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Some solution concepts for finite strategic games based on Kalai and Schmeidler's admissible set

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Abstract

We propose new solution concepts for finite strategic games based on the admissible set, a solution concept for abstract decision problems introduced by Kalai and Schmeidler. The first solution concept, called the better-response admissible set, selects the strategy profiles that cannot be excluded by a bargaining process where players consider only unilateral deviations that strictly improve their payoffs. The second solution concept, called the best-response admissible set, restricts transitions to unilateral deviations that yield the maximum possible payoff increase. We establish several properties of these solution concepts: both contain the set of Nash equilibria; they coincide with the set of Nash equilibria for all generalized ordinal potential games; they are invariant to increasing transformations of the payoffs; the best-response admissible set always selects rationalizable strategy profiles, while the better-response admissible set, in general, does not. Moreover, inspired by the properties of the admissible set and the rationale behind the definition of the Schulze voting method, we introduce special refinements of these solution concepts and establish some of their properties.

Keywords: finite strategic games; Nash equilibrium; abstract decision problem; admissible set; Schulze method; potential games.

JEL classification: C70; C72.

1 Introduction

An abstract decision problem[1] is a pair (X, R), where X is a non-empty set and R is an irreflexive relation on X. The set X represents the set of alternatives among which a society must make a choice. The relation R describes all the valid arguments that justify the transition from one alternative to another: $(x, y) \in R$ means that if the society is currently considering y, then there is a good reason to reject y and start considering x. The relation R is called the dominance relation on X and, when $(x, y) \in R$, we say that x dominates y.

A solution concept for abstract decision problems is a method for selecting, for any given abstract decision problem, some alternatives to be considered as possible outcomes of the social decision. There are many solution concepts in the literature, including: the stable set (Von Neumann and Morgenstern 1944), the core (Gillies 1953), the generalized stable set (Van Deemen 1991), the socially stable set (Delver and Monsuur 2001), the m-stable set (Peris and Subiza 2013), the w-stable set (Han and Van Deemen 2016), the supercore (Roth 1976), the admissible set (Kalai et al. 1976; Kalai and Schmeidler 1977; Schwartz 1972, 1986; Shenoy 1979, 1980), the maximum flow value set (Gori 2024). The admissible set of an abstract decision problem, which is the solution concept we are mainly focusing on in this paper, formally corresponds to the set of the maximal elements of the transitive and reflexive closure of the dominance relation (Kalai and Schmeidler 1977; Gori 2023). This set has a fascinating interpretation, as it represents the set of alternatives that cannot be excluded as final outcomes of a social bargaining process where the society has a positive probability to transition from any alternative to any other alternative dominating it. Remarkably, the admissible set contains the core (the set of alternatives that

[1] Also called abstract game (Shenoy 1979) or abstract system (Inarra et al. 2010).

are not dominated by any other alternative), and, when the set of alternatives is finite, it is nonempty. However, the admissible set is often very large, which makes it mainly useful to exclude bad alternatives rather than to select good ones.

Many decision problems can be modeled as abstract decision problems. As a notable example, given a finite strategic game, (pure or mixed) strategy profiles can be interpreted as alternatives and, once a dominance relation on the set of strategy profiles is defined, based on assumptions about players' behavior or characteristics, an abstract decision problem is associated with the game. At this point, it becomes interesting to explore whether a given classic solution concept for strategic games, such as the Nash equilibrium or the strong Nash equilibrium, coincides with or is related to the outcomes of the abstract decision problem when analyzed using a specific solution concept, such as the core or the stable set. Those connections may be valuable as they may shed light on classic solution concepts in game theory. Moreover, this approach also allows for the definition of a broader range of solution concepts for games by simply considering as solution of the game the solution of the corresponding abstract decision problem via a given solution concept for abstract decision problems.

Several authors have explored this line of research. Kalai and Schmeidler (1977) consider the mixed extension of a finite strategic game and identify a relation on the set of mixed strategy profiles such that the corresponding admissible set coincides with the set of mixed Nash equilibria. Greenberg (1989) identifies a suitable abstract decision problem associated with a game such that its (unique) stable set and its core, respectively, correspond to the set of coalition-proof Nash equilibria and the set of strong Nash equilibria. Focusing on the dominance relation on the set of strategy profiles that only accounts for single profitable deviations, called here the better-response dominance relation, Greenberg (1990) proves the existence of the stable set for any two-player game with finite strategies, and for any game with a finite number of players each having a binary strategy set; Iñarra et al. (2007) study the supercore and show its relation with the set of Nash equilibria of the game; Iñarra et al. (2014) study the stable set of the mixed extensions of two-player two-strategy games, showing, in particular, that games with a strict Nash equilibrium have infinite stable sets, and games without a strict Nash equilibrium have just a unique stable set. Iñarra et al. (2010) find a suitable refinement of the better-response dominance relation that guarantees that the set of mixed Nash equilibria of every finite strategic game always coincides with the supercore of its associated abstract decision problem. Chwe (1994) introduces a dominance relation on the set of strategy profiles that accounts for coalitional profitable deviations and incorporates farsighted behavior of players, who are then assumed to be able to understand the outcome originating by their own deviation and a chain of subsequent deviations by other players; he then considers the stable set of the corresponding abstract decision problem finding weak nonemptiness conditions. Suzuki and Muto (2005) specialize the work of Chwe (1994) to the n-player prisoners' dilemma game, while Nakanishi (2009) also examines an n-player prisoners' dilemma game, but focusing on the dominance relation that only accounts for individual profitable deviations of farsighted players. Bloch and van den Nouweland, (2021) generalize and unify existing results on stable sets in finite two-player strategic games, both for the better-response dominance relation and its version with farsighted players.

In this paper, given a finite strategic game, we consider the set of pure strategy profiles as the set of alternatives. Then, we first consider two basic dominance relations: the best-response dominance relation and the better-response dominance relation. According to the best-response dominance relation, a strategy profile dominates another if it can be obtained from the latter through a unilateral deviation by a player, provided that this deviation increases the player's payoff and the deviating strategy is one of the player's best responses to the others' strategies. According to the better-response dominance relation, as already said, a strategy profile dominates another if it can be obtained from the latter through a unilateral deviation by a player that increases the player's payoff, regardless of whether the deviating strategy is a best response to the others' strategies.[2] We then consider the corresponding admissible sets, respectively called the best-response admissible set and the better-response admissible set of the game, and analyze their properties. We emphasize that Kalai and Schmeidler (1977), in their application of the admissible set to finite strategic games, focus exclusively on mixed strategy profiles, whereas in this paper we consider only pure strategies. Moreover, to the best of our knowledge, no other application of the admissible set to finite strategic games can be found in the literature.

The best-response admissible set and the better-response admissible set are proved to be nonempty valued, to include the Nash equilibria, to be invariant under strictly increasing transformations of the payoff functions, and to select only strategy profiles in which strictly dominant strategies are played, when any exist. Moreover, the best-response admissible set is also invariant under the elimination of strategies that are never a best response, and therefore always selects rationalizable strategy profiles. On the other hand, the better-response admissible

[2]Block and van den Nouweland (2021) call this relation the myopic dominance relation.

set can include strategy profiles in which some strategies are strictly dominated. This implies that assuming that players may violate a rationality principle (by deviating without choosing a best response) may lead to strategy profiles that also violate the same rationality principle. Furthermore, both solutions coincide with the set of Nash equilibria when applied to generalized ordinal potential games.

Through various examples, it can be observed that the best-response admissible set and the better-response admissible set often select several strategy profiles.[3] Thus, in the second part of the paper, we propose some special refinements of these sets. The starting point for building the refinements is the observation that sometimes a decision problem may be modeled by a family of abstract decision problems, all having the same set of alternatives and where each dominance relation addresses specific aspects of the original problem. In this case, once a solution concept for abstract decision problems is fixed, it might be interesting to study whether there are alternatives that belong to the outcome of each abstract decision problem in the family. Of course, in principle, it may happen that no alternative satisfies this property. However, when the admissible set is considered, things work particularly well. Indeed, as proved in Gori (2023), if the dominance relations in the family of abstract decision problems can be ordered in an increasing sequence of sets, then the intersection of all the admissible sets is nonempty.

A notable example of this approach comes from the theory of social choice. Consider a voting situation where n individuals express as preferences a linear order over a finite set of alternatives. A natural dominance relation to consider on the set of alternatives is the majority relation: an alternative x dominates an alternative y if and only if the number of individuals who prefer x to y is greater than $\frac{n}{2}$. The admissible set of the corresponding abstract decision problem defines the so-called Schwartz set (Schwartz 1972). However, other natural relations can be considered. In fact, for every integer μ greater than $\frac{n}{2}$ and less than or equal to n, we can consider the μ-majority relation: x dominates y if and only if at least μ individuals prefer x to y. The family of μ-majority relations can be ordered as an increasing sequence of sets. Thus, the intersection of all the corresponding admissible sets is nonempty (and is included in the Schwartz set). This construction defines the well-known Schulze voting method, an important method rich in properties and used by several organizations (Schulze 2011, 2018; Gori 2023).

Given now a finite strategic game G, for every real number t, we define the dominance relation $D_t(G)$, where a strategy profile dominates another if it can be obtained from the latter by means of a unilateral deviation by a player, provided that such a deviation results in a payoff variation greater than t for that player, and that the deviating strategy guarantees the largest payoff among those different from the current one. We also define the dominance relation $E_t(G)$, where a strategy profile dominates another if it can be obtained from the latter by means of a unilateral deviation by a player, provided that such a deviation results in a payoff variation greater than t for that player, regardless of whether a larger payoff would be obtained by playing a different strategy. We then define the refined best-[better-]response admissible set of the game G as the intersection of the admissible sets of the abstract decision problems obtained by considering the dominance relations $D_t(G)$ $[E_t(G)]$ for nonnegative values of t. Similarly, we define the strongly refined best-[better-]response admissible set of the game G as the intersection of the admissible sets of the abstract decision problems obtained by considering the dominance relations $D_t(G)$ $[E_t(G)]$ for all values of t. We then investigate the properties of these refined solution concepts.

As obvious, for every finite strategic game, the strongly refined best-[better-]response admissible set is a subset of the refined best-[better-]response admissible set, which in turn is a subset of the best-[better-]response admissible set. Furthermore, the refined best-[better-]response admissible set always includes the set of Nash equilibria, while the strongly refined best-[better-]response admissible set may fail to include it. Notably, the strongly refined best-[better-]response admissible set is also shown to be non-empty valued, as is the refined best-[better-]response admissible set. Moreover, we provide upper bounds on the size of each of these sets under suitable conditions on the game. In particular, this allows us to show that, for almost all two-player games in which one of the players has at most three strategies, the strongly refined best-response admissible set selects a unique strategy profile; for almost all games, the strongly refined better-response admissible set selects a unique strategy profile, regardless of the number of players and the number of players' strategies. We also provide an alternative and computationally convenient definition for these sets. Further properties are described throughout the paper.

[3]We tested these solution concepts on many randomly generated two-player games, each with up to 15 strategies per player. We also observed that the best-response admissible set tends to be more selective than the better-response admissible set.

2 Preliminaries

2.1 Relations

Let X be a finite set. We set $X_d^2 = \{(x,y) \in X^2 : x = y\}$ and $X_*^2 = \{(x,y) \in X^2 : x \neq y\}$. The size of X is denoted by $|X|$.

A relation on X is a subset of X^2. Let R be a relation on X. The asymmetric part of R is the relation on X given by $\mathrm{as}(R) = \{(x,y) \in R : (y,x) \notin R\}$. We say that R is

- reflexive if, for every $x \in X$, $(x,x) \in R$;
- irreflexive if, for every $x \in X$, $(x,x) \notin R$;
- asymmetric if, for every $x, y \in X$, $(x,y) \in R$ implies $(y,x) \notin R$ (that is, $R = \mathrm{as}(R)$);
- transitive if, for every $x, y, z \in X$, $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$;
- quasi-transitive if $\mathrm{as}(R)$ is transitive;
- cyclic if there exist $m \geqslant 2$ and $x^1, \dots, x^m$ distinct elements of X such that, for every $j \in \{1, \dots, m-1\}$, $(x^j, x^{j+1}) \in R$ and $(x^m, x^1) \in R$;
- acyclic if it is not cyclic.

The set of maximal elements of R is defined as

$$\mathrm{Max}(R) = \{x \in X : (y,x) \in R \text{ implies } (x,y) \in R \text{ for all } y \in X\}.$$

Observe that $\mathrm{Max}(R) = \{x \in X : (y,x) \notin \mathrm{as}(R) \text{ for all } y \in X\}$ and $\mathrm{Max}(R) = \mathrm{Max}(\mathrm{as}(R))$. It is well known that if X is a nonempty and finite set and R is a quasi-transitive relation on X, then $\mathrm{Max}(R) \neq \varnothing$.

2.2 Abstract decision problems, the core and the admissible set

Let X be a fixed nonempty and finite set. An abstract decision problem on X is an ordered pair (X, R), where R is an irreflexive relation on X called dominance relation. If $(x,y) \in R$, we say that x dominates y (according to R).

Consider an abstract decision problem (X, R). The elements of X are interpreted as mutually exclusive alternatives among which a society has to make a choice. The dominance relation R instead represents the complete description of all and only the transitions that the society can make with positive probability from an alternative to another. Specifically, suppose that the society is currently considering an alternative y. If $\{x \in X : (x,y) \in R\} \neq \varnothing$, then the society will reject y and will begin to consider one of the alternatives in this set, each of which has a positive probability of being selected. If instead $\{x \in X : (x,y) \in R\} = \varnothing$ the society has no reason to modify its choice and y is maintained. Thus, the fact that $(x,y) \in R$ means that, should the society be considering y, it must reject y and may transition to considering x.

The core is a natural solution concept for abstract decision problems: it selects the alternatives that are dominated by no alternative.

Definition 1. *Let (X, R) be an abstract decision problem. The core of (X, R) is the set*

$$\mathbf{Co}(X, R) \coloneqq \{x \in X : (y,x) \notin R \text{ for all } y \in X\}.$$

The main flaw of the core is that it is often empty. Another remarkable solution for abstract decision problems is the admissible set introduced by Kalai and Schmeidler (1977).[4] In order to present such a solution we need some preliminary definitions. Let (X, R) be an abstract decision problem. A path in (X, R) is a sequence $(x^j)_{j=1}^m$, where $m \geqslant 2$, $x^1, \dots, x^m$ are distinct elements of X and, for every $j \in \{1, \dots, m-1\}$, $(x^j, x^{j+1}) \in R$. If $x, y \in X$ are distinct, a path from x to y in (X, R) is a path $(x^j)_{j=1}^m$ in (X, R) such that $x^1 = x$ and $x^m = y$. If there is a path $(x^j)_{j=1}^m$ from x to y in (X, R), we say that x directly or indirectly dominates y (according to R). Indeed,

[4] Equivalent definitions of the admissible set are given by Schwartz (1972, 1986), Kalai et al. (1976), and Shenoy (1979, 1980). See also Van Deemen (1997) and Gori (2023).

if $m = 2$ the path exactly describes the fact that x dominates y. If instead $m \geqslant 3$, the fact that, for every $j \in \{1, \ldots, m-1\}$, x^j dominates x^{j+1}, suggests that there is a sort of indirect domination of x over y, meaning that if y is taken into account by the society, then y must be rejected and the society might start considering x after reviewing some alternatives.

Let us denote by R^τ the reflexive and transitive closure of R, that is, the smallest reflexive and transitive relation on X containing R. It is easily seen that

$$R^\tau = \left\{(x,y) \in X^2_* : \text{ there exists a path in } (X,R) \text{ from } x \text{ to } y\right\} \cup X^2_d.$$

Thus, if $(x,y) \in R^\tau$ we have that $x = y$ or x directly or indirectly dominates y, while if $(x,y) \notin R^\tau$ we have that $x \neq y$ and x neither directly nor indirectly dominates y.

Definition 2. *Let (X,R) be an abstract decision problem. The admissible set of (X,R) is the set $\mathbf{A}(X,R) \coloneqq \mathrm{Max}(R^\tau)$.*

Thus, an alternative x belongs to the admissible set if the fact that y directly or indirectly dominates x implies that, in turn, x directly or indirectly dominates y. Interpreting R as a complete description of all and only the transitions from an alternative to another that the society can make with positive probability, the set $\mathbf{A}(X,R)$ can be viewed as the set of potential outcomes of a social bargaining process where R governs the transitions among alternatives. Indeed, from the elementary theory of finite Markov chains, we have that, starting from any alternative, with probability 1, an element of $\mathbf{A}(X,R)$ is reached after a finite number of steps. Moreover, once an element in $\mathbf{A}(X,R)$ is reached, all subsequent transitions remain within $\mathbf{A}(X,R)$.[5]

Due to the transitivity of R^τ, we have $\mathbf{A}(X,R) \neq \varnothing$. Moreover, $\mathbf{Co}(X,R) \subseteq \mathbf{A}(X,R)$. It is also important to note that if R and R' are dominance relations on X with $R \subseteq R'$, we cannot infer either $\mathbf{A}(X,R) \subseteq \mathbf{A}(X,R')$ or $\mathbf{A}(X,R') \subseteq \mathbf{A}(X,R)$. Nevertheless, as proved in Gori (2023), we certainly have $\mathbf{A}(X,R) \cap \mathbf{A}(X,R') \neq \varnothing$. As an example, consider $X = \{1,2,3,4\}$, $R = \{(1,3),(3,4)\}$ and $R' = \{(1,3),(3,4),(4,1),(4,2)\}$. Thus, $R \subseteq R'$, $\mathbf{A}(X,R) = \{1,2\}$ and $\mathbf{A}(X,R') = \{1,3,4\}$.

Let (X,R) be an abstract decision problem. We denote by $\mathscr{S}(X,R)$ the quotient set of X by the equivalence relation on X given by

$$\left\{(x,y) \in X^2 : (x,y) \in R^\tau \text{ and } (y,x) \in R^\tau\right\}.$$

Recall that $\mathscr{S}(X,R)$ is a partition of X. The elements of $\mathscr{S}(X,R)$ are called strong components of (X,R). Note that, given $Y \in \mathscr{S}(X,R)$ and $x \in X$, we have that there exists $y^* \in Y$ such that $(x,y^*) \in R^\tau$ if and only if, for every $y \in Y$, $(x,y) \in R^\tau$. We set

$$\mathscr{A}(X,R) \coloneqq \{Y \in \mathscr{S}(X,R) : (x,y) \notin R^\tau \text{ for all } x \in X \backslash Y \text{ and } y \in Y\}.$$

Thus, given $Y \in \mathscr{A}(X,R)$, $x \in X$ and $y \in Y$, if $(x,y) \in R^\tau$ then $x \in Y$ and $(y,x) \in R^\tau$. The elements of $\mathscr{A}(X,R)$ are called maximal strong components of (X,R).

The next result states a well-known fact.[6]

Theorem 1. *Let (X,R) be an abstract decision problem. Then $\mathscr{A}(X,R) \neq \varnothing$ and $\mathbf{A}(X,R) = \bigcup_{Y \in \mathscr{A}(X,R)} Y$. In particular, $\mathbf{A}(X,R) \neq \varnothing$.*

Consider, for example, the abstract decision problem (X,R), where

$$\begin{gathered} X = \{1,2,3,4,5,6,7,8,9\}, \\ R = \{(1,2),(2,3),(2,5),(3,1),(3,6),(4,5),(4,7),(5,6),(6,8),(6,9),(8,5),(9,8)\}. \end{gathered} \tag{1}$$

In Figure 1, this abstract decision problem is represented, as usual, by a directed graph: the alternatives correspond to nodes, and for any $x, y \in X$, there is an arrow from x to y if and only if $(x,y) \in R$. We have then that

$$\mathscr{S}(X,R) = \{\{1,2,3\},\{4\},\{5,6,8,9\},\{7\}\},$$

$$\mathscr{A}(X,R) = \{\{1,2,3\},\{4\}\},$$

$$\mathbf{Co}(X,R) = \{4\},$$

$$\mathbf{A}(X,R) = \{1,2,3,4\}.$$

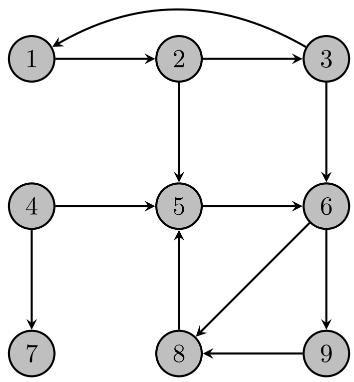

Figure 1: The abstract decision problem (X, R) in (1).

It can also be shown that, given an abstract decision problem (X, R), the set $\mathbf{Co}(X, R)$ coincides with the union of the elements of $\mathscr{A}(X, R)$ that are singletons. Since R is acyclic if and only if all the elements in $\mathscr{S}(X, R)$ are singletons, we have that if R is acyclic, then $\mathbf{Co}(X, R) = \mathbf{A}(X, R)$. This equality is remarkable since it implies that, in the bargaining process governed by R, starting from any alternative, the society will, with probability one, reach an element in the core after a finite number of transitions, at which point the bargaining process ends.[7] Of course, the fact that the dominance relation is cyclic does not exclude the possibility that the core and the admissible set may coincide. Consider, for example, the abstract decision problem (X, R), where

$$\begin{aligned} X &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \\ R &= \{(2, 3), (3, 6), (4, 5), (5, 2), (5, 6), (6, 9), (7, 8), (8, 5), (9, 8)\}, \end{aligned} \tag{2}$$

and represented in Figure 2. We have then that R is cyclic and

$$\begin{aligned} \mathscr{S}(X, R) &= \{\{1\}, \{4\}, \{7\}, \{2, 3, 5, 6, 8, 9\}\}, \\ \mathscr{A}(X, R) &= \{\{1\}, \{4\}, \{7\}\}, \\ \mathbf{Co}(X, R) &= \mathbf{A}(X, R) = \{1, 4, 7\}. \end{aligned}$$

We finally note that $\mathscr{A}(X, \varnothing) = \{\{x\} : x \in X\}$ and, therefore, $\mathbf{A}(X, \varnothing) = X$.

3 Finite strategic games

A finite strategic game, or simply a game, is a triple $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where

- I is a finite set with $|I| \geqslant 2$,
- for every $i \in I$, X_i is a nonempty and finite set,
- for every $i \in I$, u_i is a function from $\prod_{j \in I} X_j$ to $\mathbb{R}$.

Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a game. The set I is interpreted as set of players and, for every $i \in I$, the set X_i is interpreted as the set of strategies of player i. We denote by X the set $\prod_{j \in I} X_j$ and we call X the set of strategy profiles. When the strategy sets of players are not explicitly given, we denote the set of strategy profiles of G by $\pi(G)$. For every $i \in I$, we call u_i the payoff function of player i and, for our purposes, it is convenient to interpret $u_i(x)$ as the amount of money (say euros) earned (if $u_i(x) \geqslant 0$) or lost (if $u_i(x) < 0$) by player i if

[5]The set $\mathbf{A}(X, R)$ corresponds to the set of recurrent states of any homogeneous Markov chain having X as state space and $(p_{xy})_{x,y \in X}$ as transition probability matrix, where, for every $x, y \in X$ with $x \neq y$, $p_{xy} > 0$ if and only if $(y, x) \in R$.

[6]See, for instance, Kalai and Schmeidler (1977, Theorem 5) and in Gori (2023, Theorem 9).

[7]This fact is a consequence of Lemma 35(iv).

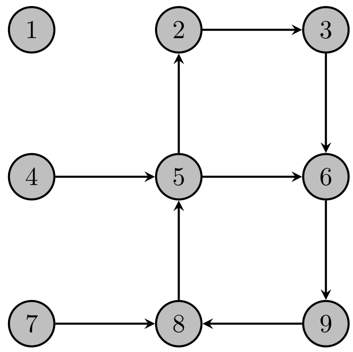

Figure 2: The abstract decision problem (X, R) in (2).

the strategy profile x occurs. We say that G is trivial if, for every $i \in I$, $|X_i| = 1$. Without loss of generality, we assume that $I \subseteq \mathbb{N}$ and, for every $i \in I$, $X_i \subseteq \mathbb{N}$.

For every $i \in I$, we denote by X_{-i} the set $\prod_{j \in I \setminus \{i\}} X_j$. For every $i \in I$ and $x \in X$, we denote by x_i the element of X_i corresponding to the component of x associated with i, and by x_{-i} the element of X_{-i} such that, for every $j \in I \setminus \{i\}$, its component associated with j equals the component of x associated with j. For every $i \in I$, $s \in X_i$ and $\sigma \in X_{-i}$, we denote by (s, σ) the element of X such that $(s, \sigma)_i = s$ and $(s, \sigma)_{-i} = \sigma$. Thus, for every $i \in I$ and $x \in X$, $x = (x_i, x_{-i})$.

A strategy profile $x^* \in X$ is a Nash equilibrium of G if, for every $i \in I$ and $s \in X_i$, $u_i(x^*) \geqslant u_i(s, x^*_{-i})$. The set of Nash equilibria of G is denoted by $\mathbf{N}(G)$. It is well known that the set of Nash equilibria of a game can be empty.

