Anonymous and neutral majority rules*

Daniela Bubboloni
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze
e-mail: daniela.bubboloni@unifi.it

Michele Gori
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127, Firenze
e-mail: michele.gori@unifi.it

October 9, 2013

Abstract

In the standard arrovian framework and under the assumptions that individual preferences and social outcomes are linear orders over the set of alternatives, we provide necessary and sufficient conditions for the existence of anonymous and neutral rules and for the existence of anonymous and neutral majority rules. We determine also general formulas for counting these rules and we explicitly determine their number in some special cases.

Keywords: Social welfare function; anonymity; neutrality; majority; linear order; group theory.

JEL classification: D71

1 Introduction

Committees are often required to provide a strict ranking of a given family of alternatives, not only to determine which alternative is top-ranked. It is possible to design many procedures to aggregate committee members' preferences on alternatives into a strict ranking of alternatives. Among them, we are interested to analyse those satisfying certain principles usually invoked by social choice theorists. The first principle is the requirement that the identities of individuals are not used to determine the social outcome so that every individual opinion influences equally the collective decision. The second one is instead the requirement that any two alternatives are equally treated. These two principles, called respectively anonymity and neutrality, simply say that individual and alternative names are immaterial. Finally, we assume that the decision process also obeys to a majority principle, that is, each time a precisely specified and large enough amount of committee members ranks an alternative over another, that ranking has to be maintained in the final decision. The paper investigate under which conditions such special collective decision procedures can be really designed.

Before starting our inquiry, we need to clarify how committee members express their preferences. Usually the format for expressing preferences are voting for one alternative or strict ranking all alternatives (being these two methods equivalent when the alternatives are two). Of course, as we decided to deal with anonymous aggregation rules, the way preferences can be expressed has to be the same for each member in the committee. Moreover, as also neutrality is required, we are forced to focus only on the strict ranking format for preferences. Indeed, when alternatives are at least three, if all members of the committee unanimously voted the same alternative, they could not strict rank the not voted alternatives without treating them impartially. In other words, neutrality is not consistent with voting an alternative only, or more generally, with each way to express preferences leaving room for indifference among two or more alternatives.

Thanks to the considerations above, the considered aggregation rules can be now easily formalized within the well known arrovian framework. We consider h individuals in a committee and n alternatives

*Daniela Bubboloni was supported by the MIUR project "Teoria dei gruppi ed applicazioni (2009)". The authors wish to thank Prof. Domenico Menicucci for providing helpful suggestions.

to be ranked, and we assume that individual preferences and decision outcomes (or social preferences) are linear orders over the set of alternatives, that is, indifference between any pair of distinct alternatives is not allowed. A preference profile is a list of h linear orders each of them associated with the name of a specific committee member and representing her preferences. A rule is a function from the set of preference profiles to the set of social preferences, and each rule represents a particular decision process allowing to determine a ranking of alternatives from any conceivable individual preferences expressed by committee members. We emphasize that committee members' preferences on alternatives are the unique information used to make the decision, and that no strategic interaction among individuals is considered. As already said, we are going to focus on anonymous and neutral majority rules. In a more formal way, a rule is anonymous if it has the same value over any pair of preference profiles such that we can get one from the other by figuring to permute individual names. A rule is instead neutral if, for every pair of preference profiles such that we can get one from the other by figuring to permute alternative names, the social preferences associated with them coincide up to the considered permutation. Finally, given an integer ν not exceeding the number of members in the committee but exceeding half of it, a ν-majority rule is a rule ranking an alternative over another one if it is preferred to the other by at least ν individuals. Of course, each anonymous and neutral majority rule cannot be independent on the irrelevant alternatives due to Arrow's impossibility theorem.

Anonymity and neutrality are principles often used in social choice literature as they are usually considered criteria able to guarantee some extent of equity and fairness. They are also two of the main properties leading to characterizations of relative and absolute majority rules. In the specific case of two alternatives and when indifference is allowed both for individual and social preferences, May (1952) characterizes the relative majority in terms of anonymity, neutrality and positive responsiveness; Asan and Sanver (2006) characterize absolute qualified majority rules in terms of anonymity, neutrality and Maskin monotonicity; Sanver (2009) presents a unified exposition of the separate characterizations of relative and absolute majority rules, all of them involving anonymity and neutrality. Anonymity and neutrality are also properties used by Asan and Sanver (2002), Woeginger (2003) and Miroiu (2004) to characterize the relative majority when the structure of society is variable, that is, the number of the individuals is not fixed. In the general case for the number of alternatives and when individual preferences are linear orders, Maskin (1995) characterizes the majority rule using anonymity, neutrality, and some maximal transitivity condition; Can and Storcken (2012) characterize ν-majority correspondences (that is, set-valued rules) in terms, among other things, of anonymity and neutrality.

In the framework of social choice functions, that is, functions which associate an unique alternative to a preference profile, Moulin proves that anonymous and neutral social choice functions exist if and only if the number of alternatives n cannot be written as sum of non-trivial divisors of the number h of individuals (Moulin, 1983, Problem 1, p.25), and also that anonymous and neutral h-majority social choice functions exist if and only if

$$\gcd(h, n!) = 1, \tag{1}$$

where $\gcd(h, n!)$ denotes the greatest common divisor between h and $n!$ (Moulin, 1983, Theorem 1, p.23). Note that the coprimality condition (1), being equivalent to the requirement that each prime divisor of h must be greater than n, is quite rarely satisfied: while for two alternatives just h odd is asked, if alternatives are three or four we need h odd and not divisible by three.

Because of the unfriendly arithmetical characterization of anonymity and neutrality, Campbell and Kelly (2011, 2013), focus on the special case of two alternatives and prove that social choice functions satisfying monotonicity and a suitable weak versions of anonymity and neutrality are consistent with the simple majority principle. We finally observe that, in a remarkable paper dealing only with majority principles, Greenberg (1979, Corollary 3) proves that ν-majority social choice functions exist if and only if

$$\nu > \frac{n-1}{n} h, \tag{2}$$

where h, n and ν have the meaning above described.

While it is known that condition (2) is necessary and sufficient also for the existence of ν-majority rules[1], at the best of our knowledge, in the literature there is no result about which conditions on the parameters h, n and ν guarantee the existence of anonymous and neutral rules and anonymous and

[1]For the sake of completeness, following Can and Storken (2012, Example 4), we prove again that fact in Propositions 8 and 9.

neutral ν-majority rules. The major contribution of the paper is just the determination of such conditions as described by Theorems A and B below[2].

Theorem A. *There exists an anonymous and neutral rule if and only if* $\gcd(h, n!) = 1$.

Theorem B. *There exists an anonymous and neutral ν-majority rule if and only if* $\gcd(h, n!) = 1$ *and* $\nu > \frac{n-1}{n}h$.

As our main Theorem A shows, a fundamental role is played by the coprimality condition $\gcd(h, n!) = 1$, which imposes severe restrictions to the existence of anonymous and neutral rules. As a consequence, though natural and appealing, anonymity and neutrality are not requirements frequently reachable by a rule. This negative result, far to be discouraging, points out new reasonable research directions which we discuss in the last section of the paper. Note also that, as already pointed out, in the context of majority there is an identical arithmetic condition for social choice functions as well as for rules (that is, condition (2)). This does not happen for the requirements of anonymity and neutrality because not being n the sum of non-trivial divisors of h is not equivalent to (1), as the case $h = 4$ and $n = 3$ shows.

Theorem B shows instead that taking together the necessary and sufficient conditions for the existence of anonymous and neutral rules (condition (1)) and for the existence of majority rules (condition (2)), we get a set of necessary and sufficient conditions for the existence of anonymous and neutral majority rules. While one of the two implication is straightforward, the other one is not obvious and a bit unexpected.

Under condition (1), there are many other interesting results which we are able to prove in the paper. First of all, we prove that there exists an anonymous and neutral rule having the property that, for every preference profile, the corresponding social preference is consistent with all majority thresholds not generating cycles for that profile, that is, consistent with transitivity (Theorem 16). Moreover, we show that Theorem B implies a generalization of the already quoted theorem by Moulin for social choice functions (Theorem 17). We also provide general formulas for the order of all the sets of rules above mentioned (Proposition 18) and give explicit methods to construct all of them. Finally, using those formulas we compute those orders when $n = 2$ and h is odd (Section 10.1), and when $n = 3$ and $h = 5$ (Section 10.2). Note that, in the different framework where alternatives are only two and individual and social preferences can express indifference between them, the problem to count anonymous and neutral rules was solved by Perry and Power (2008).

In order to get the stated results, a previous analysis of the structure of the set of preference profiles is needed. Following an approach already explored by Eğecioğlu (2009), we carry on that analysis by means of the theory of the finite symmetric groups[3]. We consider the group G whose elements are ordered pairs of permutations: the first one defined over the set of individuals and interpreted as a way to permute individual names, the second one defined instead over the set of alternatives and interpreted as a way to permute alternative names. Of course, each element in G naturally induces the function which rewrites each preference profile by permuting individual and alternative names according to the pair of permutations in the considered group element. We prove that this function is indeed a bijection from the set of preference profiles into itself, and that the bijection induced by the product of two elements in G is the composition of the bijections induced by the two elements of G (Proposition 1). Using the proper algebraic expression, we have that G acts on the set of preference profiles.

Thanks to the action of G on the set of preference profiles, many general facts from group theory can be applied and, under assumption (1), several interesting properties about the structure of the set of preference profiles stem. In particular, given any preference profile, we have that any non-trivial change in alternative names generates a new profile from which we cannot obtain again the former one changing individual names only (Propositions 3). That result is crucial as it is the key ingredient in every existence proof involving anonymity and neutrality presented in the paper. Moreover we compute the order of the orbit of each preference profile under the group action, that is, the set of all preference profiles that can be obtained by it via a permutation in individual and alternative names (Proposition 19). Further, we

[2] Observe that Theorem A is just a rephrasing of Theorem 6, while Theorem B of Theorem 13.

[3] We emphasize that the use of group theory in social choice area is not a novelty. Kelly (1991), for instance, discusses the role of symmetry in the arrovian framework trough suitable subgroups of the symmetric group. Many results proved within the topological approach to social choices developed by Chichilnisky (1980) require the use of algebraic concepts and in particular that of symmetric group. The geometric approach and the symmetry arguments introduced by Donald Saari to understand paradoxes in voting, have been recently cast in a fully algebraic framework by Daugherty et al. (2009), inaugurating what is now called algebraic voting theory.

show that there are minimal sets of preference profiles, having the same known order (Proposition 23), and whose elements generate orbits that are a partition of the whole set of preference profiles. Note that, even though the result about the order of the described minimal set is already proved by Eğecioğlu (2009, Section 4.4), our proof has a different structure and is based on a preliminary original arithmetical relation directly suggested by the economic framework (equality (18)). Other properties about the set of preference profiles, useful for the described counting procedures, are also proved and discussed along the paper (Propositions 20, 21 and 22).

