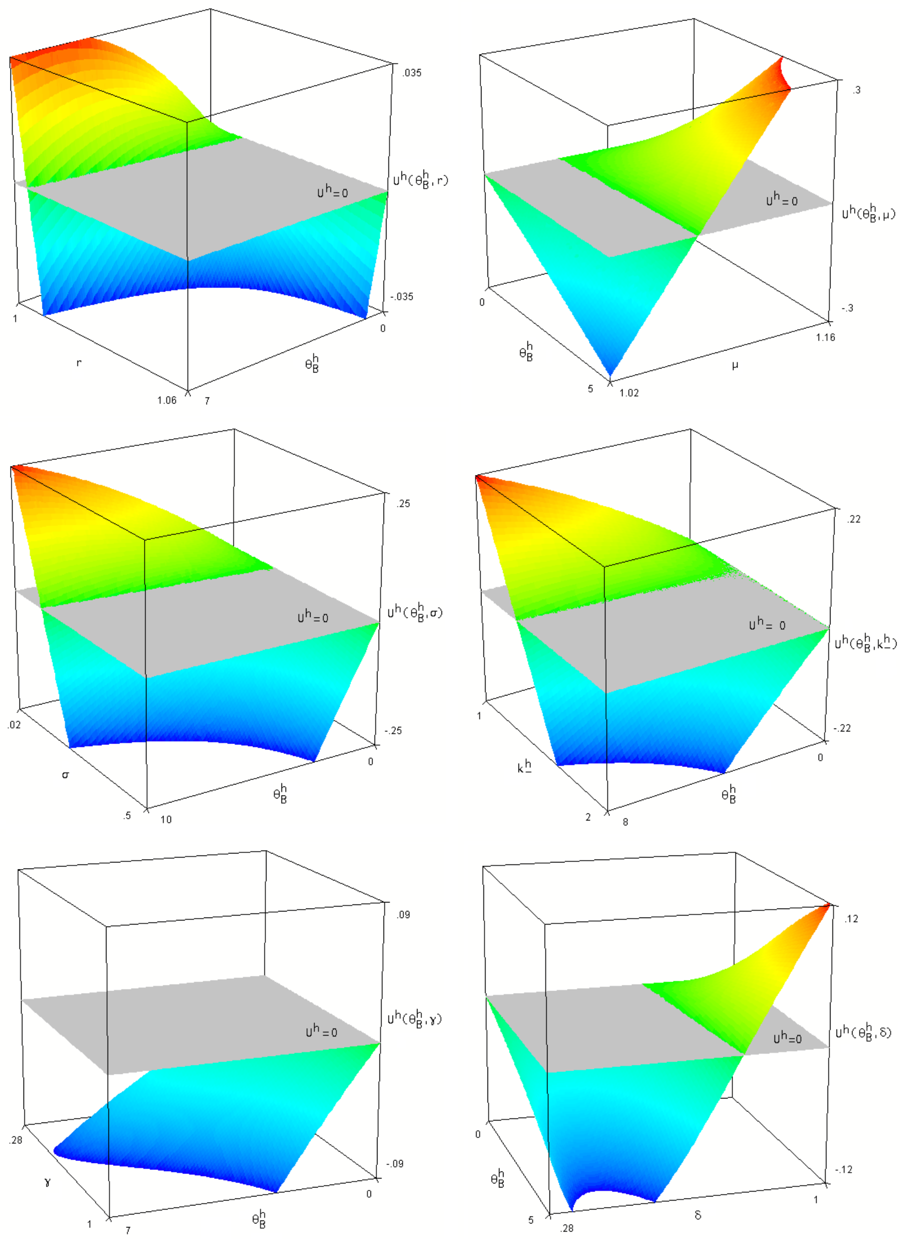

Figure 10: 3D-plots of the prospect value for the h-th CPT agent; the surface below the horizontal plane represents existence of the equilibrium.

A Proofs

Proof of Proposition 4.1. We start by explicitly writing the relevant statistic $a_1(\theta_B, \theta_C^*)$ for our CPT agent; using the integration by parts formula and the Gaussian distribution of the stock return we have:

$$a_1(\theta_B, \theta_C^*) = \int_0^{+\infty} \alpha t^{\alpha-1} w \left(1 - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(z-\mu-\epsilon\theta_C^* - \epsilon\theta_B + 1)^2}{2\sigma^2}\right\} dz\right) dt, \tag{A.1}$$

where we recall $\theta_C^* = \frac{\mu-1}{\sigma^2-2\epsilon}$; the other statistics can be explicitly similarly written. Thanks to the assumption $\alpha = \beta$ and $w_+(\cdot) \equiv w_-(\cdot)$, it is immediate to see that

$$a_1(\theta_B, \theta_C) \equiv b_2(\theta_B, \theta_C), \quad a_2(\theta_B, \theta_C) \equiv b_1(\theta_B, \theta_C), \qquad \forall\, \theta_B, \theta_C \in \mathbb{R}. \tag{A.2}$$