For every $i \in I$ and $\sigma \in X_{-i}$, we set

$$\mathcal{B}_i^G(\sigma) = \underset{s \in X_i}{\operatorname{argmax}}\, u_i(s, \sigma),$$

and, for every $i \in I$ and $x \in X$, we set

$$\mathcal{C}_i^G(x) = \underset{s \in X_i \setminus \{x_i\}}{\operatorname{argmax}}\, u_i(s, x_{-i}).$$

Thus, $\mathcal{B}_i^G(\sigma)$ is the subset of X_i corresponding to the best responses of player i when the strategies of the other players are described by σ, and $\mathcal{C}_i^G(x)$ is the subset of $X_i \setminus \{x_i\}$ corresponding to the best responses of player i when the strategies of the other players are described by x_{-i} and player i is forced to modify her strategy x_i. Note that $\mathcal{B}_i^G(\sigma) \neq \varnothing$, while $\mathcal{C}_i^G(x) \neq \varnothing$ if and only if $|X_i| \geqslant 2$. Moreover, given $x \in X$, $x \in \mathbf{N}(G)$ if and only if $x_i \in \mathcal{B}_i^G(x_{-i})$ for all $i \in I$; given $x \in X$ and $i \in I$, $x_i \notin \mathcal{B}_i^G(x_{-i})$ implies $\mathcal{B}_i^G(x_{-i}) = \mathcal{C}_i^G(x)$. Given $i \in I$ and $s \in X_i$, we say that s is never a best response in G for player i if, for every $\sigma \in X_{-i}$, $s \notin \mathcal{B}_i^G(\sigma)$; we say that s is a best response in G for player i if there exists $\sigma \in X_{-i}$ such that $s \in \mathcal{B}_i^G(\sigma)$. Observe that, for every $i \in I$, there exists an element in X_i that is a best response in G for player i.[8] Given $i \in I$ and $s \in X_i$, we say that s is strictly dominated by $s' \in X_i$ for player i if, for every $\sigma \in X_{-i}$, $u_i(s', \sigma) > u_i(s, \sigma)$; s is strictly dominated for player i if there exists $s' \in X_i$ such that s is strictly dominated by s' for player i; s is a strictly dominant strategy for player i if, for every $s' \in X_i \setminus \{s\}$, s' is strictly dominated by s for player i.

Given $(Y_i)_{i \in I}$ such that, for every $i \in I$, $\varnothing \neq Y_i \subseteq X_i$, we denote by $\mathfrak{R}(G, (Y_i)_{i \in I})$ the game $\langle I, (Y_i)_{i \in I}, (u_i)_{i \in I} \rangle$, where, for every $i \in I$, u_i is meant to be restricted to $\prod_{j \in I} Y_j$. The game $\mathfrak{R}(G, (Y_i)_{i \in I})$ is then the game obtained by eliminating from G the strategies in $X_i \setminus Y_i$ for all $i \in I$. We set $\mathfrak{F}(G) \coloneqq \mathfrak{R}(G, (Y_i)_{i \in I})$, where, for every $i \in I$, Y_i is the set of the elements of X_i that are a best response in G of player i, a set that, as already observed, is nonempty. In other words, $\mathfrak{F}(G)$ is the game obtained by eliminating from G all the strategies that are never a best response in G. We also set $\mathfrak{F}^1(G) \coloneqq \mathfrak{F}(G)$ and, for every $n \in \mathbb{N}$ with $n \geqslant 2$, $\mathfrak{F}^n(G) \coloneqq \mathfrak{F}(\mathfrak{F}^{n-1}(G))$. It is immediate to observe that

$$\text{for every } n \in \mathbb{N},\ \pi(\mathfrak{F}^{n+1}(G)) \subseteq \pi(\mathfrak{F}^n(G)), \tag{3}$$

$$\text{there exists } m \in \mathbb{N} \text{ such that, for every } n \geqslant m,\ \mathfrak{F}^n(G) = \mathfrak{F}^m(G). \tag{4}$$

[8] In fact, given $i \in I$, consider $x \in X$ that maximizes u_i: setting $s = x_i$ and $\sigma = x_{-i}$, we have $s \in \mathcal{B}_i^G(\sigma)$.

The set of rationalizable strategy profiles of G is the set[9]

$$\mathbf{Ra}(G) \coloneqq \bigcap_{n=1}^{\infty} \pi(\mathfrak{F}^n(G)).$$

Due to (3) and (4), we have $\mathbf{Ra}(G) = \pi(\mathfrak{F}^m(G))$ for a suitable $m \in \mathbb{N}$. In particular, $\mathbf{Ra}(G) \neq \varnothing$.

Following Monderer and Shapley (1996), a generalized ordinal potential for G is a function $P : X \to \mathbb{R}$ such that, for every $i \in I$, $s, s' \in X_i$ and $\sigma \in X_{-i}$, if $u_i(s, \sigma) - u_i(s', \sigma) > 0$, then $P(s, \sigma) - P(s', \sigma) > 0$; an ordinal potential for G is a function $P : X \to \mathbb{R}$ such that, for every $i \in I$, $s, s' \in X_i$ and $\sigma \in X_{-i}$, $u_i(s, \sigma) - u_i(s', \sigma) > 0$ if and only if $P(s, \sigma) - P(s', \sigma) > 0$; a potential for G is a function $P : X \to \mathbb{R}$ such that, for every $i \in I$, $s, s' \in X_i$ and $\sigma \in X_{-i}$, $u_i(s, \sigma) - u_i(s', \sigma) = P(s, \sigma) - P(s', \sigma)$; G is called a generalized ordinal potential [ordinal potential; potential] game if there exists a generalized ordinal potential [ordinal potential; potential] for G. An infinite improving walk of G is a sequence $(x^j)_{j=1}^{\infty}$ in X such that, for every $j \in \mathbb{N}$, there exists $i \in I$ such that $x_{-i}^j = x_{-i}^{j+1}$ and $u_i(x^{j+1}) - u_i(x^j) > 0$; G satisfies the finite improvement property (FIP) if G does not admit infinite improving walks. Of course, every potential game is an ordinal potential game; every ordinal potential game is a generalized ordinal potential game. Moreover, a game satisfies the FIP if and only if it is a generalized ordinal potential game.[10] Finally, each generalized ordinal potential game admits a Nash equilibrium.

In the paper, we are going to propose several examples of two-player games. For such games, we always assume that the set of players is $\{1, 2\}$, that the set of strategies of player 1 is a set of the type $\{1, \ldots, n_1\}$ with $n_1 \in \mathbb{N}$, and that the set of strategies of player 2 is a set of the type $\{1, \ldots, n_2\}$ with $n_2 \in \mathbb{N}$. As usual, we represent two-player games in matrix form with player 1 as row player and player 2 as column player. For instance, the following writing

$$G = \begin{array}{|c|c|c|} \hline 66, 0 & 6, 123 & 18, 78 \\ \hline 0, 168 & 54, 70 & 60, 8 \\ \hline \end{array} \tag{5}$$

represents a two-player game, where $X_1 = \{1, 2\}$ and $X_2 = \{1, 2, 3\}$. Recall that the first [second] number in each entry of the table refers to the value of the payoff function of player 1 [player 2] computed at the corresponding strategy profile. Thus, denoting the payoff function of player 1 [player 2] by u_1 [u_2], we deduce from the table that, for instance, $u_1(1, 1) = 66$ and $u_2(1, 2) = 123$. For the game G in (5), we have $\mathbf{N}(G) = \varnothing$.

4 Two basic dominance relations

Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a game. It is possible to interpret X as a set of alternatives among which players, as a society, have to make a choice through social bargaining. We can then consider the following dominance relations on X:[11]

$$D(G) \coloneqq \Big\{ (x, y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i},\ x_i \in \mathcal{B}_i^G(y_{-i}) \text{ and } u_i(x) > u_i(y) \Big\},$$

$$E(G) \coloneqq \Big\{ (x, y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i}, \text{ and } u_i(x) > u_i(y) \Big\}.$$

According to $D(G)$, we assume that in determining the solution of the game through a bargaining process, the society can transition from one strategy profile to another with positive probability if and only if there is a unique player who improves her payoff by the largest possible amount, given the strategies of the others. On the other hand, according to $E(G)$, we assume that the society can transition from one strategy profile to another with positive probability if and only if there is a unique player who improves her payoff, not necessarily by the maximal amount. The relation $D(G)$ reflects the assumption that players, once they know the strategies of the others, may deviate only by choosing among their best responses, each of them having positive probability to be selected. In contrast, the relation $E(G)$ models a situation in which players may face cognitive or informational limitations, leading them to deviate by choosing any strategy that increases their payoff, each of them having positive probability to be selected, even if it is not a best response.

Of course, we have $D(G) \subseteq E(G)$. Moreover, if G is trivial, then $D(G) = E(G) = \varnothing$ since $X_*^2 = \varnothing$. However, $D(G)$ and $E(G)$ can be empty even though G is not trivial. Indeed, for every game G whose payoff functions are constant, we have $D(G) = E(G) = \varnothing$.

[9] Bernheim (1984), Pearce (1984).

[10] Monderer and Shapley (1996, Lemma 2.5).

[11] These relations are present in the literature. See, for instance, Kalai and Schmeidler (1977), Kukushkin (2011), Block and van den Nouweland (2021).

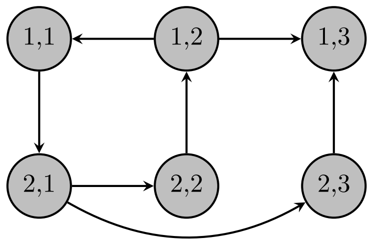

Figure 3: The abstract decision problem $(X, D(G))$ for G in (5).

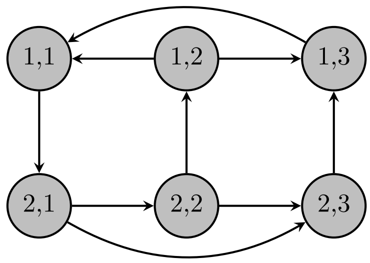

Figure 4: The abstract decision problem $(X, E(G))$ for G in (5).

Proposition 2. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a game. Suppose that, for every $i \in I$, $|X_i| \leqslant 2$. Then $D(G) = E(G)$.*

Proof. We know that $D(G) \subseteq E(G)$. Consider now $(x, y) \in E(G)$. Thus, there exists $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(x) > u_i(y)$. Since $(x, y) \in X_*^2$, we have that $x_i \neq y_i$ and then $X_i = \{x_i, y_i\}$. Thus, $x_i \in \mathcal{B}_i^G(y_{-i})$ and then $(x, y) \in D(G)$. We then get $E(G) \subseteq D(G)$, so we conclude $D(G) = E(G)$. $\square$

Once such dominance relations are introduced, one can consider the abstract decision problems $(X, D(G))$ and $(X, E(G))$. For example, if G is the game in (5), the abstract decision problems $(X, D(G))$ and $(X, E(G))$ are represented in Figures 3 and 4.

It is easy to check that, for every game G, $\mathbf{Co}(X, D(G)) = \mathbf{Co}(X, E(G)) = \mathbf{N}(G)$. We can now give the following crucial definitions.

Definition 3. *Let G be a game.*

- *The best-response admissible set of G is the set $\mathbf{D}(G) \coloneqq \mathbf{A}(X, D(G))$.*
- *The better-response admissible set of G is the set $\mathbf{E}(G) \coloneqq \mathbf{A}(X, E(G))$.*

The strategy profiles selected by the best-response admissible set are the ones that cannot be excluded as potential outcomes of a bargaining process among the players, where each player unilaterally deviates in order to increase its payoff by the largest possible amount. The strategy profiles selected by the better-response admissible set are instead the ones that cannot be excluded as potential outcomes of a bargaining process among the players, where each player unilaterally deviates in order to increase its payoff by any amount. The best-response admissible set and the better-response admissible set determine, for every finite strategic game G, a subset of its strategy profiles, and then they can be thought of as solution concepts for those games. We stress that the better-response admissible set was basically introduced by Kalai and Schmeidler (1977).[12] Note that,

[12] Kalai and Schmeidler (1977) consider the mixed extension of a finite strategic game and are interested in finding a suitable dominance relation on the set of mixed strategy profiles having the property that the corresponding admissible set (whose set of alternatives is in this case infinite) coincides with the set of Nash equilibria in mixed strategies of the game. The authors first consider the analog of the relation $E(G)$, but realize that the corresponding admissible set is, in general, very large and, even though it always contains the set of Nash equilibria, it does not coincide with it. Thus, to reach their goal, they need to consider a more sophisticated and less intuitive relation.

if G is a trivial game, then $\mathbf{D}(G) = \mathbf{E}(G) = X$. Moreover, by Proposition 2, if each player in a game G has at most two strategies, then $\mathbf{D}(G) = \mathbf{E}(G)$.

In order to assess the two solutions, we establish some of their properties.

Proposition 3. *Let G be a game. Then $\mathbf{D}(G) \neq \varnothing$, $\mathbf{E}(G) \neq \varnothing$, $\mathbf{N}(G) \subseteq \mathbf{D}(G)$, $\mathbf{N}(G) \subseteq \mathbf{E}(G)$.*

Proof. By Theorem 1, we deduce $\mathbf{D}(G) \neq \varnothing$ and $\mathbf{E}(G) \neq \varnothing$. Moreover, $\mathbf{N}(G) = \mathbf{Co}(X, D(G)) \subseteq \mathbf{D}(G)$ and $\mathbf{N}(G) = \mathbf{Co}(X, E(G)) \subseteq \mathbf{E}(G)$. □

Proposition 4. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game and let $\phi = (\phi_i)_{i\in I}$ be such that, for every $i \in I$, ϕ_i is a strictly increasing function from $\mathrm{Im}(u_i)$ to $\mathbb{R}$. Then, $\mathbf{D}(G_\phi) = \mathbf{D}(G)$ and $\mathbf{E}(G_\phi) = \mathbf{E}(G)$, where G_ϕ is the game $\langle I, (X_i)_{i\in I}, (\phi_i \circ u_i)_{i\in I}\rangle$.*

Proof. Simply note that $D(G_\phi) = D(G)$ and $E(G_\phi) = E(G)$. □

The following result shows that the best-response admissible set satisfies a significant rationality principle, namely, it is invariant under the elimination of strategies that are never a best response. Proposition 5 is proved in the appendix.

Proposition 5. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game and let $(Y_i)_{i\in I}$ be such that, for every $i \in I$, $\varnothing \neq Y_i \subseteq X_i$. Assume that, for every $i \in I$ and $s \in X_i \setminus Y_i$, s is never a best response in G for player i. Then $\mathbf{D}(G) = \mathbf{D}(\mathfrak{R}(G, (Y_i)_{i\in I}))$.*

As a consequence of Proposition 5, we deduce that the best-response admissible set always selects strategy profiles that are rationalizable, as proved in the next proposition.

Proposition 6. *Let G be a game. Then $\mathbf{D}(G) \subseteq \mathbf{Ra}(G)$.*

Proof. First, let us prove that, for every $n \in \mathbb{N}$, $\mathbf{D}(G) = \mathbf{D}(\mathfrak{F}^n(G))$. That can be easily done by induction. Indeed, by Proposition 5 applied to G, we have that $\mathbf{D}(G) = \mathbf{D}(\mathfrak{F}(G)) = \mathbf{D}(\mathfrak{F}^1(G))$. Suppose now that, $\mathbf{D}(G) = \mathbf{D}(\mathfrak{F}^n(G))$. Thus, applying now Proposition 5 to $\mathfrak{F}^n(G)$, we deduce $\mathbf{D}(\mathfrak{F}^{n+1}(G)) = \mathbf{D}(\mathfrak{F}(\mathfrak{F}^n(G))) = \mathbf{D}(\mathfrak{F}^n(G)) = \mathbf{D}(G)$.

Since, for every $n \in \mathbb{N}$, $\mathbf{D}(\mathfrak{F}^n(G)) \subseteq \pi(\mathfrak{F}^n(G))$, we deduce that, for every $n \in \mathbb{N}$, $\mathbf{D}(G) \subseteq \pi(\mathfrak{F}^n(G))$. We then conclude $\mathbf{D}(G) \subseteq \mathbf{Ra}(G)$. □

Unlike the best-response admissible set, the better-response admissible set may select strategy profiles that are not rationalizable. Indeed, consider the game G in (5) and note that for player 2 strategy 3 is strictly dominated by strategy 2. A computation shows that $\mathbf{D}(G) = \{(1,1), (1,2), (2,1), (2,2)\}$, while $\mathbf{E}(G) = \pi(G)$. In particular, $\mathbf{E}(G)$ selects strategy profiles whose some components are strictly dominated strategies. As a consequence, $\mathbf{E}(G) \nsubseteq \mathbf{Ra}(G)$, and, therefore, $\mathbf{E}$ cannot satisfy the property described in Proposition 5.

However, the next proposition shows that any strictly dominant strategy must be component of any strategy profile in $\mathbf{E}(G)$.

Proposition 7. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game, $x \in X$, $k \in I$ and $s \in X_k$ be a strictly dominant strategy for player k. If $x \in \mathbf{D}(G)$, then $x_k = s$. If $x \in \mathbf{E}(G)$, then $x_k = s$.*

Proof. Assume that $x \in \mathbf{D}(G)$. By Proposition 5, we know that $\mathbf{D}(G) = \mathbf{D}(\mathfrak{F}(G))$. Recall that $\mathfrak{F}(G) = \langle I, (Y_i)_{i\in I}, u_i\rangle$, where, for every $i \in I$, Y_i is the set of the elements of X_i that are a best response in G for player i. Thus, $Y_k = \{s\}$ and that implies $x_k = s$.

Assume now that $x \in \mathbf{E}(G)$. Suppose by contradiction that $x_k \neq s$. Thus, $((s, x_{-k}), x) \in E(G)$ and then $((s, x_{-k}), x) \in E(G)^\tau$. Since $x \in \mathbf{E}(G)$, we have $(x, (s, x_{-k})) \in E(G)^\tau$. Thus, there exists a path $(x^j)_{j=1}^m$ from x to (s, x_{-k}) in $(X, E(G))$. Since $x_k^1 \neq s$ and $x_k^m = s$, there exists $j \in \{1, \dots, m-1\}$ such that $x_k^j \neq s$ and $x_k^{j+1} = s$. Since $(x^j, x^{j+1}) \in E(G)$ and since s is a strictly dominant strategy for player k, we get a contradiction. □

If, for a game G, we have $\mathbf{D}(G) = \mathbf{N}(G)$ [$\mathbf{E}(G) = \mathbf{N}(G)$], then, by the results discussed in Section 2.2, the social bargaining process governed by $D(G)$ [$E(G)$] almost surely leads to a Nash equilibrium after a finite number of transitions. It is then an interesting problem to determine families of games for which such equalities hold. The next result shows that that happens for generalized ordinal potential games.[13]

[13]The fact that, for a generalized potential game G, the relations $D(G)$ and $E(G)$ are acyclic is a well-known fact (see, for instance, Kukushkin 2011).

Proposition 8. *Let G be a generalized ordinal potential game. Then, $D(G)$ and $E(G)$ are acyclic. As a consequence,* $\mathbf{D}(G) = \mathbf{E}(G) = \mathbf{N}(G)$.

Proof. Since G is a generalized ordinal potential game, we know that G satisfies the FIP. Suppose by contradiction that $E(G)$ is cyclic. Then there exist $m \geqslant 2$ and $x^1, \dots, x^m \in X$ such that, for every $j \in \{1, \dots, m-1\}$, $(x^j, x^{j+1}) \in E(G)$ and $(x^m, x^1) \in E(G)$. For every $j \in \mathbb{Z}$, there exists a unique element t in the set $\{1, \dots, m\}$ such that m divides $j - t$, and we denote that element by $[j]$. Consider now the sequence $(y^j)_{j=1}^{\infty}$ defined, for every $j \in \mathbb{N}$, by $y^j = x^{[-j]}$. Note that $(y^j)_{j=1}^{\infty}$ is an infinite improving walk of G. Indeed, consider $j \in \mathbb{N}$. If $[-j] \in \{2, \dots, m\}$, then $[-j-1] = [-j]-1 \in \{1, \dots, m-1\}$ and $(y^{j+1}, y^j) = (x^{[-j]-1}, x^{[-j]}) \in E(G)$. If $[-j] = 1$, then $[-j-1] = m$ and $(y^{j+1}, y^j) = (x^m, x^1) = (y, x) \in E(G)$. In both cases, in particular, there exists $i \in I$ such that $y^{j+1}_{-i} = y^j_{-i}$ and $u_i(y^{j+1}) - u_i(y^j) > 0$. That contradicts the fact that G is FIP. Thus, we conclude that $E(G)$ is acyclic. Moreover, since $D(G) \subseteq E(G)$, we also deduce that $D(G)$ is acyclic.

Since $D(G)$ and $E(G)$ are acyclic, due to the remarks made in Section 2.2, we deduce $\mathbf{A}(X, D(G)) = \mathbf{Co}(X, D(G))$ and $\mathbf{A}(X, E(G)) = \mathbf{Co}(X, E(G))$, that is, $\mathbf{D}(G) = \mathbf{N}(G)$ and $\mathbf{E}(G) = \mathbf{N}(G)$. $\square$

As can be seen through examples, both the best-response admissible set and the better-response admissible set may determine rather large sets of strategy profiles. The remainder of the paper is devoted to the definition and study of suitable refinements of these solution concepts. The approach we follow is strongly inspired by an alternative definition of the Schulze method, a well-known voting procedure appreciated for its theoretical strengths (Schulze 2011, 2018), proposed by Gori (2023). The underlying idea is that replicating a similar approach within a game-theoretical framework might lead to new solution concepts endowed with interesting properties. The strategy we will follow has already been outlined in the introduction. We first define families of dominance relations over the set of strategy profiles, each with a clear interpretation. We then consider the admissible sets associated with the corresponding abstract decision problems. Finally, we take the intersection of all these admissible sets. Each strategy profile in this intersection has the property to be considered a potential outcome of the bargaining process among players, regardless of which particular dominance relation governs it.

5 Further dominance relations

Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game. For every $t \in \mathbb{R}$, let $D_t(G)$ and $E_t(G)$ be the dominance relations on X defined as

$$D_t(G) = \Big\{(x, y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i},\, x_i \in \mathcal{C}_i^G(y) \text{ and } u_i(x) - u_i(y) > t\Big\},$$

$$E_t(G) = \Big\{(x, y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i}, \text{ and } u_i(x) - u_i(y) > t\Big\}.$$

According to $D_t(G)$, in determining the solution of the game through a bargaining process, the society can transition from one strategy profile to another with positive probability if and only if there is a unique player who deviates by choosing one of the best responses among the strategies different from the current one and, doing so, her payoff variation is greater than t. According to $E_t(G)$, in determining the solution of the game through a bargaining process, the society can transition from one strategy profile to another with positive probability if and only if there is a unique player who deviates by choosing a strategy that guarantees her a payoff variation greater than t. If $t > 0$, $D_t(G)$ and $E_t(G)$ can be interpreted as describing a situation in which all players incur a psychological cost in deviating, and this cost is quantified for each player as t euros. If instead $t < 0$, $D_t(G)$ and $E_t(G)$ can be interpreted as describing a situation in which all players derive satisfaction from having the power to change the status quo, and this satisfaction is quantified for each player as $|t|$ euros.

Finally, $D_0(G) = D(G)$ and $E_0(G) = E(G)$, where the latter equality is straightforward, while the former is proved in the following proposition.

Proposition 9. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ and $t \in \mathbb{R}_+$. Then*

$$D_t(G) = \Big\{(x, y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i},\, x_i \in \mathcal{B}_i^G(y_{-i}) \text{ and } u_i(x) - u_i(y) > t\Big\}. \tag{6}$$

In particular, $D_0(G) = D(G)$.

Proof. Let us denote by $D_t^*(G)$ the set on the right hand side of (6). We prove $D_t(G) = D_t^*(G)$ by proving the two inclusions $D_t(G) \subseteq D_t^*(G)$ and $D_t^*(G) \subseteq D_t(G)$.

Let $(x, y) \in D_t(G)$. Then there exists $i \in I$ such that $x_{-i} = y_{-i}$, $x_i \in \mathcal{C}_i^G(y)$ and $u_i(x) - u_i(y) > t$. Since $t \geqslant 0$, we have that $u_i(x) > u_i(y)$. That implies that $y_i \notin \mathcal{B}_i^G(y_{-i})$. Thus, $\mathcal{C}_i^G(y) = \mathcal{B}_i^G(y_{-i})$. Then $x_i \in \mathcal{B}_i^G(y_{-i})$ and $(x, y) \in D_t^*(G)$. We conclude that $D_t(G) \subseteq D_t^*(G)$.

Let $(x, y) \in D_t^*(G)$. Then there exists $i \in I$ such that $x_{-i} = y_{-i}$, $x_i \in \mathcal{B}_i^G(y_{-i})$ and $u_i(x) - u_i(y) > t$. Since $t \geqslant 0$, we have that $u_i(x) > u_i(y)$. That implies that $y_i \notin \mathcal{B}_i^G(y_{-i})$. Thus, $\mathcal{B}_i^G(y_{-i}) = \mathcal{C}_i^G(y)$. Then $x_i \in \mathcal{C}_i^G(y)$ and $(x, y) \in D_t(G)$. We conclude that $D_t^*(G) \subseteq D_t(G)$. □

Of course, for every $t \in \mathbb{R}$, $D_t(G) \subseteq E_t(G)$. Moreover, if G is trivial, then, for every $t \in \mathbb{R}$, $D_t(G) = E_t(G) = \varnothing$ since $X_*^2 = \varnothing$. However, given $t \in \mathbb{R}$, we may have $D_t(G) = \varnothing$ and $E_t(G) = \varnothing$ even if G is not trivial. For example, if $t \geqslant 0$ and G is a non-trivial game whose payoff functions are constant, then $D_t(G) = E_t(G) = \varnothing$. The following propositions describe some further facts about the aforementioned dominance relations.