The structure of the paper is as follows. In Section 2 we introduce basic definitions and notation. Section 3 is devoted to describe some algebraic concepts and some properties of the set of preference profiles. In Section 4 we prove the existence results about anonymous and neutral rules. In Section 5 we propose the (already known) existence results about majority rules. Section 6 is about the arithmetical characterization of the anonymous and neutral majority rules, while Section 7 is about anonymous and neutral majority social choice functions. In Section 8 we propose general formulas for counting rules. In Section 9 we propose further properties of the set of preference profiles and in Section 10 we apply them to some explicit counting of the rules in simple cases. Finally, some concluding comments are proposed in Section 11. Some technical proofs are in the Appendix.

2 Definitions and notation

2.1 Arithmetics

Along the paper the symbol $\leq$ denotes the usual linear order in $\mathbb{R}$. Given $m, n \in \mathbb{N}$, we will use the notation $m \mid n$ to say that m divides n. Given $r \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r$, we denote the greatest common divisor and the least common multiple of $\lambda_1, \ldots, \lambda_r$ by $\gcd(\lambda)$ and $\mathrm{lcm}(\lambda)$, respectively. Given $k \in \mathbb{N}$, we define the set

$$\Pi(k) = \bigcup_{r=1}^{k} \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r : \sum_{j=1}^{r} \lambda_j = k, \lambda_1 \geq \ldots \geq \lambda_r \right\}$$

whose elements are called *partitions* of k. In other words, a partition of k is a decreasing list of positive integers whose sum is k. If $\lambda \in \Pi(k)$, we write $\lambda \vdash k$ and we call each component of λ a *part* of λ. Given $\lambda \vdash k$, $r(\lambda)$ denotes the dimension of the euclidean space which λ belongs to and, for every $j \in \{1, \ldots, k\}$, $a_j(\lambda)$ counts how many parts of λ are equal to j. Observe that $\sum_{j=1}^{k} j a_j(\lambda) = k$ and $r(\lambda) = \sum_{j=1}^{k} a_j(\lambda)$. Given $k, m \in \mathbb{N}$, we define also the set $\Pi_m(k) = \{\lambda \in \Pi(k) : r(\lambda) \leq m\}$. Of course, if $m \geq k$, then $\Pi_m(k) = \Pi(k)$. If $\lambda \in \Pi_m(k)$, we write $\lambda \overset{m}{\vdash} k$.

2.2 Symmetric groups

Let X be a finite set. We denote by $|X|$ the order of X and by $\mathfrak{F}(X)$ the set of functions from X to X. Given $f_1, f_2 \in \mathfrak{F}(X)$, we denote by $f_1 f_2$ the element of $\mathfrak{F}(X)$ defined as follows: for every $x \in X$, $f_1 f_2(x) = f_1(f_2(x))$. In other words, we denote the (right-to-left) composition of two functions by juxtaposition. Given $f_1, f_2 \in \mathfrak{F}(X)$, we call $f_1 f_2$ the product between f_1 and f_2. The subset of $\mathfrak{F}(X)$ made up by the bijective functions is denoted by $\mathrm{Sym}(X)$. Under the product of functions $\mathrm{Sym}(X)$, with $X \neq \varnothing$, is a finite group called the symmetric group on X, whose neutral element is the identity function, denoted by id_X or simply by id. If $|X| = 1$ then $Sym(X) = \{id\}$ is the *trivial group*. If $U \subseteq \mathrm{Sym}(X)$ is a subgroup of $\mathrm{Sym}(X)$, we use the notation $U \leq \mathrm{Sym}(X)$. Since X is finite, $U \neq \varnothing$ is a subgroup of $\mathrm{Sym}(X)$ if and only, for every $f_1, f_2 \in U$, $f_1 f_2 \in U$. Clearly $\{id\}, \mathrm{Sym}(X) \leq \mathrm{Sym}(X)$.

Fix $k \in \mathbb{N}$ and let $K = \{1, \ldots, k\}$. We denote $\mathrm{Sym}(K)$ simply by S_k and call its elements permutations on k objects. It is well known that $|S_k| = k!$. We say that $\gamma \in S_k$ is a *cycle of length 1* if $\gamma = id$. We say that $\gamma \in S_k$ is a *cycle of length* $l \geq 2$ if there exist distinct $x_1, \ldots, x_l \in K$ such that, for every $j \in \{1, \ldots, l-1\}$, $\gamma(x_j) = x_{j+1}$ and $\gamma(x_l) = x_1$, while $\gamma(x) = x$ for any $x \in K \setminus \{x_1, \ldots, x_l\}$. In that case we write $\gamma = (x_1 \ldots x_l)$. For instance, if $k \geq 3$, (123) is the cycle of length 3 which maps 1 into 2, 2 into 3, 3 into 1 and any further element (if any) into itself. We say that a cycle is *proper* if its length is greater than 1. Two cycles γ_1, γ_2 are *disjoint* if one of them is id or if $\gamma_1 = (x_1 \ldots x_{l_1})$ and $\gamma_2 = (y_1 \ldots y_{l_2})$

with suitable $l_1, l_2 \geq 2$ and $x_1, \dots, x_{l_1}, y_1, \dots, y_{l_2} \in K$ such that $\{x_1, \dots, x_{l_1}\} \cap \{y_1, \dots, y_{l_2}\} = \varnothing$. Disjoint cycles permute, that is, making their product in any order you get the same permutation.

Let $\sigma \in S_k$. The subgroup generated by σ, denoted by $\langle\sigma\rangle$, is defined by the intersection of all the subgroups of S_k containing σ and coincide with the finite set of its integer powers, that is, $\langle\sigma\rangle = \{\sigma^j : j \in \mathbb{Z}\}$. The natural number $|\langle\sigma\rangle|$, briefly denoted by $|\sigma|$, is called the *order* of σ. If s is the minimum exponent in $\mathbb{N}$ such that $\sigma^s = id$, then $\langle\sigma\rangle = \{\sigma^j : j \in \{1, \dots, s\}\}$ and $|\sigma| = s$. Note, in particular, that the a cycle has order l if and only if has length l. By Lagrange Theorem, $|\sigma|$ always divides $k!$ and if $s' \in \mathbb{Z}$ is such that $\sigma^{s'} = id$, then $|\sigma| \mid s'$. We say that $x \in K$ is a *fix point* for $\sigma \in S_k$ if $\sigma(x) = x$.

It is well known that each $\sigma \in S_k$ splits uniquely, up to the order, into the product $\sigma = \gamma_1 \cdots \gamma_r$ of $r \geq 1$ pairwise disjoint cycles $\gamma_1, \dots, \gamma_r \in S_k$ with $|\gamma_1| + \cdots + |\gamma_r| = k$. Note that σ has as many fix points as the number of $\gamma_j = id$. Given $\lambda \vdash k$, we say that $\sigma \in S_k$ is of *type* λ if $\sigma = \gamma_1 \cdots \gamma_{r(\lambda)}$ for some $\gamma_1, \dots, \gamma_{r(\lambda)}$ pairwise disjoint cycles and, for every $j \in \{1, \dots, r(\lambda)\}$, $|\gamma_j| = \lambda_j$. Note that σ^{λ_j} has at least λ_j fix points and that $|\sigma| = \mathrm{lcm}(\lambda)$.

Given $a \in \mathbb{N}$ and $(k_1, \dots, k_a) \in \mathbb{N}^a$, the direct product $\times_{j=1}^{a} S_{k_j}$ of the groups S_{k_j} is itself a group with the usual component by component operation. For $b \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, the notation S_k^b is used to denote the direct product of b copies of S_k if $b \geq 1$, while S_k^0 stands for the trivial group.

Any other notation and basic results used for permutations are standard (see, for instance, Wielandt (1964) and Rose (1978)).

2.3 Relations

Let X be a set. A relation $\mathcal{R}$ on X is a subset of $X \times X$. We say that a relation $\mathcal{R}_2$ on X extends a relation $\mathcal{R}_1$ on X if $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

A relation $\mathcal{R}$ on X is *reflexive* if, for every $x \in X$, $(x, x) \in \mathcal{R}$; *transitive* if, for every $x, y, z \in X$, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$; *complete* if, for every $x, y \in X$, we have $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$; *antisymmetric* if, for every $x, y \in X$, if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$, then $x = y$; contains the *cycle* $x_1, \dots, x_l$ with $l \in \mathbb{N}$, $l \geq 2$ if $x_1, \dots, x_l$ are distinct element of X such that, for every $j \in \{1, \dots, l-1\}$, $(x_j, x_{j+1}) \in \mathcal{R}$ and $(x_l, x_1) \in \mathcal{R}$; is *acyclic* if contains no cycle; an *order* if it is complete and transitive; a *linear order* or *total order* if it is complete, transitive and antisymmetric. It is well known that $\mathcal{R}$ can be extended to a linear order if and only if $\mathcal{R}$ is acyclic.

The set of linear orders over X is denoted by $\mathcal{L}(X)$. If X is a finite set and $\mathcal{R} \in \mathcal{L}(X)$, then for every $Y \subseteq X$ with $Y \neq \varnothing$, there exists a unique $x \in Y$ such that, for each $y \in Y$, $(x, y) \in \mathcal{R}$; such x is called the *maximum* of Y with respect to $\mathcal{R}$.

2.4 Linear orders and permutations

Fix $n \in \mathbb{N}$ and let $N = \{1, \dots, n\}$. Consider the set of vectors with n distinct components in N given by

$$\mathcal{A}(N) = \left\{a = (a_j)_{j=1}^n \in N^n : a_{j_1} = a_{j_2} \Rightarrow j_1 = j_2\right\}$$

and think the vector $a = (a_j)_{j=1}^n \in \mathcal{A}(N)$ as the column vector

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [a_1, \dots, a_n]^T.$$

Define the function $f : \mathcal{A}(N) \to \mathcal{L}(N)$, which associates to the vector $a = (a_j)_{j=1}^n \in \mathcal{A}(N)$ the relation $\mathcal{R} \in \mathcal{L}(N)$ such that, for every $a_{j_1}, a_{j_2} \in N$, $(a_{j_1}, a_{j_2}) \in \mathcal{R}$ if and only if $j_1 \leq j_2$.

On the basis of the existence of the maximum for each each nonempty subset of N and $\mathcal{R} \in \mathcal{L}(N)$, f is easily proved to be bijective. Thus, we will identify $\mathcal{R} \in \mathcal{L}(N)$ with the vector $f^{-1}(\mathcal{R}) \in \mathcal{A}(N)$, always thought as a column vector. For instance, we identify $\{(3,3), (1,1), (2,2), (3,1), (3,2), (1,2)\} \in \mathcal{L}(\{1,2,3\})$ with $[3,1,2]^T \in \mathcal{A}(\{1,2,3\})$.

Further, we say that $\mathcal{R} \in \mathcal{L}(N)$ is *canonical* if, for every $x, y \in N$, $(x, y) \in \mathcal{R}$ if and only if $x \leq y$. The element of $\mathcal{A}(N)$ corresponding to that linear order is then $[1, \dots, n]^T$. Note also that, since f is bijective and $|\mathcal{A}(N)| = n!$, we have also $|\mathcal{L}(N)| = n!$.

Given now a relation $\mathcal{R}$ on N and $\psi \in S_n$, we define the relation $\psi\mathcal{R}$ on N as follows: for every $x, y \in N$, $(x, y) \in \psi\mathcal{R}$ if and only if $(\psi^{-1}(x), \psi^{-1}(y)) \in \mathcal{R}$.

