Optimal Capital Structure with Endogenous Default and Volatility Risk

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Abstract

This paper analyzes the capital structure of a firm in an infinite time horizon following [30] under the more general hypothesis that the firm's assets value process belongs to a fairly large class of stochastic volatility models. By applying singular perturbation theory, we fully describe the (approximate) capital structure of the firm in closed form as a *corrected* version of [30] and analyze the stochastic volatility effect on all financial variables. We propose a *corrected* version of the smooth-fit principle under volatility risk useful to determine the optimal stopping problem solution (i.e. endogenous failure level) and a *corrected* version for the Laplace transform of the stopping failure time. The numerical analysis obtained from exploiting optimal capital structure shows enhanced spreads and lower leverage ratios w.r.t. [30], improving results in a robust model-independent way.

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1 Introduction

The capital structure of a firm has been analyzed in terms of derivatives contracts since Merton's work [38]. In the corporate model proposed by Leland [30] the firm realizes its capital from both debt and equity, debt is perpetual and pays a constant coupon per instant of time, determining tax benefits proportional to coupon payments. The firm is subject to the risk of default. The default level is chosen by the firm, resulting from the managerial decision of not injecting new equity when the firm is no longer able to meet debt service requirements. Despite the simplicity of this approach, it is widely recognized that Leland's model fails to incorporate some stylized facts related to credit spreads and optimal leverage. In fact the capital structure decision is a complex issue due to many variables entering in the determinacy of corporate financing policy, i.e. riskiness of the firm, bankruptcy costs, payout, interest rates and taxes. The aim of this paper is to address the challenge of improving empirical predictions about spreads and leverage ratios inside a structural trade-off model with endogenous default by removing the classical assumption of constant volatility for the firm's assets value evolution.

Pricing and hedging problems related to equity markets suggest to introduce stochastic volatility as a fundamental feature when modeling the underlying assets value dynamics. We consider the stochastic volatility risk component of the firm's asset dynamic, in order to better capture extreme returns behavior dealing with a structural model in which the distribution of assets returns is not symmetric. Fat tailed distributions of stock returns and volatility asymmetry are among the main empirical features observed in real markets, with this last feature deeply analyzed in [6]. Economic explanations of stock market fluctuations support the consistency of countercyclical stock market volatility with rational assets evaluations, as in [36]; moreover [29] provide an analysis of the asymmetric profitability of momentum trading strategies on stocks (i.e. buying past winners and selling past losers), showing that this kind of strategies are very likely to continue when downward trends are observed in highly volatile and uncertain markets. As stressed in [15], stock prices naturally exhibit heteroskedasticity: when the price of a stock drops down, the volatility of its return usually increases, due to the so called *asymmetric volatility phenomenon*, related to the observed negative correlation between stock movements and the volatility of its return. Asymmetric volatility is usually explained by referring to the *leverage effect* and/or the *feedback effect*. *Leverage effect* means that a reduction in stock price can cause firm's leverage to increase, making the stock riskier and thus increasing its volatility. *Feedback effect* refers to a reduction in stock price observed after an increased required rate of its return due to an exogenous shock in volatility.

Shedding light on the asymmetric volatility phenomenon and its implication on risk management and corporate financing decisions, we follow a first passage structural approach to default and study the optimal capital structure of a firm extending Leland's setting [30] by assuming firm's activities value process belonging to a fairly large class of stochastic volatility models. We introduce a process describing the dynamic of the diffusion coefficient, negatively correlated with firm's assets value evolution. Volatility is driven by a one factor mean-reverting Gaussian diffusion, that is an Ornstein-Uhlenbeck process, known for its capability of capturing many stylized features of financial assets returns (i.e. heavy tailed distributions [12], [40]). In the spirit of Merton's work [38], each component of the firm's capital structure can be expressed as a coupon-paying defaultable claim written on the firm's activities value, thus the arbitrage free price of such claims can be computed as the expected value of its discounted payoff with respect to the risk neutral measure. In order to obtain explicit expressions for these defaultable claims the key tool is the Laplace transform of the stopping failure time (e.g. see [5]); but the Laplace transform is not available in closed form in our stochastic volatility model. We overcome this lack in two steps: first by considering the boundary value problems associated to the pricing partial differential equation (henceforth PDE) for these coupon-paying defaultable claims, then by carrying out a singular perturbation analysis which allows us to obtain an explicit expression of the claims prices by exploiting ideas and techniques developed in [20], [23]. Boundary conditions naturally arise depending on the specific contract, while singular perturbation theory and asymptotic expansion are applied to find *corrected* closed form solutions to the pricing problem. This enables us to completely describe the capital structure of the firm as a *corrected* version of Leland's results [30], where the correction is induced by volatility risk. All claims can be written under a common structure: starting from their corresponding value in [30], it is sufficient to correct each price through a correction term that we derive in closed form. We interpret this correction term, namely $h_\epsilon(\cdot)$, as a *default-dependent* volatility correction: it is a function of all parameters involved in the stochastic volatility model and also of the firm's distance to default *but* does not depend on the specific contract we are dealing with, i.e. this term is the same for all contracts. Starting from this last observation, we generalize our result proposing a *corrected* value for the Laplace transform of the stopping failure time under our stochastic volatility model. We then analyze the main stochastic volatility effects on the endogenous failure level derived by share holders in order to maximize equity value. Under our approach, the failure level derived from standard smooth pasting principle *is not* the solution of the optimal stopping problem, but only represents a lower bound due to limited liability of equity. Choosing that failure level would mean an early exercise of the option to default embodied in equity: [9, 10] and [34] analyze the relationship between standard smooth-fit principle and exercise time of American-style contracts, showing a failure of the traditional smooth pasting condition in some cases. Even if we do not have a closed form for the endogenous failure level, we derive it as implicit solution of a *corrected smooth pasting condition* accounting for volatility risk. Numerical results concerning optimal capital structure show spreads and leverage ratios more in line with historical norms, if compared to [30], thus confirming our intuition about the effects of introducing volatility risk into the assets dynamic: enhanced spreads and lower leverage ratios are obtained in a robust model-independent way.

Empirical results in credit risk literature have emphasized a poor job of structural models in predicting spreads, for instance [18] supposes this empirical weakness to be related to the geometric Brownian motion assumption made in the papers by [30, 31]. Thus a possible improvement to these results could ensue from introducing jumps and/or removing the assumption of constant volatility in the underlying firm value stochastic evolution. The former extension has been addressed in [26] who allow firm's assets value to make downward jumps, by supposing the dynamics of the firm's assets be driven by the exponential of a Levy process; the authors find explicit expression for the bankruptcy level, while firm's value and debt value do not have closed forms. Both [11, 14] model the firm's assets value as a double exponential jump-diffusion process and [27] study Black-Cox credit framework under the assumption that the log-leverage ratio is a time changed Brownian motion. Further [19] consider a pure jump process of the Variance-Gamma type. A jump component in the evolution of the assets dynamics is also considered in [7], where Levy processes are introduced inside an endogenous default framework and in [43], which provides a flexible model in generating various shapes of the term structure of credit spreads. Other approaches aiming at improving empirical spread predictions include the assumption of stochastic interest rates [35, 33], the introduction of incomplete information on firm's assets value [16, 17] or the assumption of uncertainty on the default level [25]. To the best of our knowledge, the extension of Leland's setting to a general class of stochastic volatility models has not been addressed. This motivates our analysis of the optimal capital structure of a firm inside a first passage structural approach to default in the spirit of [30], but assuming a stochastic volatility model for firm's assets value dynamics. Highlighting the asymmetric volatility phenomenon and its implication on risk management and corporate financing decisions, this framework is a robust way to improve empirical findings in the direction of both enhanced spreads and lower leverage ratios.

The paper is organized as follows. Section 2 describes the stochastic volatility pricing model. Section 3 provides a detailed analysis of coupon-paying defaultable claims valuation. Section 4 fully exploits each component of the firm's capital structure by providing their *corrected* values under volatility risk. Section 5 gives numerical results about the stochastic volatility effect on *optimal* capital structure, then Section 6 provides some concluding remarks.

2 A Stochastic Volatility Model of Firm's Assets Value

Due to recent crisis and turbulences observed in financial markets, a stochastic volatility pricing model seems to be the natural mathematical framework to study the credit risk associated to a firm's capital structure. In reality, the volatility process is not observable. What we observe is the stock price, from which we compute returns. Empirical studies show that stock returns usually tend towards asymmetry, since the distributions of financial asset in equity market is prone to react differently to positive or negative returns, thus motivating to move in the direction of a stochastic volatility pricing model. Removing the assumption of stock price returns being independent and Gaussian and letting the volatility be randomly varying thicken returns-distribution tails (if compared to the normal distribution), better capturing more extreme returns observed in financial markets. We consider a stochastic volatility pricing framework by assuming the assets price satisfying a specific SDE and introducing a process describing the dynamic of the diffusion coefficient, negatively correlated with firm's assets value evolution. Through simulations, [20] show that *fast mean-reversion* can be recognized in the qualitative behavior of returns time series: this is why we assume the volatility being driven by a *fast mean-reverting* one factor process of Orstein-Uhlenbeck type (henceforth OU). It is well known that mean-reversion refers to the time a process takes to return towards its long-run mean (if it exists), and is strictly related to the notion of ergodicity. Following [20], we characterize volatility by means of its *time scales* of fluctuations considering a one-parameter family of Markov processes of OU type, namely Y_t^ϵ. The introduced parameter $\epsilon > 0$ refers to the time scales of fluctuations of the OU process Y_t^ϵ. The mean-reversion can be *fast* or *slow* depending on ϵ being respectively small (short time scale) or large (long time scale). Consider for example a time horizon of one month: if the mean-reversion time is 35 days, a time scale of order one can be recognized; if the observed mean-reversion time is 5 days or 100 days, we will have respectively fast and slow mean-reversion. The volatility process (σ_t) is defined as a positive function

$$\sigma_t := f(Y_t^\epsilon) \tag{1}$$

and ϵ represents the *short* mean-reversion time scale of the *fast* OU factor Y_t^ϵ.