Moreover, the skewness of our Gaussian distribution implies $a_1(\theta_B, \theta_C^*) \geq a_2(\theta_B, \theta_C^*)$ for every $\theta_B \in [0, 1]$, with equality if and only if $\mu = 1$, i.e. when there is no skewness. To see this, just note that with a change of variable we can write

$$a_2(\theta_B,\theta_C^*) = \int_{-\infty}^{0} \alpha(-t)^{\alpha-1} w\left(\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(z-\mu-\epsilon\theta_C^*-\epsilon\theta_B+1)^2}{2\sigma^2}\right\}dz\right)dt$$
$$= \int_{0}^{+\infty} \alpha t^{\alpha-1} w\left(\int_{-\infty}^{-t} \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(z-\mu-\epsilon\theta_C^*-\epsilon\theta_B+1)^2}{2\sigma^2}\right\}dz\right)dt,$$

and we have the result thanks to the monotonicity of $w(\cdot)$[17]. Hence, using (A.2) we have

$$k_0(\theta_B,\theta_C^*) = max\left(\frac{a_1(\theta_B,\theta_C^*)}{a_2(\theta_B,\theta_C^*)}, \frac{b_1(\theta_B,\theta_C^*)}{b_2(\theta_B,\theta_C^*)}\right) = \frac{a_1(\theta_B,\theta_C^*)}{a_2(\theta_B,\theta_C^*)} \geq 1,$$

therefore equilibria with $\theta_B^* = 0$ in case *(iv)* of Theorem 3.3 are ruled out.

Now, the only relation that remains to verify in order to have an equilibrium is $k_- \geq k_0(\theta_B,\theta_C^*)$ for every choice of $\theta_B \in [0,1]$. Fortunately, calculations are somewhat simplified thanks to the first order stochastic dominance which stands behind the portfolio selection criteria in CPT. More specifically, it readily follows from First Order Stochastic Dominance that the higher is the expected stock excess return, the higher is the statistic a_1 (and vice versa for a_2, as it is a "measure" of losses). But in our situation, a higher return can be simply obtained by raising the investment in the risky asset, namely θ_B, therefore we have the relations

$$\frac{\partial a_1}{\partial \theta_B}(\theta_B,\theta_C^*) > 0, \quad \frac{\partial a_2}{\partial \theta_B}(\theta_B,\theta_C^*) < 0, \qquad \forall\, \theta_B \in [0,1],$$

which in turn imply $\frac{\partial k_0}{\partial \theta_B}(\theta_B,\theta_C^*) > 0$, $\forall\, \theta_B \in [0,1]$. As a consequence of this monotonicity property, it suffices to check the inequality $k_- \geq k_0(1,\theta_C^*)$. Replacing the parameters with their respective values and substituting the expression of θ_C^*, the right-hand side becomes a complicated function of μ, ϵ and σ^2, involving the c.d.f. of a Gaussian random variable.

A reasoning similar to the previous one shows that $k_0(1,\theta_C^*)$ is increasing even in the second argument, because a higher θ_C^* shifts the c.d.f. to the right. Recalling the dependences of θ_C^* on the market parameters, we immediately see that

$$\frac{\partial k_0(1,\cdot)}{\partial \mu} > 0\; \forall\, \mu \in [1,+\infty), \qquad \frac{\partial k_0(1,\cdot)}{\partial \epsilon} > 0\; \forall\, \epsilon \in [0,+\infty), \qquad \frac{\partial k_0(1,\cdot)}{\partial \sigma} < 0\; \forall\, \sigma \in (0,+\infty).$$

While the first and the second relations are obvious, the last one follows by a combined first order and second order stochastic dominance argument (i.e. a mean-variance one). In fact, an increase in σ^2 leads to a lower mean and a higher variance in the c.d.f. which appears in the expression of a_1 and a_2; therefore we deduce a negative dependence of k_0 on the exogenous volatility parameter.
In conclusion, we are done once we have found a triple (μ^*,ϵ^*,σ^{2*}) such that

$$k_0\left(1, \frac{\mu^*-1}{\sigma^{2*}-2\epsilon^*}\right) \leq k_-,$$

as this inequality implies existence of the equilibrium for (μ^*,ϵ^*,σ^{2*}) and any lower μ or ϵ is still compatible, just like any higher σ^2. □

Proof of Proposition 4.2. We start by explicitly writing the statistic $a_1(\theta_B,\theta_C^*)$ for our CPT agent with the assumption $\epsilon = 0$:

$$a_1(\theta_B,\theta_C^*) = \int_0^{+\infty} \alpha t^{\alpha-1} w\left(1 - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi(\sigma^2+\eta\theta_B+\eta\theta_C^*)}}\exp\left\{-\frac{(z-\mu+1)^2}{2(\sigma^2+\eta\theta_B+\eta\theta_C^*)}\right\}dz\right)dt, \tag{A.3}$$

where we recall

$$\theta_C^* = \frac{-\sigma^2+\sqrt{(\sigma^2)^2+6\eta(\mu-1)}}{3\eta}.$$

[17]For more information about the skewness effects on this kind of portfolio selection model and the relative equilibria, see [2] and [5].