Assume now that $\mathcal{R} \in \mathcal{L}(N)$. It is easily checked that, for each $\psi \in S_n$, we have $\psi\mathcal{R} \in \mathcal{L}(N)$ and that, within the column vector interpretation of the linear orders, our definition is readable as

$$\psi \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \psi(a_1) \\ \vdots \\ \psi(a_n) \end{bmatrix}.$$

Note that $\psi\mathcal{R} = \mathcal{R}$ if and only if $\psi = id$ [4]. Moreover, if $\mathcal{R} \in \mathcal{L}(N)$ and $\psi_1, \psi_2 \in S_n$, then $(\psi_1\psi_2)\mathcal{R} = \psi_1(\psi_2\mathcal{R})$. In particular, for each fixed $\mathcal{R}_0 \in \mathcal{L}(N)$, the map $g : S_n \to \mathcal{L}(N)$ defined by $g(\psi) = \psi\mathcal{R}_0$ is injective and thus also bijective. This means that for each $\mathcal{R} \in \mathcal{L}(N)$ there exists $\psi \in S_n$ such that $\mathcal{R} = \psi\mathcal{R}_0$.

If $C \subseteq \mathcal{L}(N)$ and $\psi \in S_n$, we define also the subset of relations $\psi C = \{\psi\mathcal{R} : \mathcal{R} \in C\}$. Note that if $\psi_1, \psi_2 \in S_n$, then $(\psi_1\psi_2)C = \psi_1(\psi_2 C)$. Moreover since, for each $\psi \in S_n$, the map $l : C \to \mathcal{L}(N)$ defined, for every $\mathcal{R} \in C$, by $l(\mathcal{R}) = \psi\mathcal{R}$ is injective, then

$$|\psi C| = |C|. \tag{3}$$

2.5 Individual preferences and rules

From now on, let $h, n \in \mathbb{N}$ with $h, n \geq 2$ be fixed. Let $H = \{1, \ldots, h\}$ be the set of individuals and $N = \{1, \ldots, n\}$ be the set of alternatives. A *preference* over N is an element of $\mathcal{L}(N)$. Given $p_0 \in \mathcal{L}(N)$ and $x, y \in N$, we say that x is *at least as good as* y according to p_0, if $(x, y) \in p_0$ and x is *preferred* to y according to p_0 if $(x, y) \in p_0$ and $(y, x) \notin p_0$.[5] A *preference profile* is an element of $\mathcal{L}(N)^h$. The set $\mathcal{L}(N)^h$ is denoted by $\mathcal{P}$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preference of individual i. Any $p \in \mathcal{P}$ can be identified with the matrix whose i-th column is the column vector representing the i-th component of p. Note that $|\mathcal{P}| = n!^h$. A profile $p \in \mathcal{P}$ is called *constant* if there exists $q_0 \in \mathcal{L}(N)$ such that $p_i = q_0$ for all $i \in H$. In other words, a profile is constant if all the individuals express the same preference.

A *rule* or *social welfare function* is a function from $\mathcal{P}$ to $\mathcal{L}(N)$. The set of all rules is denoted by $\mathcal{F}$.

Consider now the group $G = S_h \times S_n$. For every $(\varphi, \psi) \in G$ and $p \in \mathcal{P}$, define $p^{(\varphi,\psi)} \in \mathcal{P}$ as the preference profile such that, for every $i \in H$,

$$\left(p^{(\varphi,\psi)}\right)_i = \psi p_{\varphi^{-1}(i)}. \tag{4}$$

The profile $p^{(\varphi,\psi)}$ is thus the profile obtained by p as if alternatives and individuals were renamed according to the following rules: for every $i \in H$, individual i is renamed $\varphi(i)$; for every $x \in N$, alternative x is renamed $\psi(x)$. For instance, if $n = 3, h = 5$ and

$$p = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 2 & 2 & 1 & 2 & 3 \\ 1 & 3 & 3 & 1 & 1 \end{bmatrix}, \quad \varphi = (134)(25), \quad \psi = (12), \tag{5}$$

we have

$$p^{(\varphi,id)} = \begin{bmatrix} 3 & 2 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 3 & 3 \end{bmatrix}, \quad p^{(id,\psi)} = \begin{bmatrix} 3 & 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 2 & 2 \end{bmatrix}, \quad p^{(\varphi,\psi)} = \begin{bmatrix} 3 & 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 2 & 1 \\ 2 & 2 & 2 & 3 & 3 \end{bmatrix}$$

Since we have given no meaning to $(p_i)^{(\varphi,\psi)}$ for a single preference $p_i \in \mathcal{L}(N)$, we will write the i-th component of the profile $p^{(\varphi,\psi)}$ simply as $p_i^{(\varphi,\psi)}$, instead of $\left(p^{(\varphi,\psi)}\right)_i$, because no misleading is possible.

A rule F is said *anonymous and neutral* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$,

$$F(p^{(\varphi,\psi)}) = \psi F(p), \tag{6}$$

[4] If a relation $\mathcal{R}$ is not a linear order, the equality $\psi\mathcal{R} = \mathcal{R}$ does not necessarily imply $\psi = id$.

[5] As p_0 is a linear order, when $x \neq y$ we have that x is *preferred* to y according to p_0 if and only if $(x, y) \in p_0$.

that is, collective decisions are independent on alternative and individual names. The set of anonymous and neutral rules is denoted by $\mathcal{F}^{\mathrm{an}}$.

Given $\nu \in \mathbb{N} \cap (h/2, h]$, let us define, for every $p \in \mathcal{P}$, the set

$$C_\nu(p) = \left\{q_0 \in \mathcal{L}(N) : \forall x, y \in N, |\{i \in H : (x,y) \in p_i, (y,x) \notin p_i\}| \geq \nu \Rightarrow (x,y) \in q_0, (y,x) \notin q_0\right\},$$

that is, the set of preferences having x preferred to y whenever, according to the preference profile p, at least ν individuals prefer x to y. Since $C_\nu(p) \subseteq \mathcal{L}(N)$ we obviously have $0 \leq |C_\nu(p)| \leq n!$. Moreover, if $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ with $\nu \leq \nu'$ and $p \in \mathcal{P}$, then $C_\nu(p) \subseteq C_{\nu'}(p)$.

A rule F is said a *ν-majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_\nu(p)$. The set of ν-majority rules is denoted by $\mathcal{F}_\nu$. Of course,

$$\mathcal{F}_\nu = \times_{p \in \mathcal{P}} C_\nu(p), \tag{7}$$

and if $\nu, \nu' \in \mathbb{N} \cap (h/2, h]$ with $\nu \leq \nu'$, then $\mathcal{F}_\nu \subseteq \mathcal{F}_{\nu'}$. As already explain in the introduction, our purpose is to investigate which condition over n, h and ν are necessary and sufficient to get $\mathcal{F}^{\mathrm{an}} \neq \varnothing$, $\mathcal{F}_\nu \neq \varnothing$ and $\mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_\nu \neq \varnothing$.

3 Properties of the set of preference profiles

In the present section we begin an analysis of the structure of the preference profile set $\mathcal{P}$ to be continued in Section 9. As explained in the introduction, our approach follows the one by Eğecioğlu (2009). We start with a basic result which allows to exploit many facts from group theory.

Proposition 1. *The function $f : G \to \mathfrak{F}(\mathcal{P})$ defined, for every $(\varphi, \psi) \in G$, as*

$$f(\varphi, \psi) : \mathcal{P} \to \mathcal{P}, \quad p \mapsto p^{(\varphi,\psi)},$$

maps G into $\mathrm{Sym}(\mathcal{P})$ and is a group homomorphism from G into $\mathrm{Sym}(\mathcal{P})$.

Proof. First of all, we note that, by definition (4), we have $f(id, id) = id$. Then we show that, for every $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in G$,

$$f((\varphi_1, \psi_1)(\varphi_2, \psi_2)) = f(\varphi_1, \psi_1) f(\varphi_2, \psi_2), \tag{8}$$

that is, for every $p \in \mathcal{P}$ and (φ_1, ψ_1), $(\varphi_2, \psi_2) \in G$,

$$p^{(\varphi_1\varphi_2, \psi_1\psi_2)} = \left(p^{(\varphi_2,\psi_2)}\right)^{(\varphi_1,\psi_1)}. \tag{9}$$

Indeed, for every $i \in H$ and $(x, y) \in N$, by definition (4), we have

$$(x,y) \in p_i^{(\varphi_1\varphi_2,\psi_1\psi_2)} \quad \Leftrightarrow \quad \left((\psi_1\psi_2)^{-1}(x), (\psi_1\psi_2)^{-1}(x)\right) \in p_{(\varphi_1\varphi_2)^{-1}(i)}$$

$$\Leftrightarrow \quad \left(\psi_2^{-1}\left(\psi_1^{-1}(x)\right), \psi_2^{-1}\left(\psi_1^{-1}(y)\right)\right) \in p_{\varphi_2^{-1}\left(\varphi_1^{-1}(i)\right)} \quad \Leftrightarrow \quad \left(\psi_1^{-1}(x), \psi_1^{-1}(y)\right) \in p_{\varphi_1^{-1}(i)}^{(\varphi_2,\psi_2)}$$

$$\Leftrightarrow \quad (x,y) \in \left(p^{(\varphi_2,\psi_2)}\right)_i^{(\varphi_1,\psi_1)}.$$

As a consequence, for every $(\varphi, \psi) \in G$, we get

$$f(\varphi, \psi) f(\varphi^{-1}, \psi^{-1}) = f(\varphi^{-1}, \psi^{-1}) f(\varphi, \psi) = f(id, id) = id.$$

Thus, $f(\varphi, \psi)$ is a function of $\mathcal{P}$ into itself with inverse $f(\varphi^{-1}, \psi^{-1})$, and therefore $f(\varphi, \psi) \in \mathrm{Sym}(\mathcal{P})$. Finally note that the fact that f is a homomorphism from the group G into the group $\mathrm{Sym}(\mathcal{P})$, is now exactly the content of equality (8). ☐

Accordingly with the language of group theory, Proposition 1 states that the function $f : G \to \mathrm{Sym}(\mathcal{P})$ defined, for every $(\varphi, \psi) \in G$, as $f(\varphi, \psi) : \mathcal{P} \to \mathcal{P}$, $\quad p \mapsto p^{(\varphi,\psi)}$, is an *action* of the group G on the set $\mathcal{P}$ or equivalently that G *acts* on the set $\mathcal{P}$ by the function f. Given $p \in \mathcal{P}$ and $g \in G$ we say that the action of g on p is given by $f(g)(p) = p^g \in \mathcal{P}$.

For every $p \in \mathcal{P}$, the set $\{p^g \in \mathcal{P} : g \in G\}$ is called the *orbit* of p under the action of G and is denoted by p^G. Each orbit is nonempty and finite and it is well known that, given $p, q \in \mathcal{P}$, $p^G \cap q^G = \varnothing$ or $p^G = q^G$. Moreover $\bigcup_{p \in \mathcal{P}} p^G = \mathcal{P}$. The set $\{p^G : p \in \mathcal{P}\}$ is nonempty and finite and we denote its order by R. R can be computed, for all h, n, in terms of partitions of h and n as shown by Eğecioğlu (2009, Theorem 3.1). Any vector $(p^j)_{j=1}^R \in \mathcal{P}^R$ such that $\mathcal{P}^G = \{p^{j\,G} : j \in \{1, \ldots, R\}\}$, is called *a system of representatives* of the orbits. The set of all the systems of representatives is nonempty and denoted by $\mathfrak{S}$. If $(p^j)_{j=1}^R \in \mathfrak{S}$, then $\{p^{j\,G} : j \in \{1, \ldots, R\}\}$ is a partition[6] of $\mathcal{P}$.

