Asymptotics for the Fourier estimators * of the volatility of volatility and the leverage

Imma Valentina Curato †

Preliminary Version

Abstract

In this paper, we construct non parametric estimators of the volatility of volatility and the leverage component (covariance between the asset price and the volatility process) in the framework of one dimensional stochastic volatility model. The main feature of our estimator is that, given discrete observations of the price process, we are able to reconstruct the entire trajectory of the volatility. Thus, we handle the volatility as an observable variable and the Fourier coefficients of the volatility of volatility and the leverage processes can be computed. The estimators of the integrated quantities are easily obtained by means of the zero-Fourier coefficients. We prove consistency and feasible central limit theorems for the proposed estimators.

JEL keywords: C10,C13,C14,C15,C22

Keywords: volatility of volatility, leverage, non-parametric estimation, semi-martingale, Fourier transform, high frequency data.

1 Introduction

Stochastic volatility models are widely recognized to be able to reproduce many features of the asset returns such as time-varying volatility, volatility clustering and the leverage effect, i.e. the usual negative correlation between asset price and volatility. Many experimental studies have pointed out that parametric models, e.g Heston model and CEV models, can reproduce this stylized facts [19]. For this reason, statistical inference for stochastic volatility models mostly focuses on parametric methods. Usually, the estimations obtained are strictly connected with the model used for the asset price dynamics; a survey on early estimation methods can be found in [10]. Nevertheless, the standard maximum likelihood theory is unavailable in this context, because in most cases it is impossible

*I would like to express my deep gratitude to Maria Elvira Mancino, Arnaud Gloter, Simona Sanfelici and Maria Cristina Recchioni for the help provided and valuable comments.

†Dipartimento di Statistica e Matematica applicata all'Economia, Via Ridolfi 10, 56124 Pisa, Italy: immacurato@hotmail.it

to assess the distribution of the price process. A considerable exception is represented by the paper of [8], in which the authors obtain a Generalized Method of Moments (GMM) estimator able to overcome the problem for the estimation of the parameters of the Heston model.

In our paper we extend this line of investigation by proposing an estimator of the diffusion component of the volatility process which is supposed to be a general continuous semimartingale. This means that we work in a model free context and we do not assume the knowledge of the functional form of the volatility of variance. We construct non-parametric estimators of both the spot and the integrated variance of volatility based on the Fourier methodology introduced by [20] to estimate multivariate volatility. Moreover, assuming a stochastic dynamics for the variance of volatility allows us to construct novel estimators of the leverage process based on the same procedure. The study of the present paper will be devoted to explore the issues relative to the estimation of volatility of volatility and leverage processes focusing our attention on the asymptotic properties of the estimators.

Adding a source of randomness in the diffusion component of the volatility process is one of the directions taken by financial econometrics to better explain the asset price volatility dynamics [2] and its application in forecasting model [12],[13], [7], time variance risk premium [9], [5], [15] and inference on the spot volatility [25]. It is in this direction that the present paper should be read.

The Fourier analysis methodology has been applied for the first time to the computation of the spot volatility and covariance in [20] and has been developed in [21]. The Fourier estimation procedure allows to consider price observations which are not equally spaced and (in the multivariate case) not synchronous; it uses all the available observations and avoids any "synchronization" of the original data, because it is based on the integration of the time series of returns rather than on its differentiation. Further the spot volatility is obtained without performing any numerical derivative, which causes numerical instabilities. Finally, [22] and [23] show that the integrated estimators of volatility and covariance are robust under microstructure noise.

We extend the Fourier methodology by presenting an iterative procedure to get the Fourier coefficients of the volatility of volatility, the leverage component (asset price-volatility covariance) and the variance of the estimation errors.

Starting from the discrete observations of the price process trajectory, we compute the Fourier coefficients of the volatility. Then, we reconstruct the trajectory of volatility and can handle it as an observable variable. We iterate the procedure to compute the Fourier coefficients of the volatility of volatility and the leverage functions. Using the inverse Fourier transform we obtain spot estimators of the examined stochastic functions. We can also construct integrated estimators by means of the estimated zero Fourier coefficients. To the best of our knowledge, the Fourier estimator is the only one that allows to compute a historical estimation of the trajectory of all the stochastic functions relevant in a stochastic volatility model. In particular, [6] show that if we have estimations of the trajectory of the variance of volatility and the leverage, we have a methodology to compute for example the parameters of the diffusion process of an Heston model that overcome the problems that we have using a pure parametric estimation method. Finally, in order to produce feasible asymptotic theorems for the estimators of variance we extend a procedure that has been apply in [24] to the estimation of quarticity.

In [2] and [30] two alternative iterative procedures to compute the volatility of volatility are proposed, which are based on the quadratic variation formula. As it is highlighted in [21], the definition of the Fourier-type estimators does not depend on the features of the considered sample, in contrast with the quadratic variation type estimators, which essentially need synchronous data. For this reason, we choose the Fourier methodology in order to construct estimators that do not need any manipulation of data and are of immediate application in any scenario.

This paper is organized as follows. In the second section we describe the iterative procedure in detail. It is obtained by slightly modifying the estimators of the Fourier coefficients of the volatility of volatility and leverage functions defined in the paper [6]. In the third section we study the asymptotic properties for the estimators of the volatility of volatility. We attain the consistency of the estimators in the case of unevenly spaced price observations and a feasible central limit theorem for the integrated estimator in the case of evenly spaced price observations. In the fourth section we turn our attention to the asymptotic results obtained for the estimators of the leverage function. On finite sample, in the fifth section we test the accuracy of the asymptotic approximation of the integrated volatility of volatility estimator with numerical simulations.