Similar expressions can be found for a_2, b_1 and b_2. Now, the conclusion $k_0(\theta_B, \theta_C^*) = \frac{a_1(\theta_B,\theta_C^*)}{a_2(\theta_B,\theta_C^*)}$ is still valid and we can exploit second order stochastic dominance results. Intuitively, a raise in θ_B will only produce a volatility increase, thus reducing the prospect value of any selected terminal wealth; this is why we employ $k_0\left(0, \theta_C^*\right)$. Mathematically speaking, while holding θ_C^* fixed we would like to have

$$\frac{\partial k_0}{\partial \theta_B} < 0 \iff \frac{\partial a_1}{\partial \theta_B} a_2 - a_1 \frac{\partial a_2}{\partial \theta_B} < 0, \tag{A.4}$$

where we suppressed the arguments for notational convenience. However, we already know that $a_1(\theta_B, \theta_C^*) \geq a_2(\theta_B, \theta_C^*)$ for every $\theta_B \in [0,1]$ thanks to the skewness of the Gaussian stock return; furthermore, Proposition 3.3 implies

$$\frac{\partial a_1}{\partial \theta_B}(\theta_B, \theta_C^*) \leq \frac{\partial a_2}{\partial \theta_B}(\theta_B, \theta_C^*), \qquad \frac{\partial a_2}{\partial \theta_B}(\theta_B, \theta_C^*) > 0,$$

for every $\theta_B \in [0,1]$, as θ_B positively affects the overall variance. Therefore, inequality (A.4) is indeed fulfilled and the monotonicity of k_0 on θ_B implies that the existence of the equilibrium is finally reduced to show that

$$k_- \geq k_0(0, \theta_C^*) = k_0\left(0, \frac{-\sigma^{2*} + \sqrt{(\sigma^{2*})^2 + 6\eta^*(\mu^* - 1)}}{3\eta^*}\right) \tag{A.5}$$

holds for some choice of the market parameters (μ^*, σ^{2*},η^*) in their respective ranges and satisfying the simplified no-leverage condition $3\eta^* + 2\left(\sigma^{2*} - (\mu^* - 1)\right) \geq 0$. Once such a particular triple is found, μ can not be arbitrarily decreased because it would produce a drop in both the mean and the variance of the c.d.f. included in a_1 and a_2. However, σ^2 and η can be arbitrarily increased as they do not affect the mean of the c.d.f. but at the same time they increase the overall volatility. In fact, this last quantity is given by

$$\hat{\sigma}^2 = \sigma^2 + \eta\theta_B + \eta\theta_C^* = \frac{2}{3}\sigma^2 + \eta\theta_B + \frac{\sqrt{\sigma^4 + 6\eta(\mu - 1)}}{3},$$

and by straightforward computation we obtain

$$\frac{\partial \hat{\sigma}^2}{\partial \sigma^2} = \frac{2}{3} + \frac{\sigma^2}{\sqrt{\sigma^4 + 6\eta(\mu - 1)}} > 0,$$
$$\frac{\partial \hat{\sigma}^2}{\partial \eta} = \theta_B + \frac{2(\mu - 1)}{\sqrt{\sigma^4 + 6\eta(\mu - 1)}} > 0. \qquad \square$$

Proof of Lemma 4.2. The equivalence of the problems (4.16) and (4.19) follows from equation (2.5) replacing the power utility functions and using integration by parts formula as in Lemma 4.1. For completeness, we recall that $a_1(\theta_B)$ is given by (A.1), simply replacing $\theta_C^*(\theta_B) = \frac{(\mu-1)+\epsilon\theta_B}{\sigma^2 - 2\epsilon}$; $a_2(\theta_B)$ can be similarly obtained.

To show the ill-posedness, it is sufficient to note that as θ_B increases we have an upward shift in the drift of the excess risky asset return, which amounts to a right shift of the CDF that appears in the expressions of a_1 and a_2; this fact is a consequence of the relation $\frac{\partial \theta_C^*}{\partial \theta_B} > 0$. Therefore, a monotone convergence argument can be used to prove that $a_1(\theta_B) \uparrow +\infty$ as $\theta_B \uparrow +\infty$, whereas $a_2(\theta_B) \to 0$ as $\theta_B \uparrow +\infty$. Moreover, a_2 tends to zero faster than a power function of order β, thanks to the explicit choice of the probability distortion $w(\cdot)$ as in (3.11) and the fact that we are working with Gaussian random variables. Hence $\theta_B^\beta a_2(\theta_B)$ tends to zero as well and this concludes the ill-posedness argument.