Proposition 10. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a game. Suppose that, for every $i \in I$, $|X_i| \leqslant 2$. Then, for every $t \in \mathbb{R}$, $D_t(G) = E_t(G)$.*

Proof. We know that $D_t(G) \subseteq E_t(G)$. Consider now $(x, y) \in E_t(G)$. Thus, there exists $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(x) - u_i(y) > t$. Since $(x, y) \in X_*^2$, we have that $x_i \neq y_i$ and then $X_i = \{x_i, y_i\}$. Thus, $x_i \in \mathcal{C}_i^G(y)$ and then $(x, y) \in D_t(G)$. We then get $E_t(G) \subseteq D_t(G)$, and so $D_t(G) = E_t(G)$ follows. □

Proposition 11. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ and $t, t' \in \mathbb{R}$ with $t \leqslant t'$. Then $D_{t'}(G) \subseteq D_t(G)$ and $E_{t'}(G) \subseteq E_t(G)$.*

Proof. Let $(x, y) \in D_{t'}(G)$ and prove that $(x, y) \in D_t(G)$. We know that there exists $i \in I$ such that $x_{-i} = y_{-i}$, $x_i \in \mathcal{C}_i^G(y)$ and $u_i(x) - u_i(y) > t'$. As a consequence, $x_{-i} = y_{-i}$, $x_i \in \mathcal{C}_i^G(y)$ and $u_i(x) - u_i(y) > t$, that is, $(x, y) \in D_t(G)$. The proof of the inclusion $E_{t'}(G) \subseteq E_t(G)$ is analogous. □

Given a game $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$, we can consider, for every $t \in \mathbb{R}$, the abstract decision problems $(X, D_t(G))$ and $(X, E_t(G))$ and the corresponding admissible set $\mathbf{A}(X, D_t(G))$ and $\mathbf{A}(X, E_t(G))$, simply denoted by $\mathbf{D}_t(G)$ and $\mathbf{E}_t(G)$. Of course, since $(X, D_0(G)) = (X, D(G))$ and $(X, E_0(G)) = (X, E(G))$, we have that $\mathbf{D}_0(G) = \mathbf{D}(G)$ and $\mathbf{E}_0(G) = \mathbf{E}(G)$. Note also that if G is trivial, then, for every $t \in \mathbb{R}$, $\mathbf{D}_t(G) = \mathbf{E}_t(G) = X$. Moreover, by Proposition 10, if each player in a game G has at most two strategies, then, for every $t \in \mathbb{R}$, $\mathbf{D}_t(G) = \mathbf{E}_t(G)$.

Let us consider now $t > 0$. A strategy profile $x^* \in X$ is a t-equilibrium of G if, for every $i \in I$ and $s \in X_i$, $u_i(x) \geqslant u_i(s, x_{-i}) - t$. Of course, every Nash equilibrium of G is a t-equilibrium of G. It is easy to check that, for every $t > 0$, $\mathbf{Co}(X, D_t(G))$ and $\mathbf{Co}(X, E_t(G))$ coincide with the set of t-equilibria of G. Thus, for every $t > 0$, all the t-equilibria of G are elements of $\mathbf{D}_t(G)$ and $\mathbf{E}_t(G)$ and so, in particular, $\mathbf{N}(G) \subseteq \mathbf{D}_t(G)$ and $\mathbf{N}(G) \subseteq \mathbf{E}_t(G)$.

6 Some refinements

We are now ready to introduce some further key concepts of the paper.

Definition 4. *Let G be a game.*

- *The refined best-response admissible set of G is the set*

$$\mathbf{D}_\bullet(G) \coloneqq \bigcap_{t \in \mathbb{R}_+} \mathbf{D}_t(G).$$

- *The strongly refined best-response admissible set of G is the set*

$$\mathbf{D}_\circ(G) \coloneqq \bigcap_{t \in \mathbb{R}} \mathbf{D}_t(G).$$

- *The refined better-response admissible set of G is the set*

$$\mathbf{E}_\bullet(G) \coloneqq \bigcap_{t \in \mathbb{R}_+} \mathbf{E}_t(G).$$

- *The strongly refined better-response admissible set of G is the set*

$$\mathbf{E}_\circ(G) \coloneqq \bigcap_{t\in\mathbb{R}} \mathbf{E}_t(G).$$

Thus, the refined best-response admissible set of G selects the strategy profiles that have the property that, for every $t \in \mathbb{R}_+$, they cannot be excluded as potential outcome of a bargaining process governed by the dominance relation $D_t(G)$. Given $x \in \mathbf{D}_\bullet(G)$, we then have that x satisfies the following property: if there exist $y \in X$ and $t \in \mathbb{R}_+$ such that y directly or indirectly dominates x according to $D_t(G)$, then x directly or indirectly dominates y according to $D_t(G)$. Similar considerations can be made for the other three sets in Definition 4.

According to their definitions, the computation of the sets in Definition 4 appears complex. As explained in Section 8, when a non-trivial game is considered, it is possible to provide an equivalent definition for each of these sets that turns out particularly useful for computational purposes (Theorem 21). Of course, if G is a trivial game, then $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = X$ since, for every $t \in \mathbb{R}$, $\mathbf{D}_t(G) = \mathbf{E}_t(G) = X$. Thus, for trivial games, the computation is immediate.

The following simple proposition states some basic facts about the new solution concepts.

Proposition 12. *Let G be a game. Then $\mathbf{D}_\circ(G) \subseteq \mathbf{D}_\bullet(G) \subseteq \mathbf{D}(G)$, $\mathbf{E}_\circ(G) \subseteq \mathbf{E}_\bullet(G) \subseteq \mathbf{E}(G)$, $\mathbf{N}(G) \subseteq \mathbf{D}_\bullet(G)$, $\mathbf{N}(G) \subseteq \mathbf{E}_\bullet(G)$.*

Proof. From the definition of $\mathbf{D}_\bullet(G)$ and $\mathbf{D}_\circ(G)$, it immediately follows $\mathbf{D}_\circ(G) \subseteq \mathbf{D}_\bullet(G)$. Since $\mathbf{D}_0(G) = \mathbf{D}(G)$, we also get the inclusion $\mathbf{D}_\bullet(G) \subseteq \mathbf{D}(G)$. Finally, we know that $\mathbf{N}(G) \subseteq \mathbf{D}_0(G)$ and that, for every $t > 0$, $\mathbf{N}(G) \subseteq \mathbf{D}_t(G)$. Thus, $\mathbf{N}(G) \subseteq \mathbf{D}_\bullet(G)$. The proof for the other inclusions is analogous. □

Note that all the inclusions in Proposition 12 are in general strict. Moreover, the two sets $\mathbf{N}(G)$ and $\mathbf{D}_\circ(G)$, as well as $\mathbf{N}(G)$ and $\mathbf{E}_\circ(G)$, can be disjoint even when they are both nonempty. For example, for the two-player game

$$G = \begin{array}{|c|c|c|}\hline 0,0 & 0,0 & 0,0 \\ \hline 0,0 & 3,1 & 1,3 \\ \hline 0,0 & 1,3 & 4,2 \\ \hline \end{array}$$

we have $\mathbf{N}(G) = \{(1,1)\}$, $\mathbf{D}(G) = \mathbf{E}(G) = \{(1,1),(2,2),(2,3),(3,2),(3,3)\}$, $\mathbf{D}_\bullet(G) = \mathbf{E}_\bullet(G) = \{(1,1),(3,3)\}$, $\mathbf{D}_\circ(G) = \mathbf{E}_\circ(G) = \{(3,3)\}$.

Of course, by Propositions 6 and 12, we deduce that $\mathbf{D}_\bullet(G) \subseteq \mathbf{Ra}(G)$ and $\mathbf{D}_\circ(G) \subseteq \mathbf{Ra}(G)$. On the other hand, $\mathbf{E}_\circ$ can select strategy profiles in which some players play a strictly dominated strategy. For example, for the game G in (5), we have $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \{(1,2)\}$ and $\mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = \{(1,3)\}$. However, by Proposition 7 and 12, if a player of a game G admits a strictly dominant strategy, then that strategy is part of each element in $\mathbf{E}_\bullet(G)$ and $\mathbf{E}_\circ(G)$.

Of course, a major problem is to understand whether the solution concepts defined in Definition 4 are nonempty valued. The next theorem, which is the first main result of the paper, shows that all of them actually are. Its proof is in the appendix.

Theorem 13. *Let G be a game. Then $\mathbf{D}_\circ(G) \neq \varnothing$ and $\mathbf{E}_\circ(G) \neq \varnothing$.*

Proposition 14 shows that the sets in Definition 4 are not affected by applying to the payoff function of each player a positive affine transformation, provided that all transformations have the same multiplicative factor.

Proposition 14. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game and let $a \in \mathbb{R}$ with $a > 0$, and $b \in \mathbb{R}^I$. Then, $\mathbf{D}_\bullet(G_{a,b}) = \mathbf{D}_\bullet(G)$, $\mathbf{D}_\circ(G_{a,b}) = \mathbf{D}_\circ(G)$, $\mathbf{E}_\bullet(G_{a,b}) = \mathbf{E}_\bullet(G)$, $\mathbf{E}_\circ(G_{a,b}) = \mathbf{E}_\circ(G)$, where $G_{a,b}$ is the game $\langle I, (X_i)_{i\in I}, (au_i + b_i)_{i\in I}\rangle$.*

Proof. Let us prove that $\mathbf{D}_\bullet(G_{a,b}) = \mathbf{D}_\bullet(G)$ and $\mathbf{D}_\circ(G_{a,b}) = \mathbf{D}_\circ(G)$. First, we prove that, for every $t \in \mathbb{R}$, $D_t(G_{a,b}) = D_{\frac{t}{a}}(G)$. Let $t \in \mathbb{R}$. Consider $x, y \in X$ and the following facts:

(a) $(x,y) \in D_t(G_{a,b})$,

(b) there exists $i \in I$ such that $x_{-i} = y_{-i}$, $x_i \in \mathcal{C}_i^{G_{a,b}}(y)$, and $(au_i(x) + b_i) - (au_i(y) + b_i) > t$,

(c) there exists $i \in I$ such that $x_{-i} = y_{-i}$, $x_i \in \mathcal{C}_i^{G}(y)$, and $u_i(x) - u_i(y) > \frac{t}{a}$,

(d) $(x,y) \in D_{\frac{t}{a}}(G)$.

By definition of $D_t(G_{a,b})$ and $D_{\frac{t}{a}}(G)$, we have that (a) is equivalent to (b), and (c) is equivalent to (d). Moreover, since $a > 0$, we have $x_i \in \mathcal{C}_i^{G_{a,b}}(y)$ if and only if $x_i \in \mathcal{C}_i^{G}(y)$; $(au_i(x) + b_i) - (au_i(y) + b_i) > t$ if and only if $u_i(x) - u_i(y) > \frac{t}{a}$. Thus, (b) is equivalent to (c). That proves the equality $D_t(G_{a,b}) = D_{\frac{t}{a}}(G)$.

Observing now that $\pi(G_{a,b}) = \pi(G)$ and recalling that $a > 0$, we have

$$\mathbf{D}_\bullet(G_{a,b}) = \bigcap_{t\in\mathbb{R}_+} \mathbf{D}_t(G_{a,b}) = \bigcap_{t\in\mathbb{R}_+} \mathbf{A}(\pi(G_{a,b}), D_t(G_{a,b})) = \bigcap_{t\in\mathbb{R}_+} \mathbf{A}(\pi(G), D_{\frac{t}{a}}(G))$$

$$= \bigcap_{r\in\mathbb{R}_+} \mathbf{A}(\pi(G), D_r(G)) = \bigcap_{r\in\mathbb{R}_+} \mathbf{D}_r(G) = \mathbf{D}_\bullet(G),$$

and

$$\mathbf{D}_\circ(G_{a,b}) = \bigcap_{t\in\mathbb{R}} \mathbf{D}_t(G_{a,b}) = \bigcap_{t\in\mathbb{R}} \mathbf{A}(\pi(G_{ab}), D_t(G_{a,b})) = \bigcap_{t\in\mathbb{R}} \mathbf{A}(\pi(G), D_{\frac{t}{a}}(G))$$

$$= \bigcap_{r\in\mathbb{R}} \mathbf{A}(\pi(G), D_r(G)) = \bigcap_{r\in\mathbb{R}} \mathbf{D}_r(G) = \mathbf{D}_\circ(G).$$

The proof of the equalities $\mathbf{E}_\bullet(G_{a,b}) = \mathbf{E}_\bullet(G)$ and $\mathbf{E}_\circ(G_{a,b}) = \mathbf{E}_\circ(G)$ is analogous and then omitted. □

In general, the considered solutions are sensitive to changes in payoff functions due to the application of positive affine transformations having different multiplicative factors. Consider, for example, the two-player games

$$G = \begin{array}{|c|c|}\hline 3,1 & 1,3 \\ \hline 1,3 & 4,2 \\ \hline \end{array}, \qquad G' = \begin{array}{|c|c|}\hline 3,2 & 1,6 \\ \hline 1,6 & 4,4 \\ \hline \end{array},$$

and note that G' is obtained by G by leaving unchanged the payoff function of player 1 and doubling the payoff function of player 2. A computation shows that $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = \{(2,2)\}$ and $\mathbf{D}_\bullet(G') = \mathbf{D}_\circ(G') = \mathbf{E}_\bullet(G') = \mathbf{E}_\circ(G') = \{(2,1),(2,2)\}$.

Proposition 15. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game. Assume that, for every $i \in I$, $|X_i| \leqslant 2$. Then $\mathbf{D}_\bullet(G) = \mathbf{E}_\bullet(G)$ and $\mathbf{D}_\circ(G) = \mathbf{E}_\circ(G)$*

Proof. By Proposition 10, we know that, for every $t \in \mathbb{R}$, $\mathbf{D}_t(G) = \mathbf{E}_t(G)$. That fact immediately implies the desired equalities. □

An immediate but useful remark follows from Proposition 12: given a game G, if $\mathbf{D}(G) = \mathbf{N}(G)$, then $\mathbf{D}_\bullet(G) = \mathbf{N}(G)$; similarly, if $\mathbf{E}(G) = \mathbf{N}(G)$, then $\mathbf{E}_\bullet(G) = \mathbf{N}(G)$. Thus, if $\mathbf{D}(G) = \mathbf{N}(G)$ [$\mathbf{E}(G) = \mathbf{N}(G)$], it follows that $\mathbf{D}_\circ(G) \subseteq \mathbf{N}(G)$ [$\mathbf{E}_\circ(G) \subseteq \mathbf{N}(G)$], and therefore $\mathbf{D}_\circ(G)$ [$\mathbf{E}_\circ(G)$] provides a method to discriminate among Nash equilibria. As shown in Proposition 8, this occurs, in particular, when G is a generalized ordinal potential game.

Consider now a potential game G. Monderer and Shapley (1996, Lemma 2.7) proved that if P_1 and P_2 are potentials for G, then there exists a constant c such that, for every $x \in X$, $P_1(x) - P_2(x) = c$. As a consequence, any potential for G is maximized by the same set of strategy profiles, each of which is a Nash equilibrium. As emphasized by Monderer and Shapley, maximizing the potential provides a criterion for selecting among Nash equilibria.

Interestingly, given a potential game G, the sets $\mathbf{D}_\circ(G)$ and $\mathbf{E}_\circ(G)$ do not, in general, coincide with the set of strategy profiles that maximize the potential of G. For example, consider the two-player game

$$G = \begin{array}{|c|c|c|c|}\hline 5,5 & 4,4 & 7,7 & 6,6 \\ \hline 0,0 & 7,7 & 5,5 & 6,6 \\ \hline 7,7 & 4,4 & 1,1 & 4,4 \\ \hline 3,3 & 4,4 & 6,6 & 5,5 \\ \hline \end{array}. \qquad (7)$$

We have that G is a potential game with potential $P(x) = u_1(x)$. A computation shows that $\mathbf{D}_\circ(G) = \{(1,3),(2,2)\}$ and $\mathbf{E}_\circ(G) = \{(3,1)\}$, while the set of strategy profiles maximizing P coincides with the set $\mathbf{N}(G) = \{(1,3),(2,2),(3,1)\}$.

7 Heterogeneous and strongly heterogeneous games

Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game. Let $D_*(G)$ and $E_*(G)$ be the irreflexive relations on X defined as

$$D_*(G) \coloneqq \Big\{(x,y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i} \text{ and } x_i \in \mathcal{C}_i^G(y)\Big\},$$

$$E_*(G) \coloneqq \Big\{(x,y) \in X_*^2 : \exists\, i \in I \text{ such that } x_{-i} = y_{-i}\Big\}.$$

Of course, $D_*(G) \subseteq E_*(G)$. Moreover, $D_*(G) = \varnothing$ $[E_*(G) = \varnothing]$ if and only if G is trivial. Given $(x,y) \in D_*(G)$ $[(x,y) \in E_*(G)]$, we denote by $i(x,y)$ the unique element of I for which $x_{-i} = y_{-i}$. Thus, for every $t \in \mathbb{R}$, we have

$$D_t(G) = \Big\{(x,y) \in D_*(G) : u_{i(x,y)}(x) - u_{i(x,y)}(y) > t\Big\},$$

$$E_t(G) = \Big\{(x,y) \in E_*(G) : u_{i(x,y)}(x) - u_{i(x,y)}(y) > t\Big\}.$$

Let us also define the set

$$\mathscr{D}(G) \coloneqq \Big\{u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathbb{R} : (x,y) \in D_*(G)\Big\},$$

$$\mathscr{E}(G) \coloneqq \Big\{u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathbb{R} : (x,y) \in E_*(G)\Big\}.$$

Note that $\mathscr{D}(G) = \varnothing$ $[\mathscr{E}(G) = \varnothing]$ if and only if G is trivial. Moreover, $|\mathscr{D}(G)| \leqslant |D_*(G)|$ and $|\mathscr{E}(G)| \leqslant |E_*(G)|$. The following proposition holds.

Proposition 16. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game. Suppose that, for every $i \in I$, $|X_i| \leqslant 2$. Then, $D_*(G) = E_*(G)$ and $\mathscr{D}(G) = \mathscr{E}(G)$.*

Proof. We know that $D_*(G) \subseteq E_*(G)$. Consider now $(x,y) \in E_*(G)$. Thus, there exists $i \in I$ such that $x_{-i} = y_{-i}$. Since $(x,y) \in X_*^2$, we have that $x_i \neq y_i$ and then $X_i = \{x_i, y_i\}$. Thus, $x_i \in \mathcal{C}_i^G(y)$ and then $(x,y) \in D_*(G)$. We then get $E_*(G) \subseteq D_*(G)$, and so $D_*(G) = E_*(G)$. As an immediate consequence, we also get $\mathscr{D}(G) = \mathscr{E}(G)$. □

Definition 5. *Let G be game. We say that G is heterogeneous if $|\mathscr{D}(G)| = |D_*(G)|$. We say that G is strongly heterogeneous if $|\mathscr{E}(G)| = |E_*(G)|$.*

Observe that, since $D_*(G) \subseteq E_*(G)$, if G is strongly heterogeneous, then it is heterogeneous. Moreover, if G is a trivial game, then G is strongly heterogeneous, since $\mathscr{E}(G) = E_*(G) = \varnothing$. Observe also that if $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ is such that, for every $i \in I$, $|X_i| \leqslant 2$, then, by Proposition 16, G is heterogeneous if and only if G is strongly heterogeneous.

Consider now a game $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ such that $I = \{1,2\}$, $X_1 = \{1,2\}$ and $X_2 = \{1,2\}$. In this case, independently on the payoff functions of G, we have that $D_*(G)$ and $E_*(G)$ both coincide with the set having the following eight elements:

$$\begin{gathered}((1,1),(2,1)),((2,1),(1,1)),((1,1),(1,2)),((1,2),(1,1)),\\((1,2),(2,2)),((2,2),(1,2)),((2,1),(2,2)),((2,2),(2,1)).\end{gathered}$$

Assuming now, for example, that

$$G = \begin{array}{|c|c|}\hline 2,-2 & -3,3 \\ \hline -4,4 & 5,-5 \\ \hline\end{array}, \tag{8}$$

we have $\mathscr{D}(G) = \mathscr{E}(G) = \{-9,-8,-6,-5,5,6,8,9\}$. Thus, G is strongly heterogeneous since $|\mathscr{E}(G)| = |E_*(G)| = 8$.

The next result is the second main result of the paper. Its proof is in the appendix.

Theorem 17. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game.*

(i) *If G is heterogeneous, then $|\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))|$ and $|\mathbf{D}_\circ(G)| = |\mathscr{A}(X, D_*(G))|$.*

(ii) *If G is strongly heterogeneous, then $|\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))|$ and $|\mathbf{E}_\circ(G)| = 1$.*

Theorem 17(i) states that, given a heterogeneous game G, the number of elements in $\mathbf{D}_\bullet(G)$ equals the number of maximal strong components of $(X, D(G))$. This number is, in general, much smaller than the size of $\mathbf{D}(G)$. Indeed, by Theorem 1, $\mathbf{D}(G)$ corresponds to the union of the maximal strong components of $(X, D(G))$. Of course, $\mathbf{D}_\circ(G)$ is smaller than $\mathbf{D}_\bullet(G)$ and Theorem 17(i) explains that its size coincides with the number of maximal strong components of $(X, D_*(G))$. Similarly, given a strongly heterogeneous game G, Theorem 17(ii) states that the number of elements in $\mathbf{E}_\bullet(G)$ equals the number of maximal strong components of $(X, E(G))$. Again, this number is, in general, much smaller than the size of $\mathbf{E}(G)$. Indeed, by Theorem 1, $\mathbf{E}(G)$ corresponds to the union of the maximal strong components of $(X, E(G))$. Theorem 17(ii) also states that $\mathbf{E}_\circ(G)$ is a singleton. All the facts stated in Theorem 17 gain particular significance once noticed that there are many strongly heterogeneous games. Let us clarify this point.

Let I be a set of players and $(X_i)_{i\in I}$ be a family of strategy sets. We denote by $\mathcal{G}(I, (X_i)_{i\in I})$ the set of games whose set of players is I and where, for every $i \in I$, the set of strategies of players i is X_i. We also denote by $\mathcal{H}(I, (X_i)_{i\in I})$ [$\mathcal{SH}(I, (X_i)_{i\in I})$] the set of [strongly] heterogeneous games in $\mathcal{G}(I, (X_i)_{i\in I})$. The next proposition shows that $\mathcal{H}(I, (X_i)_{i\in I})$ and $\mathcal{SH}(I, (X_i)_{i\in I})$ are very large subsets of $\mathcal{G}(I, (X_i)_{i\in I})$. Of course, $\mathcal{SH}(I, (X_i)_{i\in I}) \subseteq \mathcal{H}(I, (X_i)_{i\in I})$. In order to formally state the proposition, we need some preliminary work.

Let φ be a bijective function from $\{1, \dots, |X|\}$ to X. Using φ, we can naturally build a bijection Ψ from $\mathcal{G}(I, (X_i)_{i\in I})$ to $\mathbb{R}^{|I||X|}$ by defining, for every $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle \in \mathcal{G}(I, (X_i)_{i\in I})$, $\Psi(G)$ as the element of $\mathbb{R}^{|I||X|}$ such that, for every $j \in \{1, \dots, |I||X|\}$, the j-th component of $\Psi(G)$ is $u_r(\varphi(j - (r-1)|X|))$, where $r \in \{1, \dots, |I|\}$ is such that $j \in \{(r-1)|X| + 1, \dots, (r-1)|X| + |X|\}$. We can then consider on $\mathcal{G}(I, (X_i)_{i\in I})$ the topology and the measure respectively induced by the euclidean topology and the Lebesgue measure on $\mathbb{R}^{|I||X|}$ through Ψ. The next proposition, proved in the appendix, holds.

Proposition 18. *Let I be a finite set with $|I| \geqslant 2$, and $(X_i)_{i\in I}$ be a family of finite and nonempty sets. Then, $\mathcal{H}(I, (X_i)_{i\in I})$ and $\mathcal{SH}(I, (X_i)_{i\in I})$ are open subsets of $\mathcal{G}(I, (X_i)_{i\in I})$ having full measure.*

We end the section by describing some consequences of Theorem 17. We know that $\mathbf{D}(G) = \mathbf{N}(G)$ implies $\mathbf{D}_\bullet(G) = \mathbf{N}(G)$. The next proposition explains, among other things, that the conditions $\mathbf{D}(G) = \mathbf{N}(G)$ and $\mathbf{D}_\bullet(G) = \mathbf{N}(G)$ are in fact equivalent for heterogeneous games.