For every $p \in \mathcal{P}$, the *stabilizer* of p in G is the subgroup of G defined by

$$\mathrm{Stab}_G(p) = \{g \in G : p^g = p\}.$$

It is well known that the order of the orbit p^G can be expressed in terms of the stabilizer of p by

$$|p^G| = \frac{|G|}{|\mathrm{Stab}_G(p)|}, \tag{10}$$

and, in particular, the order of each orbits divides $|G| = n!h!$. The orbits p^G with $\mathrm{Stab}_G(p) = \{id\} \times \{id\}$, that is, with $|p^G| = n!h!$ are called *regular*. If there are any, then they realize the maximum for the orbits order (see Proposition 22). For every $g \in G$, the subset of $\mathcal{P}$ of profiles fixed by g under its action is denoted by

$$\mathrm{Fix}_{\mathcal{P}}(g) = \{p \in \mathcal{P} : p^g = p\}.$$

Clearly $g \in \mathrm{Stab}_G(p)$ if and only if $p \in \mathrm{Fix}_{\mathcal{P}}(g)$.

The following proposition describes the order of the set of preference profiles fixed by a given element of G. The formula we get appears inside the proof of Theorem 3.1 in Eğecioğlu (2009). We propose its proof in the Appendix.

Proposition 2. *Let $\lambda \vdash h$ and $(\varphi, \psi) \in G$, with φ of type λ. Then*

$$\left|\mathrm{Fix}_{\mathcal{P}}(\varphi, \psi)\right| = \begin{cases} n!^{r(\lambda)} & \text{if } \ |\psi| \mid \gcd(\lambda) \\ 0 & \text{if } \ |\psi| \nmid \gcd(\lambda) \end{cases}$$

We are now in position to show a fundamental result on the stabilizer. Indeed, Proposition 3 below is the key ingredient in each existence theorem of the paper and it is proved under the strong coprimality condition $\gcd(h, n!) = 1$. As we will show later, that condition is very natural in the context of anonymous and neutral rules.

Proposition 3. *Let* $\gcd(h, n!) = 1$ *and $p \in \mathcal{P}$. Then* $\mathrm{Stab}_G(p) \leq S_h \times \{id\}$.

Proof. Let $(\varphi, \psi) \in \mathrm{Stab}_G(p)$ and let $\lambda \vdash h$ be the type of $\varphi \in S_h$. As $p \in \mathrm{Fix}_{\mathcal{P}}(\varphi, \psi)$, by Proposition 2, we have that, for every $j \in \{1, \ldots, r(\lambda)\}$, $|\psi| \mid \lambda_j$. Since $h = \sum_{j=1}^{r(\lambda)} \lambda_j$, we get also $|\psi| \mid h$. On the other hand, $\psi \in S_n$ implies $|\psi| \mid n!$. Hence $|\psi| \mid \gcd(h, n!) = 1$, that is, $\psi = id$ and so $\mathrm{Stab}_G(p) \leq S_h \times \{id\}$. □

4 Anonymous and neutral rules

In this section we use the action of $G = S_h \times S_n$ on $\mathcal{P}$ and the properties of its stabilizer, established in Proposition 3, to reach a fundamental result: when $\gcd(h, n!) = 1$, we can assign an anonymous and neutral rule simply by assigning freely its values on a system of representatives for the orbits.

Proposition 4. *Let* $\gcd(h, n!) = 1$. *For every $(p^j)_{j=1}^R \in \mathfrak{S}$ and $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$, there exists a unique $F \in \mathcal{F}^{\mathrm{an}}$ such that, for every $j \in \{1, \ldots, R\}$, $F(p^j) = q_j$.*

Proof. Assume $\gcd(h, n!) = 1$ and let $(p^j)_{j=1}^R \in \mathfrak{S}$ and $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$. Since $\{p^{j\,G} : j \in \{1, \ldots, R\}\}$ is a partition of $\mathcal{P}$, given $p \in \mathcal{P}$, there exist $j \in \{1, \ldots, R\}$ and $(\varphi, \psi) \in G$ such that $p = p^{j\,(\varphi, \psi)}$ even though that representation is not necessarily unique.

[6] In the paper, a partition of a nonempty set X is a family of nonempty pairwise disjoint subsets of X whose union is X.

We show that if there exist $j_1, j_2 \in \{1, \ldots, R\}$ and $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in G$ such that $p^{j_1\,(\varphi_1,\psi_1)} = p^{j_2\,(\varphi_2,\psi_2)}$, then $j_1 = j_2$ and $\psi_1 = \psi_2$. By definition of system of representatives, from $p^{j_1\,(\varphi_1,\psi_1)} = p^{j_2\,(\varphi_2,\psi_2)}$, we have immediately that $j_1 = j_2$. Thus we reduce to $p^{j_1\,(\varphi_1,\psi_1)} = p^{j_1\,(\varphi_2,\psi_2)}$ and equality (9) gives

$$\left(p^{j_1\,(\varphi_1,\psi_1)}\right)^{(\varphi_2^{-1},\psi_2^{-1})} = p^{j_1\,(\varphi_2^{-1}\varphi_1,\psi_2^{-1}\psi_1)} = p^{j_1}.$$

Thus $(\varphi_2^{-1}\varphi_1, \psi_2^{-1}\psi_1) \in \mathrm{Stab}_G(p^{j_1})$ and, by Proposition 3, we obtain $\psi_2^{-1}\psi_1 = id$, that is $\psi_1 = \psi_2$.

Consider now the rule F defined, for every $p \in \mathcal{P}$ as $F(p) = \psi q_j$, where $j \in \{1, \ldots, R\}$ and $(\varphi, \psi) \in G$ are such that $p = p^{j\,(\varphi,\psi)}$. Note that, because of our previous remark, this definition is unambiguous. Let us prove now that $F \in \mathcal{F}^{\mathrm{an}}$, seeing that for each $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$ we have $F(p^{(\varphi,\psi)}) = \psi F(p)$. Indeed, let $p = p^{j\,(\varphi_*,\psi_*)}$ for some $j \in \{1, \ldots, R\}$ and $(\varphi_*, \psi_*) \in G$. By definition of F and by the equality (9), we have

$$F(p^{(\varphi,\psi)}) = F\left(\left(p^{j\,(\varphi_*,\psi_*)}\right)^{(\varphi,\psi)}\right) = F(p^{j\,(\varphi\varphi_*,\psi\psi_*)}) = (\psi\psi_*)q_j = \psi(\psi_* q_j) = \psi F(p^{j\,(\varphi_*,\psi_*)}) = \psi F(p).$$

In order to prove the uniqueness of F, it suffices to note that if $F' \in \mathcal{F}^{\mathrm{an}}$ is such that, for every $j \in \{1, \ldots, R\}$, $F'(p^j) = q_j$, then F' must satisfy also $F'(p^{j\,(\varphi,\psi)}) = \psi q_j = F(p^{j\,(\varphi,\psi)})$ for all $(\varphi, \psi) \in G$ and thus $F'(p) = F(p)$ for all $p \in \mathcal{P}$. □

Let $\gcd(h, n!) = 1$, $(p^j)_{j=1}^R \in \mathfrak{S}$ and $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$. Later on, we denote by $F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ the unique $F \in \mathcal{F}^{\mathrm{an}}$ such that, for every $j \in \{1, \ldots, R\}$, $F(p^j) = q_j$.

Proposition 5. *Let* $\gcd(h, n!) = 1$ *and* $(p^j)_{j=1}^R \in \mathfrak{S}$*. Then the function* $f : \mathcal{L}(N)^R \to \mathcal{F}^{\mathrm{an}}$ *defined, for every* $(q_j)_{j=1}^R \in \mathcal{L}(N)^R$*, as* $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$*, is bijective. In particular,* $|\mathcal{F}^{\mathrm{an}}| = n!^R$*.*

Proof. Straightforward. □

Theorem 6. $\mathcal{F}^{\mathrm{an}} \neq \varnothing$ *if and only if* $\gcd(h, n!) = 1$*.*

Proof. From Proposition 4, it immediately follows the "if" part. In order to prove the "only if" part, assume $\gcd(h, n!) \neq 1$ so that there exists an integer c, dividing h with $2 \leq c \leq n$ and suppose, by contradiction, that there exists $F \in \mathcal{F}^{\mathrm{an}}$. Let $p_0 = [1, \ldots, n]^T$ be the canonical linear order and $m = h/c \in \mathbb{N}$. Consider $\psi = (12\ldots c) \in S_n$ and note that, being $c \geq 2$, we have $\psi \neq id$. Define $p \in \mathcal{P}$ by $p_i = \psi^{i-1}p_0$ for every $i \in H$ and $\varphi = (1\ldots c)(c+1\ldots 2c)\cdots((m-1)c+1\ldots h) \in S_h$, a product of m cycles of length c. Then, for every $i \in H$, c divides the integer $\varphi(i) - (i+1)$.

We show now that $p^{(\varphi,\psi)} = p$, that is $p^{(\varphi,\psi)}_{\varphi(i)} = p_{\varphi(i)}$ for all $i \in H$. We have $p^{(\varphi,\psi)}_{\varphi(i)} = \psi p_i = \psi^i p_0$ and $p_{\varphi(i)} = \psi^{\varphi(i)-1}p_0$. Now observe that, since $c = |\psi|$ divides $\varphi(i) - (i+1)$, we have $\psi^{\varphi(i)-(i+1)} = id$ and thus $\psi^i = \psi^{\varphi(i)-1}$. It follows also that $F(p) = F(p^{(\varphi,\psi)}) = \psi F(p)$, which implies the contradiction $\psi = id$. □

5 Majority rules

We devote this section to explain under which arithmetical conditions on h, n, ν the ν-majority rules exist. As already discussed, the results are not original. However, we think it's useful to have them expressed in terms of our notation and inside our framework (for the proofs, we follow Can and Storken, 2012, Example 4). This enables us to introduce a new type of majority rules, called minimal majority rules, which exist independently on any arithmetical condition on h and n.