We consider a firm whose (unlevered) activities value dynamic is described by process V_t^ϵ, where V_t^ϵ is interpreted as the underlying assets price of a derivative contract; process Y_t^ϵ is the fast mean-reverting factor driving the evolution of the volatility process. Since Merton's [38] work, the basic idea of the structural approach is to express each component of a firm's capital structure as a derivative contract written on V_t^ϵ, this allowing to use contingent claim valuation in order to fully describe the capital structure. Thus, in order to find these claim values, the pricing problem is addressed under a *risk neutral* probability measure $\mathbb{Q}$, where the asset's evolution follows the SDEs (as in [20]):

$$\begin{aligned} dV_t^\epsilon &= rV_t^\epsilon dt + f(Y_t^\epsilon)V_t^\epsilon dW_t, &(2)\\ dY_t^\epsilon &= \left(\frac{1}{\epsilon}(m - Y_t^\epsilon) - \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}\Lambda(Y_t^\epsilon)\right)dt + \frac{\sqrt{2}\nu}{\sqrt{\epsilon}}d\widetilde{W}_t, &(3)\end{aligned}$$

with the Brownian motions having instantaneous correlation $d\langle W, \widetilde{W}\rangle_t = \rho dt$, where $\rho < 0$ in order to capture the skew (or *leverage effect*) and $\Lambda(Y_t^\epsilon)$ defined as

$$\Lambda(Y_t^\epsilon) = \rho\frac{\mu - r}{f(Y_t^\epsilon)} + \gamma(Y_t^\epsilon)\sqrt{1-\rho^2} \tag{4}$$

representing a combined market price of risk. Parameter r is the constant risk free rate, μ the expected rate of return of firm's assets value under the physical measure[1]. The quantity $\Lambda(Y_t^\epsilon)$ in (4) is a weighted sum of the *excess return-to-risk ratio* $\frac{\mu-r}{f(Y_t^\epsilon)}$ and the *risk premium factor* or *market price of volatility risk* $\gamma(Y_t^\epsilon)$, with this last term allowing to capture the second source of randomness $\widetilde{W}_t$ driving the volatility process. We follow [23, 21] by assuming $\gamma(\cdot)$ being bounded and a function of the current level $Y_0^\epsilon = y$ of the fast factor only. Process Y_t^ϵ is ergodic: $1/\epsilon$ represents the speed of convergence of Y_t^ϵ to its unique invariant distribution with density $\Phi(y)$, thus the rate of mean reversion of the *hidden* volatility process. The long-run time average of any measurable bounded function $g(Y_t^\epsilon)$ converges almost surely (a.s.) to the deterministic average quantity

$$\langle g\rangle := \int_{\mathbb{R}} g(y)\Phi(y)dy. \tag{7}$$

[1]Under the *physical* measure $\mathbb{P}$ the dynamics of the model are described by the following SDEs in $\mathbb{R}^2$:

$$\begin{aligned} dV_t^\epsilon &= \mu V_t^\epsilon dt + f(Y_t^\epsilon)V_t^\epsilon dW_t, &(5)\\ dY_t^\epsilon &= \frac{1}{\epsilon}(m - Y_t^\epsilon)dt + \frac{\sqrt{2}\nu}{\epsilon}d\widetilde{W}_t. &(6)\end{aligned}$$

The unique invariant distribution of Y_t^ϵ is a Gaussian $\mathcal{N}(m,\nu^2)$ independent of ϵ, with density function $\Phi(y)$ given by

$$\Phi(y) := \frac{1}{\sqrt{2\pi\nu^2}} e^{-\frac{(y-m)^2}{2\nu^2}}. \tag{8}$$

The generality of the model is captured by the relation between the unobservable process Y_t^ϵ and the volatility $\sigma_t := f(Y_t^\epsilon)$, where $f(\cdot)$ is supposed to be some positive, non-decreasing function bounded above and away from zero as in [21]. As shown in [20], under fast mean-reversion the integrated squared volatility

$$\bar{\bar{\sigma}}^2 := \frac{1}{T-t}\int_t^T \sigma_s^2 ds = \frac{1}{T-t}\int_t^T f^2(Y_s^\epsilon) ds$$

is governed by the time scales of fluctuations of Y_t^ϵ and converges to a constant a.s. for $\epsilon \to 0$ (limit case of fast mean-reversion)

$$\bar{\sigma}^2 := \lim_{\epsilon\to 0} \bar{\bar{\sigma}}^2 = \int_{\mathbb{R}} f^2(y)\Phi(y)dy, \tag{9}$$

when $f^2(\cdot)$ is $\Phi-$integrable. Volatility randomly varies over the average quantity $\bar{\sigma}^2$ given in (9), driven by Y_t^ϵ fluctuations depending on the large drift coefficient $1/\epsilon$ and the large diffusion coefficient $1/\sqrt{\epsilon}$. From now on the aim is to develop a stochastic volatility pricing model for the capital structure of a firm whose underlying assets value is V_t^ϵ, without specifying a particular function $f(\cdot)$ in the definition of the volatility dynamic. This allows to present model-independent results which hold for a fairly general class of one-factor fast mean-reverting processes Y_t^ϵ.

3 Pricing Defaultable Claims under Volatility Risk

This section analyzes the pricing problem arising in the stochastic volatility model (2)-(3) for defaultable coupon-paying claims, representing the natural generalization to express corporate securities under a first passage structural approach to default. We consider any claim which continuously gives to its holder a nonnegative constant payout c (i.e. coupon stream payments) until the firm is solvent. Let x denote current assets value $x := V_0^\epsilon$. The claim default arrives when the underlying assets value V_t^ϵ reaches a certain constant level, namely x_B. We define the stopping time

$$T_B^\epsilon := \inf\{t \geq 0 : \; V_t^\epsilon = x_B\}, \tag{10}$$

moreover, since process V_t^ϵ is right continuous, it holds $V_{T_B^\epsilon}^\epsilon = x_B$. We assume

$$0 < x_B < x, \tag{11}$$

otherwise the default time T_B^ϵ would be necessarily 0. Let the price of a general claim be $P^\epsilon(t; V_t^\epsilon, Y_t^\epsilon)$, depending on parameter ϵ. We consider defaultable securities with bounded and smooth payoff at default $b(x_B)$, continuously paying a constant coupon $c \geq 0$ unless bankruptcy is declared. Since $(V_t^\epsilon, Y_t^\epsilon)$ are Markovian, the price depends on current time $t < T_B^\epsilon$, on the present value of V_t^ϵ and on the present value of Y_t^ϵ as:

$$P^\epsilon(t; V_t^\epsilon, Y_t^\epsilon) = \mathbb{E}\left[e^{-r(T_B^\epsilon - t)} b(x_B) + c \int_t^{T_B^\epsilon} e^{-rs} ds | V_t^\epsilon, Y_t^\epsilon\right], \tag{12}$$

where the expectation $\mathbb{E}[\cdot]$ is taken w.r.t. the risk neutral probability measure $\mathbb{Q}$. We assume infinite time horizon as in [30] and deal with time-independent securities, thus we are interested in prices of the form

$$P^\epsilon(x, y) = \mathbb{E}\left[e^{-rT_B^\epsilon} b(x_B) + c \int_0^{T_B^\epsilon} e^{-rs} ds | V_0^\epsilon = x, Y_0^\epsilon = y\right] \tag{13}$$

or in the most general case (i.e. for equity and total firm value) of the form

$$P^\epsilon(x, y) = ax + \mathbb{E}\left[e^{-rT_B^\epsilon} b(x_B) + c \int_0^{T_B^\epsilon} e^{-rs} ds | V_0^\epsilon = x, Y_0^\epsilon = y\right], \tag{14}$$

with $a \in \{0, 1\}$ depending on the contract. In case of constant volatility σ, the claim price $P(x)$ will depend only on the present assets value x and will be solution of an ordinary differential equation deriving from a boundary value problem of the form

$$\begin{gathered} \mathcal{L}_{BS}(\sigma) P(x) = 0, \\ P(x_B) = b(x_B), \\ \lim_{x \to \infty} P(x) < \infty, \end{gathered} \tag{15}$$

with $\mathcal{L}_{BS}(\cdot)$ representing the Black-Scholes operator for time-independent securities:

$$(\mathcal{L}_{BS}(\sigma)) g(x) := c - rg(x) + rxg'(x) + \frac{1}{2} x^2 \sigma^2 g''(x). \tag{16}$$

Under constant volatility, the Laplace transform of the stopping failure time is known in closed form (see [28]), allowing to directly compute claim prices as in [5]. In our stochastic volatility model the price $P^\epsilon(x, y)$ given in (13) depends on both the present value of assets and the current level y of the fast factor Y_t^ϵ, but our main problem is that the Laplace transform of the stopping failure time $\mathbb{E}\left[e^{-rT_B^\epsilon} | V_0^\epsilon = x, Y_0^\epsilon = y\right]$, with T_B^ϵ given in (10) is not available in closed form. And this is the main problem we face: it is well known that outside Black-Scholes model, the Laplace transform of a stopping time is available in closed form only in few cases, for example when assets dynamic is described by the exponential of a Lévy process (see [3], [26]). This is exactly what motivates the application of ideas and techniques developed in [20, 21] to overcome this difficulty and find defaultable claim prices under volatility risk. In [20] authors observe that the price $P^\epsilon(t; X_t^\epsilon, Y_t^\epsilon)$ of a contract under (2)-(3) given in (12) requires to fully estimate all parameters and functions involved in the model, which is a very complicated issue. The perturbation approach simplifies their problem by approximating the price with a quantity which depends only on few market parameters; in [23] this technique is applied for the price of a zero-coupon bond. In [24] a similar idea is applied to study default risk in a Merton-like structural model. We follow the same idea and use singular perturbation analysis to overcome the lack of a closed form for $\mathbb{E}\left[e^{-rT_B^\epsilon}|V_0^\epsilon = x, Y_0^\epsilon = y\right]$ in our stochastic volatility model. Following [20] we expand the price in powers of $\sqrt{\epsilon}$ and look for an approximation of the form:

$$P^\epsilon \approx \widetilde{P}^\epsilon := P_0 + \sqrt{\epsilon} P_1, \tag{17}$$

where P_0 is a Black-Scholes price and P_1 is the first order fast scale correction term. Applying ideas in [20] to our time-independent securities, we can state that P_0 and P_1 can be obtained as solutions of boundary value problems involving Black-Scholes operator given in (16) with, respectively, a terminal condition and a source term. In line with [20], we obtain in closed form this resulting *corrected* prices $\widetilde{P}^\epsilon$ for each defaultable claim: they correspond to some Black-Scholes prices (of the same contract) *corrected* by a term capturing the introduced sources of riskiness (the market price of volatility risk and the leverage effect ρ). When assuming a fast volatility time scaling, the first two terms P_0, P_1 of the price expansion do not directly depend on the current volatility level $f^2(y)$: we refer to [20] for all technical details. Consequently, *corrected* prices are approximations[2] of the form

$$\widetilde{P}^\epsilon(x) := P_0(x) + \sqrt{\epsilon} P_1(x). \tag{18}$$

Let $P_{BS}(x; \bar{\sigma})$ be the present Black-Scholes price of a time-independent security whose underlying assets dynamic has constant diffusion coefficient $\bar{\sigma}$ satisfying (9). Following [20] we can state that the leading order term $P_0(x) := P_{BS}(x; \bar{\sigma})$. Moreover, we must impose $\lim_{x\to\infty} P_1(x) = 0, \quad P_1(x_B) = 0$ as boundary conditions for the first order correction term: when the claim becomes riskless (as x approaches

[2] Technical details about the accuracy of this approximation are in [20], Chapter 5. When the payoff function $b(\cdot)$ is smooth and bounded, $|P^\epsilon - (P_0 + \sqrt{\epsilon} P_1)| \leq \bar{k} \cdot \epsilon$, where $\bar{k}$ is a constant which does not depend on the time scale parameter ϵ. We refer to [42] for a detailed analysis about the accuracy of the approximation in case of singular perturbation method applied to option pricing under fast and slow volatility time scaling.

∞) and when assets reaches the default barrier x_B no *correction* must be in force. In this last case, recalling Equation (14), the terminal condition is

$$P^\epsilon(x_B) = P_0(x_B) = \widetilde{P}^\epsilon(x_B), \tag{19}$$

meaning that the *corrected* price under volatility risk has the same final payoff of the corresponding Black-Scholes contract with constant volatility $\bar{\sigma}$. The following Proposition provides a general formulation for the leading order term $P_0(x)$ which will be useful for further results.