Proposition 19. *Let G be a game. If G is heterogeneous, then the following facts are equivalent:*

(i) $\mathbf{D}_\bullet(G) = \mathbf{N}(G)$,

(ii) $\mathbf{D}_\bullet(G) = \mathbf{D}(G)$,

(iii) $\mathbf{D}(G) = \mathbf{N}(G)$.

If G is strongly heterogeneous, then the following facts are equivalent:

(iv) $\mathbf{E}_\bullet(G) = \mathbf{N}(G)$,

(v) $\mathbf{E}_\bullet(G) = \mathbf{E}(G)$,

(vi) $\mathbf{E}(G) = \mathbf{N}(G)$.

The following result specializes to the case of games with two players. Recall that a two-player game $G = \langle\{1, 2\}, (X_1, X_2), (u_1, u_2)\rangle$ is strictly competitive if, for every $x, y \in X$, we have that $u_1(x) \geqslant u_1(y)$ if and only if $u_2(y) \geqslant u_2(x)$.

Proposition 20. *Let $G = \langle\{1, 2\}, (X_1, X_2), (u_1, u_2)\rangle$ be a two-player game. Then the following facts hold.*

(i) *If G is strictly competitive, then $\mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G)$.*

(ii) *If G is heterogeneous, then $|\mathbf{D}_\bullet(G)| \leqslant \min\{|X_1|, |X_2|\}$ and $|\mathbf{D}_\circ(G)| \leqslant \max\left\{1, \frac{\min\{|X_1|,|X_2|\}}{2}\right\}$.*

(iii) *If G is strongly heterogeneous, then $|\mathbf{E}_\bullet(G)| \leqslant \min\{|X_1|, |X_2|\}$ and $|\mathbf{E}_\circ(G)| = 1$.*

(iv) *If G is strongly heterogeneous and strictly competitive, then there exists $x^* \in X$ such that $\mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = \{x^*\}$.*

(v) *If G is strongly heterogeneous and strictly competitive and $\mathbf{N}(G) \neq \varnothing$, then there exists $x^* \in X$ such that $\mathbf{N}(G) = \mathbf{E}(G) = \mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = \{x^*\}$.*

By Proposition 20(ii), we deduce, in particular, that if $G = \langle \{1, 2\}, (X_1, X_2), (u_1, u_2)\rangle$ is a heterogeneous two-player game with $\min\{|X_1|, |X_2|\} \leqslant 3$, then $|\mathbf{D}_\circ(G)| = 1$. Moreover, by Propositions 15 and 20(iii), if G is a heterogeneous game in which each player has at most two strategies, then $|\mathbf{D}_\circ(G)| = 1$.

The proofs of Propositions 19 and 20 are in the appendix.

8 Computing the solutions

In this section, we describe an efficient method for computing all the solution concepts introduced in Definition 4.

Let X be a nonempty and finite set with $|X| \geqslant 2$. A network on X is a triple $N = (X, X^2_*, c)$, where c is a function from X^2_* to $\mathbb{R}$. Note that $X^2_* \neq \varnothing$ since $|X| \geqslant 2$. The set of networks on X is denoted by $\mathcal{N}$. A network solution is a function from $\mathcal{N}$ to the set of nonempty subsets of X, that is, a procedure for selecting some of the vertices of any given network on X. Network solutions (as well as procedures for ranking the vertices of a network) are extensively studied in the literature (Laslier 1997; Langville and Meyer 2012; González-Díaz et al. 2014).

Based on the Schulze method introduced by Schulze (2011), Bubboloni and Gori (2018) propose a special network solution called Schulze network solution. Let us recall its definition. Let $N = (X, X^2_*, c) \in \mathcal{N}$. A path in N is a sequence $(x^j)_{j=1}^m$, where $m \geqslant 2$, $x^1, \dots, x^m$ are distinct elements of X. Consider now $x, y \in X$ distinct. A path from x to y in N is a path $(x^j)_{j=1}^m$ in N such that $x^1 = x$ and $x^m = y$. The set of paths from x to y in N is denoted by $\Gamma(N, x, y)$. Note that $\Gamma(N, x, y)$ is nonempty and finite. Given $\gamma = (x^j)_{j=1}^m \in \Gamma(N, x, y)$, let

$$\delta^N(\gamma) \coloneqq \min\left\{c(x^j, x^{j+1}) : j \in \{1, ..., m-1\}\right\}.$$

Define then

$$s^N_{xy} \coloneqq \max\left\{\delta^N(\gamma) : \gamma \in \Gamma(N, x, y)\right\}.$$

The Schulze network solution, denoted by **Sch**, is defined, for every $N \in \mathcal{N}$, by

$$\mathbf{Sch}(N) \coloneqq \left\{x \in X : s^N_{xy} \geqslant s^N_{yx} \text{ for all } y \in X \backslash \{x\}\right\}.$$

We emphasize that an algorithm of polynomial time in the size of $|X|$ for the computation of $\mathbf{Sch}(N)$ can be deduced from the algorithm described in Schulze (2011).

Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I}\rangle$ be a non-trivial game. Thus, $|X| \geqslant 2$, $\mathscr{D}(G) \neq \varnothing$ and $\mathscr{E}(G) \neq \varnothing$. We can then consider the two numbers

$$d(G) \coloneqq \min\{-1, \min(\mathscr{D}(G)) - 1\},$$
$$e(G) \coloneqq \min\{-1, \min(\mathscr{E}(G)) - 1\}.$$

We stress here that the crucial property of $d(G)$ is that it is a negative number smaller than the minimum of $\mathscr{D}(G)$. Analogously, the crucial property of $e(G)$ is that it is a negative number smaller than the minimum of $\mathscr{E}(G)$.[14]

We then associate with G the functions $c^{\mathbf{D}_\bullet}_G : X^2_* \to \mathbb{R}$, $c^{\mathbf{D}_\circ}_G : X^2_* \to \mathbb{R}$, $c^{\mathbf{E}_\bullet}_G : X^2_* \to \mathbb{R}$, $c^{\mathbf{E}_\circ}_G : X^2_* \to \mathbb{R}$, defined,

[14] By Lemma 28 proved in the appendix, we can actually deduce the equality $e(G) = \min(\mathscr{E}(G)) - 1$.

for every $(x, y) \in X_*^2$, as

$$c_G^{\mathbf{D}_\bullet}(x,y) = \begin{cases} u_{i(x,y)}(x) - u_{i(x,y)}(y) & \text{if } (x,y) \in D_*(G) \text{ and } u_{i(x,y)}(x) - u_{i(x,y)}(y) > 0, \\ d(G) & \text{otherwise,} \end{cases}$$

$$c_G^{\mathbf{D}_\circ}(x,y) = \begin{cases} u_{i(x,y)}(x) - u_{i(x,y)}(y) & \text{if } (x,y) \in D_*(G), \\ d(G) & \text{otherwise,} \end{cases}$$

$$c_G^{\mathbf{E}_\bullet}(x,y) = \begin{cases} u_{i(x,y)}(x) - u_{i(x,y)}(y) & \text{if } (x,y) \in E_*(G) \text{ and } u_{i(x,y)}(x) - u_{i(x,y)}(y) > 0, \\ e(G) & \text{otherwise,} \end{cases}$$

$$c_G^{\mathbf{E}_\circ}(x,y) = \begin{cases} u_{i(x,y)}(x) - u_{i(x,y)}(y) & \text{if } (x,y) \in E_*(G), \\ e(G) & \text{otherwise.} \end{cases}$$

We finally associate with G the networks

$$N^{\mathbf{D}_\bullet}(G) = (X, X_*^2, c_G^{\mathbf{D}_\bullet}), \quad N^{\mathbf{D}_\circ}(G) = (X, X_*^2, c_G^{\mathbf{D}_\circ}),$$

$$N^{\mathbf{E}_\bullet}(G) = (X, X_*^2, c_G^{\mathbf{E}_\bullet}), \quad N^{\mathbf{E}_\circ}(G) = (X, X_*^2, c_G^{\mathbf{E}_\circ}).$$

We are now ready to state the third main result of the paper. The proof is in the appendix.

Theorem 21. *Let G be a non-trivial game. Then $\mathbf{D}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{D}_\bullet}(G))$, $\mathbf{D}_\circ(G) = \mathbf{Sch}(N^{\mathbf{D}_\circ}(G))$, $\mathbf{E}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{E}_\bullet}(G))$, $\mathbf{E}_\circ(G) = \mathbf{Sch}(N^{\mathbf{E}_\circ}(G))$.*

9 Two-player two-strategy games

In this section we focus on the special case of two-player two-strategy games. By Proposition 15, for every two-player two-strategy game G, we have that $\mathbf{D}_\bullet(G) = \mathbf{E}_\bullet(G)$ and $\mathbf{D}_\circ(G) = \mathbf{E}_\circ(G)$. Thus, in what follows, we refer to $\mathbf{D}_\bullet(G)$ and $\mathbf{D}_\circ(G)$ only.

9.1 Coordination games

Consider the two-player game

$$G = \begin{array}{|c|c|} \hline a_1, a_2 & d_1, c_2 \\ \hline c_1, d_2 & b_1, b_2 \\ \hline \end{array},$$

where, $a_1 > c_1$, $a_2 > c_2$, $b_1 > d_1$, $b_2 > d_2$. This game represents the general coordination game. We have $\mathbf{N}(G) = \{(1,1),(2,2)\}$. Applying Theorem 21, we get $\mathbf{D}_\bullet(G) = \{(1,1),(2,2)\}$ and

- $\mathbf{D}_\circ(G) = \{(1,1)\}$ if and only if $\min\{a_1 - c_1, a_2 - c_2\} > \min\{b_1 - d_1, b_2 - d_2\}$,
- $\mathbf{D}_\circ(G) = \{(2,2)\}$ if and only if $\min\{a_1 - c_1, a_2 - c_2\} < \min\{b_1 - d_1, b_2 - d_2\}$,
- $\mathbf{D}_\circ(G) = \{(1,1),(2,2)\}$ if and only if $\min\{a_1 - c_1, a_2 - c_2\} = \min\{b_1 - d_1, b_2 - d_2\}$.

As a particular case, consider the symmetric coordination game

$$G = \begin{array}{|c|c|} \hline a, a & d, c \\ \hline c, d & b, b \\ \hline \end{array},$$

where, $a > c$ and $b > d$. We have

- $\mathbf{D}_\circ(G) = \{(1,1)\}$ if and only if $a - c > b - d$,
- $\mathbf{D}_\circ(G) = \{(2,2)\}$ if and only if $a - c < b - d$,
- $\mathbf{D}_\circ(G) = \{(1,1),(2,2)\}$ if and only if $a - c = b - d$.

Then, $\mathbf{D}_\circ(G)$ selects between the two Nash equilibria the one from which a deviation causes a greater loss. This is equivalent to saying that $\mathbf{D}_\circ(G)$ excludes the Nash equilibrium that is strictly risk-dominated by the other, if any (Harsanyi and Selten, 1988, Lemma 5.4.4).

Consider now the following asymmetric coordination game

$$G = \begin{array}{|c|c|}\hline a_1, a_2 & d_1, c_2 \\ \hline c_1, d_2 & a_2, a_1 \\ \hline \end{array},$$

where $a_1 > a_2 > \max\{d_1, c_2\}$ and $\min\{d_1, c_2\} > \max\{c_1, d_2\}$. The game G corresponds to the Bach or Stravinsky game, where

$$X_1 = X_2 = \{1 = \text{Bach},\ 2 = \text{Stravinsky}\},$$

player 1 prefers Bach to Stravinsky, and player 2 prefers Stravinsky to Bach. We have

- $\mathbf{D}_\circ(G) = \{(1,1)\}$ if and only if $d_1 > c_2$,
- $\mathbf{D}_\circ(G) = \{(2,2)\}$ if and only if $d_1 < c_2$,
- $\mathbf{D}_\circ(G) = \{(1,1),(2,2)\}$ if and only if $d_1 = c_2$.

Thus, $\mathbf{D}_\circ$ selects as a unique strategy profile $(\text{Bach}, \text{Bach})$ [$(\text{Stravinsky}, \text{Stravinsky})$] if and only if in the scenario where each player goes to her favorite concert alone, the player who prefers Bach [Stravinsky] is more satisfied than the other.

9.2 Anti-coordination games

Consider the two-player game

$$G = \begin{array}{|c|c|}\hline a_1, a_2 & d_1, c_2 \\ \hline c_1, d_2 & b_1, b_2 \\ \hline \end{array},$$

where, $c_1 > a_1$, $d_2 > b_2$, $d_1 > b_1$, $c_2 > a_2$. This game represents the general anti-coordination game. We have $\mathbf{N}(G) = \{(1,2),(2,1)\}$. Applying Theorem 21, we get $\mathbf{D}_\bullet(G) = \{(1,2),(2,1)\}$ and

- $\mathbf{D}_\circ(G) = \{(1,2)\}$ if and only if $\min\{c_2 - a_2, d_1 - b_1\} > \min\{c_1 - a_1, d_2 - b_2\}$,
- $\mathbf{D}_\circ(G) = \{(2,1)\}$ if and only if $\min\{c_2 - a_2, d_1 - b_1\} < \min\{c_1 - a_1, d_2 - b_2\}$,
- $\mathbf{D}_\circ(G) = \{(1,2),(2,1)\}$ if and only if $\min\{c_2 - a_2, d_1 - b_1\} = \min\{c_1 - a_1, d_2 - b_2\}$.

As a particular case, assume further that $\min\{c_1 - a_1, c_2 - a_2\} > \max\{d_2 - b_2, d_1 - b_1\}$. Note that, under these additional assumptions, the game corresponds to the Chicken game supposing

$$X_1 = X_2 = \{1 = \text{Straight},\ 2 = \text{Swerve}\}.$$

We have

- $\mathbf{D}_\circ(G) = \{(1,2)\}$ if and only if $d_1 - b_1 > d_2 - b_2$,
- $\mathbf{D}_\circ(G) = \{(2,1)\}$ if and only if $d_1 - b_1 < d_2 - b_2$,
- $\mathbf{D}_\circ(G) = \{(1,2),(2,1)\}$ if and only if $d_1 - b_1 = d_2 - b_2$.

Thus, for example, the equilibrium selected by $\mathbf{D}_\circ(G)$ is $(\text{Straight}, \text{Swerve})$ if deviating from the strategy profile $(\text{Swerve}, \text{Swerve})$ is more profitable for player 1 than for player 2.

9.3 Discoordination games

Consider the game

$$G = \begin{array}{|c|c|} \hline a_1, a_2 & d_1, d_2 \\ \hline b_1, b_2 & c_1, c_2 \\ \hline \end{array},$$

where, $a_1 > b_1$, $b_2 > c_2$, $c_1 > d_1$, $d_2 > a_2$. This game can be seen as a generalization of the matching-pennies game. We have $\mathbf{N}(G) = \varnothing$. Consider the set

$$\Gamma = \{a_1 - b_1,\ b_2 - c_2,\ c_1 - d_1,\ d_2 - a_2\}$$

and assume that there exists $v^* \in \Gamma$ such that, for every $v \in \Gamma \backslash \{v^*\}$, $v^* < v$. Applying Theorem 21, we deduce that:

- if $v^* = a_1 - b_1$, then $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \{(2,1)\}$,
- if $v^* = b_2 - c_2$, then $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \{(2,2)\}$,
- if $v^* = c_1 - d_1$, then $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \{(1,2)\}$,
- if $v^* = d_2 - a_2$, then $\mathbf{D}_\bullet(G) = \mathbf{D}_\circ(G) = \{(1,1)\}$.

Thus, $\mathbf{D}_\bullet(G)$ and $\mathbf{D}_\circ(G)$ both select as unique strategy profile the one for which the profitable unilateral deviation generates the smallest payoff variation.

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A Proofs

In this appendix, we provide the proofs of Theorems 13, 17 and 21, and Propositions 5, 18, 19 and 20. Those proofs are sometimes based on some preliminary propositions.

A.1 Proof of Proposition 5

Proposition 22. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game and let $(Y_i)_{i\in I}$ be such that, for every $i \in I$, $\varnothing \neq Y_i \subseteq X_i$. Assume that, for every $i \in I$ and $s \in X_i \backslash Y_i$, s is never a best response in G for player i. Defined $G' = \mathfrak{R}(G, (Y_i)_{i\in I})$ and $Y = \pi(G') = \prod_{j\in I} Y_j$, we have that*

(i) *for every $i \in I$ and $\sigma \in \prod_{j\in I\backslash\{i\}} Y_j$, $\mathcal{B}_i^G(\sigma) = \mathcal{B}_i^{G'}(\sigma)$;*

(ii) *$D(G) \cap Y^2 = D(G')$.*

Proof. (i) Let $i \in I$ and $\sigma \in \prod_{j\in I\backslash\{i\}} Y_j$. Observe that $\sigma \in \prod_{j\in I\backslash\{i\}} X_j$. We have

$$\mathcal{B}_i^{G'}(\sigma) = \operatorname*{argmax}_{s\in Y_i} u_i(s,\sigma) = \operatorname*{argmax}_{s\in X_i} u_i(s,\sigma) = \mathcal{B}_i^G(\sigma),$$

where the second equality holds because each element in $X_i \backslash Y_i$ is never a best response in G for player i and then, in particular, it cannot be a best response to σ.

(ii) Let us first prove that $D(G) \cap Y^2 \subseteq D(G')$. Let $(x,y) \in D(G) \cap Y^2$. Then $x, y \in Y$ and there exists $i \in I$ such that

$$x_{-i} = y_{-i}, \quad x_i \in \mathcal{B}_i^G(y_{-i}), \quad u_i(x) > u_i(y). \tag{9}$$

By (i) and the fact that $y_{-i} \in \prod_{j\in I\backslash\{i\}} Y_j$, we have that (9) is equivalent to

$$x_{-i} = y_{-i}, \quad x_i \in \mathcal{B}_i^{G'}(y_{-i}), \quad u_i(x) > u_i(y).$$

Then, we conclude that $(x, y) \in D(G')$.

Next, let us prove that $D(G') \subseteq D(G) \cap Y^2$. Let $(x, y) \in D(G')$. Then $x, y \in Y$ and there exists $i \in I$ such that

$$x_{-i} = y_{-i}, \quad x_i \in \mathcal{B}_i^{G'}(y_{-i}), \quad u_i(x) > u_i(y). \tag{10}$$

By (i) and since $y_{-i} \in \prod_{j \in I \setminus \{i\}} Y_j$, we have that (10) is equivalent to

$$x_{-i} = y_{-i}, \quad x_i \in \mathcal{B}_i^{G}(y_{-i}), \quad u_i(x) > u_i(y).$$

Then, we conclude that $(x, y) \in D(G) \cap Y^2$. □

Proof of Proposition 5. Let $G' = \mathfrak{R}(G, (Y_i)_{i \in I}) = \langle I, (Y_i)_{i \in I}, (u_i)_{i \in I} \rangle$ and $Y = \pi(G') = \prod_{j \in I} Y_j$.

Let us first prove that $\mathbf{D}(G') \subseteq \mathbf{D}(G)$. Let $x \in \mathbf{D}(G')$ and prove that $x \in \mathbf{D}(G)$ showing that, for every $y \in X$ such that $(y, x) \in D(G)^\tau$, we have $(x, y) \in D(G)^\tau$.

Let $y \in X$ be such that $(y, x) \in D(G)^\tau$. Thus, there exists $m \geqslant 2$ and a sequence $(y^j)_{j=1}^m$ in X such that $y^1 = y$, $y^m = x$ and, for every $j \in \{1, \ldots, m-1\}$, $(y^j, y^{j+1}) \in D(G)$. We know that $x = y^m \in Y$. Assume, by contradiction, that there exists $j \in \{1, \ldots, m-1\}$ such that $y^j \notin Y$. We can assume that j is the largest index for which $y^j \notin Y$. Thus, $y^j \notin Y$, $y^{j+1} \in Y$ and $(y^j, y^{j+1}) \in D(G)$. We know that there exists $i \in I$ such that

$$y_{-i}^j = y_{-i}^{j+1}, \quad y_i^j \in \mathcal{B}_i^G(y_{-i}^{j+1}), \quad u_i(y^j) > u_i(y^{j+1}).$$

From $y^j \notin Y$, $y^{j+1} \in Y$ and $y_{-i}^j = y_{-i}^{j+1}$, we deduce that $y_i^j \in X_i \backslash Y_i$ and then y_i^j is never a best response in G for player i. However, we have $y_i^j \in \mathcal{B}_i^G(y_{-i}^{j+1})$, a contradiction. We then conclude that the sequence $(y^j)_{j=1}^m$ is actually a sequence in Y. As a consequence, by Proposition 22(ii), we have that, for every $j \in \{1, \ldots, m-1\}$, $(y^j, y^{j+1}) \in D(G')$, and so $(y, x) \in D(G')^\tau$. Since $x \in \mathbf{D}(G')$, we know that $(x, y) \in D(G')^\tau$. Thus, there exists $s \geqslant 2$ and a sequence $(z^j)_{j=1}^s$ in Y such that $z^1 = x$, $z^s = y$ and, for every $j \in \{1, \ldots, s-1\}$, $(z^j, z^{j+1}) \in D(G')$. In particular, the sequence $(z^j)_{j=1}^s$ is also a sequence in X and, by Proposition 22(ii), for every $j \in \{1, \ldots, s-1\}$, $(z^j, z^{j+1}) \in D(G)$. Thus, $(x, y) \in D(G)^\tau$, as desired.

Let us prove now that $\mathbf{D}(G) \subseteq \mathbf{D}(G')$. Let $x \in \mathbf{D}(G)$ and prove that $x \in \mathbf{D}(G')$ showing that $x \in Y$ and that, for every $y \in Y$ such that $(y, x) \in D(G')^\tau$, we have that $(x, y) \in D(G')^\tau$.

Assume by contradiction that $x \notin Y$. Thus, there exists $i \in I$ such that $x_i \in X_i \backslash Y_i$. Consider then $y_i \in \mathcal{B}_i^G(x_{-i})$. Since x_i is never a best response in G for player i, we have that $x_i \notin \mathcal{B}_i^G(x_{-i})$ and, in particular, $y_i \neq x_i$. Let us set $y = (y_i, x_{-i}) \in X$. Since

$$y_{-i} = x_{-i}, \quad y_i \in \mathcal{B}_i^G(x_{-i}), \quad u_i(y) > u_i(x),$$

we have that $(y, x) \in D(G)$. Thus, $(y, x) \in D(G)^\tau$ and, since $x \in \mathbf{D}(G)$, we know that $(x, y) \in D(G)^\tau$. Thus, there exists $s \geqslant 2$ and a sequence $(z^j)_{j=1}^s$ in X such that $z^1 = x$, $z^s = y$ and, for every $j \in \{1, \ldots, s-1\}$, $(z^j, z^{j+1}) \in D(G)$. Therefore, there exists $j \in \{1, \ldots, s-1\}$ such that $z_i^j = x_i$ and $z_i^{j+1} \neq x_i$. That implies, in particular, that

$$z_{-i}^j = z_{-i}^{j+1}, \quad z_i^j = x_i \in \mathcal{B}_i^G(z_{-i}^{j+1}),$$

and that is a contradiction since x_i is never a best response for player i. We then conclude that $x \in Y$.