2 The Fourier Estimator

We study estimations of the volatility of volatility and leverage functions of a wide class of stochastic volatility model. In the sequel, we call $p(t)$ the log-asset price process and $\nu(t)$ the volatility process

$$\begin{cases} dp(t) & = \sigma(t)dW(t) + a(t)dt \\ d\nu(t) & = \gamma(t)dZ(t) + b(t)dt. \end{cases} \tag{1}$$

where $\nu(t) = \sigma^2(t)$, $W(t)$ and $Z(t)$ are correlated Brownian motions on a filtered probability space $(\Omega, (\mathbb{F}_t)_{t\in[0,T]}, \mathbb{P})$, satisfying the usual conditions. We construct estimators of the process $\gamma^2(t)$, the so called volatility of volatility, and of the leverage process $\eta(t)$, which is defined by means of the It$\hat{o}$ contraction between the price and volatility

$$< dp(t), d\nu(t) >= \eta(t)dt.$$

We stress the fact that we do not assume any specific functional form of the volatility process or of the volatility of volatility process, thus we are working in a model free setting. In particular, the most common models (e.g. Heston, CEV) are included in our assumption (1).
By change of the origin of time and rescaling the unit of time, we can always reduce ourselves to the case where the time window $[0, T]$ becomes $[0, 2\pi]$. For this reason, in what follows we will consider the time window as $[0, 2\pi]$ that is the most suitable choice if we want to apply Fourier analysis. We make the following hypotheses on the processes that appear in (1):

- (H.1) $a(t)$, $b(t)$, $\sigma(t)$, $\gamma(t)$ are continuous in $[0, 2\pi]$ and adapted to the filtrazion $\mathbb{F}_t$,
- (H.2) $\forall p \geq 1$

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |a(t)|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |b(t)|^p\Big] < \infty,$$

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\sigma(t)|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\gamma(t)|^p\Big] < \infty,$$

- (H.3) $\forall p \geq 1$, the processes $\sigma, \gamma \in \mathbb{D}^{1,p}$ and

$$\mathbb{E}\Big[sup_{s,t\in[0,2\pi]}\Big|\mathrm{D}_s\sigma(t)\Big|^p\Big] < \infty, \quad \mathbb{E}\Big[sup_{s,t\in[0,2\pi]}\Big|\mathrm{D}_s\gamma(t)\Big|^p\Big] < \infty.$$

  where $\mathbb{D}^{1,p}$ is the Sobolev space of generalized derivative in the sense of Malliavin and D is the Malliavin derivative [26].

Let us turn our attention to the iterative procedure that allows us to define the Fourier coefficients of γ^2 and η. Starting from n discrete observations of $p(t)$ on the interval $[0, 2\pi]$- the sampling being either evenly spaced or unevenly spaced- we denote the discrete observed returns by $\delta_k(p) = p(t_{k+1}) - p(t_k)$ for all $k = 0, ..., n-1$. The discrete

Fourier coefficients of the return process are defined for any integer s such that $|s| \leq M$ as follows

$$c_s(dp_n) = \frac{1}{2\pi} \sum_{k=0}^{n-1} \mathrm{e}^{-\mathrm{i}st_k} \delta_k(p). \tag{2}$$

First iteration: we compute an estimation of the Fourier coefficients of volatility process using the Bohr convolution product of the Fourier coefficients of the return process for any integer h such that $h \leq L$

$$c_h(\nu_{n,M}) = \frac{2\pi}{2M+1} \sum_{|s|\leq M} c_s(dp_n) c_{h-s}(dp_n). \tag{3}$$

The numbers M and L are called *cutting frequencies* and are respectively related to the estimations of the Fourier coefficients and the spot volatility process as we see in a moment. In [21] and [11] it is shown that the estimator (3) is consistent. Using the inverse Fourier transform, the trajectory of volatility for all $t \in (0, 2\pi)$ is consistently estimated [21] by

$$\nu_{n,L,M}(t) = \sum_{|h|\leq L} \Big(1 - \frac{|h|}{L}\Big) \mathrm{e}^{\mathrm{i}ht} c_h(\nu_{n,M}). \tag{4}$$

The estimator (4) is constructed such that it preserve the positivity of the volatility function $\forall t \in (0, 2\pi)$ (see Remark 2.3 in [21]). For this reason we will use the same Fourier inversion formula to obtain spot estimators of volatility of volatility and leverage in order to preserve the sign of the examined functions.

Second iteration: we handle the volatility as an observable variable and estimate the Fourier coefficients of the process $\gamma^2(t)$ for any integer l, j such that $|l| \leq N$ and $|j| \leq N$- where N is the *cutting frequency* related to the estimations of the Fourier coefficients of the processes γ^2 and η and their trajectories.
Using the Bohr convolution product as in (3), we get the Fourier coefficients of the volatility of volatility

$$c_j(\gamma^2_{n,N,L,M}) = \frac{2\pi}{2N+1} \sum_{|l|\leq N} c_l(d\nu_{n,L,M}) c_{j-l}(d\nu_{n,L,M}). \tag{5}$$

The Fourier coefficients of the leverage are defining using the procedure developed in [21] for the computation of covariance in a multivariate setting, as follows

$$c_j(\eta_{n,N,L,M}) = \frac{2\pi}{2N+1} \sum_{|l|\leq N} c_l(d\nu_{n,L,M}) c_{j-l}(dp_n). \tag{6}$$

In the above definitions, we consider

$$c_l(d\nu_{n,L,M}) = \mathrm{i}l c_l(\nu_{n,M}) + \frac{1}{2\pi}\Big(\nu_{n,L,M}(2\pi) - \nu_{n,L,M}(0)\Big). \tag{7}$$

Remark 2.1. *From the integration by part formula applied to the Fourier coefficients of the volatility process (in continuous time)*

$$c_l(\nu) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}lt} \nu(t) dt,$$

we obtain that

$$c_l(d\nu) = \mathrm{i}l c_l(\nu) + \frac{1}{2\pi} \Big(\nu(2\pi) - \nu(0) \Big).$$

Using the estimation of the trajectory of the process $\nu(t)$ obtained in the first iteration and the definition (3), we get (7). We refer to the Appendix for further details regarding the definition of the coefficients c_l used in the asymptotic theory.