Define, for every $p \in \mathcal{P}$ and $x, y \in N$, the set

$$H(p, x, y) = \{i \in H : (x, y) \in p_i, (y, x) \notin p_i\} = \begin{cases} \varnothing & \text{if } x = y \\ \{i \in H : (x, y) \in p_i\} & \text{if } x \neq y \end{cases}$$

so that, for every $\nu \in \mathbb{N} \cap (h/2, h]$ and $p \in \mathcal{P}$,

$$C_\nu(p) = \left\{q_0 \in \mathcal{L}(N) : \forall x, y \in N, |H(p, x, y)| \geq \nu \Rightarrow (x, y) \in q_0\right\},$$

Lemma 7. *For every $p \in \mathcal{P}$, $x, y \in N$ and $(\varphi, \psi) \in G$, $H(p^{(\varphi,\psi)}, x, y) = \varphi\left(H(p, \psi^{-1}(x), \psi^{-1}(y))\right)$ and $|H(p^{(\varphi,\psi)}, x, y)| = |H(p, \psi^{-1}(x), \psi^{-1}(y))|$.*

Proof. In order to prove the first relation, note that if $x = y$ it is trivially verified, while if $x \neq y$ we have that

$$\begin{aligned} i \in H(p^{(\varphi,\psi)}, x, y) &\Leftrightarrow (x, y) \in p_i^{(\varphi,\psi)} = \psi p_{\varphi^{-1}(i)} \Leftrightarrow (\psi^{-1}(x), \psi^{-1}(y)) \in p_{\varphi^{-1}(i)} \\ &\Leftrightarrow \varphi^{-1}(i) \in H(p, \psi^{-1}(x), \psi^{-1}(y)) \Leftrightarrow i \in \varphi\left(H(p, \psi^{-1}(x), \psi^{-1}(y))\right). \end{aligned}$$

The second relation immediately follows by the fact that $\varphi \in S_h$, so that, $|H(p, \psi^{-1}(x), \psi^{-1}(y))| = |\varphi\left(H(p, \psi^{-1}(x), \psi^{-1}(y))\right)|$. □

Proposition 8. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n}h$. Then, for every $p \in \mathcal{P}$, $C_\nu(p) \neq \varnothing$. In particular, for every $p \in \mathcal{P}$, $C_h(p) \neq \varnothing$.*

Proof. Define, for every $p \in \mathcal{P}$, the set

$$\Sigma_\nu(p) = \{(x, y) \in N \times N : |H(p, x, y)| \geq \nu\} \tag{11}$$

representing a binary relation over N. Let us prove that $\Sigma_\nu(p)$ is acyclic. Assume by contradiction there exist $l \in \mathbb{N}$ with $l \geq 2$ and distinct $x_1, \ldots, x_l \in N$ such that, for every $j \in \{1, \ldots, l-1\}$, $(x_j, x_{j+1}) \in \Sigma_\nu(p)$ and $(x_l, x_1) \in \Sigma_\nu(p)$. Note that, in particular, $l \leq n$. Define also $x_{l+1} = x_1$. Then, for every $j \in \{1, \ldots, l\}$,

$$|H(p, x_j, x_{j+1})| \geq \nu > \frac{(n-1)h}{n} \geq \frac{(l-1)h}{l}. \tag{12}$$

Observe now that if it were

$$\bigcap_{j=1}^{l} H(p, x_j, x_{j+1}) = \varnothing$$

then, for every $i \in H$, we would obtain $|\{j \in \{1, \ldots, l\} : i \in H(p, x_j, x_{j+1})\}| \leq l - 1$. Thus, by (12) we would get

$$\begin{aligned} (l-1)h < \sum_{j=1}^{l} |H(p, x_j, x_{j+1})| &= |\{(j, i) \in \{1, \ldots, l\} \times H : i \in H(p, x_j, x_{j+1})\}| \\ &= \sum_{i \in H} |\{j \in \{1, \ldots, l\} : i \in H(p, x_j, x_{j+1})\}| \leq (l-1)h, \end{aligned}$$

which is a contradiction. As a consequence, we have

$$\bigcap_{j=1}^{l} H(p, x_j, x_{j+1}) \neq \varnothing$$

and there exists $i^* \in H$ such that, for every $j \in \{1, \ldots, l\}$, $(x_j, x_{j+1}) \in p_{i^*}$. Then p_{i^*} contains a cycle and cannot be a linear order, that is a contradiction. As $\Sigma_\nu(p)$ is an acyclic binary relation, it can be extended to a linear order $\widehat{\Sigma}_\nu(p)$ over N. Then $\widehat{\Sigma}_\nu(p) \in C_\nu(p)$ so that $C_\nu(p) \neq \varnothing$. □

Proposition 9. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu \leq \frac{n-1}{n}h$. Then, there exists $p \in \mathcal{P}$ such that $C_\nu(p) = \varnothing$.*

Proof. Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ with $\nu \leq \frac{n-1}{n}h$. Write $h = qn + r$, where $q \in \mathbb{N} \cup \{0\}$ and $r \in \{0, 1, \ldots, n-1\}$, and consider $p_0 = [1, \ldots, n]^T$, the canonical linear order relation on N. Let $\sigma = (12 \ldots n) \in S_n$ and define $p \in \mathcal{P}$, by $p_i = \sigma^{i-1} p_0$. Note that, for every $m \in \mathbb{N}$ and $x \in N$, there exists a unique element $\sigma^{m+i} p_0$ in the set of n linear order relations $\{\sigma^m p_0, \sigma^{m+1} p_0, \ldots, \sigma^{m+n-1} p_0\}$ such that $(\sigma(x), x) \in \sigma^{m+i} p_0$. Then, for every $x \in N$, $|H(p, x, \sigma(x))| \geq (n-1)q + \max\{0, r-1\}$. Moreover,

$$(n-1)q + \max\{0, r-1\} = (n-1)q + \left\lfloor \frac{n-1}{n} r \right\rfloor = \left\lfloor \frac{n-1}{n}(qn + r) \right\rfloor \geq \nu$$

It follows that, for every $x \in N$, $|H(p, x, \sigma(x))| \geq \nu$ and so the relation $\Sigma_\nu(p)$, defined in (11), contains the cycle $1, 2, \ldots, n$. Thus $\Sigma_\nu(p)$ cannot be extended to a linear order, that is, $C_\nu(p) = \varnothing$. □

By the definition of $\mathcal{F}_\nu$ and Propositions 8 and 9, the following result immediately follows.

Theorem 10. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then $\mathcal{F}_\nu \neq \varnothing$ if and only if $\nu > \frac{n-1}{n}h$.*

The role of the condition $C_\nu(p) \neq \varnothing$ in the two previous results, suggests to consider a majority rule in which, given a profile, the corresponding social preference supports the will of the largest possible number of individuals consistent with transitivity of the social preference.

For every $p \in \mathcal{P}$, let us define

$$\nu(p) = \min\{\nu \in \mathbb{N} \cap (h/2, h] : C_\nu(p) \neq \varnothing\}.$$

Of course, $\nu(p)$ is well defined as, by Proposition 8, for every $p \in \mathcal{P}$, $C_h(p) \neq \varnothing$ and $\nu(p) \in \mathbb{N} \cap (h/2, h]$. A rule F is said a *minimal majority rule* if, for every $p \in \mathcal{P}$, $F(p) \in C_{\nu(p)}(p)$. The set of minimal majority rules is denoted by $\mathcal{F}_{\min}$. Thus

$$\mathcal{F}_{\min} = \times_{p \in \mathcal{P}} C_{\nu(p)}(p).$$

and, by definition, $\mathcal{F}_{\min} \neq \varnothing$ independently by any arithmetic condition on h, n. Note that, for every $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n}h$, we have $\mathcal{F}_{\min} \subseteq \mathcal{F}_\nu \subseteq \mathcal{F}_h$.

6 Anonymous and neutral majority rules

The tools developed in the previous sections enable us to reach a complete description of the *anonymous and neutral majority rules*, that is of the rules in $\mathcal{F}^{\mathrm{an}}_\nu = \mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_\nu$: we can characterize in terms of h, n, ν not only when those rules actually exist (Theorem 13), but also we can give a method to construct all of them (Proposition 12), assigning their values on a system of representatives for the orbits. In particular we count how many anonymous and neutral majority rules there are for each h, n, ν. We do that expanding the procedure given for the anonymous and neutral rules (Proposition 5), through the friendly behaviour of the set C_ν with respect to the action of G (Lemma 11). Those ideas are as well used to deal with the set $\mathcal{F}^{\mathrm{an}}_{\min} = \mathcal{F}^{\mathrm{an}} \cap \mathcal{F}_{\min}$ of the *anonymous and neutral minimal majority rules* (Proposition 15 and Theorem 16).

Lemma 11. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, $C_\nu(p^{(\varphi,\psi)}) = \psi C_\nu(p)$ and, $|C_\nu(p^{(\varphi,\psi)})| = |C_\nu(p)|$. In particular, $\nu(p) = \nu(p^{(\varphi,\psi)})$.*

Proof. We prove first that, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$,

$$\psi C_\nu(p) \subseteq C_\nu(p^{(\varphi,\psi)}). \tag{13}$$

Let us consider then $p \in \mathcal{P}$, $(\varphi, \psi) \in G$, $q_0 \in C_\nu(p)$ and show that $\psi q_0 \in C_\nu(p^{(\varphi,\psi)})$. Assume that $x, y \in N$ are such that $|H(p^{(\varphi,\psi)}, x, y)| \geq \nu$. By Lemma 7, we have $|H(p, \psi^{-1}(x), \psi^{-1}(y))| \geq \nu$ and since $q_0 \in C_\nu(p)$, we have $(\psi^{-1}(x), \psi^{-1}(y)) \in q_0$, that is, $(x, y) \in \psi q_0$. Then $\psi q_0 \in C_\nu(p^{(\varphi,\psi)})$.

We are left to show that, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, we have also $\psi C_\nu(p) \supseteq C_\nu(p^{(\varphi,\psi)})$ that is $\psi^{-1} C_\nu(p^{(\varphi,\psi)}) \subseteq C_\nu(p)$. This is immediately seen because, by (13) and by (9), we have

$$\psi^{-1} C_\nu(p^{(\varphi,\psi)}) \subseteq C_\nu\left(\left(p^{(\varphi,\psi)}\right)^{(\varphi^{-1},\psi^{-1})}\right) = C_\nu(p).$$

Finally, by (3), $|C_\nu(p)| = |\psi C_\nu(p)|$ and since $C_\nu(p^{(\varphi,\psi)}) = \psi C_\nu(p)$ we get also $|C_\nu(p^{(\varphi,\psi)})| = |C_\nu(p)|$. In particular $\nu(p) = \nu(p^{(\varphi,\psi)})$. □

Proposition 12. *Let $\gcd(h, n!) = 1$, $\nu \in \mathbb{N} \cap (h/2, h]$ such that $\nu > \frac{n-1}{n}h$, and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then the function $f : \times_{j=1}^R C_\nu(p^j) \to \mathcal{F}^{\mathrm{an}}$ defined, for every $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$, as $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ has nonempty domain, is injective and its image is $\mathcal{F}^{\mathrm{an}}_\nu$. In particular, $|\mathcal{F}^{\mathrm{an}}_\nu| = \prod_{j=1}^R |C_\nu(p^j)| \neq 0$.*

Proof. By Proposition 8, the condition $\nu > \frac{n-1}{n}h$ implies that f has nonempty domain; by Proposition 4, the definition of f is well posed; the injectivity of f is trivial.

Let us prove now that $Im(f) \subseteq \mathcal{F}^{\mathrm{an}}_\nu$. Let $F \in Im(f)$ and show that $F \in \mathcal{F}^{\mathrm{an}}_\nu$. Surely $F \in \mathcal{F}^{\mathrm{an}}$ and there exists $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$ such that $F = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$. Consider now any $p \in \mathcal{P}$. Then there are $j \in \{1, \dots, R\}$ and $(\varphi, \psi) \in G$ such that $p = p^{j\,(\varphi,\psi)}$. As, for every $j \in \{1, \dots, R\}$, we know that $F(p^j) \in C_\nu(p^j)$, using Lemma 11 we have that

$$F(p) = F(p^{j\,(\varphi,\psi)}) = \psi F(p^j) \in \psi C_\nu(p^j) = C_\nu(p^{j\,(\varphi,\psi)}) = C_\nu(p).$$

Then, for every $p \in \mathcal{P}$, $F(p) \in C_\nu(p)$, that is, $F \in \mathcal{F}_\nu$.