Proposition 3.1 *Let V_t be the geometric Brownian motion defined by*

$$dV_t = rV_t dt + \bar{\sigma} V_t dW_t \tag{20}$$

and

$$T_B = \inf\{t \geq 0 : V_t = x_B\}, \tag{21}$$

with $\bar{\sigma}$ given in (9). Under the stochastic volatility model (2)-(3), consider a defaultable claim with price P^ϵ given in (14). The leading order term $P_0(x)$ of its price approximation $\widetilde{P}^\epsilon(x)$ in (18) has the following probabilistic representation

$$P_0(x) = ax + \mathbb{E}\left[e^{-rT_B} b(x_B) + c\int_0^{T_B} e^{-rs} ds | V_0 = x\right], \tag{22}$$

and can be written under the general form:

$$P_0(x) = k(x) + l(x_B)\left(\frac{x_B}{x}\right)^\lambda, \tag{23}$$

with

$$k(x) := ax + \frac{c}{r}, \tag{24}$$

$$l(x_B) := b(x_B) - \frac{c}{r}, \tag{25}$$

$$\lambda = \frac{2r}{\bar{\sigma}^2}, \tag{26}$$

where $k(x)$ is the riskless part of the Black-Scholes price of the claim under constant effective volatility $\bar{\sigma}$; $a \in \{0,1\}$ depending on the claim, c is the constant continuous coupon paid by the contract and $b(x_B)$ is the payoff of the claim at default.

The following Proposition shows how to determine the first order fast scale correction terms $P_1(x)$ capturing the stochastic volatility effect on defaultable claim prices.

Proposition 3.2 *Under the stochastic volatility model (2)-(3), the first-order fast scale correction terms $P_1(x)$ have the following probabilistic representation*

$$P_1(x) = \mathbb{E}\left[c_{DP}(x, x_B; \rho, \bar{\sigma}) \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \tag{27}$$

with V_t given in (20), T_B given in (21) and

$$c_{DP}(x, x_B; \rho, \bar{\sigma}) := \frac{l(x_B)H(\rho, \bar{\sigma}) {x_B}^{\lambda} \log \frac{x}{x_B}}{{x_B}^{\lambda} - x^{\lambda}}. \tag{28}$$

The correction term $P_1(x)$ can be written under the general form:

$$P_1(x) = l(x_B)H(\rho, \bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^{\lambda} \log \frac{x}{x_B} \tag{29}$$

with $l(x_B)$ given in (25) and

$$\begin{aligned} H(\rho, \bar{\sigma}) &= \frac{4r}{\bar{\sigma}^4}\left(\frac{\nu}{\sqrt{2}}\langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2} v_3\right), && (30) \\ v_3 &= \frac{\rho}{\sqrt{2}} \nu \langle f \phi' \rangle, && (31) \\ \phi'(y) &:= \frac{1}{\nu^2 \Phi(y)} \int_{-\infty}^{y} \left(f(z)^2 - \bar{\sigma}^2\right) \Phi(z) dz, && (32) \end{aligned}$$

where $f(\cdot)$, $\Lambda(\cdot)$, $\langle \cdot \rangle$, $\Phi(y)$, $\bar{\sigma}^2$ are given in (1), (4), (7)-(9) and $\rho < 0, r, \nu > 0$.

Previous results give us the first two terms for the price expansion of a coupon-paying defaultable claim written on firm's activities value under (2)-(3). Looking at Equation (17), we can think of $P_1(x)$ as a claim which pays to its holder a continuous coupon equal to $c_{DP}(x, x_B; \rho, \bar{\sigma})$ given in (28) unless default arrives. We then interpret the term $c_{DP}(x, x_B; \rho, \bar{\sigma})$ as a continuous *default-dependent payment stream* accounting for volatility risk in a robust model-independent way. It directly depends on both the default boundary x_B and the distance to default $\log \frac{x}{x_B}$. It continuously *corrects* the price accounting for volatility randomness depending on ρ and $\bar{\sigma}$; it is independent on the time scale parameter ϵ.

Proposition 3.3 *Consider $H(\rho, \bar{\sigma})$ given in (30) and v_3 given by (31). Assuming*

$$0 < v_3 < \frac{\nu}{2\sqrt{2}} \langle \Lambda \phi' \rangle, \tag{33}$$

with $\nu > 0$, we have

$$H(\rho, \bar{\sigma}) = \frac{2r}{\bar{\sigma}^4}\sqrt{2}\nu \left(\langle \Lambda \phi' \rangle + \frac{2r}{\bar{\sigma}^2}\rho \langle f \phi' \rangle \right) > 0. \tag{34}$$

As a consequence of Proposition 3.3, the sign of the first order fast scale correction term $P_1(x)$ depends only on the specific boundary conditions of the contract as the following Proposition shows.

Proposition 3.4 *Consider the default-dependent payment stream $c_{DP}(x, x_B; \rho, \bar{\sigma})$ given in (28) and the first order fast scale correction term $P_1(x)$ given in (29). Under constraint (11), the following holds*

$$c_{DP}(x, x_B; \rho, \bar{\sigma}) \cdot l(x_B) < 0, \tag{35}$$

$$P_1(x) \cdot l(x_B) > 0, \tag{36}$$

with $l(x_B)$ given in (25).

Remark 3.5 *Following [20], (Chapter 5), we can directly write approximate prices of our defaultable time-independent securities $\widetilde{P}^{\epsilon}(x)$ as solution of the following problem*

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0(x) + \sqrt{\epsilon}P_1(x)) = V_2 x^2 {P_0}''(x) + V_3 x^3 {P_0}'''(x), \tag{37}$$

with

$$V_2 = \sqrt{\epsilon} v_2, \tag{38}$$

$$V_3 = \sqrt{\epsilon} v_3, \tag{39}$$

$$v_2 = \left(2v_3 - \frac{\nu}{\sqrt{2}} \langle \Lambda \phi' \rangle \right), \tag{40}$$

and v_3 given by (31), $\mathcal{L}_{BS}(\cdot)$ by (16), where coefficients V_2, V_3 correspond exactly to the notation used in [20] (Equations 5.39-5.40, page 95 where parameter α in the book corresponds to our $1/\epsilon$). Calibrating V_2, V_3 from market data suggests to assume these small parameters being respectively: $V_2 < 0$, $V_3 > 0$ as shown in [23], which is equivalent to constraint (33). Typically coefficient $V_2 < 0$, representing a correction for the price in terms of volatility level, while $V_3 > 0$ captures the skew effect related to the third moment of stock prices returns (see [20] and [22] for a detailed analysis about parameters calibration). In case of V_3 given in (37) vanishing (as $\rho \to 0$), the uncorrelated scenario becomes a pure Black-Scholes setting with constant volatility equal to the corrected effective volatility

$$\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2} > \bar{\sigma}, \tag{41}$$

meaning that V_2 corrects claims values for the market price of volatility risk. As in [20] we interpret the r.h.s. of Equation (37) as a source term*; differently from their result, this quantity does not directly enter in the probabilistic representation of the $P_1(x)$ term (27), but the default-dependent payment $c_{DP}(x, x_B; \rho, \bar{\sigma})$ in (28) can be written as function of this source term. Equation (37) is equivalent to*

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0(x) + \sqrt{\epsilon}P_1(x)) = -\sqrt{\epsilon}S(x, x_B; \rho, \bar{\sigma}),$$

thus, the default-dependent payment can be written as

$$c_{DP}(x, x_B; \rho, \bar{\sigma}) = \frac{2S(x, x_B; \rho, \bar{\sigma})x^{-\lambda}}{({x_B}^{\lambda} - x^{\lambda})(2r + \bar{\sigma}^2)} \log \frac{x}{x_B}.$$

This last formulation underlines that the default-dependent payment *represents an* asymptotic correction *for the price which may be positive or negative depending on the specific $P_0(x)$ claim we are dealing with, since it involves its derivatives w.r.t. x, through its dependence on the source term $S(x, x_B; \rho, \bar{\sigma})$.*

3.1 Asymptotic Price Correction Under Volatility Risk

This subsection gives a general structure for all coupon-paying defaultable claim prices under fast volatility scaling, when only the first two terms P_0, P_1 of the expansion are considered, proposing an economic interpretation of the resulting *corrected* pricing formula. Under fast volatility time scaling, the general form for defaultable coupon-paying claim prices accounts for volatility risk through a *default-dependent correction* defined below.

Proposition 3.6 *Under the stochastic volatility model (2)-(3),* corrected prices *$\widetilde{P}^{\epsilon}$ of coupon-paying defaultable claims given in (18) have the following probabilistic representation*

$$\widetilde{P}^{\epsilon}(x) = ax + \mathbb{E}\left[e^{-rT_B}b(x_B) + (c + \sqrt{\epsilon} \cdot c_{DP}(x, x_B; \rho, \bar{\sigma})) \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \quad (42)$$

with $a \in \{0, 1\}$, and $c_{DP}(x, x_B; \rho, \bar{\sigma})$ given in (28). Equivalently, $\widetilde{P}^{\epsilon}$ can be written under the general form

$$\widetilde{P}^{\epsilon}(x) = k(x) + l(x_B) \left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x, x_B; \rho, \bar{\sigma}), \quad (43)$$

with

$$h_{\epsilon}(x, x_B; \rho, \bar{\sigma}) := 1 + \sqrt{\epsilon} H(\rho, \bar{\sigma}) \log \frac{x}{x_B} \quad (44)$$

being a default-dependent correction for the price due to volatility risk. Coefficient $H(\rho, \bar{\sigma})$ *is given by (30) and* $b(x_B), c, k(x), l(x_B)$ *are given in Proposition 3.1. Under constraint (11) we have*

$$h_\epsilon(x, x_B; \rho, \bar{\sigma}) > 1.$$

Recalling the structure of $P_{BS}(x; \bar{\sigma})$ given in (23), we interpret $\widetilde{P}^\epsilon(x)$ given by (43) as a *corrected* (w.r.t. Black-Scholes) pricing formula under fast volatility time scaling. We define $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ as a *default-dependent correction* for the price capturing the main effects of stochastic volatility for $x > x_B$ in a robust model-independent way: we are not specifying a particular function $f(\cdot)$ for the diffusion coefficient, thus results hold for a large class of processes and depend on the volatility time scale parameter ϵ. The term $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ depends on all parameters describing the introduced randomness in volatility: it is increasing w.r.t. the distance to default $\log \frac{x}{x_B}$ and w.r.t. the time scale ϵ; in the limiting case of $\epsilon \to 0$ it reaches its lower bound $h_\epsilon(x, x_B; \rho, \bar{\sigma}) \to 1$, thus $\widetilde{P}^\epsilon(x) \to P_{BS}(x; \bar{\sigma})$. Notice that (23) and (43) have a common structure, leading to consider $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ as a correction to Black-Scholes claim price, with this default-dependent term being an increasing function of $H(\rho, \bar{\sigma})$.