Remark 2.2. *The choice of the cutting frequencies M, N, L is fundamental to conduct a coherent harmonic analysis. First, we recall that in order to avoid aliasing effects in the reconstruction of the trajectory of the process $\nu(t)$ in the time domain, we have to choose M such that $M/n < 1/2$ (Nyquist frequency). It is also necessary that the rate N/M and L/M are carefully chosen in order to obtain the asymptotic properties in the next sections. Throughout the paper, we will use the parameters α and β defined by the following relations*

$$N = M^{\alpha}, \quad L = M^{\beta},$$

where $0 < \alpha, \beta < 1$, in order to identify the rate between the cutting frequencies.

From the second iteration we have the necessary tools to construct several estimators of the stochastic processes appearing in (1). The estimator of spot volatility of volatility is obtained by the Fourier inversion formula as follows

$$\hat{\gamma}^2(t) = \sum_{|j| \le N} \Big(1 - \frac{|j|}{N} \Big) \mathrm{e}^{\mathrm{i}jt} c_j(\gamma^2_{n,N,L,M}). \tag{8}$$

The estimator of the leverage function is given by

$$\hat{\eta}(t) = \sum_{|j| \le N} \Big(1 - \frac{|j|}{N} \Big) \mathrm{e}^{\mathrm{i}jt} c_j(\eta_{n,N,L,M}). \tag{9}$$

Now, we turn our attention to the estimators of the integrated quantities. The integrated volatility of volatility and the integrated leverage are defined as

$$\gamma^{[2]} = \int_0^{2\pi} \gamma^2(t) dt, \text{ and } \eta^{[1]} = \int_0^{2\pi} \eta(t) dt. \tag{10}$$

By means of the zero Fourier coefficients we obtain the following estimators of the quantities (10)

$$2\pi c_0(\gamma^2_{n,N,L,M}), \tag{11}$$

in the case of the volatility of volatility and

$$2\pi c_0(\eta_{n,N,L,M}) \tag{12}$$

for the leverage.
We will return to the properties of this estimators in the next sections.

3 Volatility of volatility computation

Throughout this section, we demonstrate the asymptotic properties of the estimators (8) and (11). Before stating the results, some comments are needed. Given a continuous function φ defined in $[0, 2\pi]$, denote by $\omega_\varphi(\tau)$ the modulus of continuity defined as

$$\omega_\varphi(\tau) := \sup_{|\theta-\theta'|<\tau} |\varphi(\theta) - \varphi(\theta')|$$

and let $c_k(\varphi) := \mathrm{F}(\varphi)(k)$, where F denote the Fourier transform, then it holds

$$\sup_{t\in I} \Big| \sum_{|k|<L} \Big(1 - \frac{|k|}{L}\Big) \mathrm{e}^{\mathrm{i}kt} c_k(\varphi) - \varphi(t)\Big| \leq \omega_\varphi\Big(\frac{4}{L}\Big), \tag{13}$$

for all I compact set such that $I \subset [0, 2\pi]$ (see the paper [31] and [28]). Assuming that the dynamics of volatility process $\sigma^2(t)$ is described by a continuous semimartingale (1), driven by the Brownian motion Z, means that the approximation (13) becomes in L_2-norm at least

$$E\Big[\Big| \sum_{|k|<L} \Big(1 - \frac{|k|}{L}\Big) \mathrm{e}^{\mathrm{i}kt} c_k(\nu) - \nu(t)\Big|^2\Big] = O\Big(\frac{1}{L^\lambda}\Big),$$

for all $t \in (0, 2\pi)$, where $\lambda/2$ is the Hölder coefficient of the Brownian motion $Z(t)$ $(0 < \lambda < 1)$. We underline this properties because the choice of the rate N/M and L/M depends on the regularity of the examined volatility process as we see in a moment.
Finally, we remark that Fourier methodology can be applied only if we have high frequencies observations of the price process and that the consistency properties of the estimators (8) and (11) are obtained in the general case of unevenly spaced price observations whereas the central limit theorems associated with (11) in the case of evenly spaced price observations. Hereafter, let the interval $[0, 2\pi]$ sampled in n points such that $0 = t_0 < ... < t_k < ... < t_n = 2\pi$, we define $\rho(n) = \max_{k=0,...,n-1} |t_{k+1} - t_k|$.

3.1 Consistency theorems

We obtain the following consistency theorem for the estimator of the spot volatility of volatility.

Theorem 3.1. *Let $\hat{\gamma}^2(t)$ defined in (8). We assume that hypotheses (H) and the following relations*

$$\frac{N^4}{M} \to 0, \quad \frac{N^2L^2}{M} \to 0, \quad \frac{N^2}{L^\lambda} \to 0, \quad M\rho(n) \to a$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then we have the convergence in probability

$$\lim_{n,N,L,M\to\infty} \sup_{0<t<2\pi} |\hat{\gamma}^2(t) - \gamma^2(t)| = 0. \tag{14}$$

Proof. The first step of the proof shows the consistency of the Fourier coefficients defined in (5). The limit in probability (14) simply follows by the convergence of Fejer summation for continuous functions. □

Concerning the estimator of the integrated function, we attain the following result

Theorem 3.2. *Let $2\pi\hat{c}_0(\gamma^2_{n,N,L,M})$ be the integrated estimator of volatility of volatility. We assume that hypotheses (H) and the following relations*

$$\frac{N^4}{M} \to 0, \quad \frac{N^2L^2}{M} \to 0, \quad \frac{N^2}{L^\lambda} \to 0, \quad M\rho(n) \to a$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then

$$2\pi c_0(\gamma^2_{n,N,L,M}) \xrightarrow{\mathbb{P}} \gamma^{[2]}.$$

Therefore, it is clear that assessing the consistency of all the Fourier coefficients of the volatility of volatility process is all we need to complete the proofs of theorems 3.1 and 3.2. The following theorem is the main result of this section

Theorem 3.3. *For all $|j| \le N$, let $\hat{c}_j(\gamma^2_{n,N,L,M})$ be the Fourier coefficient of the volatility of volatility process defined in (5). We assume that hypotheses (H) and the following relations*

$$\frac{N^3}{M} \to 0 \quad \frac{N^2L^2}{M} \to 0 \quad \frac{N^2}{L^\lambda} \to 0 \quad M\rho(n) \to a \tag{15}$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then

$$c_j(\gamma^2_{n,N,L,M}) \xrightarrow{\mathbb{P}} c_j(\gamma^2). \tag{16}$$

We point out that the above consistency properties are unaffected by the presence of the drift components of the semi-martingale $p(t)$ and $\nu(t)$ that give negligible contribution to the limit in probability. We refer the reader to the Appendix for the proof of theorem 3.3.