To prove (4.20), we observe that a_1 and a_2 are continuous functions of θ_B. Therefore, as $\theta_B \downarrow 0$, the c.d.f. that appears in those statistics converges; note that also θ_C^* converges to a constant depending on the market parameters. To conclude, we use Lemma 3.1 which states that a_1 and a_2 are well-defined and strictly positive, if considered as functions of the market parameters. $\square$

Proof of Lemma 4.3. We show (4.27) only for $\hat{\mu}$, the other for $\hat{\sigma}^2$ being similar. Recalling equation (4.1) and using (4.23) we obtain

$$\begin{aligned}\frac{\partial \hat{\mu}}{\partial \theta_B}(\theta_B, \theta_C^*(\theta_B)) &= \epsilon \left(1 + \frac{\partial \theta_C^*}{\partial \theta_B}(\theta_B)\right) \\ &= \epsilon \left(\frac{2\sqrt{(\sigma^2 - 2\epsilon + \eta\theta_B)^2 + 6\eta(\mu - 1 + \epsilon\theta_B)} + \eta\theta_B + \epsilon + \sigma^2}{3\sqrt{(\sigma^2 - 2\epsilon + \eta\theta_B)^2 + 6\eta(\mu - 1 + \epsilon\theta_B)}} \right),\end{aligned}$$

which is always positive thanks to Assumption 4.1 as long as $\epsilon > 0$ and it is null when $\epsilon = 0$.

To prove equations (4.26) and (4.28), we recall that $V(\theta_B) = \theta_B^\alpha a_1(\theta_B) - k_- \theta_B^\beta a_2(\theta_B)$ and

$$\begin{aligned} a_1(\theta_B) &= \int_0^{+\infty} \alpha t^{\alpha-1} w \left(1 - \int_{-\infty}^t \frac{1}{\sqrt{2\pi\hat{\sigma}^2(\theta_B, \theta_C^*(\theta_B))}} \exp\left\{-\frac{(z - \hat{\mu}(\theta_B, \theta_C^*(\theta_B)) + 1)^2}{2\hat{\sigma}^2(\theta_B, \theta_C^*(\theta_B))}\right\} dz\right) dt, \\ a_2(\theta_B) &= \int_{-\infty}^0 \beta(-t)^{\beta-1} w \left(\int_{-\infty}^t \frac{1}{\sqrt{2\pi\hat{\sigma}^2(\theta_B, \theta_C^*(\theta_B))}} \exp\left\{-\frac{(z - \hat{\mu}(\theta_B, \theta_C^*(\theta_B)) + 1)^2}{2\hat{\sigma}^2(\theta_B, \theta_C^*(\theta_B))}\right\} dz\right) dt. \end{aligned}$$

The case $\epsilon = 0$ can be proved following a monotone convergence argument similar to the one previously used for Lemma 4.2. However, we will follow a slightly different approach, analyzing $V(\cdot)$ for any $\epsilon \geq 0$ and finally seeing what happens when we have an equality.
At first, we note that the c.d.f. which appears in $a_1(\cdot)$ (and similarly for $a_2(\cdot)$) can be written as

$$\int_{-\infty}^{\frac{t - (\hat{\mu}(\theta_B, \theta_C^*(\theta_B)) - 1)}{\sqrt{\hat{\sigma}^2(\theta_B, \theta_C^*(\theta_B))}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

Now, replacing $\theta_C^*(\theta_B)$ by using (4.23), for any given $t \geq 0$ we can explicitly compute the upper extreme of the previous integral as

$$\zeta(\theta_B) \equiv -\frac{\sqrt{3}\left(\epsilon\Delta(\theta_B) + \epsilon(2\eta\theta_B + 2\epsilon - \sigma^2) + 3\eta(\mu - 1) - 3t\eta\right)}{3\eta\sqrt{\Delta(\theta_B) + 2(\eta\theta_B + \epsilon + \sigma^2)}},$$

where $\Delta(\theta_B) := \sqrt{(\sigma^2 - 2\epsilon + \eta\theta_B)^2 + 6\eta(\mu - 1 + \epsilon\theta_B)}$.

It is easy to see that for any fixed t, $\lim_{\theta_B \to +\infty} \zeta(\theta_B) = -\infty$ if $\epsilon > 0$, whereas $\lim_{\theta_B \to +\infty} \zeta(\theta_B) = 0$ if $\epsilon = 0$. Intuitively, using a monotone convergence argument, one can argue that $a_1(\theta_B) \to +\infty$ and $\theta_B^\beta a_2(\theta_B) \to 0$ as $\theta_B \to +\infty$ (the presence of the factor θ_B^β is not relevant thanks to the explicit choice of the probability distortion $w(\cdot)$ as in (3.11) and the fact that we are working with Gaussian random variables).