Remark 3.7 *An equivalent formulation for Equation (43) is*

$$\widetilde{P}^\epsilon(x) := k(x) + (P_0(x) - k(x))h_\epsilon(x, x_B; \rho, \bar{\sigma}), \tag{45}$$

with $P_0(x) := P_{BS}(x; \bar{\sigma})$ *given by (23),* $k(x)$ *in (24) and* $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ *given by (44). We observe that* $k(x)$ *in (24) represents the riskless part of the Black-Scholes price of the defaultable claim, evaluated under constant volatility* $\bar{\sigma}$ *in (9). Equation (45) underlines the role of the* default-dependent correction *in affecting only the defaultable (risky) part of the contract. We observe that each* corrected price $\widetilde{P}^\epsilon(x)$ *of the form (23) satisfies also the condition*

$$\lim_{x \to \infty} \frac{\widetilde{P}^\epsilon(x)}{x} < \infty,$$

required under infinite horizon to avoid bubbles (see [11]). Observe each claim with price P^ϵ *given in (14). The asymptotic approximation given by the corrected price* $\widetilde{P}^\epsilon$ *satisfies (19) as:*

$$P^\epsilon(x_B) = \widetilde{P}^\epsilon(x_B) = a x_B + b(x_B),$$

with $a \in \{0, 1\}$ *and* $b(x_B)$ *the payoff of the claim at default, meaning the approximation does not modify the payoff at default.*

Observe that the *default-dependent correction* $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ given in (44) does not depend on the specific claim we are dealing with, meaning that this correction term is only related to the stochastic volatility assumption, thus to volatility risk: the specific boundary conditions of each claim do not appear in (44). This observation leads us to interpret this *default-dependent correction* as a tool to approximate the Laplace transform of the stopping failure time T_B^ϵ in (10), as shown in the Proposition below. This idea is similar to the one proposed in [4] for perpetual real-options, where a closed-form for the Laplace transform of the optimal stopping time is provided.

Proposition 3.8 *Let us consider the stopping (failure) time T_B^ϵ defined in (10). Under the stochastic volatility model (2)-(3), the Laplace transform of the stopping (failure) time T_B^ϵ can be approximate as*

$$\mathbb{E}\left[e^{-rT_B^\epsilon}|V_0^\epsilon = x, Y_0^\epsilon = y\right] \approx \widetilde{LT}^\epsilon(T_B^\epsilon; x, x_B) = \left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \tag{46}$$

with $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ given in (44).

We consider our result about the approximation of the Laplace transform of the stopping (failure) time T_B^ϵ and propose an alternative interpretation of the *corrected* price $\widetilde{P}^\epsilon(x)$ w.r.t. the one proposed in Proposition 3.6. Equation (46) gives a way to directly compute *corrected* prices alternative but *equivalent* to the approach of differential equations shown in Propositions 3.1 and 3.2.

Corollary 3.9 *Consider a claim with price P^ϵ given in (14) under the stochastic volatility model (2)-(3) and T_B^ϵ defined in (10). This price can directly be written as*

$$P^\epsilon(x, y) = k(x) + l(x_B)\mathbb{E}\left[e^{-rT_B^\epsilon}|V_0^\epsilon = x, Y_0^\epsilon = y\right], \tag{47}$$

thus its approximation $\widetilde{P}^\epsilon(x)$ given in (18) directly computed as

$$\widetilde{P}^\epsilon(x) = k(x) + l(x_B)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}),$$

with $k(x)$ given in (24), $l(x_B)$ in (25) and $\mathbb{E}\left[e^{-rT_B^\epsilon}|V_0^\epsilon = x, Y_0^\epsilon = y\right]$ in (46).

Proposition 3.9 gives us a way to directly compute the price of a defaultable claim in (14) by interpreting its approximation $\widetilde{P}^\epsilon(x)$ in (47) as a *corrected* version of the corresponding Black-Scholes price, where the default-dependent correction directly affects *only* the probability of x reaching x_B. The correction does not affect nor the final payoff $b(x_B)$, neither the constant coupon paid by the contract c: the *corrected* price $\widetilde{P}^{\epsilon}(x)$ given in (47) has exactly the same structure of the Black-Scholes price given in (23). Figure 1 shows a comparison between

$$LT(T_B; x, x_B) := \left(\frac{x_B}{x}\right)^{\lambda}, \quad \lambda = \frac{2r}{\bar{\sigma}^2}, \tag{48}$$

with T_B given in (21), and $\widetilde{LT}^{\epsilon}(T_B^{\epsilon}; x, x_B)$ given in (46). This comparison underlines how the stochastic volatility assumption affects the Laplace transform of the stopping (failure) time before the default barrier x_B is touched (i.e. $\widetilde{LT}^{\epsilon}$) w.r.t. the corresponding value under a constant volatility setting with volatility $\bar{\sigma}$ (i.e. LT).

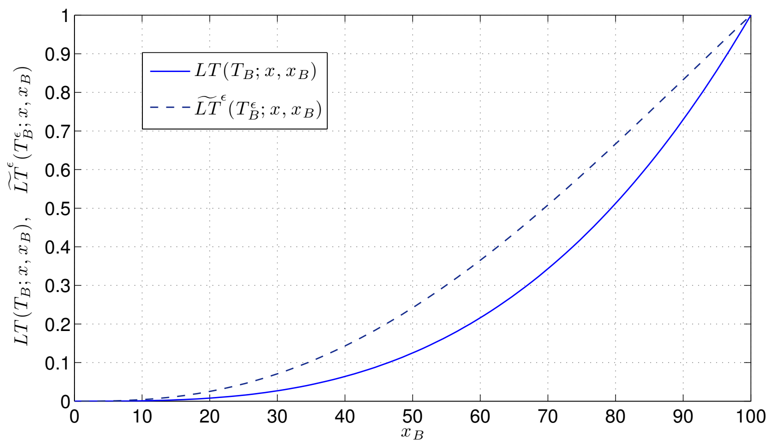

Figure 1: *Corrected* **Laplace transform of the stopping failure time.** The plot shows $\widetilde{LT}^{\epsilon}(T_B^{\epsilon}; x, x_B), LT(T_B; x, x_B)$ given by (46)-(48) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $x = 100$, $V_3 = 0.003$, $V_2 = 2V_3$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$.

4 Firm's Capital Structure under Volatility Risk

Following structural models approach the capital structure of a firm is analyzed in terms of derivative contracts. As in [30] we consider an infinite time horizon and a firm issuing both equity and debt. The firm issues debt and debt is perpetual. Debt holders receive a constant coupon C per instant of time: from issuing debt the firm obtains tax deductions proportional to coupon payments. The corporate tax rate $\tau, 0 \leq \tau < 1$, is assumed to be unique and does not vary in time, thus the firm can benefit of a constant tax-sheltering value of interest payments τC. The firm is subject to default risk and the strict priority rule holds, generating the trade-off between taxes and bankruptcy costs. Default is endogenously triggered by firm's activities value crossing a constant level x_B, corresponding to firm's incapability of covering its debt obligations (due to equity limited liability). We deal with a stationary and time homogeneous debt structure as proposed by Leland [30], where the infinite horizon assumption can be interpreted as a continuous rolling-over of debt. Following [30, 38] contingent claim valuation can be used and each component of the capital structure is expressed as a defaultable coupon-paying claim on the underlying assets represented by firm's activities value (see also [5]). When bankruptcy occurs at time T_B^ϵ given in (10), a fraction α ($0 \leq \alpha \leq 1$) of firm value is lost (i.e. due to bankruptcy procedures): debt holders receive the rest and stockholders nothing, meaning the strict priority rule holds. Equity E^ϵ, debt D^ϵ, tax benefits TB^ϵ, bankruptcy costs BC^ϵ can be written as follows

$$\begin{aligned} E^\epsilon(x,y) &= x - \mathbb{E}\left[e^{-rT_B^\epsilon}x_B + (1-\tau)\int_0^{T_B^\epsilon} e^{-rs}Cds | V_0^\epsilon = x, Y_0^\epsilon = y\right], && (49) \\ D^\epsilon(x,y) &= \mathbb{E}\left[(1-\alpha)e^{-rT_B^\epsilon}x_B + \int_0^{T_B^\epsilon} e^{-rs}Cds | V_0^\epsilon = x, Y_0^\epsilon = y\right], && (50) \\ TB^\epsilon(x,y) &= \mathbb{E}\left[\tau C\int_0^{T_B^\epsilon} e^{-rs}ds | V_0^\epsilon = x, Y_0^\epsilon = y\right], && (51) \\ BC^\epsilon(x,y) &= \mathbb{E}\left[e^{-rT_B^\epsilon}\alpha x_B | V_0^\epsilon = x, Y_0^\epsilon = y\right], && (52) \end{aligned}$$

where T_B^ϵ is the stopping time given in (10) and $\mathbb{E}[\cdot]$ denotes the expectation w.r.t. the risk neutral probability measure. The total value of the (levered) firm v^ϵ can be obtain as the sum of equity and debt $E^\epsilon + D^\epsilon$ or, equivalently, as current assets value x plus tax benefits of debt less bankruptcy costs, $x + TB^\epsilon - BC^\epsilon$. Both formulations lead to:

$$v^\epsilon(x,y) = x + \mathbb{E}\left[\tau\int_0^{T_B^\epsilon} e^{-rs}Cds - e^{-rT_B^\epsilon}\alpha x_B | V_0^\epsilon = x, Y_0^\epsilon = y\right]. \qquad (53)$$

The value of each capital-structure claim can be seen as the price of a defaultable contract having the same structure of (14). Thus, each of them can be approximate as in (18) where the P_0 and P_1 terms have the structure described in Propositions 3.1-3.2. Observe that E^ϵ and v^ϵ are claims having an option embodied contract. The following Proposition provides the P_0 leading order terms for each specific claim describing the capital structure of the firm, i.e. equity (E), debt (D), tax benefits (TB), bankruptcy costs (BC) and total value of the firm (v), corresponding to those ones computed under a pure Leland [30] setting with constant *effective volatility* $\bar{\sigma}$, i.e. a Black-Scholes setting under infinite horizon with constant *effective volatility* $\bar{\sigma}$ given in (9).

Proposition 4.1 *Under the stochastic volatility model (2)-(3), the capital structure of the firm has the following P_0 terms:*

$$P_0{}^E(x) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^\lambda, \tag{54}$$

$$P_0{}^D(x) = \frac{C}{r} + \left((1-\alpha)x_B - \frac{C}{r}\right)\left(\frac{x_B}{x}\right)^\lambda, \tag{55}$$

$$P_0{}^{TB}(x) = \frac{\tau C}{r} - \frac{\tau C}{r}\left(\frac{x_B}{x}\right)^\lambda, \tag{56}$$

$$P_0{}^{BC}(x) = \alpha x_B \left(\frac{x_B}{x}\right)^\lambda, \tag{57}$$

$$P_0{}^v(x) = x + \frac{\tau C}{r} - \left(\frac{\tau C}{r} + \alpha x_B\right)\left(\frac{x_B}{x}\right)^\lambda. \tag{58}$$

The first order fast scale correction terms P_1 accounting for volatility risk are explicitly given in the Proposition below for each capital structure-claim.