As we have seen in the previous section, the identification of the rate N/M, L/M are fundamental. Basically, this is due to the shift of the data analysis in the frequency domain. The asymptotic theory developed above give us a range in which the admissible rate can vary in order to obtain a consistent estimator. M has to be chosen proportional to the number of observations. Recalling that $N = M^\alpha$ and $L = M^\beta$, the hyphoteses (15) imply that the parameters α and β must satisfy the following relations

$$\begin{cases} 4\alpha - 1 < 0 \\ 2\alpha + 2\beta - 1 < 0 \\ 2\alpha - \beta\lambda < 0 \\ 0 < \alpha, \beta < 1. \end{cases}$$

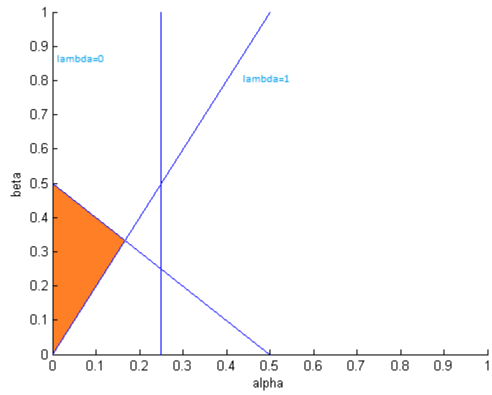

Figure 1: The orange area identify the range in which the parameters α and β can vary under the hypothesis (15). The admissible area is bounded by two lines that depends on the regularity of the path of the volatility process. In all the cases, for $0 < \lambda < 1$, a possible choice of the parameters α and β exists.

The orange area, that we obtain in the Figure 1, identify the possible choice of the parameters α and β.

3.2 Feasible central limit theorem for the integrated estimator

Considering the $j = 0$ Fourier coefficient in (16) we have a consistent estimator of the integrated volatility of volatility (11). In this section, we study the asymptotic error distribution of this estimator. The result have two parts. The first gives the asymptotic distribution of the estimator of the integrated volatility of volatility.

Theorem 3.4. *We assume that hypotheses (H) and the following relations*

$$\frac{N^5}{M} \to 0 \quad \frac{N^3 L^2}{M} \to 0 \quad \frac{N^3}{L^\lambda} \to 0 \quad \frac{M}{n} \to a \tag{17}$$

with $a \in (0, \frac{1}{2})$ hold true as $n, N, L, M \to \infty$. Then

$$\frac{\sqrt{N}\left(2\pi c_0(\gamma^2_{n,N,L,M}) - \gamma^{[2]}\right)}{\sqrt{\pi \gamma^{[4]}}} \xrightarrow{d} N(0,1), \quad \textit{where } \gamma^{[4]} = \int_0^{2\pi} \gamma^4(s) ds \ . \tag{18}$$

In the sequel, we refer to the $\gamma^{[4]}$, the integrated fourth moment of γ, as the quarticity of the volatility of volatility. Of course, the problem with this theory is that $\gamma^{[4]}$ is unknown. We solve this problem defining a consistent estimator of the quarticity based on the Fourier methodology. Following [24], we can define an estimator of quarticity as follows

$$2\pi c_0(\gamma^4_{n,Q,N,L,M}), \tag{19}$$

where

$$c_0(\gamma^4_{n,Q,N,L,M}) = \sum_{|j| \le Q} c_j(\gamma^2_{n,N,L,M}) c_{-j}(\gamma^2_{n,N,L,M})$$

is the zero-Fourier coefficient of the function $\gamma^4(s)$ and the coefficients $c_j(\gamma^2)$ for all $|j| \leq Q$ are defined in (5). We obtain the following theorem.

Theorem 3.5. *We assume that hypotheses (H) and the following relations*

$$\frac{Q^2N^4}{M} \to 0 \quad \frac{Q^2N^2L^2}{M} \to 0 \quad \frac{Q^2N^2}{L^\lambda} \to 0 \quad \frac{M}{n} \to 0 \tag{20}$$

with $a \in (0, \frac{1}{2})$ hold true as $n, N, L, M \to \infty$. Then

$$2\pi c_0(\gamma^4_{n,Q,N,L,M}) \xrightarrow{\mathbb{P}} \gamma^{[4]}$$

An implication of this is that we can obtain a feasible asymptotic distribution for the estimation error

$$\frac{\sqrt{N}\Big(2\pi c_0(\gamma^2_{n,N,L,M}) - \gamma^{[2]}\Big)}{\sqrt{2\pi^2 c_0(\gamma^4_{n,Q,N,L,M})}} \xrightarrow{d} N(0,1) \tag{21}$$

The proofs of this results can be found in the Appendix. In particular, the proof of Theorem 3.4 is based on the stable convergence of the continuous martingale processes [16], [17]. Stable convergence is a stronger way of convergence than the weak convergence, but it is weaker than convergence in probability. We can not prove directly that (18) holds, therefore we have to use this theory in order to attain a proof of the central limit theorem 3.4. This result is obtained by means of analytical tools involving stochastic integrals, Malliavin calculus, delta sequences theory and martingale inequalities.