Now, let's fix a diverging sequence $\{\theta_B^n\}_{n\in\mathbb{N}}$ and let's compute $\lim_{n\to+\infty} a_1(\theta_B^n, \theta_C^*(\theta_B^n))$. Note that this is just a sequence of well-defined strictly positive real numbers, as Lemma 3.1 states. Moreover, the previous limit exists because the sequence $\{a_1(\theta_B^n, \theta_C^*(\theta_B^n))\}_{n\in\mathbb{N}}$ becomes eventually monotone increasing. To prove this fact, one can compute

$$\frac{d\zeta}{d\theta_B}(\theta_B) = -\frac{\sqrt{3}\left(2\Delta(\theta_B) + \eta\theta_B + \epsilon + \sigma^2\right)\left[\epsilon\Delta(\theta_B) + \epsilon(2\eta\theta_B + 2\epsilon + 5\sigma^2) - 3\eta(\mu - 1) + 3\eta t\right]}{6\Delta(\theta_B)\left[\Delta(\theta_B) + 2(\eta\theta_B + \epsilon + \sigma^2)\right]^{3/2}}.$$

Hence we can choose $\bar{n} \in \mathbb{N}$ such that $\forall\, n > \bar{n}$, $\forall\, t \geq 0$ we have $\frac{d\zeta}{d\theta_B}(\theta_B^n) < 0$. Now we can conclude that $a_1(\theta_B) \to +\infty$ for any choice of $\epsilon \geq 0$, whereas $\theta_B^\beta a_2(\theta_B) \to +\infty$ if $\epsilon = 0$ and $\theta_B^\beta a_2(\theta_B) \to 0$ if $\epsilon > 0$. Therefore, (4.28) follows immediately; on the other hand, (4.26) holds because $0 < \alpha < \beta \leq 1$ and $\zeta(\theta_B) \to 0$ as $\theta_B \to +\infty$ if $\epsilon = 0$. □

Proof of Proposition 4.3. Suppose that the market parameters as well as λ_1, λ_2 are fixed accordingly to Assumptions 4.1 and 4.4. Now we restrict our attention to the case $\eta > 0$, the case $\eta = 0$ being fully analyzed in the proof of Corollary 4.1.

Let agent 2 select a specific $\theta_2 \geq 0$ as her investment level; using (4.6) as the expression of the terminal wealth of agent 1, standard computations lead to the following optimization problem for our EU agent 1:

$$\max_{\theta_1 \in [0, +\infty)} 1 - \exp\left\{\frac{\eta\lambda_1^2}{2}\theta_1^3 + \left(\frac{\lambda_1^2(\eta\theta_2 + \sigma^2)}{2} - \epsilon\lambda_1\right)\theta_1^2 + \left(-\lambda_1\left(\epsilon\theta_2 + \mu - 1\right)\right)\theta_1 - \lambda_1\right\}, \qquad \text{(A.6)}$$

which is obviously equivalent to minimize the third-degree polynomial which appears as the argument of the exponential. Note that at the same time, agent 2 faces a similar problem and first order conditions for *internal* equilibria can be easily obtained:

$$\begin{cases} f_1(\theta_1,\theta_2) := 3\eta\lambda_1\theta_1^2 + 2\theta_1\left[\lambda_1(\sigma^2+\eta\theta_2)-2\epsilon\right] - 2(\mu-1+\epsilon\theta_2) = 0, \\ f_2(\theta_1,\theta_2) := 3\eta\lambda_2\theta_2^2 + 2\theta_2\left[\lambda_2(\sigma^2+\eta\theta_1)-2\epsilon\right] - 2(\mu-1+\epsilon\theta_1) = 0. \end{cases} \tag{A.7}$$

Now, for any fixed $\theta_2 \geq 0$ there exists a unique minimum point $\theta_1 \geq 0$ for the aforementioned third-degree polynomial; this follows by the fact that $f_1(\theta_1,\cdot)$ is nothing but a convex parabola with $f_1(0,\cdot) \leq 0$. Not only, we can also explicitly compute that global minimum point over $[0,+\infty)$ as

$$\theta_1(\theta_2) = \frac{-\left[\lambda_1(\sigma^2+\eta\theta_2)-2\epsilon\right] + \sqrt{\left[\lambda_1(\sigma^2+\eta\theta_2)-2\epsilon\right]^2 + 6\eta\lambda_1(\mu-1+\epsilon\theta_2)}}{3\eta\lambda_1} \geq 0, \tag{A.8}$$

which represents our candidate equilibrium strategy for θ_1. Therefore, we can replace θ_1 with $\theta_1(\theta_2)$ in the second equation of (A.7) to obtain the necessary condition $f_2(\theta_1(\theta_2),\theta_2) = 0$. The existence of a non negative root to this equation is clearly guaranteed if we observe that $f_2(\theta_1,\cdot)$ is a continuous function as well as $\theta_1(\cdot)$; besides this, $f_2(\theta_1(0),0) \leq 0$ and $\lim_{\theta_2\to+\infty} f_2(\theta_1(\theta_2),\theta_2) = +\infty$. Hence, repeating the same argument reversing the roles of θ_1 and θ_2, we have proved well-posedness and the consequent existence of an equilibrium.