Consider now $y \in Y$ such that $(y, x) \in D(G')^\tau$, and prove that $(x, y) \in D(G')^\tau$. Since $(y, x) \in D(G')^\tau$, we know that there exists $m \geqslant 2$ and a sequence $(y^j)_{j=1}^m$ in Y such that $y^1 = y$, $y^m = x$ and, for every $j \in \{1, \ldots, m-1\}$, $(y^j, y^{j+1}) \in D(G')$. In particular, $(y^j)_{j=1}^m$ is a sequence in X such that $y^1 = y$, $y^m = x$ and, by Proposition 22(ii), for every $j \in \{1, \ldots, m-1\}$, $(y^j, y^{j+1}) \in D(G)$. Thus, we have that $(y, x) \in D(G)^\tau$. Since $x \in \mathbf{D}(G)$, we know that $(x, y) \in D(G)^\tau$. Thus, there exists $s \geqslant 2$ and a sequence $(z^j)_{j=1}^s$ in X such that $z^1 = x$, $z^s = y$ and, for every $j \in \{1, \ldots, s-1\}$, $(z^j, z^{j+1}) \in D(G)$. We know that $z^s = y \in Y$. Assume, by contradiction, that there exists $j \in \{1, \ldots, s-1\}$ such that $z^j \notin Y$. We can assume that j is the largest index for which $z^j \notin Y$. Thus, $z^j \notin Y$, $z^{j+1} \in Y$ and $(z^j, z^{j+1}) \in D(G)$. We know that there exists $i \in I$ such that

$$z_{-i}^j = z_{-i}^{j+1}, \quad z_i^j \in \mathcal{B}_i^G(z_{-i}^{j+1}), \quad u_i(z^j) > u_i(z^{j+1}).$$

From $z^j \notin Y$, $z^{j+1} \in Y$, and $z_{-i}^j = z_{-i}^{j+1}$, we deduce that $z_i^j \in X_i \backslash Y_i$. Then, z_i^j is a never best response in G for player i. However, we have $z_i^j \in \mathcal{B}_i^G(z_{-i}^{j+1})$, a contradiction. We then conclude then that the sequence $(z^j)_{j=1}^m$ is actually a sequence in Y and therefore, by Proposition 22(ii), we have that, for every $j \in \{1, \ldots, s-1\}$, $(z^j, z^{j+1}) \in D(G')$. Thus, $(x, y) \in D(G')^\tau$, as desired. □

A.2 Proof of Theorem 13

Lemma 23. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game, $i \in I$ and $\sigma \in X_{-i}$ with $|\mathcal{B}_i^G(\sigma)| \geqslant 2$. Then $0 \in \mathscr{D}(G)$*

Proof. Let $s, s' \in \mathcal{B}_i^G(\sigma)$ be such that $s \neq s'$. Thus, $s' \in \mathcal{C}_i^G((s,\sigma))$ and then $((s',\sigma),(s,\sigma)) \in D_*(G)$. Since $u_i(s',\sigma) = u_i(s,\sigma)$, we conclude that $u_i(s',\sigma) - u_i(s,\sigma) = 0 \in \mathscr{D}(G)$. □

Lemma 24. *Let G be a game and $t \in \mathscr{D}(G)$ with $t < 0$. Then $-t \in \mathscr{D}(G)$.*

Proof. Let $(x,y) \in D_*(G)$ be such that $u_i(x) - u_i(y) = t$, where $i = i(x,y)$. Thus, $x_{-i} = y_{-i}$, $x_i \neq y_i$ and $x_i \in \mathcal{C}_i^G(y)$. Since $u_i(x) < u_i(y)$, we deduce that $y_i \in \mathcal{B}_i^G(x_{-i})$ and then $y_i \in \mathcal{C}_i^G(x)$. We then have $(y,x) \in D_*(G)$, and so $u_i(y) - u_i(x) = -t \in \mathscr{D}(G)$. □

Lemma 25. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game, $i \in I$ be such that $|X_i| \geqslant 2$, and $\sigma \in X_{-i}$ be such that $|\mathcal{B}_i^G(\sigma)| = 1$. Then there exists $t > 0$ such that $t, -t \in \mathscr{D}(G)$.*

Proof. Let $s \in X_i$ such that $\mathcal{B}_i^G(\sigma) = \{s\}$. Since $|X_i| \geqslant 2$, we have that there exists $s' \in \mathcal{C}_i^G((s,\sigma))$ and then $((s',\sigma),(s,\sigma)) \in D_*(G)$. Since $s' \neq s$, we deduce that $u_i(s',\sigma) < u_i(s,\sigma)$ and $u_i(s',\sigma) - u_i(s,\sigma) \in \mathscr{D}(G)$. However, we surely have $s \in \mathcal{C}_i^G((s',\sigma))$ and then also $((s,\sigma),(s',\sigma)) \in D_*(G)$. As a consequence, $u_i(s,\sigma) - u_i(s',\sigma) \in \mathscr{D}(G)$. Setting then $t = u_i(s,\sigma) - u_i(s',\sigma) > 0$, we get $t, -t \in \mathscr{D}(G)$. □

Lemma 26. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a heterogeneous game, $i \in I$ and $\sigma \in X_{-i}$. Then, $|\mathcal{B}_i^G(\sigma)| = 1$.*

Proof. Assume by contradiction that there are $s, s' \in \mathcal{B}_i^G(\sigma)$ with $s \neq s'$. Then $((s',\sigma),(s,\sigma)) \in D_*(G)$ and $((s,\sigma),(s',\sigma)) \in D_*(G)$. However, $u_i(s',\sigma) - u_i(s,\sigma) = u_i(s,\sigma) - u_i(s',\sigma) = 0$. Thus, $|\mathscr{D}(G)| < |D_*(G)|$. That contradicts the fact that G is heterogeneous. □

Lemma 27. *Let G be a non-trivial game. Then the following facts hold true:*

(i) *if $|\mathscr{D}(G)| = 1$, then $\mathscr{D}(G) = \{0\}$;*

(ii) *if $|\mathscr{D}(G)| \geqslant 2$, then there exists $t > 0$ such that $t \in \mathscr{D}(G)$;*

(iii) *if G is heterogeneous, then there exists $t > 0$ such that $t, -t \in \mathscr{D}(G)$.*

Proof. We set $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$.

(i) Assume that $|\mathscr{D}(G)| = 1$. Since G is non-trivial, there exists $i \in I$ such that $|X_i| \geqslant 2$. Consider then $\sigma \in X_{-i}$. By Lemma 25, we know that it cannot be $|\mathcal{B}_i^G(\sigma)| = 1$. Thus, $|\mathcal{B}_i^G(\sigma)| \geqslant 2$ and by Lemma 23, $0 \in \mathscr{D}(G)$. As a consequence, $\mathscr{D}(G) = \{0\}$.

(ii) Assume that $|\mathscr{D}(G)| \geqslant 2$. Suppose by contradiction that, for every $t \in \mathscr{D}(G)$, $t \leqslant 0$. By Lemma 24, none of the elements of $\mathscr{D}(G)$ can be negative. Thus, $\mathscr{D}(G) = \{0\}$, and then $|\mathscr{D}(G)| = 1$, a contradiction.

(iii) Assume that G is heterogeneous. Since G is non-trivial, there exists $i \in I$ such that $|X_i| \geqslant 2$. Consider then any $\sigma \in X_{-i}$. Since G is heterogeneous, by Lemma 26, $|\mathcal{B}_i^G(\sigma)| = 1$. By Lemma 25 we conclude that there exists $t > 0$ such that $t, -t \in \mathscr{D}(G)$. □

Lemma 28. *Let G be a game. Then the following facts hold true:*

(i) *if $t \in \mathscr{E}(G)$, then $-t \in \mathscr{E}(G)$;*

(ii) *if $|\mathscr{E}(G)| = 1$, then $\mathscr{E}(G) = \{0\}$;*

(iii) *if $|\mathscr{E}(G)| \geqslant 2$, then there exists $t \in \mathscr{E}(G)$ such that $t > 0$;*

(iv) *if G is strongly heterogeneous, then $0 \notin \mathscr{E}(G)$.*

(v) *if G is non-trivial and strongly heterogeneous, then there exists $t > 0$ such that $t, -t \in \mathscr{E}(G)$.*

Proof. (i) Assume that $t \in \mathscr{E}(G)$. Then there exists $(x, y) \in E_*(G)$ such that $u_{i(x,y)}(x) - u_{i(x,y)}(y) = t$. Since $(y, x) \in E_*(G)$ and $i(y, x) = i(x, y)$ we deduce that $-t = u_{i(y,x)}(y) - u_{i(y,x)}(x) \in \mathscr{E}(G)$.

(ii) Assume that $|\mathscr{E}(G)| = 1$. Thus, there exists $t \in \mathbb{R}$ such that $\mathscr{E}(G) = \{t\}$. Suppose, by contradiction, that $t \neq 0$. Then, we have $-t \neq t$. By (i) we know that $-t \in \mathscr{E}(G)$ and so $|\mathscr{E}(G)| \geqslant 2$, a contradiction.

(iii) Assume that $|\mathscr{E}(G)| \geqslant 2$. Thus, there exists $t \in \mathscr{E}(G)$ such that $t \neq 0$. If $t > 0$, then we have completed the proof. If instead $t < 0$, then, by (i), we know that $-t \in \mathscr{E}(G)$ and since $-t > 0$ we have completed the proof.

(iv) Assume that G is strongly heterogeneous. Suppose by contradiction that $0 \in \mathscr{E}(G)$. Thus, there exists $(x, y) \in E_*(G)$ such that $u_{i(x,y)}(x) - u_{i(x,y)}(y) = 0$. Since $i(x, y) = i(y, x)$, we have $u_{i(y,x)}(y) - u_{i(y,x)}(x) = 0$. Then (x, y) and (y, x) are two elements of $E_*(G)$ associated with the same element of $\mathscr{E}(G)$. Thus, $|\mathscr{E}(G)| < |E_*(G)|$ and G is not strongly heterogeneous, a contradiction.

(v) Assume that G is non-trivial and strongly heterogeneous. Thus, $\mathscr{E}(G) \neq \varnothing$ and, by (iv), we know that $0 \notin \mathscr{E}(G)$. Thus, there exists $t \in \mathscr{E}(G)$ with $t \neq 0$. Thus, we conclude applying (i). □

Proposition 29. *Let G be a non-trivial game. Then, for every $t < \min(\mathscr{D}(G))$, $D_t(G) = D_*(G)$, $D_t(G) = D_{d(G)}(G)$ and $\mathbf{D}_t(G) = \mathbf{D}_{d(G)}(G)$.*

Proof. Since G is non-trivial, $\mathscr{D}(G)$ is nonempty (and finite), and so $\min(\mathscr{D}(G))$ is well defined. Let $t < \min(\mathscr{D}(G))$. We know that $D_t(G) \subseteq D_*(G)$. Consider now $(x, y) \in D_*(G)$. Since $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G)$ and $t < \min(\mathscr{D}(G))$, we have $u_{i(x,y)}(x) - u_{i(x,y)}(y) > t$. Thus, we conclude that $(x, y) \in D_t(G)$. That also proves that $D_*(G) \subseteq D_t(G)$. Thus, $D_t(G) = D_*(G)$.

Since, in particular, $d(G) < \min(\mathscr{D}(G))$, we have $D_t(G) = D_*(G) = D_{d(G)}(G)$. Therefore, $\mathbf{D}_t(G) = \mathbf{A}(\pi(G), D_t(G)) = \mathbf{A}(\pi(G), D_{d(G)}(G)) = \mathbf{D}_{d(G)}(G)$. □

Proposition 30. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game. Then $\mathscr{A}(X, E_*(G)) = \{X\}$ and $\mathbf{A}(X, E_*(G)) = X$.*

Proof. If G is trivial, then $E_*(G) = \varnothing$ and the desired equalities have already been observed.

Assume now that G is non-trivial. Let us prove $\mathscr{A}(X, E_*(G)) = \{X\}$. This is equivalent to prove that, for every $x, y \in X$ with $x \neq y$, $(x, y) \in E_*(G)^\tau$. Consider then $x, y \in X$ with $x \neq y$ and prove that there is a path from x to y in $(X, E_*(G))$. For every $j \in \{0, \dots, |I|\}$, let $x^j \in X$ be defined as follows: for every $i \in I$ with $i > j$, $x^j_i = x_i$; for every $i \in I$ with $i \leqslant j$, $x^j_i = y_i$. We have $x^0 = x$ and $x^{|I|} = y$. Moreover, for every $j \in \{1, \dots, |I|\}$, $x^{j-1}_{-j} = x^j_{-j}$. Then, for every $j \in \{1, \dots, |I|\}$, we have $x^{j-1} = x^j$ or $(x^{j-1}, x^j) \in E_*(G)$. Thus, the sequence $(x^j)_{j=0}^{|I|}$ admits a subsequence that is a path from x to y in $(X, E_*(G))$. Thus, $\mathscr{A}(X, E_*(G)) = \{X\}$. From $\mathscr{A}(X, E_*(G)) = \{X\}$, it immediately follows that $\mathbf{A}(X, E_*(G)) = X$. □

Proposition 31. *Let G be a non-trivial game. Then*

$$\mathbf{D}_\bullet(G) = \bigcap_{t\in(\mathscr{D}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{D}_t(G), \qquad \mathbf{D}_\circ(G) = \bigcap_{t\in\mathscr{D}(G)\cup\{d(G)\}} \mathbf{D}_t(G).$$

Proof. Since G is non-trivial, we have $D_*(G) \neq \varnothing$ and $\mathscr{D}(G) \neq \varnothing$. Assume first that $|\mathscr{D}(G)| = 1$. Thus, by Lemma 27(i), we have $\mathscr{D}(G) = \{0\}$.

- If $t \in (-\infty, 0)$, then, by Proposition 29, $\mathbf{D}_t(G) = \mathbf{D}_{d(G)}(G)$.
- If $t \in [0, +\infty)$, then $D_t(G) = \varnothing$, and so $\mathbf{D}_t(G) = \pi(G) = \mathbf{D}_0(G)$. Indeed, assume by contradiction that there exists $(x, y) \in D_t(G)$. Thus, $(x, y) \in D_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > t$. Since $(x, y) \in D_*(G)$ implies $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G)$, we get $u_{i(x,y)}(x) - u_{i(x,y)}(y) = 0$. We then deduce the contradiction $0 > t$.

Then, we have

$$\begin{aligned}\mathbf{D}_\circ(G) &= \bigcap_{t\in\mathbb{R}} \mathbf{D}_t(G) = \left(\bigcap_{t\in(-\infty,0)} \mathbf{D}_t(G)\right) \cap \left(\bigcap_{t\in[0,+\infty)} \mathbf{D}_t(G)\right)\\ &= \mathbf{D}_{d(G)}(G) \cap \mathbf{D}_0(G) = \bigcap_{t\in\mathscr{D}(G)\cup\{d(G)\}} \mathbf{D}_t(G),\end{aligned}$$

and

$$\mathbf{D}_\bullet(G) = \bigcap_{t\in\mathbb{R}_+} \mathbf{D}_t(G) = \mathbf{D}_0(G) = \bigcap_{t\in(\mathscr{D}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{D}_t(G).$$

Assume now that $|\mathscr{D}(G)| \geqslant 2$. Let $\mathscr{D}(G) = \{t_j\}_{j=1}^k$, where $k \geqslant 2$, and $t_j < t_{j+1}$ for all $j \in \{1,\dots,k-1\}$.

- If $t \in (-\infty, t_1)$, then, by Proposition 29, $\mathbf{D}_t(G) = \mathbf{D}_{d(G)}(G)$.
- Let $a, b \in \mathbb{R}$ be such that $a < b$ and $(a,b) \cap \mathscr{D}(G) = \varnothing$. Then, for every $t \in [a,b)$, $D_t(G) = D_a(G)$, and so $\mathbf{D}_t(G) = \mathbf{D}_a(G)$. Indeed, let $t \in [a,b)$, we know, by Proposition 11, that $D_t(G) \subseteq D_a(G)$. Consider now $(x,y) \in D_a(G)$. Then $(x,y) \in D_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > a$. Assume by contradiction that $(x,y) \notin D_t(G)$. Since $(x,y) \in D_*(G)$ it must be $u_{i(x,y)}(x) - u_{i(x,y)}(y) \leqslant t$. Since $t < b$, we deduce that $a < u_{i(x,y)}(x) - u_{i(x,y)}(y) < b$. Thus, $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G) \cap (a,b)$, a contradiction. We conclude then that $(x,y) \in D_t(G)$. We then get the inclusion $D_a(G) \subseteq D_t(G)$. Thus, the equality $D_t(G) = D_a(G)$ follows.
- If $t \in [t_k, +\infty)$, then $D_t(G) = \varnothing$, and so $\mathbf{D}_t(G) = \pi(G) = \mathbf{D}_{t_k}(G)$. Indeed, assume by contradiction that there exists $(x,y) \in D_t(G)$. Thus, $(x,y) \in D_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > t$. Since $(x,y) \in D_*(G)$ implies $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G)$, we get $u_{i(x,y)}(x) - u_{i(x,y)}(y) \leqslant t_k$. We then deduce the contradiction $t_k > t$.

Then, we have

$$\mathbf{D}_\circ(G) = \bigcap_{t\in\mathbb{R}} \mathbf{D}_t(G) = \left(\bigcap_{t\in(-\infty,t_1)} \mathbf{D}_t(G)\right) \cap \left(\bigcap_{j\in\{1,\dots,k-1\}} \left(\bigcap_{t\in[t_j,t_{j+1})} \mathbf{D}_t(G)\right)\right) \cap \left(\bigcap_{t\in[t_k,+\infty)} \mathbf{D}_t(G)\right)$$

$$= \mathbf{D}_{d(G)}(G) \cap \left(\bigcap_{j\in\{1,\dots,k-1\}} \mathbf{D}_{t_j}(G)\right) \cap \mathbf{D}_{t_k}(G) = \mathbf{D}_{d(G)}(G) \cap \left(\bigcap_{j\in\{1,\dots,k\}} \mathbf{D}_{t_j}(G)\right) = \bigcap_{t\in\mathscr{D}(G)\cup\{d(G)\}} \mathbf{D}_t(G).$$

In order to prove the desired result for $\mathbf{D}_\bullet(G)$, observe that, since $|\mathscr{D}(G)| \geqslant 2$, by Lemma 27(ii), we know that there exists a positive element in $\mathscr{D}(G)$. Let j^* be the smallest element of the set $\{1,2,\dots,k\}$ for which $t_{j^*} > 0$. Thus, $\mathscr{D}(G) \cap (0, t_{j^*}) = \varnothing$ and then, for every $t \in [0, t_{j^*})$, $\mathbf{D}_t(G) = \mathbf{D}_0(G)$.

If $j^* = k$, then $\mathscr{D}(G)$ as a unique positive element, say t^*. Thus,

$$\mathbf{D}_\bullet(G) = \bigcap_{t\in\mathbb{R}_+} \mathbf{D}_t(G) = \left(\bigcap_{t\in[0,t^*)} \mathbf{D}_t(G)\right) \cap \left(\bigcap_{t\in[t^*,+\infty)} \mathbf{D}_t(G)\right)$$

$$= \mathbf{D}_0(G) \cap \mathbf{D}_{t^*}(G) = \bigcap_{t\in(\mathscr{D}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{D}_t(G).$$

If instead $j^* < k$, then

$$\mathbf{D}_\bullet(G) = \bigcap_{t\in\mathbb{R}_+} \mathbf{D}_t(G) = \left(\bigcap_{t\in[0,t_{j^*})} \mathbf{D}_t(G)\right) \cap \left(\bigcap_{j\in\{j^*,\dots,k-1\}} \left(\bigcap_{t\in[t_j,t_{j+1})} \mathbf{D}_t(G)\right)\right) \cap \left(\bigcap_{t\in[t_k,+\infty)} \mathbf{D}_t(G)\right)$$

$$= \mathbf{D}_0(G) \cap \left(\bigcap_{j\in\{j^*,\dots,k-1\}} \mathbf{D}_{t_j}(G)\right) \cap \mathbf{D}_{t_k}(G) = \mathbf{D}_0(G) \cap \bigcap_{j\in\{j^*,\dots,k\}} \mathbf{D}_{t_j}(G) = \bigcap_{t\in(\mathscr{D}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{D}_t(G).$$

□

Proposition 32. *Let G be a non-trivial game. Then*

$$\mathbf{E}_\bullet(G) \coloneqq \bigcap_{t\in(\mathscr{E}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{E}_t(G), \qquad \mathbf{E}_\circ(G) \coloneqq \bigcap_{t\in\mathscr{E}(G)} \mathbf{E}_t(G).$$

Proof. Since G is non-trivial, we have $E_*(G) \neq \varnothing$ and $\mathscr{E}(G) \neq \varnothing$. Assume first that $|\mathscr{E}(G)| = 1$. Thus, by Lemma 28(ii), we have $\mathscr{E}(G) = \{0\}$.

- If $t \in (-\infty, 0)$, then $E_t(G) = E_*(G)$, and so, by Proposition 30, $\mathbf{E}_t(G) = \pi(G)$. Indeed, we know that $E_t(G) \subseteq E_*(G)$. Consider now $(x,y) \in E_*(G)$. Thus, $u_{i(x,y)}(x) - u_{i(x,y)}(y) = 0 > t$ and then $(x,y) \in E_t(G)$. We then deduce $E_*(G) \subseteq E_t(G)$, and the equality $E_t(G) = E_*(G)$ follows.
- If $t \in [0, +\infty)$, then $E_t(G) = \varnothing$, and so $\mathbf{E}_t(G) = \pi(G) = \mathbf{E}_0(G)$. Indeed, assume by contradiction that there exists $(x,y) \in E_t(G)$. Thus, $(x,y) \in E_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > t$. Since $(x,y) \in E_*(G)$ implies $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{E}(G)$, we get $u_{i(x,y)}(x) - u_{i(x,y)}(y) = 0$. We then deduce the contradiction $0 > t$.

Thus, for every $t \in \mathbb{R}$, $\mathbf{E}_t(G) = \pi(G)$. Then, we have

$$\mathbf{E}_\circ(G) = \bigcap_{t \in \mathbb{R}} \mathbf{E}_t(G) = \bigcap_{t \in \mathbb{R}} \pi(G) = \pi(G) = \mathbf{E}_0(G) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G),$$

and

$$\mathbf{E}_\bullet(G) = \bigcap_{t \in \mathbb{R}_+} \mathbf{E}_t(G) = \bigcap_{t \in \mathbb{R}_+} \pi(G) = \pi(G) = \mathbf{E}_0(G) = \bigcap_{t \in \mathscr{E}(G) \cap \mathbb{R}_+} \mathbf{E}_t(G).$$

Assume now that $|\mathscr{E}(G)| \geqslant 2$. Let $\mathscr{E}(G) = \{t_j\}_{j=1}^k$, where $k \geqslant 2$, and $t_j < t_{j+1}$ for all $j \in \{1, \dots, k-1\}$.

- If $t \in (-\infty, t_1)$, then $E_t(G) = E_*(G)$, and so, by Proposition 30, $\mathbf{E}_t(G) = \pi(G)$. Indeed, we know that $E_t(G) \subseteq E_*(G)$. Consider now $(x,y) \in E_*(G)$. Thus, $u_{i(x,y)}(x) - u_{i(x,y)}(y) \geqslant t_1 > t$ and then $(x,y) \in E_t(G)$. We then conclude $E_*(G) \subseteq E_t(G)$, and the equality $E_t(G) = E_*(G)$ follows.
- Let $a, b \in \mathbb{R}$ be such that $a < b$ and $(a,b) \cap \mathscr{E}(G) = \varnothing$. Then, for every $t \in [a,b)$, $E_t(G) = E_a(G)$, and so $\mathbf{E}_t(G) = \mathbf{E}_a(G)$. Indeed, let $t \in [a,b)$, we know, by Proposition 11, that $E_t(G) \subseteq E_a(G)$. Consider now $(x,y) \in E_a(G)$. Then $(x,y) \in E_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > a$. Assume by contradiction that $(x,y) \notin E_t(G)$. Since $(x,y) \in E_*(G)$ it must be $u_{i(x,y)}(x) - u_{i(x,y)}(y) \leqslant t$. Since $t < b$, we deduce that $a < u_{i(x,y)}(x) - u_{i(x,y)}(y) < b$. Thus, $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{E}(G) \cap (a,b)$, a contradiction. We then conclude that $(x,y) \in E_t(G)$. Thus, we get the inclusion $E_a(G) \subseteq E_t(G)$, and so the equality $E_t(G) = E_a(G)$ follows.
- If $t \in [t_k, +\infty)$, then $E_t(G) = \varnothing$, and so $\mathbf{E}_t(G) = \pi(G) = \mathbf{E}_{t_k}(G)$. Indeed, assume by contradiction that there exists $(x,y) \in E_t(G)$. Thus, $(x,y) \in E_*(G)$ and $u_{i(x,y)}(x) - u_{i(x,y)}(y) > t$. Since $(x,y) \in E_*(G)$ implies $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{E}(G)$, we get $u_{i(x,y)}(x) - u_{i(x,y)}(y) \leqslant t_k$. We then deduce the contradiction $t_k > t$.

Then, we have

$$\begin{aligned}\mathbf{E}_\circ(G) = \bigcap_{t \in \mathbb{R}} \mathbf{E}_t(G) &= \left(\bigcap_{t \in (-\infty, t_1)} \mathbf{E}_t(G) \right) \cap \left(\bigcap_{j \in \{1, \dots, k-1\}} \left(\bigcap_{t \in [t_j, t_{j+1})} \mathbf{E}_t(G) \right) \right) \cap \left(\bigcap_{t \in [t_k, +\infty)} \mathbf{E}_t(G) \right) \\ &= \pi(G) \cap \left(\bigcap_{j \in \{1, \dots, k-1\}} \mathbf{E}_{t_j}(G) \right) \cap \mathbf{E}_{t_k}(G) = \pi(G) \cap \left(\bigcap_{j \in \{1, \dots, k\}} \mathbf{E}_{t_j}(G) \right) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G).\end{aligned}$$

In order to prove the desired result for $\mathbf{E}_\bullet(G)$, observe that, since $|\mathscr{E}(G)| \geqslant 2$, by Lemma 28(iii), we know that there exists a positive element in $\mathscr{E}(G)$. Let j^* be the smallest element of the set $\{1, 2, \dots, k\}$ for which $t_{j^*} > 0$. Thus, $\mathscr{E}(G) \cap (0, t_{j^*}) = \varnothing$ and then, for every $t \in [0, t_{j^*})$, $\mathbf{E}_t(G) = \mathbf{E}_0(G)$.