In order to prove that $\mathcal{F}_\nu^{\mathrm{an}} \subseteq Im(f)$, let $F \in \mathcal{F}_\nu^{\mathrm{an}}$ and define, for every $j \in \{1 \dots, R\}$, $q_j = F(p^j)$. Then we immediately have $(q_j)_{j=1}^R \in \times_{j=1}^R C_\nu(p^j)$ and $F = f\left((q_j)_{j=1}^R\right)$, so that $F \in Im(f)$. □

Theorem 13. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. Then $\mathcal{F}_\nu^{\mathrm{an}} \neq \varnothing$ if and only if $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n} h$.*

Proof. The "if" part follows from Proposition 12. The "only if" part follows instead from Theorems 6 and 10. □

In particular, by Proposition 8, we immediately get the following corollary which is analogous to the result proved by Moulin (1983, Theorem 1, p.23) for social choice functions.

Corollary 14. *$\mathcal{F}_h^{\mathrm{an}} \neq \varnothing$ if and only if $\gcd(h, n!) = 1$.*

Proposition 15. *Let $\gcd(h, n!) = 1$ and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then the function $f : \times_{j=1}^R C_{\nu(p^j)}(p^j) \to \mathcal{F}^{\mathrm{an}}$ defined, for every $(q_j)_{j=1}^R \in \times_{j=1}^R C_{\nu(p^j)}(p^j)$, as $f\left((q_j)_{j=1}^R\right) = F\left[(p^j)_{j=1}^R, (q_j)_{j=1}^R\right]$ has nonempty domain, is injective and its image is $\mathcal{F}_{\min}^{\mathrm{an}}$. In particular, $|\mathcal{F}_{\min}^{\mathrm{an}}| = \prod_{j=1}^R |C_{\nu(p^j)}(p^j)| \neq 0$.*

Proof. by definition of $\nu(p)$ for $p \in \mathcal{P}$ we get immediately that f has nonempty domain and is injective. In order to prove that $Im(f) = \mathcal{F}_{\min}^{\mathrm{an}}$ we have only to observe that, from Lemma 11, $\nu(p) = \nu(p^j)$ for all $j \in \{1, \dots, R\}$ and $p \in p^{j\,G}$. Then we conclude using the same argument of Proposition 12. □

Theorem 16. *$\mathcal{F}_{\min}^{\mathrm{an}} \neq \varnothing$ if and only if $\gcd(h, n!) = 1$.*

Proof. The "if" part follows from Proposition 15. The "only if" part follows instead from Theorem 6. □

7 Anonymous and neutral majority social choice functions

The last existence result of the paper is about social choice functions: Theorem 17 below is an immediate consequence of Theorem 13, and generalizes Theorem 1 in Moulin (1983, p.23). We recall that a *social choice function* is a function $f : \mathcal{P} \to N$; it is called *anonymous and neutral* if, for every $p \in \mathcal{P}$ and $(\varphi, \psi) \in G$, $f(p^{(\varphi,\psi)}) = \psi(f(p))$; given $\nu \in \mathbb{N} \cap (h/2, h]$, it is called a *$\nu$-majority* if, for every $p \in \mathcal{P}$ and $x \in N \setminus \{f(p)\}$, we have $|H(p, x, f(p))| < \nu$.

Theorem 17. *Let $\nu \in \mathbb{N} \cap (h/2, h]$. There exists an anonymous and neutral ν-majority social choice function if and only if $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n} h$.*

Proof. The "only if" part follows Theorem 1 in Moulin (1983, p.23) and Corollary 3 in Greenberg (1979). In order to prove the "if" part assume that $\gcd(h, n!) = 1$ and $\nu > \frac{n-1}{n} h$ and, using Theorem 13, consider $F \in \mathcal{F}_\nu^{\mathrm{an}}$. Consider then the social choice function $f : \mathcal{P} \to N$ defined as follows: for every $p \in \mathcal{P}$, $f(p)$ is the unique maximum of the linear order $F(p)$. It is immediate to verify that f is an anonymous and neutral ν-majority social choice function. □

8 General formulas for counting the rules

Summarizing the content of theorems and propositions proved in Sections 4, 5 and 6, we get the following proposition which provides formulas for the order of all the sets of rules defined along the paper.

Proposition 18. *Let $\nu \in \mathbb{N} \cap (h/2, h]$ and $(p^j)_{j=1}^R \in \mathfrak{S}$. Then*

$$|\mathcal{F}| = n!^{(n!^h)} \tag{18.1}$$

$$|\mathcal{F}_\nu| = \prod_{j=1}^R |C_\nu(p^j)|^{|p^{j\,G}|} \tag{18.2}$$

$$|\mathcal{F}_{\min}| = \prod_{j=1}^R |C_{\nu(p^j)}(p^j)|^{|p^{j\,G}|} \tag{18.3}$$

$$|\mathcal{F}^{\rm an}| = \begin{cases} n!^R & \text{if } \gcd(h, n!) = 1 \\ 0 & \text{if } \gcd(h, n!) \neq 1 \end{cases} \tag{18.4}$$

$$|\mathcal{F}^{\rm an}_\nu| = \begin{cases} \prod_{j=1}^R |C_\nu(p^j)| & \text{if } \gcd(h, n!) = 1 \text{ and } \nu > \frac{n-1}{n}h \\ 0 & \text{if } \gcd(h, n!) \neq 1 \text{ or } \nu \leq \frac{n-1}{n}h \end{cases} \tag{18.5}$$

$$|\mathcal{F}^{\rm an}_{\min}| = \begin{cases} \prod_{j=1}^R |C_{\nu(p^j)}(p^j)| & \text{if } \gcd(h, n!) = 1 \\ 0 & \text{if } \gcd(h, n!) \neq 1 \end{cases} \tag{18.6}$$

Proof. In order to get (18.1), simply note that $|\mathcal{P}| = n!^h$. By definition (7) of $\mathcal{F}_\nu$ and Lemma 11, we have

$$|\mathcal{F}_\nu| = \prod_{p \in \mathcal{P}} |C_\nu(p)| = \prod_{j=1}^R \prod_{p \in p^j G} |C_\nu(p)| = \prod_{j=1}^R |C_\nu(p^j)|^{|p^j G|}$$

and (18.2) is proved. An analogous argument proves (18.3). Formula (18.4) follows from Proposition 5 and Theorem 6. Formula (18.5) follows from Proposition 12 and Theorem 13. Finally, (18.6) follows from Proposition 15 and Theorem 16. □

What Proposition 18 essentially says is that, when $\gcd(h, n!) = 1$, if we are able to determine $(p^j)_{j=1}^R \in \mathfrak{S}$ and compute, for every $j \in \{1, \ldots, R\}$ and $\nu \in \mathbb{N} \cap (h/2, h]$, the order of $p^{j\,G}$ and $C_\nu(p^j)$, then we obtain the order of the sets of rules above described. Of course, computing the mentioned objects is very hard in general. However, in the next section we are going to prove some further properties of the set of preference profile that will greatly simplify the task.

9 Further properties of the set of preference profiles

9.1 The block type

Let us introduce first some useful notation. For every $r \in \{1, \ldots, h\}$, define the nonempty set

$$\mathfrak{B}_r(H) = \left\{B = (B_1, \ldots, B_r) \in (2^H)^r : \{B_j\}_{j=1}^r \text{ is a partition of } H,\ |B_1| \geq \ldots \geq |B_r|\right\}$$

If $B = (B_1, \ldots, B_r) \in \mathfrak{B}_r(H)$, then $b = (|B_1|, \ldots, |B_r|) \vdash h$ and $r(b) = r$. Conversely, given $b \vdash h$ with $r(b) = r$, there exists $(B_1, \ldots, B_r) \in \mathfrak{B}_r(H)$ with $|B_i| = b_i$. Indeed, there might be more than one element in $\mathfrak{B}_r(H)$ having that property. Anyway among them one is surely the most natural: simply consider, for every $j \in \{1, \ldots, r\}$, the non-empty subset of H given by

$$\overline{B}_j(b) = \left\{ i \in H \ :\ \left(\sum_{m=1}^{j-1} b_m\right) + 1 \leq i \leq \left(\sum_{m=1}^{j} b_m\right)\right\}$$

where $\sum_{m=1}^0 b_m$ is assumed to be 0. Then $(\overline{B}_1(b), \ldots, \overline{B}_r(b)) \in \mathfrak{B}_r(H)$ will be called the *canonical partition* of H associated to b. Next, for $r \in \{1, \ldots, n!\}$, define the nonempty set

$$\mathcal{L}(N)^r_* = \{q = (q_1, \ldots, q_r) \in \mathcal{L}(N)^r : j_1 \neq j_2 \text{ implies } q_{j_1} \neq q_{j_2}\}.$$

Define also the nonempty set

$$\mathfrak{R} = \bigcup_{r=1}^{\min\{h,n!\}} \mathfrak{B}_r(H) \times \mathcal{L}(N)^r_*,$$

whose elements are given by (B, q), where $B = (B_1, \dots, B_r) \in \mathfrak{B}_r(H)$, $q = (q_1, \dots, q_r) \in \mathcal{L}(N)^r_*$ and $r \in \{1, \dots, \min\{h, n!\}\}$. Consider now the function

$$f : \mathfrak{R} \to \mathcal{P} \tag{14}$$

which associates with each $((B_1, \dots, B_r), (q_1, \dots, q_r)) \in \mathfrak{R}$ the unique preference profile p such that, for every $j \in \{1, \dots, r\}$ and $i \in B_j$, $p_i = q_j$. Observe that this definition is well posed because $\{B_j\}_{j=1}^r$ is a partition of H. Moreover, the function f is surjective. Yet, that map is not injective because, for instance, the image of f is not affected by a permutation of the B_j of equal size and an identical permutation of the corresponding q_j. We explicitly construct a right inverse

$$\Theta : \mathcal{P} \to \mathfrak{R} \tag{15}$$

for f. For every $p \in \mathcal{P}$, consider the $r \leq \min\{h, n!\}$ distinct $q_j \in \mathcal{L}(N)$ such that $\{q_j : j \in \{1, \dots, r\}\} = \{p_i : i \in \{1, \dots, h\}\}$; define, for every $j \in \{1, \dots, r\}$, the nonempty set $B_j = \{i \in H : p_i = q_j\}$ and number them in order to have $|B_1| \geq \ldots \geq |B_r|$ and $\min B_{j_1} < \min B_{j_2}$ when $|B_{j_1}| = |B_{j_2}|$ for some $j_1 < j_2$. Now define $B = (B_1, \dots, B_r) \in \mathfrak{B}_r(H)$, $q = (q_1, \dots, q_r) \in \mathcal{L}(N)^r_*$ and $\Theta(p) = (B, q)$. Clearly $f\Theta = id_{\mathcal{P}}$.