Proposition 4.2 *Under the stochastic volatility model (2)-(3), the capital structure of the firm has the following first order fast scale correction terms P_1:*

$$P_1{}^E(x) = \left(\frac{(1-\tau)C}{r} - x_B\right) H(\rho,\bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B} \tag{59}$$

$$P_1{}^D(x) = \left((1-\alpha)x_B - \frac{C}{r}\right) H(\rho,\bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B} \tag{60}$$

$$P_1{}^{TB}(x) = -\frac{\tau C}{r} H(\rho,\bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B} \tag{61}$$

$$P_1{}^{BC}(x) = \alpha x_B H(\rho,\bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B} \tag{62}$$

$$P_1{}^v(x) = -\left(\alpha x_B + \frac{\tau C}{r}\right) H(\rho,\bar{\sigma}) \cdot \left(\frac{x_B}{x}\right)^\lambda \log\frac{x}{x_B} \tag{63}$$

where $H(\rho,\bar{\sigma})$ is given in (30). The P_1 terms can also be written under the following probabilistic representation

$$P_1{}^E(x) = \mathbb{E}\left[\frac{\left(\frac{(1-\tau)C}{r} - x_B\right) H(\rho,\bar{\sigma}) x_B{}^\lambda \log\frac{x}{x_B}}{x_B{}^\lambda - x^\lambda} \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \tag{64}$$

$$P_1{}^D(x) = \mathbb{E}\left[\frac{\left((1-\alpha)x_B - \frac{C}{r}\right) H(\rho,\bar{\sigma}) x_B{}^\lambda \log \frac{x}{x_B}}{x_B{}^\lambda - x^\lambda} \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \quad (65)$$

$$P_1{}^{TB}(x) = \mathbb{E}\left[\frac{\frac{\tau C}{r} H(\rho,\bar{\sigma}) x_B{}^\lambda \log \frac{x}{x_B}}{x^\lambda - x_B{}^\lambda} \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \quad (66)$$

$$P_1{}^{BC}(x) = \mathbb{E}\left[\frac{\alpha x_B H(\rho,\bar{\sigma}) x_B{}^\lambda \log \frac{x}{x_B}}{x_B{}^\lambda - x^\lambda} \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \quad (67)$$

$$P_1{}^{v}(x) = \mathbb{E}\left[\frac{\left(\alpha x_B + \frac{\tau C}{r}\right) H(\rho,\bar{\sigma}) x_B{}^\lambda \log \frac{x}{x_B}}{x^\lambda - x_B{}^\lambda} \int_0^{T_B} e^{-rs} ds | V_0 = x\right], \quad (68)$$

with V_t given in (20) and T_B in (21).

Approximate prices $\widetilde{P}^\epsilon(x) := P_0(x) + \sqrt{\epsilon} P_1(x)$ for each defaultable capital structure claim can be alternatively (directly) derived by using Proposition 3.9, through our approximation (46) of the Laplace transform of the stopping (failure) time T_B^ϵ given in (10).

Proposition 4.3 *Under the stochastic volatility model (2)-(3) approximate prices (18) for each capital structure defaultable claims are given by*

$$\widetilde{E}^\epsilon(x) = x - \frac{(1-\tau)C}{r} + \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \quad (69)$$

$$\widetilde{D}^\epsilon(x) = \frac{C}{r} + \left((1-\alpha)x_B - \frac{C}{r}\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B, \rho, \bar{\sigma}), \quad (70)$$

$$\widetilde{TB}^\epsilon(x) = \frac{\tau C}{r} - \frac{\tau C}{r}\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \quad (71)$$

$$\widetilde{BC}^\epsilon(x) = \alpha x_B \left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \quad (72)$$

$$\widetilde{v}^\epsilon(x) = x + \frac{\tau C}{r} - \left(\frac{\tau C}{r} + \alpha x_B\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}), \quad (73)$$

with $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ given in (44) and $\lambda = 2r/\bar{\sigma}^2$.

From now on we will denote each *corrected* price of a capital structure claim as function of both x, x_B, i.e. $\widetilde{E}^\epsilon(x, x_B)$, which will be useful to stress their dependence on the failure level, thus their nature of being defaultable-claims. We will do the same for the P_0 and P_1 terms of each contract.

4.1 Equity

Look at equity claim *corrected* price $\widetilde{E}^\epsilon$ given in (69): we can recognize equity structure of being the sum of two different components as

$$\widetilde{E}^\epsilon(x, x_B) = k(x) + \widetilde{DF}^\epsilon(x, x_B), \tag{74}$$

where $k(x)$ corresponds to (24)

$$k(x) = x - \frac{(1-\tau)C}{r},$$

and

$$\widetilde{DF}^\epsilon(x, x_B) = \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}) \tag{75}$$

being the first order approximation for the *option to default* embodied in equity

$$DF^\epsilon(x, y) = \mathbb{E}\left[\left(\frac{(1-\tau)C}{r} - x_B\right) e^{-rT_B^\epsilon} | V_0^\epsilon = x, Y_0^\epsilon = y\right], \tag{76}$$

with $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ given in (44). Observe that

$$DF_L(x, x_B) := \left(\frac{(1-\tau)C}{r} - x_B\right)\left(\frac{x_B}{x}\right)^\lambda, \tag{77}$$

is the *option to default* embodied in equity in Leland [30] setting with constant volatility $\bar{\sigma}$. Considering Equation (74) we interpret *corrected* equity claim price $\widetilde{E}^\epsilon(x, x_B)$ as the price of a contract whose value derives from two different sources, a riskless part and a defaultable-risky part, as observed in Remark 3.7. From an economic point of view $k(x)$ is equity value without risk of default and unless limit of time, and it is exactly the same value we have in a pure Leland [30] framework, since $k(x)$ doesn't depend on the failure level x_B. As a consequence, the stochastic volatility assumption does not produce any effect on it, this term being independent of the probability of x reaching the barrier x_B. Function $k(x)$ is always positive under

$$x > \frac{(1-\tau)C}{r}. \tag{78}$$

The *corrected* price $\widetilde{DF}^\epsilon(x, x_B)$ in (75) directly depends on firm's current assets value x, on coupon payments C, on the failure level x_B and also on all parameters describing the volatility fast mean reverting process, since $\widetilde{DF}^\epsilon(x, x_B)$ is exactly the defaultable contract embodied in equity. Function $\widetilde{DF}^{\epsilon}(x, x_B)$ must have positive value due to its option-like nature, thus we impose

$$x_B < \frac{(1-\tau)C}{r}. \tag{79}$$

We define $\widetilde{DF}^{\epsilon}(x, x_B)$ as the *corrected price* of the option to default embodied in equity having positive value $\forall x \geq x_B$, with $x_B \in [0, \frac{(1-\tau)C}{r}]$. Notice that constraint (78) and (79) are the same we have in a pure Leland model [30]. No more constraints are needed when considering separately the two components $k(x), \widetilde{DF}^{\epsilon}(x, x_B)$, since random volatility fluctuations do not produce any effect when i) there is no risk of default, i.e. on $k(x)$ value; ii) at default, i.e. when $x = x_B$, thus on the payoff obtained by equity holders from the option to default.

4.1.1 First Order Fast-Scale Equity Correction

We now focus on the first-order fast scale correction term ${P_1}^E(x, x_B)$ given in (59) and its influence on approximate equity-claim behavior. Look at its probabilistic representation in (64). From Proposition 3.4 we can state that under constraint (79), ${P_1}^E(x, x_B) \geq 0, \forall x \geq x_B$, meaning it always has the effect of increasing equity claim price, as Figure 2 shows. The present value of one unit of money obtained at default is greater than in a constant volatility setting with $\bar{\sigma}$, i.e.

$$\mathbb{E}\left[e^{-rT_B^{\epsilon}} | V_0^{\epsilon} = x, Y_0^{\epsilon} = y\right] > \left(\frac{x_B}{x}\right)^{\lambda},$$

due to an increased likelihood of default, thus holding equity claim requires a higher compensation. The *default dependent payment* for equity claim $c_{DP}^E(x, x_B; \rho, \bar{\sigma})$ capturing volatility risk is given by

$$c_{DP}^E(x, x_B; \rho, \bar{\sigma}) = \frac{\left(\frac{(1-\tau)C}{r} - x_B\right) H(\rho, \bar{\sigma}) {x_B}^{\lambda} \log \frac{x}{x_B}}{{x_B}^{\lambda} - x^{\lambda}}. \tag{80}$$

The following Proposition analyzes this higher compensation required, showing its dependence on current assets value x.

Proposition 4.4 *The first order fast-scale correction term ${P_1}^E(x, x_B)$ given in (59) for equity claim increases corrected equity $\widetilde{E}^{\epsilon}(x, x_B)$ in (69) for $x > x_B$. The maximum correction effect is achieved when the distance to default satisfies the condition*

$$\log \frac{x}{x_B} = \frac{\bar{\sigma}^2}{2r}. \tag{81}$$

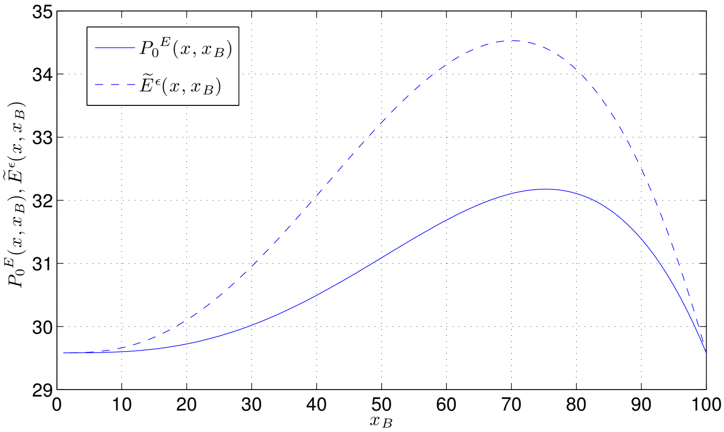

Figure 2: *Corrected* **Equity.** The plot shows $P_0{}^E(x, x_B)$ in (54) and *corrected* equity $\widetilde{E}^\epsilon(x, x_B)$ in (69) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$.

4.1.2 Endogenous Failure Level

Considering a first passage structural approach to default requires an analysis of the endogenous failure level chosen by equity holders. The economic insight behind this problem is related to share holders maximization of equity value. As in Leland [30], equity's limited liability prevents equity holders from choosing an arbitrary small failure level, making as natural constraint on x_B

$$\frac{\partial \widetilde{E}^\epsilon(x, x_B)}{\partial x}\Big|_{x=x_B} \geq 0 \quad \forall x \geq x_B, \tag{82}$$

which guarantees equity being a non-negative and increasing function of firm's current assets value x for $x \geq x_B$. The following Proposition analyzes the failure level satisfying condition (82) under the stochastic volatility model (2)-(3).

Proposition 4.5 *Consider $\widetilde{E}^\epsilon(x, x_B)$ given in (69). Under $\lambda > \sqrt{\epsilon}H(\rho, \bar{\sigma})$, the endogenous failure level chosen by equity holders in order to maximize $x_B \mapsto \widetilde{E}^\epsilon(x, x_B)$ belongs to:*

$$\left[\overline{x}_B{}^\epsilon, \frac{(1-\tau)C}{r}\right], \tag{83}$$

where

$$\overline{x}_B{}^\epsilon := \frac{(1-\tau)C}{r} \frac{\lambda - \sqrt{\epsilon}H(\rho, \bar{\sigma})}{1 + \lambda - \sqrt{\epsilon}H(\rho, \bar{\sigma})} > 0, \quad \lambda = \frac{2r}{\bar{\sigma}^2}, \tag{84}$$

with $\overline{x}_B{}^{\epsilon}$ solution of the standard smooth-pasting condition:

$$\frac{\partial \widetilde{E}^{\epsilon}(x, x_B)}{\partial x}|_{x=x_B} = 0, \tag{85}$$

and $H(\rho, \bar{\sigma})$ given in (30).

Notice that this lower bound $\overline{x}_B{}^{\epsilon} > 0$ in (84) for the endogenous failure level is function of all parameters involved in the volatility diffusion process, due to its dependence on $H(\rho, \bar{\sigma})$ and ϵ but it is also affected by economic variables defining the capital structure (i.e. coupon payments, risk free, corporate tax rate).