If $j^* = k$, then $\mathscr{E}(G)$ as a unique positive element, say t^*. Thus,

$$\mathbf{E}_\bullet(G) = \bigcap_{t \in \mathbb{R}_+} \mathbf{E}_t(G) = \left(\bigcap_{t \in [0, t^*)} \mathbf{E}_t(G) \right) \cap \left(\bigcap_{t \in [t^*, +\infty)} \mathbf{E}_t(G) \right) = \mathbf{E}_0(G) \cap \mathbf{E}_{t^*}(G) = \bigcap_{t \in (\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{E}_t(G).$$

If instead $j^* < k$, then

$$\mathbf{E}_\bullet(G) = \bigcap_{t \in \mathbb{R}_+} \mathbf{E}_t(G) = \left(\bigcap_{t \in [0, t_{j^*})} \mathbf{E}_t(G) \right) \cap \left(\bigcap_{j \in \{j^*, \dots, k-1\}} \left(\bigcap_{t \in [t_j, t_{j+1})} \mathbf{E}_t(G) \right) \right) \cap \left(\bigcap_{t \in [t_k, +\infty)} \mathbf{E}_t(G) \right)$$

$$= \mathbf{E}_0(G) \cap \left(\bigcap_{j \in \{j*, \dots, k-1\}} \mathbf{E}_{t_j}(G) \right) \cap \mathbf{E}_{t_k}(G) = \mathbf{E}_0(G) \cap \bigcap_{j \in \{j*, \dots, k\}} \mathbf{E}_{t_j}(G) = \bigcap_{t \in (\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{E}_t(G).$$

□

Proof of Theorem 13. If G is trivial, then we know that $\mathbf{D}_\circ(G) = \mathbf{E}_\circ(G) = \pi(G) \neq \varnothing$. Thus, we can assume then that G is non-trivial.

Let us first prove that $\mathbf{D}_\circ(G) \neq \varnothing$. Since G is non-trivial, $\mathscr{D}(G) \neq \varnothing$. Let $\mathscr{D}(G) \cup \{d(G)\} = \{t_j\}_{j=1}^k$, where $k \geqslant 2$ and $t_j < t_{j+1}$ for all $j \in \{1, \dots, k-1\}$. Of course, $t_1 = d(G)$. By Proposition 11, we know that, for every $j \in \{1, \dots, k-1\}$, $D_{t_j}(G) \supseteq D_{t_{j+1}}(G)$. Let us define, for every $j \in \{1, \dots, k\}$, $R_j = D_{t_{k-j+1}}(G)$. By Proposition 31, we have

$$\mathbf{D}_\circ(G) = \bigcap_{t \in \mathscr{D}(G) \cup \{d(G)\}} \mathbf{D}_t(G) = \bigcap_{j=1}^k \mathbf{A}(\pi(G), R_j).$$

Since $(R_j)_{j=1}^k$ is a sequence of irreflexive relations on X such that, for every $j \in \{2, \dots, k\}$, $R_{j-1} \subseteq R_j$, by Theorem 3 in Gori (2023), we deduce that $\mathbf{D}_\circ(G) \neq \varnothing$.

Let us now prove that $\mathbf{E}_\circ(G) \neq \varnothing$. Since G is non-trivial, $\mathscr{E}(G) \neq \varnothing$. If $|\mathscr{E}(G)| = 1$, by Lemma 28(ii), we know that $\mathscr{E}(G) = \{0\}$. Thus, by Proposition 32, $\mathbf{E}_\circ(G) = \mathbf{E}_0(G) \neq \varnothing$.

Assume then that $|\mathscr{E}(G)| \geqslant 2$. Thus, $\mathscr{E}(G) = \{t_j\}_{j=1}^k$, where $k \geqslant 2$ and $t_j < t_{j+1}$ for all $j \in \{1 \dots, k-1\}$. By Proposition 11, we know that, for every $j \in \{1, \dots, k-1\}$, $E_{t_j}(G) \supseteq E_{t_{j+1}}(G)$. Let us define, for every $j \in \{1, \dots, k\}$, $R_j = E_{t_{k-j+1}}(G)$. By Proposition 32, we have

$$\mathbf{E}_\circ(G) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G) = \bigcap_{j=1}^k \mathbf{A}(\pi(G), R_j).$$

Since $(R_j)_{j=1}^k$ is a sequence of irreflexive relations on X such that, for every $j \in \{2, \dots, k\}$, $R_{j-1} \subseteq R_j$, by Theorem 3 in Gori (2023), we deduce that $\mathbf{E}_\circ(G) \neq \varnothing$. □

A.3 Proof of Theorem 17

Theorem 33. *Let X be a nonempty and finite set, $\alpha \in \mathbb{N}$, $(R_i)_{i=0}^\alpha$ be a sequence of irreflexive relations on X. Assume that $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$ and, for every $i \in \{1, \dots, \alpha\}$, $R_{i-1} \subseteq R_i$ and $|R_i \backslash R_{i-1}| = 1$. Then,*

$$\left| \bigcap_{i=0}^\alpha \mathbf{A}(X, R_i) \right| = |\mathscr{A}(X, R_\alpha)|.$$

Proof. Consider the sequence $((\widehat{y}_i, \widehat{z}_i))_{i=1}^\alpha$ in X_*^2 where, for every $i \in \{1, \dots, \alpha\}$, $(\widehat{y}_i, \widehat{z}_i)$ is the unique element in $R_i \backslash R_{i-1}$. We can then consider the sequence of functions $(\nu_i)_{i=0}^\alpha$ defined in the statement of Theorem 10 in Gori (2023). By Theorem 10 in Gori (2023), we know that:

(a1) $\{\nu_\alpha(Y) : Y \in \mathscr{A}(X, R_0)\} \backslash \{\varnothing\} = \mathscr{A}(X, R_\alpha)$;

(b1) for every $Y, Y' \in \mathscr{A}(X, R_0)$ distinct with $\nu_\alpha(Y) \neq \varnothing$ and $\nu_\alpha(Y') \neq \varnothing$, we have $\nu_\alpha(Y) \neq \nu_\alpha(Y')$;

(c1) $\displaystyle\bigcap_{i=0}^\alpha A(X, R_i) = \bigcup_{Y \in \mathscr{A}(X, R_0), \nu_\alpha(Y) \neq \varnothing} Y.$

Since $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$, the aforementioned properties are equivalent to the following ones:

(a2) $\{\nu_\alpha(\{x\}) : x \in X\} \backslash \{\varnothing\} = \mathscr{A}(X, R_\alpha)$;

(b2) for every $x, y \in X$ distinct with $\nu_\alpha(\{x\}) \neq \varnothing$ and $\nu_\alpha(\{y\}) \neq \varnothing$, we have $\nu_\alpha(\{x\}) \neq \nu_\alpha(\{y\})$;

(c2) $\displaystyle\bigcap_{i=0}^\alpha A(X, R_i) = \bigcup_{x \in X, \nu_\alpha(\{x\}) \neq \varnothing} \{x\}.$

Consider now the function F from $\{x \in X : \nu_\alpha(\{x\}) \neq \varnothing\}$ to $\mathscr{A}(X, R_\alpha)$ defined, for every $x \in X$, by $F(x) = \nu_\alpha(\{x\})$. The function F is well defined by (a2). Moreover, using (a2), we deduce that F is surjective, and, using (b2), we deduce that F is injective. Thus, from (c2) and the fact that F is bijective, we get

$$\left|\bigcap_{i=0}^{\alpha} A(X, R_i)\right| = \left|\bigcup_{x\in X, \nu_\alpha(\{x\})\neq\varnothing} \{x\}\right| = |\{x \in X, \nu_\alpha(\{x\}) \neq \varnothing\}| = |\mathscr{A}(X, R_\alpha)|.$$

$\square$

Proof of Theorem 17(i). If G is trivial, then $|X| = 1$ and we know that $\mathbf{D}_\circ(G) = \mathbf{D}_\bullet(G) = X$. Moreover, $D(G) = D_*(G) = \varnothing$, and $\mathscr{A}(X, \varnothing) = \{\{x\} : x \in X\}$. Thus, $|\mathbf{D}_\circ(G)| = |\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))| = |\mathscr{A}(X, D_*(G))| = 1$. Assume now that G is non-trivial. Thus, $D(G) \neq \varnothing$. Moreover, since G is heterogeneous, we know that $|\mathscr{D}(G)| = |D_*(G)|$ and, by Lemma 27(iii), $\mathscr{D}(G)$ has a positive element and a negative element.

Let us prove that $|\mathbf{D}_\circ(G)| = |\mathscr{A}(X, D_*(G))|$. Assume then that $\mathscr{D}(G) = \{t_j\}_{j=1}^{k}$, where $\alpha = |\mathscr{D}(G)| \geqslant 2$ and, for every $j \in \{1, \ldots, \alpha - 1\}$, $t_j < t_{j+1}$. Let us set $t_0 = d(G)$. By Proposition 31, we have that

$$\mathbf{D}_\circ(G) = \bigcap_{t\in\mathscr{D}(G)\cup\{d(G)\}} \mathbf{D}_t(G) = \bigcap_{j=0}^{\alpha} \mathbf{D}_{t_j}(G).$$

Defining, for every $j \in \{0, \ldots, \alpha\}$, $R_j = D_{t_{\alpha-j}}(G)$, we get that

$$\mathbf{D}_\circ(G) = \bigcap_{j=0}^{\alpha} \mathbf{A}(X, R_j).$$

By Proposition 29, we know that $R_\alpha = D_*(G)$. Moreover, $R_0 = D_{t_\alpha}(G) = \varnothing$ and then $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$. Finally, for every $j \in \{1, \ldots, \alpha\}$, $R_{j-1} = D_{t_{\alpha-j+1}}(G) \subseteq D_{t_{\alpha-j}}(G) = R_j$. Thus, we complete the proof showing that, for every $j \in \{1, \ldots, \alpha\}$, $|R_j \backslash R_{j-1}| = 1$. Indeed, we can apply Theorem 33 and deduce that $|\mathbf{D}_\circ(G)| = |\mathscr{A}(X, D_*(G))|$.

Consider then $j \in \{1, \ldots, \alpha\}$. Note that $t_{\alpha-j+1} \neq t_0 = d(G)$, and so $t_{\alpha-j+1} \in \mathscr{D}(G)$. We know that there exists $(x, y) \in D_*(G)$ such that

$$u_{i(x,y)}(x) - u_{i(x,y)}(y) = t_{\alpha-j+1} > t_{\alpha-j}.$$

Thus, $(x, y) \in D_{t_{\alpha-j}}(G) = R_j$ and $(x, y) \notin D_{t_{\alpha-j+1}}(G) = R_{j-1}$. Thus, $|R_j \backslash R_{j-1}| \geqslant 1$.

Assume by contradiction that $|R_j \backslash R_{j-1}| \geqslant 2$. Then, there are $(x, y), (x', y') \in R_j = D_{t_{\alpha-j}}(G)$ with $(x, y) \neq (x', y')$ such that $(x, y), (x', y') \notin R_{j-1} = D_{t_{\alpha-j+1}}(G)$. Thus,

$$t_{\alpha-j+1} \geqslant u_{i(x,y)}(x) - u_{i(x,y)}(y) > t_{\alpha-j}, \quad t_{\alpha-j+1} \geqslant u_{i(x',y')}(x') - u_{i(x',y')}(y') > t_{\alpha-j}.$$

Since $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G)$ and $u_{i(x',y')}(x') - u_{i(x',y')}(y') \in \mathscr{D}(G)$, it must be

$$t_{\alpha-j+1} = u_{i(x,y)}(x) - u_{i(x,y)}(y) = u_{i(x',y')}(x') - u_{i(x',y')}(y').$$

But that contradicts the fact that $|\mathscr{D}(G)| = |D_*(G)|$, since there are two elements of $D_*(G)$ that correspond to the same element of $\mathscr{D}(G)$.

Let us prove now that $|\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))|$. Note that $|(\mathscr{D}(G) \cap \mathbb{R}_+) \cup \{0\}| \geqslant 2$. Assume then that $(\mathscr{D}(G) \cap \mathbb{R}_+) \cup \{0\} = \{t_j\}_{j=0}^{\alpha}$, where $\alpha \geqslant 1$ and, for every $j \in \{1, \ldots, \alpha - 1\}$, $t_j < t_{j+1}$. Of course, $t_0 = 0$. By Proposition 31, we know that

$$\mathbf{D}_\bullet(G) = \bigcap_{t\in(\mathscr{D}(G)\cap\mathbb{R}_+)\cup\{0\}} \mathbf{D}_t(G) = \bigcap_{j=0}^{\alpha} \mathbf{D}_{t_j}(G).$$

Defining, for every $j \in \{0, \ldots, \alpha\}$, $R_j = D_{t_{\alpha-j}}(G)$, we get that

$$\mathbf{D}_\bullet(G) = \bigcap_{j=0}^{\alpha} \mathbf{A}(X, R_j).$$

We have that $R_\alpha = D_{t_0}(G) = D_0(G) = D(G)$. Moreover , $R_0 = D_{t_\alpha}(G) = \varnothing$ and then $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$. Furthermore, for every $j \in \{1, \dots, \alpha\}$, $R_{j-1} = D_{t_{\alpha-j+1}}(G) \subseteq D_{t_{\alpha-j}}(G) = R_j$. We complete the proof showing that, for every $j \in \{1, \dots, \alpha\}$, $|R_j \backslash R_{j-1}| = 1$. Indeed, we can apply Theorem 33 and deduce that $|\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))|$.

Consider then $j \in \{1, \dots, \alpha\}$. Note that $t_{\alpha-j+1} \neq t_0 = 0$ and then $t_{\alpha-j+1} \in \mathscr{D}(G)$. We know that there exists $(x, y) \in D_*(G)$ such that

$$u_{i(x,y)}(x) - u_{i(x,y)}(y) = t_{\alpha-j+1} > t_{\alpha-j}.$$

Thus, $(x, y) \in D_{t_{\alpha-j}}(G) = R_j$ and $(x, y) \notin D_{t_{\alpha-j+1}}(G) = R_{j-1}$. Thus, $|R_j \backslash R_{j-1}| \geqslant 1$.

Assume by contradiction that $|R_j \backslash R_{j-1}| \geqslant 2$. Then, there are $(x, y), (x', y') \in R_j = D_{t_{\alpha-j}}(G)$ with $(x, y) \neq (x', y')$ such that $(x, y), (x', y') \notin R_{j-1} = D_{t_{\alpha-j+1}}(G)$. Thus,

$$t_{\alpha-j+1} \geqslant u_{i(x,y)}(x) - u_{i(x,y)}(y) > t_{\alpha-j}, \quad t_{\alpha-j+1} \geqslant u_{i(x',y')}(x') - u_{i(x',y')}(y') > t_{\alpha-j}.$$

Since $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{D}(G)$ and $u_{i(x',y')}(x') - u_{i(x',y')}(y') \in \mathscr{D}(G)$, it must be

$$t_{\alpha-j+1} = u_{i(x,y)}(x) - u_{i(x,y)}(y) = u_{i(x',y')}(x') - u_{i(x',y')}(y').$$

But that contradicts the fact that $|\mathscr{D}(G)| = |D_*(G)|$, since there are two elements of $D_*(G)$ that correspond to the same element of $\mathscr{D}(G)$. □

The next proof is analogous to the previous one. For completeness, we write it down.

Proof of Theorem 17(ii). If G is trivial, then $|X| = 1$ and we know that $\mathbf{E}_\circ(G) = \mathbf{E}_\bullet(G) = X$. Moreover, $E(G) = E_*(G) = \varnothing$, and $\mathscr{A}(X, \varnothing) = \{\{x\} : x \in X\}$. Thus, $|\mathbf{E}_\circ(G)| = |\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))| = |\mathscr{A}(X, E_*(G))| = 1$. Assume now that G is non-trivial. Thus, $E(G) \neq \varnothing$. Moreover, since G is strongly heterogeneous, we know that $|\mathscr{E}(G)| = |E_*(G)|$ and, by Lemma 28(v), $\mathscr{E}(G)$ has a positive element and a negative element.

Let us prove that $|\mathbf{E}_\circ(G)| = |\mathscr{A}(X, E_*(G))|$ so that, by Proposition 30, we can conclude that $|\mathbf{E}_\circ(G)| = 1$. Assume then that $\mathscr{E}(G) = \{t_j\}_{j=1}^\alpha$, where $\alpha = |\mathscr{E}(G)| \geqslant 2$ and, for every $j \in \{1, \dots, \alpha - 1\}$, $t_j < t_{j+1}$. Let us set $t_0 = e(G)$. Note that $E_{t_0}(G) = E_*(G)$ and, by Proposition 30, $\mathbf{E}_{t_0}(G) = X$. By Proposition 32, we have that

$$\mathbf{E}_\circ(G) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G) = \bigcap_{j=0}^{\alpha} \mathbf{E}_{t_j}(G).$$

Defining, for every $j \in \{0, \dots, \alpha\}$, $R_j = E_{t_{\alpha-j}}(G)$, we get that

$$\mathbf{E}_\circ(G) = \bigcap_{j=0}^{\alpha} \mathbf{A}(X, R_j).$$

We have that $R_\alpha = E_{t_0}(G) = E_*(G)$. Moreover, $E_0 = E_{t_\alpha}(G) = \varnothing$ and then $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$. Finally, for every $j \in \{1, \dots, \alpha\}$, $R_{j-1} = E_{t_{\alpha-j+1}}(G) \subseteq E_{t_{\alpha-j}}(G) = R_j$. Thus, we complete the proof showing that, for every $j \in \{1, \dots, \alpha\}$, $|R_j \backslash R_{j-1}| = 1$. Indeed, we can apply Theorem 33 and deduce that $|\mathbf{E}_\circ(G)| = |\mathscr{A}(X, E_*(G))|$.

Consider then $j \in \{1, \dots, \alpha\}$. Note that $t_{\alpha-j+1} \neq t_0 = e(G)$, and so $t_{\alpha-j+1} \in \mathscr{E}(G)$. We know that there exists $(x, y) \in E_*(G)$ such that

$$u_{i(x,y)}(x) - u_{i(x,y)}(y) = t_{\alpha-j+1} > t_{\alpha-j}.$$

Thus, $(x, y) \in E_{t_{\alpha-j}}(G) = R_j$ and $(x, y) \notin E_{t_{\alpha-j+1}}(G) = R_{j-1}$. Thus, $|R_j \backslash R_{j-1}| \geqslant 1$.

Assume by contradiction that $|R_j \backslash R_{j-1}| \geqslant 2$. Then, there are $(x, y), (x', y') \in R_j = E_{t_{\alpha-j}}(G)$ with $(x, y) \neq (x', y')$ such that $(x, y), (x', y') \notin R_{j-1} = E_{t_{\alpha-j+1}}(G)$. Thus,

$$t_{\alpha-j+1} \geqslant u_{i(x,y)}(x) - u_{i(x,y)}(y) > t_{\alpha-j}, \quad t_{\alpha-j+1} \geqslant u_{i(x',y')}(x') - u_{i(x',y')}(y') > t_{\alpha-j}.$$

Since $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{E}(G)$ and $u_{i(x',y')}(x') - u_{i(x',y')}(y') \in \mathscr{E}(G)$, it must be

$$t_{\alpha-j+1} = u_{i(x,y)}(x) - u_{i(x,y)}(y) = u_{i(x',y')}(x') - u_{i(x',y')}(y').$$

But that contradicts the fact that $|\mathscr{E}(G)| = |E_*(G)|$, since there are two elements of $E_*(G)$ that correspond to the same element of $\mathscr{E}(G)$.

Let us prove now that $|\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))|$. Note that $|(\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}| \geqslant 2$. Assume then that $(\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\} = \{t_j\}_{j=0}^{\alpha}$, where $\alpha \geqslant 1$ and, for every $j \in \{1, \ldots, \alpha - 1\}$, $t_j < t_{j+1}$. Of course, $t_0 = 0$. By Proposition 32, we know that

$$\mathbf{E}_\bullet(G) = \bigcap_{t \in (\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{E}_t(G) = \bigcap_{j=0}^{\alpha} \mathbf{E}_{t_j}(G).$$

Defining, for every $j \in \{0, \ldots, \alpha\}$, $R_j = E_{t_{\alpha-j}}(G)$, we get that

$$\mathbf{E}_\bullet(G) = \bigcap_{j=0}^{\alpha} \mathbf{A}(X, R_j).$$

We have that $R_\alpha = E_{t_0}(G) = E_0(G) = E(G)$. Moreover, $R_0 = E_{t_\alpha}(G) = \varnothing$ and then $\mathscr{A}(X, R_0) = \{\{x\} : x \in X\}$. Furthermore, for every $j \in \{1, \ldots, \alpha\}$, $R_{j-1} = E_{t_{\alpha-j+1}}(G) \subseteq E_{t_{\alpha-j}}(G) = R_j$. We complete the proof showing that, for every $j \in \{1, \ldots, \alpha\}$, $|R_j \backslash R_{j-1}| = 1$. Indeed, we can apply Theorem 33 and deduce that $|\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))|$.

Consider then $j \in \{1, \ldots, \alpha\}$. Note that $t_{\alpha-j+1} \neq t_0 = 0$ and then $t_{\alpha-j+1} \in \mathscr{E}(G)$. We know that there exists $(x, y) \in E_*(G)$ such that

$$u_{i(x,y)}(x) - u_{i(x,y)}(y) = t_{\alpha-j+1} > t_{\alpha-j}.$$

Thus, $(x, y) \in E_{t_{\alpha-j}}(G) = R_j$ and $(x, y) \notin E_{t_{\alpha-j+1}}(G) = R_{j-1}$. Thus, $|R_j \backslash R_{j-1}| \geqslant 1$.

Assume by contradiction that $|R_j \backslash R_{j-1}| \geqslant 2$. Then, there are $(x, y), (x', y') \in R_j = E_{t_{\alpha-j}}(G)$ with $(x, y) \neq (x', y')$ such that $(x, y), (x', y') \notin R_{j-1} = E_{t_{\alpha-j+1}}(G)$. Thus,

$$t_{\alpha-j+1} \geqslant u_{i(x,y)}(x) - u_{i(x,y)}(y) > t_{\alpha-j}, \quad t_{\alpha-j+1} \geqslant u_{i(x',y')}(x') - u_{i(x',y')}(y') > t_{\alpha-j}.$$

Since $u_{i(x,y)}(x) - u_{i(x,y)}(y) \in \mathscr{E}(G)$ and $u_{i(x',y')}(x') - u_{i(x',y')}(y') \in \mathscr{E}(G)$, it must be

$$t_{\alpha-j+1} = u_{i(x,y)}(x) - u_{i(x,y)}(y) = u_{i(x',y')}(x') - u_{i(x',y')}(y').$$

But that contradicts the fact that $|\mathscr{E}(G)| = |E_*(G)|$, since there are two elements of $E_*(G)$ that correspond to the same element of $\mathscr{E}(G)$. □

A.4 Proof of Proposition 18

Proof of Proposition 18. For simplicity in what follows we set $\mathcal{G} = \mathcal{G}(I, (X_i)_{i \in I})$, $\mathcal{H} = \mathcal{H}(I, (X_i)_{i \in I})$ and $\mathcal{SH} = \mathcal{SH}(I, (X_i)_{i \in I})$. If, for every $i \in I$, $|X_i| = 1$, then $\mathcal{G} = \mathcal{H} = \mathcal{SH}$ and the theorem is true.

Assume then that there exists $i \in I$ such that $|X_i| \geqslant 2$. Let

$$\widetilde{D} = \Big\{ \big((x, y), (x', y')\big) \in D_*(G)^2 : (x, y) \neq (x', y') \Big\},$$

$$\widetilde{E} = \Big\{ \big((x, y), (x', y')\big) \in E_*(G)^2 : (x, y) \neq (x', y') \Big\},$$

and note that $\varnothing \neq \widetilde{D} \subseteq \widetilde{E}$. Moreover, we have that

$$\mathcal{H} = \bigcap_{((x,y),(x',y')) \in \widetilde{D}} \left\{ \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle \in \mathcal{G} : u_{i(x,y)}(x) - u_{i(x,y)}(y) \neq u_{i(x',y')}(x') - u_{i(x',y')}(y') \right\},$$

and

$$\mathcal{SH} = \bigcap_{((x,y),(x',y')) \in \widetilde{E}} \left\{ \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle \in \mathcal{G} : u_{i(x,y)}(x) - u_{i(x,y)}(y) \neq u_{i(x',y')}(x') - u_{i(x',y')}(y') \right\}.$$

Consider now $((x, y), (x', y')) \in \widetilde{E}$. Observe, in particular, that $x \neq y$, $x' \neq y'$ and $(x, y) \neq (x', y')$. There are then $j_1, j_2, j_3, j_4 \in \{1, \ldots, |I||S|\}$ with $j_1 \neq j_2$, $j_3 \neq j_4$ and $(j_1, j_2) \neq (j_3, j_4)$ such that

$$\Psi\Big(\{ \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle \in \mathcal{G} : u_{i(x,y)}(x) - u_{i(x,y)}(y) \neq u_{i(x',y')}(x') - u_{i(x',y')}(y') \} \Big)$$

$$= \{v \in \mathbb{R}^{|I||S|} : v_{j_1} - v_{j_2} \neq v_{j_3} - v_{j_4}\} = \mathbb{R}^{|I||S|} \setminus \{v \in \mathbb{R}^{|I||S|} : v_{j_1} - v_{j_2} - v_{j_3} + v_{j_4} = 0\}.$$

Observe now that $j_1 \notin \{j_2, j_3\}$ or $j_2 \notin \{j_1, j_4\}$. Indeed, assume by contradiction that they both fail and so $j_1 \in \{j_2, j_3\}$ and $j_2 \in \{j_1, j_4\}$. Since $j_1 \neq j_2$, it must be $j_1 = j_3$ and $j_2 = j_4$. Thus, we get $(j_1, j_2) = (j_3, j_4)$, a contradiction. We deduce then that the equation

$$v_{j_1} - v_{j_2} - v_{j_3} + v_{j_4} = 0$$

is an equation of the type $av = 0$, where $a \in \mathbb{R}^{|I||S|}$ with $a \neq 0$. Indeed, if $j_1 \notin \{j_2, j_3\}$, then the coefficient of v_{j_1} cannot be zero; if $j_2 \notin \{j_1, j_4\}$, then the coefficient of v_{j_2} cannot be zero.