In order to use this asymptotic result we need to assess the rate N/M, L/M and Q/M. Recall that we select the number of frequencies M proportional to the number of data. We define the parameters α, β and δ as follows

$$N = M^\alpha \quad L = M^\beta \quad Q = M^\delta,$$

where $0 < \alpha, \beta, \delta < 1$. The hypotheses (17) and (20) imply that

$$\begin{cases} 5\alpha - 1 < 0 \\ 3\alpha + 2\beta - 1 < 0 \\ 3\alpha - \beta\lambda < 0 \\ 0 < \alpha, \beta < 1. \end{cases} \bigcup \begin{cases} 2\delta + 4\alpha - 1 < 0 \\ 2\delta + 2\alpha + 2\beta - 1 < 0 \\ 2\delta + 2\alpha - \beta\lambda < 0 \\ 0 < \alpha, \beta, \delta < 1. \end{cases}$$

If we choose $\delta = 1/16$, the green area in the Figure 2 gives an example of the admissible choice of the parameters α and β.

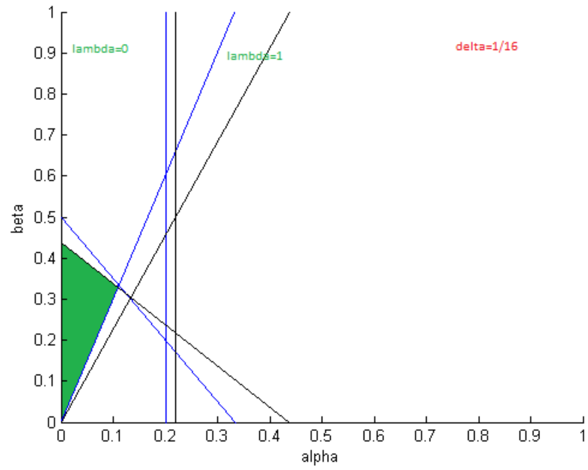

Figure 2: The green area identify the range in which the parameters α and β can vary under the hyphoteses (20) and (17) and supposing that $\delta = 1/16$. In particular, the blue lines depend on the hypotheses (17) and the black line on (20). The admissible area is bounded by two lines that depends on the regularity of the path of the volatility process. In all the cases, for $0 < \lambda < 1$, a possible choice of the parameters α and β exists.

4 Leverage computation

In this section, we prove the asymptotic properties of the estimators (9) and (12). The proofs of the following results use the same philosophy of the proofs for the computation of the volatility of volatility. For this reason, in what follows we only emphasize the main differences between the two computations and we refer the reader to the Appendix for the proofs of the below theorems.

We start with the consistency properties of the Fourier coefficients of the leverage function $\eta(t)$ in the general case of unevenly spaced price observations.

Theorem 4.1. *For all $|j| \leq N$, let $\hat{c}_j(\eta_{n,N,L,M})$ be the Fourier coefficient of the leverage process defined in (5). We assume that hypotheses (H) and the following relations*

$$\frac{N^2}{M} \to 0 \quad \frac{L^2}{M} \to 0 \quad M\rho(n) \to a \tag{22}$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then

$$c_j(\eta_{n,N,L,M}) \xrightarrow{\mathbb{P}} c_j(\eta). \tag{23}$$

We define $N = M^\alpha$ and $L = M^\beta$. The hypothesis (22) imply that the parameters α and β must satisfy the following constraints

$$\begin{cases} 2\alpha - 1 < 0 \\ 2\beta - 1 < 0. \end{cases}$$

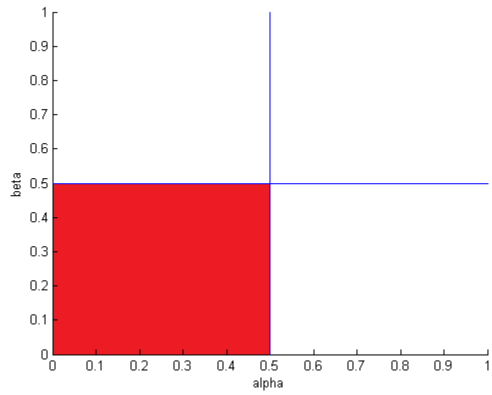

Figure 3: The red area identify the range in which the parameters α and β can vary under the hypotheses (22).

These conditions show that the admissible area in which we can choose the parameters α and β is larger than the admissible area of Theorem 3.3 and not depend by the regularity of volatility path (Figure 3).

As we have seen in the previous section, the Theorem 4.1 imply that the following consistency theorems hold.

Theorem 4.2. *Let $\hat{\eta}(t)$ defined in (6). We assume that hypotheses (H) and the following relations*

$$\frac{N^2}{M} \to 0 \quad \frac{L^2}{M} \to 0 \quad M\rho(n) \to a$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then we have the following convergence in probability

$$\lim_{n,N,L,M\to\infty} \sup_{0<t<2\pi} |\hat{\eta}(t) - \eta(t)| = 0.$$

Theorem 4.3. *Let $2\pi\hat{c}_0(\eta_{n,N,L,M})$ the integrated leverage estimator. We assume that hypotheses (H) and the following relations*

$$\frac{N^2}{M} \to 0 \quad \frac{L^2}{M} \to 0 \quad M\rho(n) \to a$$

with $a \in (0, \frac{1}{2})$ hold true as $N, L, M \to \infty$ and $\rho(n) \to 0$. Then

$$2\pi c_0(\eta_{n,N,L,M}) \xrightarrow{\mathbb{P}} \eta^{[1]}.$$

Concerning the integrated estimator (12), we can construct a feasible asymptotic distribution for the estimation error supposing that the observed price process data are evenly spaced. The result have two parts. The first gives the asymptotic distribution of the integrated leverage estimator.