Remark 4.6 *As limiting case, when $\epsilon \to 0$, the lower bound given in (84) converges to the lower bound arising from a pure Leland [30] framework, namely x_{BL}, with constant effective volatility $\bar{\sigma}$:*

$$x_{BL} := \frac{(1-\tau)C}{r} \frac{\lambda}{1+\lambda}, \quad \lambda = \frac{2r}{\bar{\sigma}^2}. \tag{86}$$

Under (2)-(3), the solution $\overline{x}_B{}^{\epsilon}$ of the standard smooth pasting condition reduces as the speed of mean reversion $1/\epsilon$ increases, since the application $\sqrt{\epsilon} \mapsto \overline{x}_B{}^{\epsilon}$ is decreasing, meaning that equity holders can choose an endogenous failure level which is lower that in Leland [30] setting with constant effective volatility $\bar{\sigma}$. The lower bound $\overline{x}_B{}^{\epsilon}$ in (84) is also decreasing w.r.t. $H(\rho, \bar{\sigma}) > 0$.

Remark 4.6 is useful to formulate the optimal stopping problem faced by equity holders in this stochastic volatility model as

$$\max_{x_B \in [\overline{x}_B{}^{\epsilon}, \frac{(1-\tau)C}{r}]} \widetilde{E}^{\epsilon}(x, x_B), \tag{87}$$

with $\overline{x}_B{}^{\epsilon}$ given in (84) and $\widetilde{E}^{\epsilon}(x, x_B)$ in (69). Assuming constant volatility allows to directly apply the smooth pasting condition to equity value, thus obtaining the endogenous failure level solution of the optimal stopping problem: equity holders will always choose the lowest admissible failure level due to limited liability of equity (see also [37] footnote 60). An example of this is analyzed in [5] and an analogous relation exists when considering American-style options under Black-Scholes model (see also [20]). Assuming volatility fluctuations driven by a fast scaling mean reverting process, the *standard smooth pasting condition* applied to *corrected* equity claim price as $\frac{\partial \widetilde{E}^{\epsilon}(x,x_B)}{\partial x}|_{x=x_B} = 0$ gives only a lower bound $\overline{x}_B{}^{\epsilon} > 0$ for the endogenous failure level, which guarantes equity being an increasing function of firm's current assets value x.

Proposition 4.7 *The endogenous failure level solution of the optimal stopping problem*

$$\max_{x_B \in [\overline{x}_B{}^{\epsilon}, \frac{(1-\tau)C}{r}]} \widetilde{E}^{\epsilon}(x, x_B), \text{ where } \overline{x}_B{}^{\epsilon} = \frac{(1-\tau)C}{r} \frac{\lambda - H(\rho,\bar{\sigma})\sqrt{\epsilon}}{1 + \lambda - H(\rho,\bar{\sigma})\sqrt{\epsilon}},$$

is $\widetilde{x_B}^{\epsilon}$*, solution of*

$$h_{\epsilon}(x, x_B; \rho, \bar{\sigma}) \left(\frac{\lambda(1-\tau)C}{r} - (\lambda+1)x_B \right) - \sqrt{\epsilon} H(\rho,\bar{\sigma}) \left(\frac{(1-\tau)C}{r} - x_B \right) = 0, \quad (88)$$

with $h_{\epsilon}(x, x_B; \rho, \bar{\sigma})$ *given in (44),* $\lambda = 2r/\bar{\sigma}^2$*.*

Even if $\widetilde{x_B}^{\epsilon}$ is given only as implicit solution of (88), it is possible to stress some of its important features. As in Leland [30] setting, the endogenous failure level chosen by equity holders $\widetilde{x_B}^{\epsilon}$ does not depend on bankruptcy costs, since the strict priority rule holds, but instead depends on coupon, risk free rate and corporate tax rate. Coeteris paribus, it is increasing w.r.t the coupon level C and decreasing w.r.t. the corporate tax rate τ. Under volatility risk, equity holders choose the default barrier maximizing equity value depending on both the market price of volatility and the leverage effect, through coefficient $H(\rho, \bar{\sigma})$ in Equation (88). Differently from the optimal default boundary in [30], the endogenous failure level $\widetilde{x_B}^{\epsilon}$ derived inside a structural model framework under volatility risk depends on initial firm's assets value, due to the term $h_{\epsilon}(x, x_B; \rho, \bar{\sigma})$ in (88). We interpret this dependence as related to the standard smooth-pasting condition $\frac{\partial \widetilde{E}^{\epsilon}(x,x_B)}{\partial x}|_{x=x_B} = 0$ 'failure', since $\overline{x}_B{}^{\epsilon}$ given in (84) only represents a lower bound for the optimal stopping problem solution $\widetilde{x_B}^{\epsilon}$. An upper bound for the endogenous failure level $\widetilde{x_B}^{\epsilon}$ is provided by the following Proposition.

Proposition 4.8 *The endogenous failure level* $\widetilde{x_B}^{\epsilon}$ *solution of (88) satisfies*

$$\widetilde{x_B}^{\epsilon} < x_{BL} := \frac{(1-\tau)C}{r} \frac{\lambda}{1+\lambda}, \quad \lambda = \frac{2r}{\bar{\sigma}^2}. \quad (89)$$

Remark 4.9 *Consider* $\overline{x}_B{}^{\epsilon}, x_{BL}, \widetilde{x_B}^{\epsilon}$ *given in (84), (86), (88). From previous results we have* $\overline{x}_B{}^{\epsilon} < \widetilde{x_B}^{\epsilon} < x_{BL}$*, with these three points coinciding only in case* $\epsilon \to 0$*, i.e. when volatility risk disappears.*

We now propose an equivalent formulation of (88) in order to define a *corrected smooth-pasting condition* inside this stochastic volatility pricing model with a fast scale mean-reverting one factor process driving volatility.

Proposition 4.10 *Corrected Smooth-Pasting Condition.*
At point $\widetilde{x_B}^{\epsilon}$ implicit solution of (88), the following 'corrected smooth-pasting condition' holds:

$$\frac{\partial {P_0}^E(x,x_B)}{\partial x}|_{x=x_B} h_\epsilon(x,x_B;\rho,\bar{\sigma}) + \sqrt{\epsilon}\frac{\partial {P_1}^E(x,x_B)}{\partial x}|_{x=x_B} = 0, \tag{90}$$

with ${P_0}^E$ given in (54) and ${P_1}^E$ in (59).

Equation (90) presents the endogenous failure level $\widetilde{x_B}^{\epsilon}$ as solution of a *corrected smooth-pasting condition*, accounting for the introduced randomness in volatility fluctuations. Finding the failure level solution of $\frac{\partial \widetilde{DF}^{\epsilon}(x,x_B)}{\partial x_B} = 0$, with $\widetilde{DF}^{\epsilon}$ given in (75), is thus equivalent to find the solution of a *corrected smooth-pasting condition* given by (90): the traditional smooth-fit principle must be *corrected* due to the introduced *default-dependent volatility risk*. As observed in Proposition 4.4, the first order correction ${P_1}^E(x,x_B)$ for the price given in (59) achieves its maximum when the distance to default satisfies (81). Figure 2 shows the behavior of both approximate equity $\widetilde{E}^{\epsilon}(x,x_B)$ and ${P_0}^E(x,x_B)$ w.r.t. a constant failure level x_B: in line with Remark 4.9, function $\widetilde{E}^{\epsilon}(x,x_B)$ achieves its maximum *before* ${P_0}^E(x,x_B)$, meaning at a failure level $\widetilde{x_B}^{\epsilon}$ lower than x_{BL} in (86).

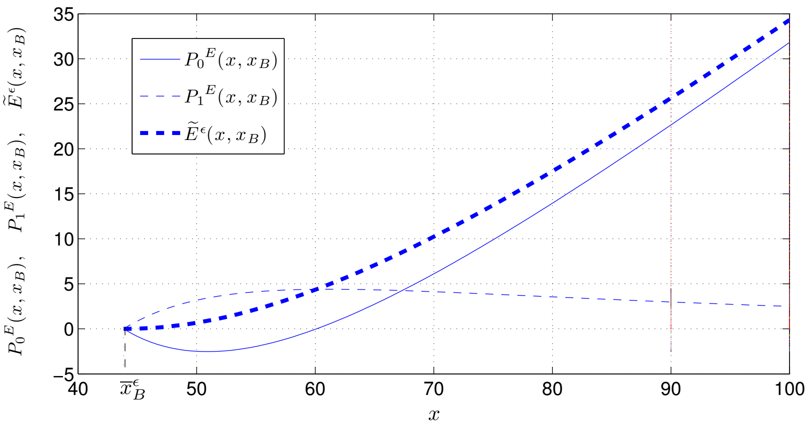

Figure 3: P_0^E, P_1^E **and** *corrected* **equity claim price.** The plot shows *corrected* equity value $\widetilde{E}^{\epsilon}(x,x_B)$ in (69), and ${P_0}^E(x,x_B), {P_1}^E(x,x_B)$ terms given in (54), (59), as function of current assets value x. The support of each function is $[\overline{x}_B{}^{\epsilon}, x]$, with $\overline{x}_B{}^{\epsilon}$ given by (84). Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $\rho < 0$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$. The endogenous failure level $\widetilde{x_B}^{\epsilon}$ is determined as implicit solution of (88) for $x = 100$.

Remark 4.11 *Recall Equation (90). We can interpret its solution $\widetilde{x_B}^\epsilon$ as the failure level satisfying*

$$\frac{h_\epsilon(x, \widetilde{x_B}^\epsilon; \rho, \bar{\sigma})}{\sqrt{\epsilon}} = -\frac{\Delta_{{P_0}^E(\widetilde{x_B}^\epsilon)}}{\Delta_{{P_1}^E(\widetilde{x_B}^\epsilon)}}, \tag{91}$$

where

$$\Delta_{{P_0}^E(\widetilde{x_B}^\epsilon)} := \frac{\partial {P_0}^E(x, x_B)}{\partial x}|_{(x=\widetilde{x_B}^\epsilon, x_B=\widetilde{x_B}^\epsilon)}, \quad \Delta_{{P_1}^E(\widetilde{x_B}^\epsilon)} := \frac{\partial {P_1}^E(x, x_B)}{\partial x}|_{(x=\widetilde{x_B}^\epsilon, x_B=\widetilde{x_B}^\epsilon)},$$

denote the Greek-Delta corresponding to the first derivative of ${P_0}^E, {P_1}^E$ w.r.t. x evaluated at point $x = \widetilde{x_B}^\epsilon$. The terms ${P_0}^E$ and ${P_1}^E$ are given in (54)-(59). Equivalently we can write (90) as

$$\log \frac{\widetilde{x_B}^\epsilon}{x} = \frac{1}{\sqrt{\epsilon} H(\rho, \bar{\sigma})} \frac{\Delta_{\widetilde{E}^\epsilon(\widetilde{x_B}^\epsilon)}}{\Delta_{{P_0}^E(\widetilde{x_B}^\epsilon)}}, \tag{92}$$

with

$$\Delta_{\widetilde{E}^\epsilon(\widetilde{x_B}^\epsilon)} := \frac{\partial \widetilde{E}^\epsilon(x, x_B)}{\partial x}|_{(x=\widetilde{x_B}^\epsilon, x_B=\widetilde{x_B}^\epsilon)}$$

and $\widetilde{E}^\epsilon$ given in (69).