As a consequence, for every $((x, y), (x', y')) \in \widetilde{E}$, the set

$$\Psi\Big(\{\langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle \in \mathcal{G} : u_{i(x,y)}(x) - u_{i(x,y)}(y) \neq u_{i(x',y')}(x') - u_{i(x',y')}(y')\}\Big)$$

is a hyper-plane in $\mathbb{R}^{|I||S|}$ and then it is closed in $\mathbb{R}^{|I||S|}$ and has null measure in $\mathbb{R}^{|I||S|}$. Thus, the set

$$\{\langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle \in \mathcal{G} : u_{i(x,y)}(x) - u_{i(x,y)}(y) \neq u_{i(x',y')}(x') - u_{i(x',y')}(y')\}$$

is closed in $\mathcal{G}$ and has null measure in $\mathcal{G}$. Recalling that $\varnothing \neq \widetilde{D} \subseteq \widetilde{E}$, we then conclude that $\mathcal{H}$ and $\mathcal{SH}$ are open subsets of $\mathcal{G}$ having full measure in $\mathcal{G}$. □

A.5 Proof of Proposition 19

Proposition 34. *Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$ be a game and $x \in X$. Then the three following facts are equivalent:*

(i) $x \in \mathbf{N}(G)$,

(ii) $\{x\} \in \mathscr{A}(X, D(G))$,

(iii) $\{x\} \in \mathscr{A}(X, E(G))$.

Proof. (i) ⇒ (ii) Assume that $x \in \mathbf{N}(G)$. If, by contradiction, $\{x\} \notin \mathscr{A}(X, D(G))$, then there exists $y \in X$ such that $(y, x) \in D(G)$. Thus, there exists $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(y) > u_i(x)$. That contradicts the fact that $x \in \mathbf{N}(G)$.

(ii) ⇒ (iii) Assume that $\{x\} \in \mathscr{A}(X, D(G))$. If, by contradiction, $\{x\} \notin \mathscr{A}(X, E(G))$, then, there exists $y \in Y$ such that $(y, x) \in E(G)$. Thus, there exists $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(y) > u_i(x)$. Considering now $z_i \in \mathcal{B}_i^G(x_{-i})$, we have that $z_i \neq x_i$. Setting $z = (z_i, x_{-i})$, we have that $u_i(z) \geqslant u_i(y) > u_i(x)$. Thus, $(z, x) \in D(G)$ and that implies that $\{x\} \notin \mathscr{A}(X, D(G))$, a contradiction.

(iii) ⇒ (i) Assume that $\{x\} \in \mathscr{A}(X, E(G))$. If, by contradiction, $x \notin \mathbf{N}(G)$, then there exists $y \in X$ and $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(y) > u_i(x)$. Thus, $(y, x) \in E(G)$ and so $\{x\} \notin \mathscr{A}(X, E(G))$, a contradiction. □

Proof of Proposition 19. Let $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$. Assume first that G is heterogeneous and prove the equivalence of (i), (ii), and (iii).

(i) ⇒ (ii) Assume that $\mathbf{D}_\bullet(G) = \mathbf{N}(G)$. By Proposition 34, we know that $\{\{x\} : x \in \mathbf{N}(G)\} \subseteq \mathscr{A}(X, D(G))$. Of course, $|\{\{x\} : x \in \mathbf{N}(G)\}| = |\mathbf{N}(G)|$. Moreover, by Theorem 17, $|\mathscr{A}(X, D(G))| = |\mathbf{D}_\bullet(G)| = |\mathbf{N}(G)|$. We deduce then that

$$\mathscr{A}(X, D(G)) = \{\{x\} : x \in \mathbf{N}(G)\}.$$

Using now Theorem 1, we then conclude that $\mathbf{D}(G) = \mathbf{N}(G) = \mathbf{D}_\bullet(G)$.

(ii) ⇒ (iii) Assume now that $\mathbf{D}_\bullet(G) = \mathbf{D}(G)$. By Theorem 1, we know that $\mathbf{D}(G) = \bigcup_{Y\in\mathscr{A}(X,D(G))} Y$. Moreover, since G is heterogeneous, by Theorem 17, we also know that $|\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))|$. Since $\mathbf{D}_\bullet(G) = \mathbf{D}(G)$, it must be

$$|\mathscr{A}(X, D(G))| = \Bigg|\bigcup_{Y\in\mathscr{A}(X,D(G))} Y\Bigg|.$$

Since the elements of $\mathscr{A}(X, D(G))$ are pairwise disjoint, that implies that, for every $Y \in \mathscr{A}(X, D(G))$, $|Y| = 1$. Thus, by Proposition 34, we conclude that $\mathbf{D}(G) \subseteq \mathbf{N}(G)$. Since, by Proposition 3, $\mathbf{N}(G) \subseteq \mathbf{D}(G)$, we conclude that $\mathbf{D}(G) = \mathbf{N}(G)$.

(iii) $\Rightarrow$ (i) Assume that $\mathbf{D}(G) = \mathbf{N}(G)$. By Proposition 3, we know that $\mathbf{N}(G) \subseteq \mathbf{D}_\bullet(A) \subseteq \mathbf{D}(G)$. Thus, we conclude that $\mathbf{D}_\bullet(G) = \mathbf{D}(G)$.

Assume now that G is strongly heterogeneous and prove the equivalence of (iv), (v), and (vi). The proof is completely analogous to the previous one. However, for the sake of completeness, we propose it below.

(iv) $\Rightarrow$ (v) Assume that $\mathbf{E}_\bullet(G) = \mathbf{N}(G)$. By Proposition 34, we know that $\{\{x\} : x \in \mathbf{N}(G)\} \subseteq \mathscr{A}(X, E(G))$. Of course, $|\{\{x\} : x \in \mathbf{N}(G)\}| = |\mathbf{N}(G)|$. Moreover, by Theorem 17, $|\mathscr{A}(X, E(G))| = |\mathbf{E}_\bullet(G)| = |\mathbf{N}(G)|$. We deduce then that

$$\mathscr{A}(X, E(G)) = \{\{x\} : x \in \mathbf{N}(G)\}.$$

Using now Theorem 1, we then conclude that $\mathbf{E}(G) = \mathbf{N}(G) = \mathbf{E}_\bullet(G)$.

(v) $\Rightarrow$ (vi) Assume now that $\mathbf{E}_\bullet(G) = \mathbf{E}(G)$. By Theorem 1, we know that $\mathbf{E}(G) = \bigcup_{Y \in \mathscr{A}(X,E(G))} Y$. Moreover, since G is strongly heterogeneous, by Theorem 17, we also know that $|\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))|$. Since $\mathbf{E}_\bullet(G) = \mathbf{E}(G)$, it must be

$$|\mathscr{A}(X, E(G))| = \left| \bigcup_{Y \in \mathscr{A}(X,E(G))} Y \right|.$$

Since the elements of $\mathscr{A}(X, E(G))$ are pairwise disjoint, that implies that, for every $Y \in \mathscr{A}(X, E(G))$, $|Y| = 1$. Thus, by Proposition 34, we conclude that $\mathbf{E}(G) \subseteq \mathbf{N}(G)$. Since, by Proposition 3, $\mathbf{N}(G) \subseteq \mathbf{E}(G)$, we conclude that $\mathbf{E}(G) = \mathbf{N}(G)$.

(vi) $\Rightarrow$ (iv) Assume that $\mathbf{E}(G) = \mathbf{N}(G)$. By Proposition 3, we know that $\mathbf{N}(G) \subseteq \mathbf{E}_\bullet(A) \subseteq \mathbf{E}(G)$. Thus, we conclude that $\mathbf{E}_\bullet(G) = \mathbf{E}(G)$. $\square$

A.6 Proof of Proposition 20

The following lemma collects some basic facts about the maximal strong components of an abstract decision problem.

Lemma 35. *Let (X, R) be an abstract decision problem. The following facts hold.*

(i) *Let $Y \in \mathscr{A}(X, R)$, $y \in Y$ and $x \in X$. If $(x, y) \in R^\tau$, then $x \in Y$ and $(y, x) \in R^\tau$.*

(ii) *Let $Y, Z \in \mathscr{A}(X, R)$ with $Y \neq Z$, $y \in Y$ and $z \in Z$. Then $(y, z) \notin R^\tau$.*

(iii) *Let $Y, Z \in \mathscr{A}(X, R)$ with $Y \neq Z$, $y \in Y$, $z \in Z$ and $x \in X$. Then $(x, y) \notin R^\tau$ or $(x, z) \notin R^\tau$.*

(iv) *For every $x \in X$, there exists $y \in \mathbf{A}(X, R)$ such that $(y, x) \in R^\tau$.*

(v) *Assume that $\mathscr{A}(X, R) = \{Y\}$. Then, for every $y \in Y$ and $x \in X$, $(y, x) \in R^\tau$.*

Proof. (i) Straightforward.

(ii) If, by contradiction, $(y, z) \in R^\tau$, then $y \in Z$. We know that $Y \neq Z$ implies $Y \cap Z = \varnothing$. Since $y \in Y \cap Z$, we get a contradiction.

(iii) Assume, by contradiction, that $(x, y) \in R^\tau$ and $(x, z) \in R^\tau$. Thus, by (i), $x \in Y \cap Z$. Since $Y \neq Z$ implies $Y \cap Z = \varnothing$, we get a contradiction.

(iv) First of all, for every $x \in X$, let us denote by $S(x)$ the unique element in $\mathscr{S}(X, R)$ which x belongs to and set

$$T(x) = \{y \in X \backslash S(x) : (y, x) \in R^\tau\}.$$

Note that, $T(x) = \varnothing$ if and only if $x \in \mathbf{A}(X, R)$. For every $x \in X \backslash \mathbf{A}(X, R)$, pick an element in $T(x)$ and denote it by $d(x)$; for every $y \in \mathbf{A}(X, R)$, set $d(y) = y$. Of course, for every $x \in X \backslash \mathbf{A}(X, R)$, $d(x) \neq x$ and $(d(x), x) \in R^\tau$.

Consider now $x^* \in X$ and prove that there exists $y^* \in \mathbf{A}(X, R)$ such that $(y^*, x^*) \in R^\tau$. If $x^* \in \mathbf{A}(X, R)$, simply take $y^* = x^*$. Assume then that $x^* \in X \backslash \mathbf{A}(X, R)$. Consider the sequence $(x^j)_{j=1}^\infty$ of X recursively defined as follows: $x^1 = x^*$ and, for every $j \in \mathbb{N}$, $x^{j+1} = d(x^j)$. Let us prove that there exists $m \in \mathbb{N}$ such that $x^m \in \mathbf{A}(X, R)$. Assume by contradiction that, for every $m \in \mathbb{N}$, $x^m \in X \backslash \mathbf{A}(X, R)$. Since $X \backslash \mathbf{A}(X, R)$ is finite and, for every $m \in \mathbb{N}$, $x^{m+1} \neq x^m$, there exists $j, k \in \mathbb{N}$ with $k \geqslant 2$ such that $x^j = x^{j+k}$ and, $x^j, \ldots, x^{j+k-1}$ are distinct. We then have, for every $l \in \{1, \ldots, k-1\}$, $(x^{j+l}, x^{j+l-1}) \in R^\tau$. By transitivity of R^τ, we deduce that $(x^{j+k-1}, x^j) \in R^\tau$. Moreover, $(x^j, x^{j+k-1}) = (x^{j+k}, x^{j+k-1}) \in R^\tau$. Thus, $S(x^j) = S(x^{j+k-1})$. However, since $x^j = x^{j+k} = d(x^{j+k-1})$, we must have $x^j \notin S(x^{j+k-1})$, a contradiction.

Thus, there exists $m \in \mathbb{N}$ such that $x^m \in \mathbf{A}(X, R)$. Consider the minimum m for which that is true. Of course, since $x^1 \in X \backslash \mathbf{A}(X, R)$, we have $m \geqslant 2$. Thus, we have $x^1, \ldots, x^{m-1} \in X \backslash \mathbf{A}(X, R)$ and $x^m \in \mathbf{A}(X, R)$. As a consequence, for every $j \in \{1, \ldots, m-1\}$, we get $(x^{l+1}, x^l) \in R^\tau$, and then, by transitivity of R^τ, $(x^m, x^1) = (x^m, x^*) \in R^\tau$. Since $x^m \in Y$, we have $(y^*, x^m) \in R^\tau$ and, using again the transitivity of R^τ, we conclude that $(y^*, x^*) \in R^\tau$, as desired.

(v) Consider now $y^* \in Y$ and $x^* \in X$ and prove that $(y^*, x^*) \in R^\tau$. By (iv), we know that there exists $z^* \in Y$ such that $(z^*, x^*) \in R^\tau$. Since $(y^*, z^*) \in R^\tau$, by transitivity of R^τ, we conclude that $(y^*, x^*) \in R^\tau$, as desired. $\square$

Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a game. For every $Y \subseteq X$ and $i \in I$, we set

$$\mathbb{P}_i(Y) = \Big\{ \sigma \in X_{-i} : \text{there exists } s \in X_i \text{ such that } (s, \sigma) \in Y \Big\}.$$

Proposition 36. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a heterogeneous game and let $Y, Z \in \mathscr{A}(X, D(G))$ with $Y \neq Z$. Then, for every $i \in I$, we have $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) = \varnothing$.*

Proof. Assume by contradiction that there exists $i \in I$ such that $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) \neq \varnothing$. Consider then $\sigma \in \mathbb{P}_i(Y) \cap \mathbb{P}_i(Z)$. Thus, there are $s, s' \in X_i$ such that $(s, \sigma) \in Y$ and $(s', \sigma) \in Z$. Since $Y, Z \in \mathscr{A}(X, D(G))$ with $Y \neq Z$, we have that $Y \cap Z = \varnothing$. Thus, $s \neq s'$. Since G is heterogeneous, by Lemma 26, there exists $\widehat{s} \in X_i$ such that $\mathcal{B}_i^G(\sigma) = \{\widehat{s}\}$. If $s = \widehat{s}$, then $((s, \sigma), (s', \sigma)) \in D(G)$. However, by Lemma 35(ii), $((s, \sigma), (s', \sigma)) \notin D(G)$, a contradiction. If $s' = \widehat{s}$, then $((s', \sigma), (s, \sigma)) \in D(G)$. However, by Lemma 35(ii), $((s', \sigma), (s, \sigma)) \notin D(G)$, a contradiction. If $s \neq \widehat{s}$ and $s' \neq \widehat{s}$, then $((\widehat{s}, \sigma), (s, \sigma)) \in D(G)$ and $((\widehat{s}, \sigma), (s', \sigma)) \in D(G)$. However, by Lemma 35(iii), $((\widehat{s}, \sigma), (s, \sigma)) \notin D(G)$ or $((\widehat{s}, \sigma), (s', \sigma)) \notin D(G)$, a contradiction. $\square$

The proof of the next proposition is completely analogous to that of Proposition 36. For the sake of completeness, we propose it.

Proposition 37. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a heterogeneous game and let $Y, Z \in \mathscr{A}(X, D_*(G))$ with $Y \neq Z$. Then, for every $i \in I$, we have $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) = \varnothing$.*

Proof. Assume by contradiction that there exists $i \in I$ such that $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) \neq \varnothing$. Consider then $\sigma \in \mathbb{P}_i(Y) \cap \mathbb{P}_i(Z)$. Thus, there are $s, s' \in X_i$ such that $(s, \sigma) \in Y$ and $(s', \sigma) \in Z$. Since $Y, Z \in \mathscr{A}(X, D_*(G))$ with $Y \neq Z$, we have $Y \cap Z = \varnothing$. Thus, $s \neq s'$. Since G is heterogeneous, by Lemma 26, there exists $\widehat{s} \in X_i$ such that $\mathcal{B}_i^G(\sigma) = \{\widehat{s}\}$. If $s = \widehat{s}$, then $((s, \sigma), (s', \sigma)) \in D_*(G)$. However, by Lemma 35(ii), $((s, \sigma), (s', \sigma)) \notin D_*(G)$, a contradiction. If $s' = \widehat{s}$, then $((s', \sigma), (s, \sigma)) \in D_*(G)$. However, by Lemma 35(ii), $((s', \sigma), (s, \sigma)) \notin D_*(G)$, a contradiction. If $s \neq \widehat{s}$ and $s' \neq \widehat{s}$, then $((\widehat{s}, \sigma), (s, \sigma)) \in D_*(G)$ and $((\widehat{s}, \sigma), (s', \sigma)) \in D_*(G)$. However, by Lemma 35(iii), $((\widehat{s}, \sigma), (s, \sigma)) \notin D_*(G)$ or $((\widehat{s}, \sigma), (s', \sigma)) \notin D_*(G)$, a contradiction. $\square$

Proposition 38. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a strongly heterogeneous game and let $Y, Z \in \mathscr{A}(X, E(G))$ with $Y \neq Z$. Then, for every $i \in I$, we have $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) = \varnothing$.*

Proof. Assume by contradiction that there exists $i \in I$ such that $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) \neq \varnothing$. Consider then $\sigma \in \mathbb{P}_i(Y) \cap \mathbb{P}_i(Z)$. Thus, there are $s, s' \in X_i$ such that $(s, \sigma) \in Y$ and $(s', \sigma) \in Z$. Since $Y, Z \in \mathscr{A}(X, E(G))$ with $Y \neq Z$, we have $Y \cap Z = \varnothing$. Thus, $s \neq s'$. Since G is strongly heterogeneous, by Lemma 28(iv), we get $u_i(s, \sigma) \neq u_i(s', \sigma)$ and so $((s, \sigma), (s', \sigma)) \in E(G)$ or $((s', \sigma), (s, \sigma)) \in E(G)$. However, by Lemma 35(ii), $((s, \sigma), (s', \sigma)) \notin E(G)$ and $((s', \sigma), (s, \sigma)) \notin E(G)$, a contradiction. $\square$

Proposition 39. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a heterogeneous game. Then,*

$$|\mathscr{A}(X, D(G))| \leqslant \min\{|X_{-i}| : i \in I\}.$$

Proof. Consider $i \in I$. For every $Y \in \mathscr{A}(X, D(G))$, since $Y \neq \varnothing$, $|\mathbb{P}_i(Y)| \geqslant 1$. Thus, by Proposition 36, we have that

$$|\mathscr{A}(X, D(G))| \leqslant \sum_{Y \in \mathscr{A}(X,D(G))} |\mathbb{P}_i(Y)| = \left| \bigcup_{Y \in \mathscr{A}(X,D(G))} \mathbb{P}_i(Y) \right| \leqslant |X_{-i}|.$$

As a consequence, $|\mathscr{A}(X, D(G))| \leqslant \min\{|X_{-i}| : i \in I\}$. □

Proposition 40. *Let $G = \langle \{1, 2\}, (X_1, X_2), (u_1, u_2) \rangle$ be a heterogeneous two-player game with $|X_1| \geqslant 2$ and $|X_2| \geqslant 2$. Then,*

$$|\mathscr{A}(X, D_*(G))| \leqslant \frac{\min\{|X_1|, |X_2|\}}{2}.$$

Proof. Let $Y \in \mathscr{A}(X, D_*(G))$ and prove that $|\mathbb{P}_2(Y)| \geqslant 2$. Since $Y \neq \varnothing$, there exist $s^* \in X_1$ and $t^* \in X_2$ such that $(s^*, t^*) \in Y$. Since $|X_1| \geqslant 2$, $\mathcal{C}_1^G(s^*, t^*) \neq \varnothing$. Let $\widehat{s} \in \mathcal{C}_1^G(s^*, t^*)$. Thus, $((\widehat{s}, t^*), (s^*, t^*)) \in D_*(G)$ and so also $(\widehat{s}, t^*) \in Y$. As a consequence, $\widehat{s}, s^* \in \mathbb{P}_2(Y)$. In particular, $|\mathbb{P}_2(Y)| \geqslant 2$. An analogous reasoning allows also to show that $|\mathbb{P}_1(Y)| \geqslant 2$.

By Proposition 37, for every $i \in \{1, 2\}$, we have that

$$|\mathscr{A}(X, D_*(G))| \leqslant \sum_{Y \in \mathscr{A}(X,D_*(G))} \frac{|\mathbb{P}_i(Y)|}{2} = \frac{1}{2} \sum_{Y \in \mathscr{A}(X,D_*(G))} |\mathbb{P}_i(Y)| = \frac{1}{2} \left| \bigcup_{Y \in \mathscr{A}(X,D_*(G))} \mathbb{P}_i(Y) \right| \leqslant \frac{|X_{-i}|}{2}.$$

As a consequence, $|\mathscr{A}(X, D_*(G))| \leqslant \frac{\min\{|X_1|,|X_2|\}}{2}$. □

Proposition 41. *Let $G = \langle I, (X_i)_{i \in I}, (u_i)_{i \in I} \rangle$ be a strongly heterogeneous game. Then,*

$$|\mathscr{A}(X, E(G))| \leqslant \min\{|X_{-i}| : i \in I\}.$$

Proof. Consider $i \in I$. For every $Y \in \mathscr{A}(X, E(G))$, since $Y \neq \varnothing$, $|\mathbb{P}_i(Y)| \geqslant 1$. Thus, by Proposition 38, we have that

$$|\mathscr{A}(X, E(G))| \leqslant \sum_{Y \in \mathscr{A}(X,E(G))} |\mathbb{P}_i(Y)| = \left| \bigcup_{Y \in \mathscr{A}(X,E(G))} \mathbb{P}_i(Y) \right| \leqslant |X_{-i}|.$$

As a consequence, $|\mathscr{A}(X, E(G))| \leqslant \min\{|X_{-i}| : i \in I\}$. □

Lemma 42. *Let $G = \langle \{1, 2\}, (X_1, X_2), (u_1, u_2) \rangle$ be a strictly competitive two-player game. Let R be the relation on X defined as*

$$R = \{(x, y) \in E_*(G) : u_{i(x,y)}(x) - u_{i(x,y)}(y) \geqslant 0\}. \tag{11}$$

Then $|\mathscr{A}(X, R)| = 1$.

Proof. Assume by contradiction that there are $Y, Z \in \mathscr{A}(X, R)$ with $Y \neq Z$. First, note that, $\mathbb{P}_1(Y) \cap \mathbb{P}_1(Z) = \varnothing$ and $\mathbb{P}_2(Y) \cap \mathbb{P}_2(Z) = \varnothing$. Indeed, assume by contradiction that there exists $i \in \{1, 2\}$ such that $\mathbb{P}_i(Y) \cap \mathbb{P}_i(Z) \neq \varnothing$. Consider then $\sigma \in \mathbb{P}_i(Y) \cap \mathbb{P}_i(Z)$. Thus, there are $s, s' \in X_i$ such that $(s, \sigma) \in Y$ and $(s', \sigma) \in Z$. Since $Y, Z \in \mathscr{A}(X, R)$ with $Y \neq Z$, we have $Y \cap Z = \varnothing$, and so $s \neq s'$. Since $u_i(s, \sigma) - u_i(s', \sigma) \geqslant 0$ or $u_i(s', \sigma) - u_i(s, \sigma) \geqslant 0$, we get $((s, \sigma), (s', \sigma)) \in R$ or $((s', \sigma), (s, \sigma)) \in R$. However, by Lemma 35(ii), $((s, \sigma), (s', \sigma)) \notin R$ and $((s', \sigma), (s, \sigma)) \notin R$, a contradiction.

Consider now $(s_1, s_2) \in Y$ and $(t_1, t_2) \in Z$. Of course, they are distinct since $Y \cap Z = \varnothing$. Since $\mathbb{P}_1(Y) \cap \mathbb{P}_1(Z) = \varnothing$ and $\mathbb{P}_2(Y) \cap \mathbb{P}_2(Z) = \varnothing$, we also have that $s_1 \neq t_1$ and $s_2 \neq t_2$. Observe now that $(s_1, t_2) \notin Y \cup Z$. Indeed, assume by contradiction that $(s_1, t_2) \in Y \cup Z$: if $(s_1, t_2) \in Y$, then, since $(t_1, t_2) \in Z$, we deduce that $t_2 \in \mathbb{P}_1(Y) \cap \mathbb{P}_1(Z) = \varnothing$, a contradiction; if instead $(s_1, t_2) \in Z$, then, since $(s_1, s_2) \in Y$, we deduce that $s_1 \in \mathbb{P}_2(Y) \cap \mathbb{P}_2(Z) = \varnothing$, a contradiction. Similarly, we can prove that $(t_1, s_2) \notin Y \cup Z$. Of course, due to $s_1 \neq t_1$ and $s_2 \neq t_2$, we also have $(s_1, t_2) \neq (t_1, s_2)$.