Theorem 4.4. *(CLT) We assume that hypotheses (H) and the following relations*

$$\frac{N^3}{M} \to 0 \quad \frac{L^2 N}{M} \to 0 \quad \frac{N}{L^\lambda} \to 0 \quad \frac{M}{n} \to a \tag{24}$$

with $a \in (0, \frac{1}{2})$ hold true as $n, N, L, M \to \infty$. Then

$$\frac{\sqrt{N}\Big(2\pi c_0(\eta_{n,N,L,M}) - \eta^{[1]}\Big)}{\sqrt{\pi \eta^{[2]}}} \xrightarrow{d} N(0,1), \quad \text{where } \eta^{[2]} = \int_0^{2\pi} \sigma^2(s)\gamma^2(s) + \eta^2(s) ds.$$

We refer to $\eta^{[2]}$ as the variance of the leverage estimation error and we call the spot variance of the leverage the process $\eta^{(2)}(s) = \sigma^2(s)\gamma^2(s) + \eta^2(s)$. As in the previous section we define a consistent estimator of $\eta^{[2]}$ using the Fourier analysis as

$$2\pi c_0(\eta^{(2)}_{n,Q,N,L,M}), \tag{25}$$

where

$$c_0(\eta^{(2)}_{n,Q,N,L,M}) = \sum_{|j| \le Q} c_j(\gamma^2_{n,N,L,M}) c_{-j}(\nu_{n,M}) + c_j(\eta_{n,N,L,M}) c_{-j}(\eta_{n,N,L,M})$$

is the zero-Fourier coefficient of the function $\eta^{(2)}(s)$ and the coefficients $c_j(\gamma^2)$ for all $|j| \le Q$ are defined in (6). We can prove that this estimator is consistent.

Theorem 4.5. *We assume that hypotheses (H) and the following relations*

$$\frac{Q^2 N^4}{M} \to 0 \quad \frac{Q^2 N^2 L^2}{M} \to 0 \quad \frac{Q^2 N^2}{L^\lambda} \to 0 \quad \frac{M}{n} \to 0 \tag{26}$$

with $a \in (0, \frac{1}{2})$ hold true as $n, N, L, M \to \infty$. Then

$$2\pi c_0(\eta^{(2)}_{n,Q,N,L,M}) \xrightarrow{\mathbb{P}} \eta^{[2]}$$

Under the hypotheses (24) and (26) we have that the admissible choice of the parameters α,β and δ (recall that $Q = M^\delta$) can be derived by the following relations

$$\begin{cases} 3\alpha - 1 < 0 \\ \alpha + 2\beta - 1 < 0 \\ \alpha - \beta\lambda < 0 \\ 0 < \alpha, \beta < 1. \end{cases} \bigcup \begin{cases} 2\delta + 4\alpha - 1 < 0 \\ 2\delta + 2\alpha + 2\beta - 1 < 0 \\ 2\delta + 2\alpha - \beta\lambda < 0 \\ 0 < \alpha, \beta, \delta < 1. \end{cases}$$

Finally, we can obtain a feasible asymptotic distribution for the estimation error

$$\frac{\sqrt{N}\Big(2\pi c_0(\eta_{n,N,L,M}) - \eta^{[1]}\Big)}{\sqrt{2\pi^2 c_0(\eta^{(2)}_{n,Q,N,L,M})}} \xrightarrow{d} N(0,1). \tag{27}$$

We observe that the admissible area for the choice of the parameters α and β (Figure 4) is the same as in the hypotheses of the Figure 2.

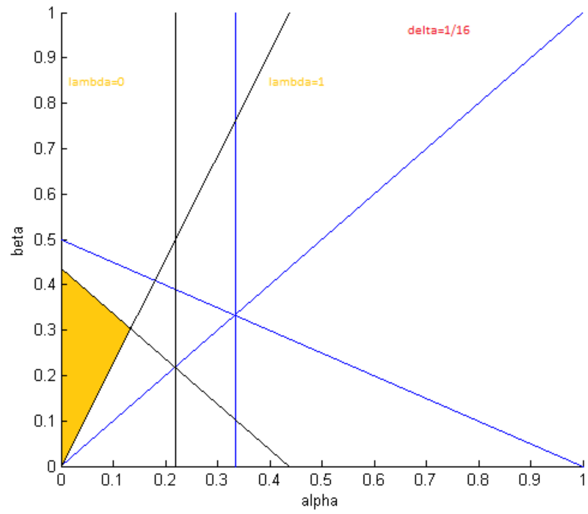

Figure 4: The yellow area identify the range in which the parameters α and β can vary under the hyphoteses (24) and (26) and supposing that $\delta = 1/16$. In particular, the blue lines depend on the hypotheses (24) and the black line on (26). The admissible area is bounded by two lines that depends on the regularity of the path of the volatility process. In all the cases, for $0 < \lambda < 1$, a possible choice of the parameters α and β exists.

5 Numerical study

In this section, we prove that applying a logarithmic transformation to the statistic (21), we can obtain an asymptotic approximation of the distribution error suitable to construct confidence intervals. The statistic (21) approximate the distribution $N(0,1)$ only for very large value of n. We use the asymptotic theory developed in the section 3 to find a new statistic that works better on finite sample. Sometimes, the features of the statistic suggests the most suitable transformation [29]. In our case, the behaviour of (18) is similar to the realized volatility estimator developed by Barndorff-Nielsen and Shepard [3] and by Andersen and Bollerslev [1]. In the paper [4], the asymptotic error distribution of the realized volatility estimator is empirically investigated and is found out that the logarithmic transformation improves the accuracy. Under a logarithmic transformation, we can prove that

$$\frac{\sqrt{N}\Big(\log(2\pi\hat{c}_0(\gamma^2_{n,N,L,M})) - \log(\gamma^{[2]})\Big)}{\sqrt{\frac{\pi\gamma^{[4]}}{(\gamma^{[2]})^2}}} \xrightarrow{d} N(0,1), \tag{28}$$

because the Theorem 3.4 holds. Using the Fourier estimators (11) and (19) we obtain the following feasible statistic

$$\frac{\sqrt{N}\Big(\log(2\pi\hat{c}_0(\gamma^2_{n,N,L,M}) - log(\gamma^{[2]})\Big)}{\sqrt{\frac{2\hat{c}_0(\gamma^4_{n,Q,N,L,M})}{(\hat{c}_0(\gamma^2_{n,N,L,M}))^2}}} \xrightarrow{d} N(0,1). \tag{29}$$