We call $\Theta(p) = (B, q)$ the *block representation* of p and $b(p) = (|B_1|, \dots, |B_r|)$ the partition of h associated to p. Of course, $b(p) \overset{n!}{\vdash} h$. Note that $b(p)$ does not depend on the choice of the right inverse Θ of f but only on p, because if $((B'_1, \dots, B'_{r'}), (q'_1, \dots, q'_{r'})) \in \mathfrak{R}$ is any element in $f^{-1}(p)$ then $r = r'$ and $|B'_j| = |B_j|$ for all $j \in \{1, \dots, r\}$.

Given $b \in \Pi_{n!}(h)$, we say that $p \in \mathcal{P}$ has *block type* b if $b(p) = b$. The set of preference profiles having block type b is denoted by $\mathcal{P}(b)$. Clearly, for each $b \in \Pi_{n!}(h)$, the set $\mathcal{P}(b)$ is nonempty and if $b_1, b_2 \in \Pi_{n!}(h)$, with $b_1 \neq b_2$, then $\mathcal{P}(b_1)$,$\mathcal{P}(b_2)$ are disjoint. In other words $\{\mathcal{P}(b) : b \in \Pi_{n!}(h)\}$ is a partition of $\mathcal{P}$ into $|\Pi_{n!}(h)|$ subsets. Moreover, for each $b \in \Pi_{n!}(h)$, $p \in \mathcal{P}(b)$ implies $p^G \subseteq \mathcal{P}(b)$. In particular $\mathcal{P}(b)$ is union of orbits. A simple but useful consequence is that if $p^1 \in \mathcal{P}(b_1)$ and $p^2 \in \mathcal{P}(b_2)$ for some $b_1, b_2 \in \Pi_{n!}(h)$, with $b_1 \neq b_2$, then p^1 and p^2 do not belong to the same orbit. The block representation $((B_1, \dots, B_r), (q_1, \dots, q_r))$ of p is said *canonical* if q_1 is canonical and $(B_1, \dots, B_r) = (\overline{B}_1(b), \dots, \overline{B}_r(b))$ is the canonical partition of H. Of course, the block representation of a preference profile is not necessarily canonical. However, in each orbit we can find a profile having a canonical block representation: such preference profile is called a *canonical block representative* of the orbit.

We illustrate the above concepts with an example. Let $n = 3$ and $h = 11$ and consider

$$p = \begin{bmatrix} 2 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 3 & 3 & 2 \\ 1 & 2 & 3 & 3 & 2 & 1 & 1 & 2 & 2 & 2 & 1 \\ 3 & 3 & 1 & 1 & 1 & 3 & 3 & 3 & 1 & 1 & 3 \end{bmatrix}.$$

Then the block representation of p is

$$((\{1, 6, 7, 11\}, \{5, 9, 10\}, \{2, 8\}, \{3, 4\}), ([2, 1, 3]^T, [3, 2, 1]^T, [1, 2, 3]^T, [2, 3, 1]^T)),$$

and that representation is not canonical. Considering now

$$\varphi_1 = (6\ 2\ 8\ 9)(3\ 10\ 7)(4\ 11) \in S_{11}, \quad \varphi_2 = (2\ 10\ 7\ 3\ 8\ 11\ 4\ 9\ 6) \in S_{11}, \quad \psi = (1\ 2) \in S_3.$$

we have that

$$p^{(\varphi_1,\psi)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 2 & 2 \end{bmatrix},$$

$$p^{(\varphi_2,\psi)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 3 & 3 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \end{bmatrix}.$$

Of course, $p^{(\varphi_1,\psi)}$ and $p^{(\varphi_2,\psi)}$ belong to p^G. Moreover, they respectively have the following canonical block representations

$$\big((\{1,2,3,4\},\{5,6,7\},\{8,9\},\{10,11\}),([1,2,3]^T,[3,1,2]^T,[1,3,2]^T,[2,1,3]^T)\big),$$

$$\big((\{1,2,3,4\},\{5,6,7\},\{8,9\},\{10,11\}),([1,2,3]^T,[3,1,2]^T,[2,1,3]^T,[1,3,2]^T)\big),$$

so that $p^{(\varphi_1,\psi)}$ and $p^{(\varphi_2,\psi)}$ are both canonical block representatives of p^G. Note also that p, $p^{(\varphi_1,\psi)}$ and $p^{(\varphi_2,\psi)}$ have the same block type $(4,3,2,2)$.

9.2 Orbits when $\gcd(h,n!)=1$

As shown by the following propositions, the concept of block type allows to get a deeper insight into the properties of the orbits. In fact, Proposition 19 shows that the order of the orbit of a preference profile depends only on its block type. Proposition 20 provides instead a formula for counting the orbits contained in the set of preference profiles having the same block type. Finally, Propositions 21 and 22 characterize, respectively, the orbits of minimal and maximal order in terms of block type. Our conclusion is that there always exists just one orbit of minimal order $n!$, while the regular orbits, that is those of maximal order $n!h!$, exist if and only if $h<n!$. The proofs of those propositions are in the Appendix.

Proposition 19. *Let* $\gcd(h,n!)=1$, $b\overset{n!}{\vdash} h$ *and* $p\in\mathcal{P}(b)$. *Then* $\mathrm{Stab}_G(p)$ *is isomorph to* $\times_{j=1}^{h} S_j^{a_j(b)}$ *and*

$$|p^G|=\frac{n!\,h!}{\prod_{j=1}^{h}\ j!^{\,a_j(b)}}. \tag{16}$$

Proposition 20. *Let* $\gcd(h,n!)=1$ *and* $b\overset{n!}{\vdash} h$. *Then* $\mathcal{P}(b)$ *is the union of*

$$\frac{\binom{n!}{r(b)}r(b)!}{n!\prod_{j=1}^{h}a_j(b)!} \tag{17}$$

orbits of equal order.

Proposition 21. *Let* $\gcd(h,n!)=1$ *and* $b\overset{n!}{\vdash} h$.

i) If $r(b)=1$, *then* $\mathcal{P}(b)$ *is made up of one orbit of order* $n!$.

ii) If $p\in\mathcal{P}(b)$ *and* $|p^G|=n!$, *then* $r(b)=1$.

In particular, there is only one orbit of order $n!$ *and it is constituted by all the preference profiles where all individuals express the same preference.*

Proposition 22. *Let* $\gcd(h,n!)=1$ *and* $b\overset{n!}{\vdash} h$.

i) If $r(b)=h$, *then* $\mathcal{P}(b)$ *is made up of* $\frac{1}{n!}\binom{n!}{h}$ *regular orbits.*

ii) If $p\in\mathcal{P}(b)$ *and* p^G *is regular, then* $r(b)=h$.

In particular, there exists a regular orbit if and only if $h<n!$ *and just one regular orbit if and only if* $h=n!-1$. *Moreover, a preference profile has a regular orbit if and only if all individual preferences are distinct.*

Note that, Propositions 21 and 22 imply in particular that if two preference profiles have orbits of minimum size ($n!$) or of maximum size ($n!h!$), then they have the same block type. However, that does not hold true, in general, for each pair of preference profiles having orbits of the same size. Consider, for instance, $n=4$ and $h=19$ and let

$$b=(1,1,2,2,2,2,3,3,3)\overset{4!}{\vdash}19,\quad b'=(1,1,1,1,1,2,2,3,3,4)\overset{4!}{\vdash}19.$$

Using Proposition 19, it is easy to check that both the orbits in $\mathcal{P}(b)$ and those in $\mathcal{P}(b')$ have order $\frac{19!}{2^4 \cdot 3^2}$.

Let us propose now a simple formula for the number R of orbits under the assumption $\gcd(h, n!) = 1$. That formula, already proved by Eğecioğlu (2009, Section 4.4)[7], is obtained here in a different manner, using Proposition 20 and the following equality, holding for every $m, k \in \mathbb{N}$:

$$\binom{m+k-1}{k-1} = \sum_{b \overset{k}{\vdash} m} \frac{\binom{k}{r(b)} r(b)!}{\prod_{j=1}^{m} a_j(b)\,!} \tag{18}$$

Equality (18) is an original arithmetic relation linking a classical balls-in-boxes counting with partitions. We leave its proof in the Appendix.

Proposition 23. *Let* $\gcd(h, n!) = 1$. *Then the number* R *of orbits in* $\mathcal{P}$ *is*

$$R = \frac{1}{n!} \binom{h+n!-1}{n!-1} \tag{19}$$

Proof. Since $\mathcal{P}$ is partitioned by the $\mathcal{P}(b)$, where $b \in \Pi_{n!}(h)$, by Proposition 20, we have that

$$R = \frac{1}{n!} \sum_{b \overset{n!}{\vdash} h} \frac{\binom{n!}{r(b)} r(b)!}{\prod_{j=1}^{h} a_j(b)\,!}. \tag{20}$$

To conclude it is enough to apply equality (18) with $k = n!$ and $m = h$. □

We emphasize how the arithmetical condition $\gcd(h, n!) = 1$, joined with the group theory approach enables us to reach a wide knowledge of the orbits of $\mathcal{P}$ under the action of G: namely we have a complete information about the number of orbits (Theorem 23), their order, the multiplicities of those with the same block type (Propositions 19 and 20) and we can identify both the minimal and the regular orbits (Propositions 21 and 22). This goes very far beyond the results found in Eğecioğlu (2009) under $\gcd(h, n!) = 1$, and it will be widely applied in the next section.

10 Counting the rules

Thanks to the result proved in the previous section and to Proposition 18, we explicit compute here the order of all the sets of rules considered in the paper, in the special cases when $n = 2$ and h odd, and when $n = 3$ and $h = 5$.