Since $Y, Z \in \mathscr{A}(X, R)$ and since $(s_1, t_2) \notin Y \cup Z$, and $(t_1, s_2) \notin Y \cup Z$, using Lemma 35(i), we have then that

$$u_1(s_1, s_2) > u_1(t_1, s_2), \quad u_1(t_1, t_2) > u_1(s_1, t_2), \quad u_2(s_1, s_2) > u_2(s_1, t_2), \quad u_2(t_1, t_2) > u_2(t_1, s_2).$$

Thus, using the fact that G is strictly competitive, we deduce that

$$u_1(s_1, s_2) > u_1(t_1, s_2), \quad u_1(t_1, t_2) > u_1(s_1, t_2), \quad u_1(s_1, s_2) < u_1(s_1, t_2), \quad u_1(t_1, t_2) < u_1(t_1, s_2).$$

As a consequence, we get

$$u_1(s_1, s_2) < u_1(s_1, t_2) < u_1(t_1, t_2) < u_1(t_1, s_2) < u_1(s_1, s_2),$$

that implies the contradiction $u_1(s_1, s_2) < u_1(s_1, s_2)$. □

Proof of Proposition 20. (i) Assume that G is strictly competitive. Consider R defined in (11). By Lemma 42, we know that there exists $Y \subseteq X$ such that $\mathscr{A}(X, R) = \{Y\}$, and so $Y = \mathbf{A}(X, R)$. Let us prove now that, for every $t < 0$, we have that $Y \subseteq \mathbf{E}_t(G)$. Let $t < 0$, we know that $R \subseteq E_t(G)$. As a consequence, we have that Y is included in a strong component Y' of $(X, E_t(G))$. Let us prove next that Y' is the unique element in $\mathscr{A}(X, E_t(G))$. Let $Z \in \mathscr{A}(X, E_t(G))$ and suppose by contradiction that $Y' \neq Z$. Consider $x^* \in Y \subseteq Y'$ and $z \in Z$. Since $Y', Z \in \mathscr{S}(X, E_t(G))$ and $Y' \neq Z$, we have that $Y' \cap Z = \varnothing$ and so $x^* \neq z$. Moreover, since $x^* \in Y$ and $\mathscr{A}(X, R) = \{Y\}$, by Lemma 35(v), we have that there exists a path from x^* to z in (X, R). Thus, the same path is also a path from x^* to z in $(X, E_t(G))$. Thus, it must be $Z = Y'$, a contradiction. As a consequence, $\mathscr{A}(X, E_t(G)) = \{Y'\}$ and $\mathbf{E}_t(G) = Y' \supseteq Y$, as desired.

Let us now prove that there exists $t < 0$ such that $Y = \mathbf{E}_t(G)$. Assume first that $|\mathscr{E}(G)| = 1$. Then, by Lemma 28(ii), we know that $\mathscr{E}(G) = \{0\}$. Thus, $R = X_*^2$ and $E_t(G) = X_*^2$ for all $t < 0$. We then get $Y = \mathbf{E}_t(G)$ for all $t < 0$. Assume now that $\mathscr{E}(G) \neq \{0\}$. Then, by Lemma 28(i), $\mathscr{E}(G)$ contains a negative number. Let t^* be the maximum of the set $\mathscr{E}(G) \cap (-\infty, 0)$. Surely $R \subseteq E_{t^*}(G)$. Let us prove the opposite inclusion. Assume that $(x, y) \in \mathbf{E}_{t^*}(G)$. Then there exists $i \in I$ such that $x_{-i} = y_{-i}$ and $u_i(x) - u_i(y) > t^*$. If $u_i(x) - u_i(y) < 0$, then we get the contradiction $u_i(x) - u_i(y) \in \mathscr{E}(G) \cap (t^*, 0)$. Thus, it must be $u_i(x) - u_i(y) \geqslant 0$ and then $(x, y) \in R$. Thus, we conclude that $R = E_{t^*}(G)$ and so $Y = \mathbf{E}_{t^*}(G)$.

We then have

$$\mathbf{E}_\circ(G) = \bigcap_{t \in \mathbb{R}} \mathbb{E}_t(G) = \left(\bigcap_{t \in \mathbb{R}_+} \mathbf{E}_t(G) \right) \cap \left(\bigcap_{t \in (-\infty, 0)} \mathbf{E}_t(G) \right) = \mathbf{E}_\bullet(G) \cap Y.$$

In order to prove that $\mathbf{E}_\circ(G) = \mathbf{E}_\bullet(G)$, it is enough to show that $\mathbf{E}_0(G) \subseteq Y$. Assume by contradiction that there exists $(s_1, s_2) \in \mathbf{E}_0(G) \backslash Y$. Let us first prove that any strategy profile of the type (t_1, s_2) with $t_1 \in X_1 \backslash \{s_1\}$ cannot be element of Y. In fact, assume by contradiction that $(t_1, s_2) \in Y$ with $t_1 \in X_1 \backslash \{s_1\}$. If $u_1(s_1, s_2) \geqslant u_1(t_1, s_2)$, then $((s_1, s_2), (t_1, s_2)) \in R$ and then $(s_1, s_2) \in Y$, a contradiction. If instead $u_1(s_1, s_2) < u_1(t_1, s_2)$, then $((t_1, s_2), (s_1, s_2)) \in E_0(G)$. Since $(s_1, s_2) \in \mathbf{E}_0(G)$ there exists a path from (s_1, s_2) to (t_1, s_2) in $(X, E_0(G))$. Since that path is also a path from (s_1, s_2) to (t_1, s_2) in (X, R), we deduce that $(s_1, s_2) \in Y$, a contradiction. With analogous reasoning we can also prove that any strategy profile of the type (s_1, t_2) with $t_2 \in X_2 \backslash \{s_2\}$ cannot be an element of Y.

Consider now $(t_1, t_2) \in Y$. Thus, $s_1 \neq t_1$ and $s_2 \neq t_2$. Since $(t_1, s_2), (s_1, t_2) \notin Y$, we must have $u_1(t_1, t_2) > u_1(s_1, t_2)$ and $u_2(t_1, t_2) > u_2(t_1, s_2)$. Moreover, since G is strictly competitive, from the last inequality we get $u_1(t_1, t_2) < u_1(t_1, s_2)$.

Assume now by contradiction that $u_1(t_1, s_2) > u_1(s_1, s_2)$, that is, $((t_1, s_2), (s_1, s_2)) \in E_0(G)$. Thus, $\gamma = ((t_1, t_2), (t_1, s_2), (s_1, s_2))$ is a path from (t_1, t_2) to (s_1, s_2) in $E_0(G)$. Since $(s_1, s_2) \in \mathbf{E}_0(G)$, there exists a path from (s_1, s_2) to (t_1, s_2) in $(X, E_0(G))$. Since that path is also a path from (s_1, s_2) to (t_1, s_2) in (X, R), we deduce that $(s_1, s_2) \in Y$, a contradiction. Thus, it must be $u_1(t_1, s_2) \leqslant u_1(s_1, s_2)$. An analogous reasoning shows that it must be $u_2(s_1, t_2) \leqslant u_2(s_1, s_2)$ and, using the fact that G is strictly competitive we get $u_1(s_1, t_2) \geqslant u_1(s_1, s_2)$.

Using the inequalities previously proved, we get the chain

$$u_1(t_1, t_2) < u_1(t_1, s_2) \leqslant u_1(s_1, s_2) \leqslant u_1(s_1, t_2) < u_1(t_1, t_2).$$

We then deduce the contradiction $u_1(t_1, t_2) < u_1(t_1, t_2)$.

(ii) Assume that G is heterogeneous. From Theorem 17 and Proposition 39, we conclude that

$$|\mathbf{D}_\bullet(G)| = |\mathscr{A}(X, D(G))| \leqslant \min\{|X_1|, |X_2|\}.$$

Let us now prove that $|\mathbf{D}_\circ(G)| \leqslant \max\{1, \frac{\min\{|X_1|,|X_2|\}}{2}\}$. By Theorem 17, we know $|\mathbf{D}_\circ(G)| = |\mathscr{A}(X, D_*(G))|$. Thus, we are going to prove that $|\mathscr{A}(X, D_*(G))| \leqslant \max\{1, \frac{\min\{|X_1|,|X_2|\}}{2}\}$.

If $|X_1| \geqslant 2$ and $|X_2| \geqslant 2$, then we immediately conclude using Proposition 40. Assume now that $|X_1| \geqslant 2$ and $|X_2| = 1$. Consider the unique element $\sigma \in X_2$. By Lemma 26, we know that there exists $\widehat{s} \in X_1$ such that $\mathcal{B}_i^G(\sigma) = \{\widehat{s}\}$. Thus, for every $s \in X_1 \backslash \{\widehat{s}\}$, we have $((\widehat{s}, \sigma), (s, \sigma)) \in D_*(G)$. As a consequence, $|\mathscr{A}(X, D_*(G))| = 1 = \max\{1, \frac{\min\{|X_1|,|X_2|\}}{2}\}$. The same argument proves that the desired inequality is true if $|X_1| = 1$ and $|X_2| \geqslant 2$.

Finally, if G is trivial, then we have

$$|\mathscr{A}(X, D_*(G))| = |\{X\}| = 1 = \max\left\{1, \frac{\min\{|X_1|, |X_2|\}}{2}\right\}.$$

(iii) Assume that G is strongly heterogeneous. From Theorem 17 and Proposition 41, we easily deduce $|\mathbf{E}_\bullet(G)| = |\mathscr{A}(X, E(G))| \leqslant \min\{|X_1|, |X_2|\}$ and $|\mathbf{E}_\circ(G)| = 1$.

(iv) Assume that G is strongly heterogeneous and strictly competitive. Then the result follows from (i) and (iii).

(v) Assume that G is strongly heterogeneous and strictly competitive and $\mathbf{N}(G) \neq \varnothing$. Since $\mathbf{N}(G) \subseteq \mathbf{E}_\bullet(G)$ and, by (iv), $|\mathbf{E}_\bullet(G)| = 1$, we conclude that $\mathbf{N}(G) = \mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G)$. By Proposition 19, we also get $\mathbf{N}(G) = \mathbf{E}_\bullet(G) = \mathbf{E}_\circ(G) = \mathbf{E}(G)$. □

A.7 Proof of Theorem 21

Proof of Theorem 21. Assume that $G = \langle I, (X_i)_{i\in I}, (u_i)_{i\in I}\rangle$.

Let us first prove the equality $\mathbf{D}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{D}_\bullet}(G))$. By Theorem 4 in Gori (2023), we have

$$\mathbf{Sch}(N^{\mathbf{D}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\bullet}(G))),$$

where

$$\Sigma_\mu(N^{\mathbf{D}_\bullet}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\bullet}(x, y) \geqslant \mu\}.$$

We have $\mathrm{Im}(c_G^{\mathbf{D}_\bullet}) = (\mathscr{D}(G) \cap (0, +\infty)) \cup \{d(G)\}$. Assume first that $|\mathrm{Im}(c_G^{\mathbf{D}_\bullet})| = 1$. Thus, $\mathrm{Im}(c_G^{\mathbf{D}_\bullet}) = \{d(G)\}$ and, since G is non-trivial, by Lemma 27, it must be $\mathscr{D}(G) = \{0\}$. As a consequence, for every $t \in \mathbb{R}_+$, we have $D_t(G) = \varnothing$ and so $\mathbf{D}_t(G) = X$. Thus, $\mathbf{D}_\bullet(G) = X$. On the other hand,

$$\mathbf{Sch}(N^{\mathbf{D}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\bullet}(G))) = \mathbf{A}(X, \Sigma_{d(G)}(N^{\mathbf{D}_\bullet})) = \mathbf{A}(X, X_*^2) = X.$$

We then conclude that $\mathbf{D}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{D}_\bullet}(G))$.

Assume now that $|\mathrm{Im}(c_G^{\mathbf{D}_\bullet})| \geqslant 2$. Let $k \geqslant 1$ and $t_0, \ldots, t_k \in \mathbb{R}$ distinct be such that, for every $j \in \{1, \ldots, k\}$, $t_{j-1} < t_j$ and $\mathrm{Im}(c_G^{\mathbf{D}_\bullet}) = \{t_0, \ldots, t_k\}$. Thus, we have

- $t_0 = d(G) < 0$;
- $t_1 > 0$;
- $(\mathscr{D}(G) \cap \mathbb{R}_+) \cup \{0\} = \{0\} \cup \{t_1, \ldots, t_k\}$;
- $\Sigma_{t_0}(N^{\mathbf{D}_\bullet}(G)) = X_*^2$, and so $\mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{D}_\bullet}(G))) = X$;
- $\Sigma_{t_1}(N^{\mathbf{D}_\bullet}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\bullet}(x, y) \geqslant t_1\} = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\bullet}(x, y) > 0\} = D_0(G)$, and so $\mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{D}_\bullet}(G))) = \mathbf{D}_0(G)$;
- $D_{t_k}(G) = \varnothing$ and so $\mathbf{D}_{t_k}(G) = X$.

If $k = 1$, then, by Proposition 31, we have

$$\mathbf{Sch}(N^{\mathbf{D}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\bullet}(G))) = \mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{D}_\bullet}(G))) \cap \mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{D}_\bullet}(G)))$$

$$= X \cap \mathbf{D}_0(G) = \mathbf{D}_{t_1}(G) \cap \mathbf{D}_0(G) = \bigcap_{t \in (\mathscr{D}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{D}_t(G) = \mathbf{D}_\bullet(N).$$

If $k \geqslant 2$, for every $j \in \{2, \ldots, k\}$, we have

$$\Sigma_{t_j}(N^{\mathbf{D}_\bullet}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\bullet}(x, y) \geqslant t_j\} = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\bullet}(x, y) > t_{j-1}\} = D_{t_{j-1}}(G).$$

Thus, by Proposition 31,

$$\mathbf{Sch}(N^{\mathbf{D}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\bullet}(G))) = \bigcap_{j=0}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^{\mathbf{D}_\bullet}(G)))$$

$$= X \cap \mathbf{D}_0(G) \cap \bigcap_{j=2}^{k} \mathbf{A}(X, D_{t_{j-1}}(G)) = \mathbf{D}_0(G) \cap \bigcap_{j=1}^{k-1} \mathbf{D}_{t_j}(G)$$

$$= \mathbf{D}_0(G) \cap \bigcap_{j=1}^{k} \mathbf{D}_{t_j}(G) = \bigcap_{t \in (\mathscr{D}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{D}_t(G) = \mathbf{D}_\bullet(N),$$

as desired.

Next, let us prove the equality $\mathbf{D}_\circ(G) = \mathbf{Sch}(N^{\mathbf{D}_\circ}(G))$. By Theorem 4 in Gori (2023), we have

$$\mathbf{Sch}(N^{\mathbf{D}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\circ}(G))),$$

where

$$\Sigma_\mu(N^{\mathbf{D}_\circ}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\circ}(x, y) \geqslant \mu\}.$$

We have $\mathrm{Im}(c_G^{\mathbf{D}_\circ}) = \mathscr{D}(G) \cup \{d(G)\}$, and note that $|\mathrm{Im}(c_G^{\mathbf{D}_\circ})| \geqslant 2$ since G is non-trivial. Let $k \geqslant 1$ and $t_0, \ldots, t_k \in \mathbb{R}$ distinct be such that, for every $j \in \{1, \ldots, k\}$, $t_{j-1} < t_j$ and $\mathrm{Im}(c_G^{\mathbf{D}_\circ}) = \{t_0, \ldots, t_k\}$. Thus, we have

- $t_0 = d(G) < 0$;
- $\Sigma_{t_0}(N^{\mathbf{D}_\circ}(G)) = X_*^2$, and so $\mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{D}_\circ}(G))) = X$;
- $\Sigma_{t_1}(N^{\mathbf{D}_\circ}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\circ}(x, y) \geqslant t_1\} = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\circ}(x, y) > t_0\} = D_*(G)$, and so, by Proposition 29, $\mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{D}_\circ}(G))) = \mathbf{D}_{d(G)}(G) = \mathbf{D}_{t_0}(G)$;
- $D_{t_k}(G) = \varnothing$ and so $\mathbf{D}_{t_k}(G) = X$.

If $k = 1$, then, by Proposition 31, we have

$$\mathbf{Sch}(N^{\mathbf{D}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\circ}(G))) = \mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{D}_\circ}(G))) \cap \mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{D}_\circ}(G)))$$

$$= X \cap \mathbf{D}_{d(G)}(G) = \mathbf{D}_{d(G)}(G) \cap \mathbf{D}_{t_1}(G) = \bigcap_{t \in \mathscr{D}(G) \cup \{d(G)\}} \mathbf{D}_t(G) = \mathbf{D}_\circ(G).$$

If $k \geqslant 2$, then, for every $j \in \{2, \ldots, k\}$, we have

$$\Sigma_{t_j}(N^{\mathbf{D}_\circ}(G)) = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\circ}(x, y) \geqslant t_j\} = \{(x, y) \in X_*^2 : c_G^{\mathbf{D}_\circ}(x, y) > t_{j-1}\} = D_{t_{j-1}}(G).$$

Thus, by Proposition 31, we finally get

$$\mathbf{Sch}(N^{\mathbf{D}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{D}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{D}_\circ}(G))) = \bigcap_{j=0}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^{\mathbf{D}_\circ}(G)))$$

$$= X \cap \mathbf{D}_{d(G)}(G) \cap \bigcap_{j=2}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^{\mathbf{D}_\circ}(G))) = \mathbf{D}_{d(G)}(G) \cap \bigcap_{j=2}^{k} \mathbf{A}(X, D_{t_{j-1}}(G))$$

$$= \mathbf{D}_{d(G)}(G) \cap \bigcap_{j=1}^{k-1} \mathbf{D}_{t_j}(G) = \bigcap_{j=0}^{k} \mathbf{D}_{t_j}(G) = \bigcap_{t \in \mathscr{D}(G) \cup \{d(G)\}} \mathbf{D}_t(G) = \mathbf{D}_\circ(N),$$

as desired.

Let us now prove the equality $\mathbf{E}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{E}_\bullet}(G))$. By Theorem 4 in Gori (2023), we have

$$\mathbf{Sch}(N^{\mathbf{E}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\bullet}(G))),$$

where

$$\Sigma_\mu(N^{\mathbf{E}_\bullet}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) \geqslant \mu\}.$$

We have $\mathrm{Im}(c_G^{\mathbf{E}_\bullet}) = (\mathscr{E}(G) \cap (0,+\infty)) \cup \{e(G)\}$. Assume first that $|\mathrm{Im}(c_G^{\mathbf{E}_\bullet})| = 1$. Thus, $\mathrm{Im}(c_G^{\mathbf{E}_\bullet}) = \{e(G)\}$ and, since G is non-trivial, by Lemma 28, it must be $\mathscr{E}(G) = \{0\}$. As a consequence, for every $t \in \mathbb{R}_+$, we have $E_t(G) = \varnothing$ and so $\mathbf{E}_t(G) = X$. Thus, $\mathbf{E}_\bullet(G) = X$. On the other hand,

$$\mathbf{Sch}(N^{\mathbf{E}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\bullet}(G))) = \mathbf{A}(X, \Sigma_{e(G)}(N^{\mathbf{E}_\bullet})) = \mathbf{A}(X, X_*^2) = X.$$

We then conclude that $\mathbf{E}_\bullet(G) = \mathbf{Sch}(N^{\mathbf{E}_\bullet}(G))$.

Assume now that $|\mathrm{Im}(c_G^{\mathbf{E}_\bullet})| \geqslant 2$. Let $k \geqslant 1$ and $t_0, \dots, t_k \in \mathbb{R}$ distinct be such that, for every $j \in \{1, \dots, k\}$, $t_{j-1} < t_j$ and $\mathrm{Im}(c_G^{\mathbf{E}_\bullet}) = \{t_0, \dots, t_k\}$. Thus, we have

- $t_0 = e(G) < 0$;
- $t_1 > 0$;
- $(\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\} = \{0\} \cup \{t_1, \dots, t_k\}$;
- $\Sigma_{t_0}(N^{\mathbf{E}_\bullet}(G)) = X_*^2$, and so $\mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{E}_\bullet}(G))) = X$;
- $\Sigma_{t_1}(N^{\mathbf{E}_\bullet}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) \geqslant t_1\} = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) > 0\} = E_0(G)$, and so $\mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{E}_\bullet}(G))) = \mathbf{E}_0(G)$;
- $E_{t_k}(G) = \varnothing$ and so $\mathbf{E}_{t_k}(G) = X$.

If $k = 1$, then, by Proposition 32, we have

$$\mathbf{Sch}(N^{\mathbf{E}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\bullet}(G))) = \mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{E}_\bullet}(G))) \cap \mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{E}_\bullet}(G)))$$

$$= X \cap \mathbf{E}_0(G) = \mathbf{E}_{t_1}(G) \cap \mathbf{E}_0(G) = \bigcap_{t \in (\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{E}_t(G) = \mathbf{E}_\bullet(N).$$

If $k \geqslant 2$, for every $j \in \{2, \dots, k\}$, we have

$$\Sigma_{t_j}(N^{\mathbf{E}_\bullet}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) \geqslant t_j\} = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) > t_{j-1}\} = E_{t_{j-1}}(G).$$

Thus, by Proposition 32,

$$\mathbf{Sch}(N^{\mathbf{E}_\bullet}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\bullet})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\bullet}(G))) = \bigcap_{j=0}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^{\mathbf{E}_\bullet}(G)))$$

$$= X \cap \mathbf{E}_0(G) \cap \bigcap_{j=2}^{k} \mathbf{A}(X, E_{t_{j-1}}(G)) = \mathbf{E}_0(G) \cap \bigcap_{j=1}^{k-1} \mathbf{E}_{t_j}(G)$$

$$= \mathbf{E}_0(G) \cap \bigcap_{j=1}^{k} \mathbf{E}_{t_j}(G) = \bigcap_{t \in (\mathscr{E}(G) \cap \mathbb{R}_+) \cup \{0\}} \mathbf{E}_t(G) = \mathbf{E}_\bullet(N),$$

as desired.

Let us finally prove the equality $\mathbf{E}_\circ(G) = \mathbf{Sch}(N^{\mathbf{E}_\circ}(G))$. By Theorem 4 in Gori (2023), we have

$$\mathbf{Sch}(N^{\mathbf{E}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\circ}(G))),$$

where

$$\Sigma_\mu(N^{\mathbf{E}_\circ}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\circ}(x,y) \geqslant \mu\}.$$

We have $\mathrm{Im}(c_G^{\mathbf{E}_\circ}) = \mathscr{E}(G) \cup \{e(G)\}$, and note that $|\mathrm{Im}(c_G^{\mathbf{E}_\circ})| \geqslant 2$ since G is non-trivial. Let $k \geqslant 1$ and $t_0, \dots, t_k \in \mathbb{R}$ distinct be such that, for every $j \in \{1, \dots, k\}$, $t_{j-1} < t_j$ and $\mathrm{Im}(c_G^{\mathbf{E}_\circ}) = \{t_0, \dots, t_k\}$. Thus, we have

- $t_0 = e(G) < 0$;
- $\mathscr{E}(G) = \{t_1, \dots, t_k\}$;
- $\Sigma_{t_0}(N^{\mathbf{E}_\circ}(G)) = X_*^2$, and so $\mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{E}_\circ}(G))) = X$;
- $\Sigma_{t_1}(N^{\mathbf{E}_\bullet}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) \geqslant t_1\} = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\bullet}(x,y) > t_0\} = E_*(G)$, and so, by Proposition 30, $\mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{E}_\bullet}(G))) = X$;
- $E_{t_k}(G) = \varnothing$ and so $\mathbf{E}_{t_k}(G) = X$.

If $k = 1$, then, by Proposition 32, we have

$$\mathbf{Sch}(N^{\mathbf{E}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\circ}(G))) = \mathbf{A}(X, \Sigma_{t_0}(N^{\mathbf{E}_\circ}(G))) \cap \mathbf{A}(X, \Sigma_{t_1}(N^{\mathbf{E}_\circ}(G)))$$

$$= X \cap X = X = \mathbf{E}_{t_1}(G) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G) = \mathbf{E}_\circ(G).$$

If $k \geqslant 2$, then, for every $j \in \{2, \dots, k\}$, we have

$$\Sigma_{t_j}(N^{\mathbf{E}_\circ}(G)) = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\circ}(x,y) \geqslant t_j\} = \{(x,y) \in X_*^2 : c_G^{\mathbf{E}_\circ}(x,y) > t_{j-1}\} = E_{t_{j-1}}(G).$$

Thus, by Proposition 32, we finally get

$$\mathbf{Sch}(N^{\mathbf{E}_\circ}(G)) = \bigcap_{\mu \in \mathrm{Im}(c_G^{\mathbf{E}_\circ})} \mathbf{A}(X, \Sigma_\mu(N^{\mathbf{E}_\circ}(G))) = \bigcap_{j=0}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^\circ(G)))$$

$$= X \cap X \cap \bigcap_{j=2}^{k} \mathbf{A}(X, \Sigma_{t_j}(N^\circ(G))) = X \cap \bigcap_{j=2}^{k} \mathbf{A}(X, E_{t_{j-1}}(G))$$

$$= X \cap \bigcap_{j=1}^{k-1} \mathbf{E}_{t_j}(G) = \bigcap_{j=1}^{k} \mathbf{E}_{t_j}(G) = \bigcap_{t \in \mathscr{E}(G)} \mathbf{E}_t(G) = \mathbf{E}_\circ(N),$$

as desired. □

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