In the sequel, we will study the accuracy in approximating standard normal distribution of (29). In the statistic (29) five different parameters appear, the number of observations n that depends on the data we have and the *cutting frequencies* M, N, L and Q that we have to choose. The optimization of the *cutting frequencies* of the estimator (11) is the main issue of the paper [14]. In this paper, the finite sample properties of the estimator (11) are studied, even in the presence of market microstructure effects. The bias and the mean square error of the integrated estimator are computed and a procedure to optimize the *cutting frequencies* based on the minimization of the MSE is developed. We will use this procedure to choose the cutting frequencies (other example of the application of this methodology can be found in [22], [23] and [24]) in our simulations. We generate (through simple Euler Monte Carlo discretization) different high frequency evenly spaced data (price and variance) over a daily trading period of $T = 6$ hours assuming the following dynamics

$$\begin{cases} dp(t) & = \sqrt{\nu(t)}dW(t) \\ d\nu(t) & = k(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dZ(t) \end{cases}$$

where $\nu(t)$, $t > 0$ is the volatility process, k, θ and ξ are constants and W and Z are two correlated Brownian motions. In particular k is the so called mean reversion parameter, θ is the long-run mean variance and ξ is the local volatility parameter. We choose $v(0) > 0$ and $2k\theta/\xi^2 - 1 > 0$, which guarantees that $v(t)$ remains positive (with probability one) for $t > 0$. We fix the value of the parameters k, θ, ξ and ρ (the correlation parameter between the Brownian motion W and Z) as follows

$$k = 0.03, \quad \theta = 0.25, \quad \xi = 0.1, \quad \rho = 0.2.$$

We simulate 4 different scenarios,

$$\begin{aligned} &\text{Scenario 1:} \quad n = 2160, \text{data frequency 10 seconds,} \\ &\text{Scenario 2:} \quad n = 4320, \text{data frequency 5 seconds,} \\ &\text{Scenario 3:} \quad n = 21600, \text{data frequency 1 second,} \\ &\text{Scenario 4:} \quad n = 43200, \text{data frequency 1/2 second.} \end{aligned}$$

For each scenario, we start the optimization procedure of the *cutting frequencies* with the following initial conditions:

$$M = 800, \quad N = 50, \quad L = 50, \quad Q = 10.$$

The samples of the statistic (29) is computed with 1000 daily replications.
In all the scenarios, the simulations show that the estimated finite sample coverage (setting the nominal level at 95.0) of statistic (29) is near to the sample coverage of a standard normal distribution. The logarithmic transformation seems to work well in order to produce a feasible statistic that can work in a finite sample.
As we can see from the QQplots (Figure 5), in the scenario 1 the quantiles of the standard

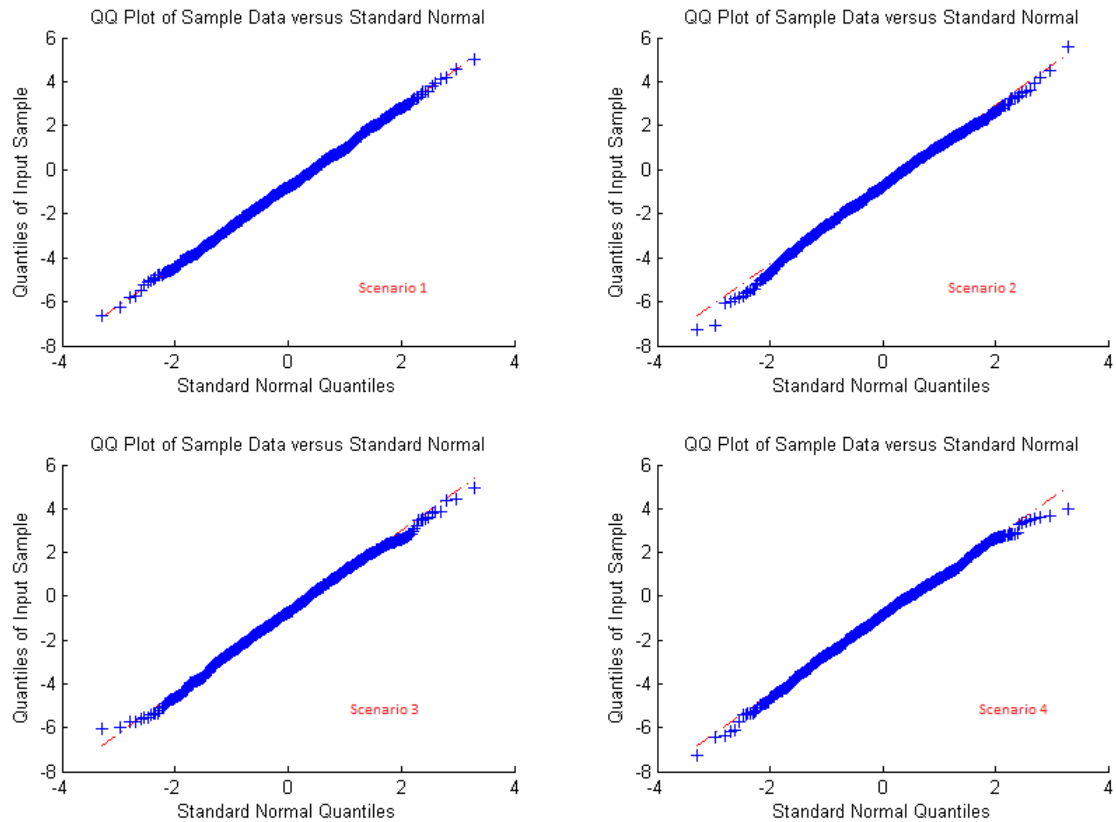

Figure 5: QQplots of the 4 simulated Scenarios.

normal distribution are better approximated respect to the other scenarios. This happens because of the maximum frequencies M,N,L and Q that have been used to set up the optimization procedure. In this simulations, we want to highlight that if the inputs M, N,L and Q are chosen according to the asymptotic theory, that has been developed in the section 3, the accuracy of the estimation improves. We know that the cutting frequency M has to be chosen proportional to the number of observations n in order to ensure the asymptotic result (18). The cutting frequency M has to be only respect the upper bond indicating by the Nyquist frequency. In the simulations, $M = 800$ is a suitable choice for the scenario 1. For the other ones, we should start the optimization procedure using more higher inputs for M,N,L and Q. In the paper [14], after providing a feasible method to optimize the choice of the cutting frequencies, simulations of the statistic (29) are conducted in order to investigate the influence of the parameters M, N, L and Q on the accuracy of the estimations, even in the presence of microstructure noise. The simulation study, that we conducted in this section, is only a starting point but it underlines the importance of further study regarding this type of statistic and the possible future applications of our estimators.