i) Recall that the lower bound $\overline{x}_B{}^\epsilon$ in (84) solution of $\frac{\partial \widetilde{E}^\epsilon(x, x_B)}{\partial x}|_{x=x_B} = 0$ is independent of current assets value x. At that point, we have:

$$\sqrt{\epsilon} = -\frac{\Delta_{{P_0}^E(\overline{x}_B{}^\epsilon)}}{\Delta_{{P_1}^E(\overline{x}_B{}^\epsilon)}}. \tag{93}$$

ii) The two equivalent formulations (91)-(92) for equation (90) are useful to understand the economic optimality of the endogenous failure level $\widetilde{x_B}^\epsilon$ and its dependence on firm's activities value x, which is a new feature w.r.t. [30], where the endogenous failure level is instead x-independent. The left hand side of both equations is the only one depending on x. Choosing $\widetilde{x_B}^\epsilon$ means choosing the endogenous level corresponding to the optimal exercise time: due to current assets value x, before and after $\widetilde{x_B}^\epsilon$ the correction for the price $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ does not exactly compensate the ratio between the instantaneous variations (due to an instantaneous variation in x) in the first order correction term and the Δ of the corresponding Black-Scholes contract ${P_0}^E(x, x_B)$. Equation (92) suggests the same dependence relating the distance to default and the ratio between the instantaneous variations in approximate claim $\widetilde{E}^\epsilon(x, x_B)$ and, again, the Δ-sensitivity of ${P_0}^E(x, x_B)$. The endogenous failure level $\widetilde{x_B}^\epsilon$ can be seen as an equilibrium level which increases as current assets value

x rises. Looking at Equation (91), we interpret the correction term $h_\epsilon(x, \widetilde{x_B}^\epsilon; \rho, \bar{\sigma})$ as a default-dependent elasticity measure at equilibrium satisfying

$$\frac{1}{\sqrt{\epsilon}} + H(\rho, \bar{\sigma}) \log \frac{x}{\widetilde{x_B}^\epsilon} = -\frac{\Delta_{P_0{}^E(\widetilde{x_B}^\epsilon)}}{\Delta_{P_1{}^E(\widetilde{x_B}^\epsilon)}}. \tag{94}$$

By simply applying $\frac{\partial \widetilde{E}^\epsilon(x,x_B)}{\partial x}|_{x=x_B} = 0$ gives a failure level $\overline{x}_B{}^\epsilon$ which is independent of firm's current activities value x corresponding to a *non-optimal exercise* of the option to default embodied in equity. We are assuming fast mean reversion and short time scale parameter ϵ, thus we have

$$-\frac{\Delta_{P_0{}^E(\widetilde{x_B}^\epsilon)}}{\Delta_{P_1{}^E(\widetilde{x_B}^\epsilon)}} > \sqrt{\epsilon} = -\frac{\Delta_{P_0{}^E(\overline{x}_B{}^\epsilon)}}{\Delta_{P_1{}^E(\overline{x}_B{}^\epsilon)}},$$

meaning that at point $\overline{x}_B{}^\epsilon$ the ratio between the two *Deltas* is too small to exercise the option. Under volatility risk, assuming $\rho < 0$ makes the distribution of stock price returns not symmetric, thickening the *left tail*: this is why it is not optimal to exercise at the standard smooth pasting level but choosing a failure level greater than this. The *correction for the price* represented by $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ is a default-dependent correction function of current assets value x which cannot be ignored by the optimal exercise rule. This is analogous to [35], where the pricing of American options is addressed when both volatility and interest rates are stochastic. The authors derive a proxy for the optimal exercise rule, showing that it corresponds to the moneyness (measured in standard deviations) reaching a certain level: behind our Equation (92) the same idea can be recognized by interpreting the distance to default as a log-moneyness measure. Recent literature contains an increasing number of papers ([1], [2], [8], [13], [41]) showing the failure of the standard smooth fit principle in some optimal stopping problems, suggesting this failure being related to the assumption of discontinous jumps in the underlying assets dynamics. Closed form solutions for some optimal stopping problems depending on a diffusion with jumps are derived in [39], while [1], Theorem 6, propose a characterization of the smooth-fit principle when a general Lévy process for the underlying assets dynamics is introduced. Optimal stopping under infinite horizon is then considered in [32], providing examples of the value function not exhibiting smooth pasting at the optimal stopping boundary.

Remark 4.12 *Inside our framework the riskiness of the firm is taken into account from two different points of view: i) its market price through Λ, ii) the leverage effect through ρ. Even in the uncorrelated case $\rho = 0$ the endogenous failure level $\widetilde{x_B}^\epsilon$ depends on x and is different from the lower bound $\overline{x}_B{}^\epsilon$ given in (84), since the* *correction for the volatility level due to coefficient V_2 in (38) is still in force. Only when $\epsilon \to 0$ the dependence of the endogenous failure level $\widetilde{x_B}^\epsilon$ on x disappears (and also in case $\rho = 0, \Lambda = 0$).*

4.2 Debt, Tax Benefits, Bankruptcy Costs

We now analyze the stochastic volatility effect on debt, tax benefits and bankruptcy costs by first recalling the expressions of their *corrected* claim prices given by (70)-(72):

$$
\begin{aligned}
\widetilde{D}^\epsilon(x, x_B) &= \frac{C}{r} + \left((1-\alpha)x_B - \frac{C}{r}\right)\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}),\\
\widetilde{TB}^\epsilon(x, x_B) &= \frac{\tau C}{r} - \frac{\tau C}{r}\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}),\\
\widetilde{BC}^\epsilon(x, x_B) &= \alpha x_B\left(\frac{x_B}{x}\right)^\lambda h_\epsilon(x, x_B; \rho, \bar{\sigma}),
\end{aligned}
$$

with $h_\epsilon(x, x_B; \rho, \bar{\sigma})$ given in (44). Due (only) to the infinite horizon assumption, debt holders receive $\frac{C}{r}$ without limit of time in the event of no default and $(1-\alpha)x_B - \frac{C}{r}$ in case of x reaching x_B, meaning $\frac{C}{r}$ being riskless part of debt value. The *corrected*

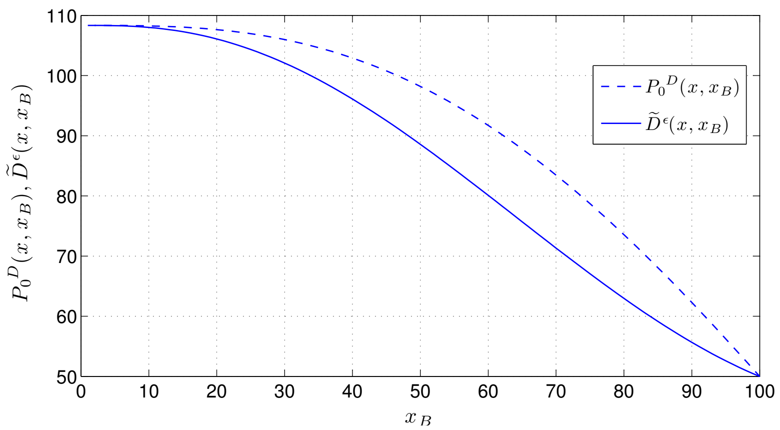

Figure 4: *Corrected* **Debt.** The plot shows the $P_0{}^D(x)$ term in (55) and *corrected* debt value $\widetilde{D}^\epsilon(x, x_B)$ in (70) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$.

tax benefits-claim $\widetilde{TB}^\epsilon(x, x_B)$ has a *downward correction*: its first-order fast scale correction term ${P_1}^{TB}(x, x_B)$ in (61) is negative for all values $x > x_B$, due to the volatility risk influence on the likelihood of default. *Corrected* bankruptcy cost-claim $\widetilde{BC}^{\epsilon}(x, x_B)$ is a defaultable capital structure security which does not pay any constant continuous coupon c: it only gives αx_B to its holder in the event of default. Its present value is higher than in [30], due to the increased present value of 1 unit of money at default under (2)-(3).

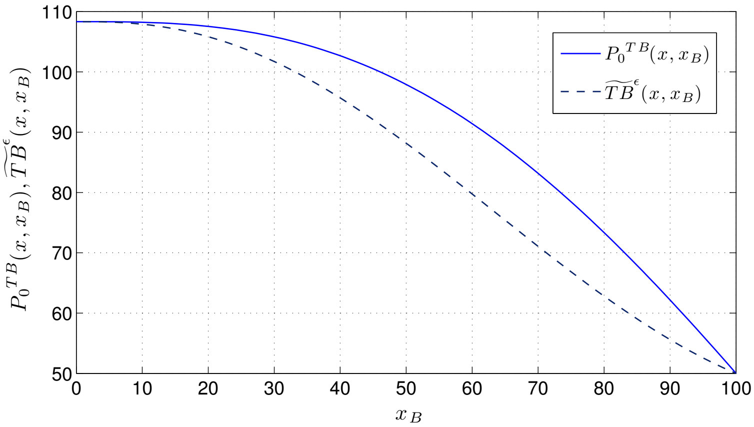

Figure 5: *Corrected* **Tax Benefits of Debt.** The plot shows the ${P_0}^{TB}(x)$ term in (56) and *corrected* tax benefits of debt $\widetilde{TB}^{\epsilon}(x, x_B)$ given in (71) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$.

4.3 Credit Spreads and Leverage Ratios

We now question wether stochastic volatility can increase credit spreads and reduce leverage ratios w.r.t. results predicted by Leland [30] model. Under (2)-(3) credit spreads are defined as $R^{\epsilon}(x, y) - r$, with

$$R^{\epsilon}(x, y) := \frac{C}{D^{\epsilon}(x, y)},$$

where $D^{\epsilon}(x, y)$ is given in (50). We denote with $\widetilde{R}^{\epsilon}$:

$$\widetilde{R}^{\epsilon}(x, x_B) := \frac{C}{\widetilde{D}^{\epsilon}(x, x_B)}, \tag{95}$$

thus $\widetilde{R}^\epsilon(x, x_B) - r$ is the *corrected* credit spread in our stochastic volatility model, with $\widetilde{D}^\epsilon(x, x_B)$ given in (70). Its corresponding value in Leland [30] setting with constant effective volatility $\bar{\sigma}$ is

$$P_0^R(x, x_B) - r := \frac{C}{P_0^D(x, x_B)} - r, \tag{96}$$

with $P_0^D(x, x_B)$ given in (55). For a fixed coupon C, randomness in volatility moves credit spreads exactly in the expected direction, rising them before the default time in order to compensate investors for the new source of risk, as Figure 6 shows. In order to study leverage ratios, we consider the stochastic volatility effect on them by analyzing the behavior of *corrected* leverage ratios defined as:

$$\widetilde{L}^\epsilon(x, x_B) := \frac{\widetilde{D}^\epsilon(x, x_B)}{\widetilde{v}^\epsilon(x, x_B)}, \tag{97}$$

with $\widetilde{D}^\epsilon(x, x_B)$ given in (70) and $\widetilde{v}^\epsilon(x, x_B)$ given in (73).