Next, we are going to prove that if one agent's risky demand is null, then the only strategy of the other trader (possibly) compatible with an equilibrium is not to invest too. To fix ideas, suppose $\theta_2^* = 0$. Then, by (A.8) it is immediate to see that $\theta_1^* > 0$ only if $\mu > 1$. At this point, we have to check if $\theta_2^* = 0$ is the best reply to $\theta_1^*(0)$. However, this can not be true as $f_2(\theta_1^*(0),0) < 0$. Hence, equilibria with only one null demand are ruled out and the remaining equilibria are just the *internal* or the *boundary* ones. □

Proof of Corollary 4.1. For the sake of clarity, we will conduct our analysis by the side of agent 1. From the first order conditions in (A.7) it is easily obtained

$$\theta_1(\theta_2) = \frac{\mu-1+\epsilon\theta_2}{\lambda_1\sigma^2-2\epsilon}, \tag{A.9}$$

which is always non negative thanks to our assumptions. Now, if $\theta_2^* = 0$ then a necessary and sufficient condition to have an equilibrium with $\theta_1^* = 0$ is $\mu = 1$. In fact, first order condition becomes

$$(\mu-1) + \epsilon\theta_1(0) = 0;$$

hence, $\theta_1^* > 0$ is not compatible with such an equilibrium and $\theta_1^* = 0$ requires $\mu = 1$. Finally, by the second order conditions we find that (4.31) effectively ensures that the objective functions are indeed maximized for both investors. On the other hand, if $\theta_2^* > 0$ then the equilibrium is the solution of the following linear system

$$\begin{cases} \theta_1^*(\lambda_1\sigma^2-2\epsilon) - (\mu-1+\epsilon\theta_2^*) = 0, \\ \theta_2^*(\lambda_2\sigma^2-2\epsilon) - (\mu-1+\epsilon\theta_1^*) = 0, \end{cases} \tag{A.10}$$

which simply gives the optimal policies of (4.32). □

Proof of Corollary 4.2. If $\lambda_1 = \lambda_2 = \lambda$, then (A.7) shows that $f_1(\theta_1,\theta_2) = f_2(\theta_2,\theta_1)$ for any choice of θ_1,θ_2. At the same time, for (θ_1^*,θ_2^*) to be an *internal* equilibrium, we must have $f_1(\theta_1^*,\theta_2^*) = f_2(\theta_1^*,\theta_2^*) = 0$; hence, it follows that a necessary condition is $f_2(\theta_1^*,\theta_2^*) = f_2(\theta_2^*,\theta_1^*) = 0$, which in turn implies $\theta_1^* = \theta_2^*$ because $f_2(\cdot,\cdot)$ is not symmetric in its arguments. As a consequence, the equilibrium strategy for both agents will be the positive root of $f_1(\theta_1^*,\theta_1^*) = 0$, which is nothing but (4.33). In particular, if $\mu = 1$ it reduces to the null-investment solution.

The next step is to show that $\theta_1^* = \theta_2^* = 0$ implies $\mu = 1$; but this follows immediately by (A.8), as $\mu > 1$ would surely induce a positive risky investment. The converse is obviously not true as (A.8) shows, e.g. in the case of strictly positive ϵ and η.

Finally, if $\mu > 1$ we look at the dependence of the demand from the CARA coefficient. We have already seen that $\lambda_1 = \lambda_2$ implies $\theta_1^* = \theta_2^*$. Now, suppose $\lambda_1 > \lambda_2$ and $\theta_1^* \geq \theta_2^*$, the opposite case being almost identical. As $\mu > 1$, we will have $\theta_1^*, \theta_2^* > 0$, i.e. *internal* equilibria; therefore, (A.7) must hold. Moreover, thanks to (4.31) we can compute

$$\begin{aligned} f_1(\theta_1,\theta_2) - f_2(\theta_1,\theta_2) &= 3\eta(\lambda_1\theta_1^2 - \lambda_2\theta_2^2) + 2\eta\theta_1\theta_2(\lambda_1 - \lambda_2) + 2\sigma^2(\lambda_1\theta_1 - \lambda_2\theta_2) - 2\epsilon(\theta_1 - \theta_2) \\ &> 2(\theta_1 - \theta_2)(\sigma^2\lambda_2 - \epsilon) > 0 \end{aligned}$$

which is not compatible with an equilibrium. Hence, it follows $\theta_1^* < \theta_2^*$ as we wanted. □

Proof of Proposition 5.1. To fix ideas, let's choose the i-th agent as our representative EU investor. Her goal is to maximize the expected utility of her terminal wealth, which as usual is given by

$$X_C^i = (W^i - \theta_C^i)r + \theta_C^i R = W^i r + \theta_C^i\left(\frac{D}{p} - r\right). \tag{A.11}$$