6 Appendix

Available on request.

References

[1] T. Andersen and Bollerslev. *Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts. International Economic Review*, 39:885–905, 1998.

[2] O. Barndoff-Nielsen and Veraart A. *Stochastic volatility of volatility in continuous time. Technical report*, 2009.

[3] O.E. Barndorff-Nielsen and N. Shepard. *Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models. Journal of the Royal Statistical Society, Series B*, 64:253–280, 2002.

[4] O.E. Barndorff-Nielsen and N. Shepard. *How accurate is the symptotic approximation to the distribution of realised variance? Identification and Inference for Econometric Models. A Festschrift for Tom Rothenberg, (edited by Donald W.K. Andrews and James H.Stock), Cambridge University Press*, pages 306–331, 2005.

[5] O.E. Barndorff-Nielsen and A.E.D Veraart. *Stochastic volatility of volatility and variance risk premia. Journal of Financial Econometrics*, Forthcoming, 2012.

[6] E. Barucci and M.E. Mancino. *Computation of volatility in stochasticvolatility models with high frequency data. International journal of theoretical and applied finance*, 15(5):767–787, 2010.

[7] Litvinova J. Pigorsch C. Bollerslev, T. and G. Tauchen. *A discrete-time model for daily SP 500 returns and realised variations: jumps and leverage effects. Journal of Econometrics*, 150 (2):151–166, 2009.

[8] T. Bollerslev and H. Zhou. *Estimating stochastic volatility diffusion using conditional moments of integrated volatility. J. Econometrics*, 109:33–65, 2002.

[9] Tauchen G. Bollerslev, T. and H. Zhou. *Expected stock returns and variance risk premia. Review of Financial Studies*, 22(11):4463–4492, 2009.

[10] M. Chernov and Ghysels E. *Estimation of stochastic volatility models for the purpose of option pricing. Abu-Mostafa, Y., Le Baron B., Lo A., Weigend A. (Eds) Computational Finance 1999, MIT Press, Cambridge*, pages 567–581, 2000.

[11] E. Clément and A. Gloter. *Limit theorems in the Fourier transform method for the estimation of multivariate volatility. Stochastic Processes and their Applications*, 121:10971124, 2000.

[12] F. Corsi. *A simple long memory model for realised volatility. Technical report, University of Southern Switzerland*, 2004.

[13] Mittnik S. Pigorsch C. Corsi, F. and U. Pigorsch. *The volatility of realised volatility. Econometric reviews*, 27(1-3):46–78, 2008.

[14] Mancino M.E. Curato, I.V. and S. Sanfelici. *Estimating volatility of volatility in the presence of market microstructure effects. Forthcoming.*

[15] I. Drechsler and A. Yaron. *What's vol got to do with it. AFA 2009 San Francisco Meeting Paper*, 2008.

[16] J. Jacod. *On continuous conditional Gaussian martingales and stable convergence in law. Seminarie de Probabilites XXXI*, pages 232–246, 1997.

[17] J. Jacod and P. Protter. *Asymptotic error distributions for the euler method for stochastic differential equations. The Annals of probability*, 26(1):267–307, 1998.

[18] J. Jacod and N.A. Shiryaev. *Limit theorems for stochastic processes. Springer*, 2003.

[19] C. Jones. *The dynamics of stochastic volatility: evidence from underlying and option markets. J. Econometrics*, 116:181–224, 2003.

[20] P. Malliavin and M.E. Mancino. *Fourier series method for measurement of multivariate volatilities. Finance and Stochastics*, 4:49–61, 2002.

[21] P. Malliavin and M.E. Mancino. *A Fourier transform method for nonparametric estimation of volatility. The Annals of Statistics*, 37(4):1983–2010, 2009.

[22] M.E. Mancino and S. Sanfelici. *Robustness of Fourier estimator of integrated volatility in the presence of microstructure noise. Computational statistics and data analysis*, 52(6):2966–2989, 2008.

[23] M.E. Mancino and S. Sanfelici. *Estimating covariance via Fourier method in the presence of asynchronous trading and microstructure noise. Journal of financial econometrics*, 9(2):367–408, 2011.

[24] M.E. Mancino and S. Sanfelici. *Estimation of quarticity with high frequency data. Quantitative finance*, 12:607–622, 2012.

[25] P. Mykland and L. Zhang. *Inference for continuous semimartingales observed at high frequency: A general approach. Econometrica*, 77 (5):1403–1445, 2009.

[26] D. Nualart. *The Malliavin Calculus and Related Topics, 2nd edition. Springer-Verlag*, 2006.

[27] M. Podolskij and M. Vetter. *Understanding limit theorems for semimartingales: a short survey. Statistica Nederlandica*, 64(3):329–351, 2010.

[28] De La Vallée Poussin. *Lecons sur l'approximation d'une variable réelle. Gauthier-Villars Paris*, 1919.

[29] T.J. Rothenberg. *Approximating the Distribution of Econometric Estimators and Test Statistics. Handbook of Econometrics, Vol.2 (ed. by Z.Griliches and M:D: Intriligator) Amsterdam:Elsevier*, pages 881–935.

[30] M. Vetter. *Estimation of integrated volatility of volatility with applications to goodness-of-fit testing. Working paper.*

[31] M. Zamansky. *Classes de saturation de certains procédés d'approximation des séries de Fourier des fonctions continues et applications à quelques problémes d'approximation. Annales scientifiques de l'E.N.S 3- séries, tome 66*, pages 19–93, 1949.