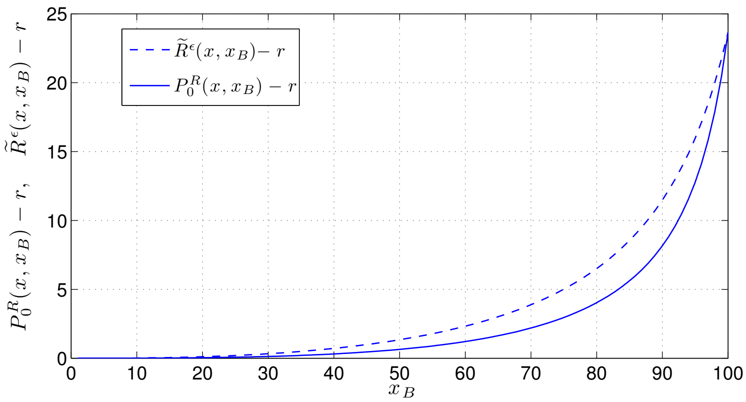

Figure 6: *Corrected* **Credit Spreads.** The plot shows *corrected* credit spreads $\widetilde{R}^\epsilon(x, x_B) - r$ in (95) and $P_0^R(x, x_B) - r$ in (96) as functions of the failure level $x_B \in [0, x]$. Base case parameters values are: $\Lambda = 0$, $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $C = 6.5$, $V_3 = 0.003$, $V_2 = 2V_3$, $x = 100$. Recall $\Lambda(\cdot)$ is given in (4), coefficients V_2, V_3 are given by (38), (39) and $\rho < 0$.

5 Optimal Capital Structure

We numerically derive the optimal coupon chosen by equity holders in order to maximize the *corrected* total value of the firm under volatility risk. Recall constraint

(78): the optimal coupon $\widetilde{C}^{\epsilon *}$ is solution of the following problem

$$\max_{C\in\left[0,\frac{xr}{(1-\tau)}\right]} \widetilde{v}^{\epsilon}(x,x_B) \tag{98}$$

with $\widetilde{v}^{\epsilon}(x,x_B)$ given in (73) as

$$\widetilde{v}^{\epsilon}(x,x_B) = x + \frac{\tau C}{r} - \left(\alpha x_B + \frac{\tau C}{r}\right)\left(\frac{x_B}{x}\right)^{\lambda} h_{\epsilon}(x,x_B;\rho,\bar{\sigma}),$$

and $h_{\epsilon}(x,x_B;\rho,\bar{\sigma})$ in (44). The aim of our analysis is now to describe the whole *corrected-optimal* capital structure value under volatility risk, considering Leland model [30] with constant effective volatility $\bar{\sigma}$ in (9) as a benchmark. For each *corrected* price $\widetilde{P}^{\epsilon}(x,x_B;C)$ corresponding to a capital-structure claim price, we denote with $\widetilde{P}^{\epsilon *} := \widetilde{P}^{\epsilon}(x,\widetilde{x_B}^{\epsilon *};\widetilde{C}^{\epsilon *})$ its optimal value, obtained by replacing the failure level x_B with the solution of the optimal stopping problem $\widetilde{x_B}^{\epsilon *}$ in (88) and the coupon C with its optimal value $\widetilde{C}^{\epsilon *}$. The aim is to analyze the influence of both sources of risk induced by the model on all financial variables at their optimal level: at first we only consider the *skew effect*, captured by ρ, then its joint influence with the *correction for the volatility level*. In this last case we consider the difference between the corrected effective volatility σ^* and the average volatility $\bar{\sigma}$ in (9), with their relation given by (41). As noted in [20], markets data suggest the corrected effective volatility being higher than the average volatility, this is why we assume $\sigma^* > \bar{\sigma}$.

Table 1 shows how corporate decisions about optimal capital structure are influenced by the introduction of a negative correlation ρ between assets value dynamics (2) and the fast factor process Y_t^{ϵ} driving volatility fluctuations (3). We leave ρ varying from $\rho = -0.05$ to $\rho = -0.1$, in order to capture its effects on corporate financing decisions. When assuming $\Lambda = 0$ (i.e. zero correction for the volatility level, with $\Lambda(\cdot)$ given in (4)), numerical results show that only the *skew effect* induced by $\rho < 0$ produces a quantitative significant impact on corporate decisions. Skewness in the underlying dynamics makes debt less attractive: optimal coupon, debt, total value of the firm and leverage ratios drop down. And in some cases, this reduction is quantitatively significant. Only a slightly negative correlation $\rho = -0.05$ brings down leverage of around 8% w.r.t. Leland [30] predictions (row 1, Table 1), while a 15%-reduction is achieved with $\rho = -0.1$. The maximum *corrected* total value of the firm is also reduced, due to the combined effect on equity and debt. The increase in optimal equity value is more than compensated by the reduction in optimal debt. The coupon level maximizing *corrected* total firm value is decreasing with the *skew effect*, generating a downward jump in optimal debt from 96.3 in case

ρ	$\widetilde{C}^{\epsilon *}$	$\widetilde{D}^{\epsilon *}$	$\widetilde{R}^{\epsilon *}$	$\widetilde{R}^{\epsilon *} - r$	$\widetilde{E}^{\epsilon *}$	$\widetilde{x_B}^{\epsilon *}$	$\widetilde{v}^{\epsilon *}$	$\widetilde{L}^{\epsilon *}$
0	6.501	96.274	6.753 %	75.255	32.168	52.820	128.442	74.956 %
-0.05	5.910	84.158	7.022%	102.197	39.581	44.916	123.739	68.012 %
-0.06	5.741	81.452	7.049%	104.862	41.417	43.413	122.870	66.292 %
-0.07	5.569	78.796	7.067%	106.744	43.255	41.961	122.051	64.560 %
-0.08	5.397	76.233	7.080%	107.991	45.054	40.575	121.287	62.853 %
-0.09	5.230	73.792	7.087%	108.748	46.787	39.264	120.578	61.198 %
-0.1	5.203	73.398	7.088%	108.835	47.068	39.053	120.465	60.929 %

Table 1: **Skew effect on optimal capital structure.** The table shows financial variables at their optimal level when only the *skew effect* is considered, i.e. $\rho < 0, \Lambda = 0$. The first row of the table reports Leland [30] results as benchmark, as particular case of $\rho = 0, \Lambda = 0$. We consider $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$ (Leland's base case values). Recall $V_3 := \sqrt{\epsilon}\rho\frac{\sqrt{2}}{2}\nu\langle f\phi'\rangle$. We consider $V_3 = -0.06\rho$, $V_2 = 2V_3$, see also [23]. $\widetilde{L}^{\epsilon *}$, $\widetilde{R}^{\epsilon *}$ are in percentage (%), $\widetilde{R}^{\epsilon *} - r$ in basis points (bps).

$\rho = 0$ to 73.4 in case $\rho = -0.1$. Despite lower leverage ratios, yield spreads are increasing with $|\rho|$. Letting $\rho = 0$, and introducing the correction for the volatility level will bring the model to a pure Black-Scholes setting with constant *effective volatility* σ^*. This is not the case we are interested in. As a second step we consider both sources of risk associated to firm value. Table 2 shows how financial variables are modified when also the *correction for the volatility level* is considered in a framework with negative leverage effect, i.e. $\rho < 0$. As example we consider a negative correlation $\rho = -0.05$ and a gap between σ^* and $\bar{\sigma}$ of $1\%, 2\%$, respectively. The skew effect and the volatility level correction seem to represent an interesting feature to develop applied to credit risk models. Optimal financing decisions move w.r.t. a pure Leland [30] model where volatility is constant: when both sources of risk are considered, their joint influence is quantitatively strong. The optimal amount of debt is reduced and leverage ratios can drop down from 75% to 62% only with a slightly negative correlation $\rho = -0.05$ and a volatility level correction of 2%. Numerical results emphasize interesting insights arising from a model where asymmetry in assets returns distribution and a volatility level correction coexist. This suggests a possible direction to follow aiming at improving empirical predictions inside a structural model with endogenous bankruptcy. The mean-reverting process describing the evolution of assets volatility makes possible to capture how prices are modified due to the market's perception of firm's credit risk: there is uncertainty about the volatility level and its evolution over time, making the firm becoming a riskier activity. Investors will require higher compensations: yield spreads must be

σ^*	$\widetilde{C}^{\epsilon *}$	$\widetilde{D}^{\epsilon *}$	$\widetilde{R}^{\epsilon *}$	$\widetilde{R}^{\epsilon *} - r$	$\widetilde{E}^{\epsilon *}$	$\widetilde{x_B}^{\epsilon *}$	$\widetilde{v}^{\epsilon *}$	$\widetilde{L}^{\epsilon *}$
$\bar{\sigma}$	6.501	96.274	6.753%	75.255	32.168	52.820	128.442	74.956 %
$\bar{\sigma} + 0.01$	5.701	80.266	7.103 %	110.252	42.011	42.955	123.260	65.120 %
$\bar{\sigma} + 0.02$	5.597	77.350	7.236 %	123.597	43.495	41.887	124.041	62.358 %

Table 2: **Skew effect and volatility level correction: influence on optimal capital structure.** The table shows financial variables at their optimal level when $\rho = -0.05$ and also a volatility correction is considered. Recall that $\sigma^* = \sqrt{\bar{\sigma}^2 - 2V_2}$. We consider $r = 0.06$, $\bar{\sigma} = 0.2$, $\alpha = 0.5$, $\tau = 0.35$, $V_3 = 0.003$. $\widetilde{L}^{\epsilon *}$, $\widetilde{R}^{\epsilon *}$ are in percentage (%), $\widetilde{R}^{\epsilon *} - r$ in basis points (bps).

higher despite lower leverage ratios.

6 Conclusions

The focus in this paper is to deal with a credit risk stochastic volatility pricing model following a first passage structural approach to default. The capital structure of a firm is analyzed in an infinite horizon framework following Leland's idea [30] but assuming firm's assets value belonging to a fairly large class of stochastic volatility models as in [20]. Volatility is driven by a one factor fast mean reverting process of OU type and characterized by means of its time scales of fluctuations. We first address the pricing problem of coupon-paying defaultable claims written on firm's assets value, then apply our results to each specific claim defining the firm's capital structure. Singular perturbation techniques and asymptotic expansion are applied following [20]: this enables us to find *corrected* closed form solutions to our pricing problem for each coupon-paying defaultable claim, then for each corporate security. We completely describe the firm's capital structure as a *corrected* version of Leland's results [30], with this correction induced by volatility risk. We find in closed form a *default-dependent correction* due to volatility risk which does not depend on the specific contract we are dealing with, but is instead general and common to all claims. This correction depends on all parameters involved in the stochastic volatility model and also on the firm's current distance to default: it acts only on the defaultable part of the contract before the failure barrier is touched, i.e. neither when the contract is riskless, nor when default arrives. Considering the *default-dependent correction*'s independence on the specific claim conditions, we then extend our results and propose in closed form a *corrected* value for the Laplace transform of the stopping failure time under volatility risk. The Laplace transform of the stopping failure time is otherwise not available in closed form under our stochastic volatility model, but our *corrected* closed form allows to directly compute defaultable claim prices. Equity holders face the problem of optimizing equity value w.r.t. the failure level: standard smooth-fit principle does not give a failure level solution of the optimal stopping problem, but only a lower bound for the endogenous default boundary due to limited liability of equity. Choosing that failure level is not optimal since it would mean an early exercise of the option to default embodied in equity. We do not derive the endogenous failure level in closed form, but as implicit solution of a *corrected smooth pasting condition* involving our *default-dependent correction* term, thus depending also on current assets value.

Optimal spreads increase and optimal leverage ratios reduce w.r.t. Leland's results [30] as our numerical analysis about optimal capital structure shows. The market perception of the credit risk associated to the firm is captured by *corrected* prices: despite lower leverage, enhanced spreads are predicted by the model, since the required compensation for risk increases, thus incorporating the *asymmetric volatility phenomenon* on corporate financing decisions in a robust model-independent way.

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