Note that the unitary price p that the i-th agent has to take into consideration is determined by the market clearing condition combined with the Equilibrium Conjecture; more precisely, it must be true that

$$W^{C-i} + \theta_C^i = np.$$

Now we can substitute for p in (A.11), use the explicit form of $u_C^i(\cdot)$ and the Gaussian distribution of the risky asset dividend to obtain

$$\mathbb{E}\left[u_C^i\left(X_C^i\right)\right] = \mathbb{E}\left[1 - \exp\left\{-\lambda^i W^i r - \lambda^i\theta_C^i\left(Z\frac{\sigma}{p} + \frac{\mu}{p} - r\right)\right\}\right], \tag{A.12}$$

where $Z \sim \mathcal{N}(0,1)$. Let's ignore for the moment the constraint on the initial endowment given by (5.11). Then, the optimization problem of our trader becomes

$$\max_{\theta_C^i \in [0, W^i]} \mathbb{E}\left[u_C^i\left(X_C^i\right)\right];$$

after tedious (but not difficult) computations, the previous objective function reduces to

$$1 - \exp\left\{-\tfrac{\lambda^i n W^{C-i}[2\theta_C^i(\lambda^i n\sigma^2 - \mu) + W^{C-i}(\lambda^i n\sigma^2 - 2\mu)]}{2(\theta_C^i + W^{C-i})^2} - \lambda^i\left(W^i r - \theta_C^i r - \tfrac{\lambda^i n^2\sigma^2}{2} + n\mu\right)\right\}. \tag{A.13}$$

At this point we have to check that our conjecture leads to a sustainable equilibrium; for this, we have to show that

$$\frac{\partial\mathbb{E}\left[u_C^i\left(X_C^i\right)\right]}{\partial\theta_C^i} > 0, \quad \forall\, \theta_C^i \in (0, W^i). \tag{A.14}$$

Using (A.13), we can explicitly compute

$$\frac{\partial\mathbb{E}\left[u_C^i\left(X_C^i\right)\right]}{\partial\theta_C^i} = -\exp\{\cdots\}\left[\lambda^i r + \frac{\lambda^i n W^{C-i}\left(\theta_C^i(\lambda^i n\sigma^2 - \mu) - \mu W^{C-i}\right)}{(\theta_C^i + W^{C-i})^3}\right], \tag{A.15}$$

where the dots substitute a cumbersome expression. After some manipulation, we find that the condition which assures the existence of the suggested equilibrium becomes

$$P(\theta_C^i) > 0, \quad \forall\, \theta_C^i \in (0, W^i), \tag{A.16}$$

where $P(\cdot)$ is a third degree polynomial defined by

$$P(x) := -rx^3 - 3rW^{C-i}x^2 - W^{C-i}(\lambda^i n^2\sigma^2 - n\mu + 3rW^{C-i})x + {W^{C-i}}^2(n\mu - rW^{C-i}). \tag{A.17}$$

Now, it is immediate to check that $P(0) > 0$. In fact, we have

$$\begin{aligned} P(0) &= {W^{C-i}}^2(n\mu - rW^{C-i}) \\ &= {W^{C-i}}^2(n\mu - rW^C + rW^i) \\ &> {W^{C-i}}^2 rW^i > 0, \end{aligned}$$

thanks to (5.10). Furthermore, we state that $P(W^i) \geq 0$ is necessary and sufficient in order to verify (A.16). To see this, we note that $\lim_{x\to+\infty} P(x) = -\infty$; combining this with $P(0) > 0$, we deduce the existence of at least one strictly positive real solution to the equation $P(x) = 0$. Let $\overline{W^i} > 0$ be that solution; it is now sufficient to note that $\overline{W^i}$ actually is the *unique* strictly positive solution. In fact, it is straightforward to compute the unique inflexion point of $P(\cdot)$ as $-W^{C-i} < 0$. Hence, the other two roots of $P(\cdot)$ are real and negative or they are complex conjugate. Consequently, we have $P(\theta_C^i) > 0$ $\forall\ \theta_C^i \in [0, \overline{W^i}]$ and repeating all the previous argument adding the constraint $W^i \in [0, \overline{W^i}]$ concludes. To obtain the upper bound $\overline{W^i}$ it is sufficient to explicitly compute the positive root of $P(\cdot)$; using the *ad hoc* formula for cubic equations, we find (5.11). □

Proof of Proposition 5.2. Let's fix the h-th *behavioral* investor as our representative agent. For completeness, we now write down the analogous to (A.11) for the terminal wealth X_B^h:

$$X_B^h = (W^h - \theta_B^h)r + \theta_B^h R = W^h r + \theta_B^h\left(\frac{D}{p} - r\right); \tag{A.18}$$

however, in this case the price p is given by the condition

$$W^C + \theta_B^h = np,$$

which takes into account the Equilibrium Conjecture. What really cares to our trader is the c.d.f. of the excess risky return; thanks to the Normal distribution of R it can be written as

$$F(x) = \mathbb{P}\{R - r \leq x\} = \mathcal{N}\left(\frac{(x+r)p - \mu}{\sigma}\right),$$

where $\mathcal{N}(\cdot)$ is the c.d.f. of a standard Gaussian random variable. The next step is to use equation (2.5) in order to obtain her value function; exploiting the "risk-neutrality" assumption in (5.2) and the usual integration by parts, we have

$$\begin{aligned}
U^h(\theta_B^h) &= \int_0^{+\infty} \theta_B^h x\, d[-w_+^h(1-F(x))] - \int_{-\infty}^0 (-k_-^h\theta_B^h x)\, d[w_-^h(F(x))] \\
&= \theta_B^h\left[\int_0^{+\infty} w_+^h(1-F(x))\, dx - k_-^h \int_0^{+\infty} w_-^h(F(-x))\, dx\right] \\
&= \theta_B^h f^h\left(r, \mu, \sigma, W^C, n, \theta_B^h, k_-^h\right),
\end{aligned}$$

where $f^h(\cdots)$ is obviously defined by the last equality. Therefore, the problem of the h-th *behavioral* agent is nothing but

$$\max_{\theta_B^h \geq 0} \theta_B^h f^h\left(r, \mu, \sigma, W^C, n, \theta_B^h, k_-^h\right). \tag{A.19}$$

Now, the following properties are easy to check:

(i) the value function is null for the choice $\theta_B^h = 0$;

(ii) $f^h(\cdots)$ is strictly decreasing in k_-^h;

(iii) $f^h(\cdots)$ is strictly decreasing in r, whereas it is strictly increasing in μ;

(iv) for every choice of $c > 0$, we have

$$f^h\left(r, c\mu, c\sigma, cW^C, cn, c\theta_B^h, k_-^h\right) = f^h\left(r, \mu, \sigma, W^C, n, \theta_B^h, k_-^h\right). \tag{A.20}$$

Briefly, (i) and (ii) are straightforward by the definition of the value function and $f^h(\cdots)$. Then, (iii) is a direct consequence of the monotone increasing property of the probability weightings $w_\pm(\cdot)$; finally, the positive homogeneity property (iv) can be directly verified using the explicit expression of $F(\cdot)$. We recall that the economic interpretation of (iii) and (iv) can be found in Remark 5.2.

Thanks to (i) and (ii), our result will be proved if we show the existence of a lower threshold $\overline{k_-^h} > 0$ such that

$$\sup_{\theta_B^h > 0} f^h\left(r, \mu, \sigma, W^C, n, \theta_B^h, \overline{k_-^h}\right) \leq 0. \tag{A.21}$$

To better understand the way $\overline{k_-^h}$ can be computed, let's fix a particular k_-^h. Using the explicit form of $f^h(\cdots)$, our requirement becomes

$$\sup_{\theta_B^h>0} f^{h+}(\cdots) - k_-^h f^{h-}(\cdots) \le 0, \tag{A.22}$$

where we have implicitly defined

$$f^{h+} = \int_0^{+\infty} w_+^h \left(1 - \mathcal{N}\left(\frac{(x+r)\frac{W^C+\theta_B^h}{n} - \mu}{\sigma}\right)\right) dx\,,$$
$$f^{h-} = \int_0^{+\infty} w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C+\theta_B^h}{n} - \mu}{\sigma}\right)\right) dx\,.$$

Now, one can check that

$$\frac{\partial f^{h+}}{\partial \theta_B^h} = -\int_0^{+\infty} {w_+^h}'[1-\mathcal{N}(\cdots)]\phi(\cdots)\left(\frac{x+r}{n\sigma}\right) dx < 0, \tag{A.23}$$

where the dots substitute the arguments of $\mathcal{N}(\cdot)$, and $\phi(\cdot)$ is the probability density function of a standard Gaussian random variable. Obviously, the previous inequality implies that f^{h+} as a function of θ_B^h attains its supremum at $\theta_B^h = 0$. Moreover, for any fixed $\theta_B^h > 0$, we have

$$\begin{aligned}\int_0^r w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C}{n} - \mu}{\sigma}\right)\right) dx &< \int_0^r w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C+\theta_B^h}{n} - \mu}{\sigma}\right)\right) dx \\ &< \int_0^{+\infty} w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C+\theta_B^h}{n} - \mu}{\sigma}\right)\right) dx = f^{h-}(\cdots).\end{aligned} \tag{A.24}$$

Turning back to the original estimation (A.21), we find

$$\begin{aligned}\sup_{\theta_B^h>0} f^h(\cdots) &= \sup_{\theta_B^h>0} \left[f^{h+}(\cdots) - k_-^h f^{h-}(\cdots)\right] \\ &\le \sup_{\theta_B^h>0} f^{h+}(\cdots) - k_-^h \inf_{\theta_B^h>0} f^{h-}(\cdots) \\ &\le f^{h+}(\cdots)\big|_{\theta_B^h=0} - k_-^h \inf_{\theta_B^h>0} \int_0^r w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C}{n} - \mu}{\sigma}\right)\right) dx \\ &= \int_0^{+\infty} w_+^h \left(1 - \mathcal{N}\left(\frac{(x+r)\frac{W^C}{n} - \mu}{\sigma}\right)\right) dx - k_-^h \int_0^r w_-^h \left(\mathcal{N}\left(\frac{(-x+r)\frac{W^C}{n} - \mu}{\sigma}\right)\right) dx,\end{aligned} \tag{A.25}$$

where we recall that $\frac{W^C}{n} = p^*$. In conclusion, (A.21) will be assured if we choose $k_-^h \ge \overline{k_-^h}$ as expressed in (5.14). □

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