10.1 The case $n = 2$ and h odd

Let $n = 2$ and h odd. Let $k \in \mathbb{N}$ such that $h = 2k+1$. First of all, we note that $\mathbb{N} \cap (h/2, h] = \{k+1, \ldots, h\}$ and that

$$\Pi_{2!}(h) = \{(k+j, k+1-j) : j \in \{1, \ldots, k\}\} \cup \{(2k+1)\}.$$

In particular $|\Pi_{2!}(h)| = k+1$. On the other hand, by Theorem 23, the number of orbits is also $R = k+1$ so that, for every $b \overset{2!}{\vdash} h$, $\mathcal{P}(b)$ is made up by exactly one orbit and we can build a system of representatives of the orbits simply choosing, for every $b \overset{2!}{\vdash} h$, a preference profile having block type b. Then, a system of representatives of the orbits is given by $(p^j)_{j=1}^{k+1}$, where, for every $j \in \{1, \ldots, k+1\}$ and $i \in \{1, \ldots, h\}$,

$$p_i^j = \begin{cases} [1,2]^T & \text{if } i \leq k+j \\ [2,1]^T & \text{if } i > k+j \end{cases} \tag{21}$$

Note that those representatives have a canonical block representation. For every $j \in \{1, \ldots, k\}$ we have $b(p^j) = (k+j, k+1-j)$, with $k+j > k+1-j$ and $b(p^{k+1}) = (2k+1)$. Thus, using Proposition 19, we

[7] Eğecioğlu (2009) claims that this result was previously proved by Giritligil and Doğan (An Impossibility Result on Anonymous and Neutral Social Choice Functions, preprint). However, we could not find that reference.

get $|p^{j\,G}| = \frac{2(2k+1)!}{(k+j)!(k+1-j)!} = 2(2k+1)\dots(k+2-j)$ for all $j \in \{1,\dots,k+1\}$. Moreover, it is immediate to verify that, for every $\nu \in \mathbb{N} \cap (h/2, h]$,

$$C_\nu(p^j) = \begin{cases} \mathcal{L}(\{1,2\}) & \text{if } j < \nu - k \\ \{[1,2]^T\} & \text{if } j \geq \nu - k \end{cases}$$

By Proposition 18, applied to $n = 2$, $h = 2k+1$ and $\nu \in \{k+1,\dots,h\}$ we finally obtain

$$|\mathcal{F}| = 2^{2^{2k+1}}, \quad |\mathcal{F}_\nu| = \begin{cases} 1 & \text{if } \nu = k+1 \\ 4^{\sum_{j=1}^{\nu-k-1}(2k+1)\dots(k+2-j)} & \text{if } \nu \geq k+2 \end{cases}$$

$$|\mathcal{F}_{\min}| = 1, \quad |\mathcal{F}^{\text{an}}| = 2^{k+1}, \quad |\mathcal{F}^{\text{an}}_\nu| = 2^{\nu-k-1}, \quad |\mathcal{F}^{\text{an}}_{\min}| = 1.$$

In particular,

$$\mathcal{F}_{k+1} = \mathcal{F}_{\min} = \mathcal{F}^{\text{an}}_{k+1} = \mathcal{F}^{\text{an}}_{\min} = \{F_{maj}\},$$

where

$$F_{maj} : \mathcal{P} \to \mathcal{L}(\{1,2\}), \quad F_{maj}(p) = \begin{cases} [1,2]^T & \text{if } |H(p,1,2)| \geq k+1 \\ [2,1]^T & \text{if } |H(p,2,1)| \geq k+1 \end{cases}$$

is the simple majority rule.

10.2 The case $n = 3$ and $h = 5$

Let us devote now to carefully analyse the special case when $n = 3$ and $h = 5$. Thus $N = \{1,2,3\}$ and since $3! = 6$, we have $(3!, 5) = 1$, $\mathbb{N} \cap (\frac{h}{2}, h] = \{3,4,5\}$ and $\nu > \frac{n-1}{n}h$ if and only if $\nu \in \{4,5\}$. Moreover, we have that

$$\Pi_{3!}(5) = \{(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)\}.$$

By Proposition 23, we know that the number of orbits in $\mathcal{P}$ is 42, and by Propositions 19 and 20 we have that:

- $\mathcal{P}(5)$ is one orbit of order 6,
- $\mathcal{P}(4,1)$ is the union of 5 orbits of order 30,
- $\mathcal{P}(3,2)$ is the union of 5 orbits of order 60,
- $\mathcal{P}(3,1,1)$ is the union of 10 orbits of order 120,
- $\mathcal{P}(2,2,1)$ is the union of 10 orbits of order 180,
- $\mathcal{P}(2,1,1,1)$ is the union of 10 orbits of order 360,
- $\mathcal{P}(1,1,1,1,1)$ is one (regular) orbit of order 720.

Consider now the following preference profiles all of them having a canonical block representation

$$p^1 = \begin{bmatrix} 1&1&1&1&1\\ 2&2&2&2&2\\ 3&3&3&3&3 \end{bmatrix}, p^2 = \begin{bmatrix} 1&1&1&1&2\\ 2&2&2&2&1\\ 3&3&3&3&3 \end{bmatrix}, p^3 = \begin{bmatrix} 1&1&1&1&3\\ 2&2&2&2&2\\ 3&3&3&3&1 \end{bmatrix}, p^4 = \begin{bmatrix} 1&1&1&1&1\\ 2&2&2&2&3\\ 3&3&3&3&2 \end{bmatrix}$$

$$p^5 = \begin{bmatrix} 1&1&1&1&2\\ 2&2&2&2&3\\ 3&3&3&3&1 \end{bmatrix}, p^6 = \begin{bmatrix} 1&1&1&1&3\\ 2&2&2&2&1\\ 3&3&3&3&2 \end{bmatrix}, p^7 = \begin{bmatrix} 1&1&1&2&2\\ 2&2&2&1&1\\ 3&3&3&3&3 \end{bmatrix}, p^8 = \begin{bmatrix} 1&1&1&3&3\\ 2&2&2&2&2\\ 3&3&3&1&1 \end{bmatrix}$$

$$p^9 = \begin{bmatrix} 1&1&1&1&1\\ 2&2&2&3&3\\ 3&3&3&2&2 \end{bmatrix}, p^{10} = \begin{bmatrix} 1&1&1&2&2\\ 2&2&2&3&3\\ 3&3&3&1&1 \end{bmatrix}, p^{11} = \begin{bmatrix} 1&1&1&3&3\\ 2&2&2&1&1\\ 3&3&3&2&2 \end{bmatrix}, p^{12} = \begin{bmatrix} 1&1&1&2&3\\ 2&2&2&1&2\\ 3&3&3&3&1 \end{bmatrix}$$

$$p^{13} = \begin{bmatrix} 1&1&1&2&1\\ 2&2&2&1&3\\ 3&3&3&3&2 \end{bmatrix}, p^{14} = \begin{bmatrix} 1&1&1&2&2\\ 2&2&2&1&3\\ 3&3&3&3&1 \end{bmatrix}, p^{15} = \begin{bmatrix} 1&1&1&2&3\\ 2&2&2&1&1\\ 3&3&3&3&2 \end{bmatrix}, p^{16} = \begin{bmatrix} 1&1&1&3&1\\ 2&2&2&2&3\\ 3&3&3&1&2 \end{bmatrix}$$

$$p^{17} = \begin{bmatrix} 1&1&1&3&2\\ 2&2&2&2&3\\ 3&3&3&1&1 \end{bmatrix}, p^{18} = \begin{bmatrix} 1&1&1&3&3\\ 2&2&2&2&1\\ 3&3&3&1&2 \end{bmatrix}, p^{19} = \begin{bmatrix} 1&1&1&1&2\\ 2&2&2&3&3\\ 3&3&3&2&1 \end{bmatrix}, p^{20} = \begin{bmatrix} 1&1&1&1&3\\ 2&2&2&3&1\\ 3&3&3&2&2 \end{bmatrix}$$

$$p^{21}=\begin{bmatrix}1&1&1&2&3\\2&2&2&3&1\\3&3&3&1&2\end{bmatrix}, p^{22}=\begin{bmatrix}1&1&2&2&3\\2&2&1&1&2\\3&3&3&3&1\end{bmatrix}, p^{23}=\begin{bmatrix}1&1&2&2&1\\2&2&1&1&3\\3&3&3&3&2\end{bmatrix}, p^{24}=\begin{bmatrix}1&1&3&3&2\\2&2&2&2&1\\3&3&1&1&3\end{bmatrix}$$

$$p^{25}=\begin{bmatrix}1&1&3&3&1\\2&2&2&2&3\\3&3&1&1&2\end{bmatrix}, p^{26}=\begin{bmatrix}1&1&1&1&2\\2&2&3&3&1\\3&3&2&2&3\end{bmatrix}, p^{27}=\begin{bmatrix}1&1&1&1&3\\2&2&3&3&2\\3&3&2&2&1\end{bmatrix}, p^{28}=\begin{bmatrix}1&1&2&2&2\\2&2&3&3&1\\3&3&1&1&3\end{bmatrix}$$

$$p^{29}=\begin{bmatrix}1&1&2&2&3\\2&2&3&3&2\\3&3&1&1&1\end{bmatrix}, p^{30}=\begin{bmatrix}1&1&3&3&3\\2&2&1&1&2\\3&3&2&2&1\end{bmatrix}, p^{31}=\begin{bmatrix}1&1&3&3&2\\2&2&1&1&3\\3&3&2&2&1\end{bmatrix}, p^{32}=\begin{bmatrix}1&1&2&3&1\\2&2&1&2&3\\3&3&3&1&2\end{bmatrix}$$

$$p^{33}=\begin{bmatrix}1&1&2&3&2\\2&2&1&2&3\\3&3&3&1&1\end{bmatrix}, p^{34}=\begin{bmatrix}1&1&2&3&3\\2&2&1&2&1\\3&3&3&1&2\end{bmatrix}, p^{35}=\begin{bmatrix}1&1&2&1&2\\2&2&1&3&3\\3&3&3&2&1\end{bmatrix}, p^{36}=\begin{bmatrix}1&1&2&1&3\\2&2&1&3&1\\3&3&3&2&2\end{bmatrix}$$

$$p^{37}=\begin{bmatrix}1&1&2&2&3\\2&2&1&3&1\\3&3&3&1&2\end{bmatrix}, p^{38}=\begin{bmatrix}1&1&3&1&2\\2&2&2&3&3\\3&3&1&2&1\end{bmatrix}, p^{39}=\begin{bmatrix}1&1&3&1&3\\2&2&2&3&1\\3&3&1&2&2\end{bmatrix}, p^{40}=\begin{bmatrix}1&1&3&2&3\\2&2&2&3&1\\3&3&1&1&2\end{bmatrix}$$

$$p^{41}=\begin{bmatrix}1&1&1&2&3\\2&2&3&3&1\\3&3&2&1&2\end{bmatrix}, p^{42}=\begin{bmatrix}1&2&3&1&2\\2&1&2&3&3\\3&3&1&2&1\end{bmatrix}$$

Using the fact that two preferences profiles having different block type cannot be in the same orbit and simple algebraic arguments, it can be proved that:

- p^1 is a representative of the unique orbit in $\mathcal{P}(5)$,
- p^j for $2 \leq j \leq 6$ are representatives of the 5 orbits in $\mathcal{P}(4,1)$,
- p^j for $7 \leq j \leq 11$ are representatives of the 5 orbits in $\mathcal{P}(3,2)$,
- p^j for $12 \leq j \leq 21$ are representatives of the 10 orbits in $\mathcal{P}(3,1,1)$,
- p^j for $22 \leq j \leq 31$ are representatives of the 10 orbits in $\mathcal{P}(2,2,1)$,
- p^j for $32 \leq j \leq 41$ are representatives of the 10 orbits in $\mathcal{P}(2,1,1,1)$,
- p^{42} is a representative of the unique orbit in $\mathcal{P}(1,1,1,1,1)$.

In order to give to the reader the possibility to check what affirmed above, we illustrate why p^j for $22 \leq j \leq 31$ are representatives of the 10 orbits in $\mathcal{P}(2,2,1)$. The other facts can be similarly checked. First of all, let $p_0 = [1,2,3]^T$ be the canonical linear order on N and write all the columns of p^j in the form ψp_0 for suitable $\psi \in S_3$. We find the following equalities:

$$p^{22} = [p_0, p_0, (12)p_0, (12)p_0, (13)p_0], \quad p^{23} = [p_0, p_0, (12)p_0, (12)p_0, (23)p_0]$$

$$p^{24} = [p_0, p_0, (13)p_0, (13)p_0, (12)p_0], \quad p^{25} = [p_0, p_0, (13)p_0, (13)p_0, (23)p_0]$$

$$p^{26} = [p_0, p_0, (23)p_0, (23)p_0, (12)p_0], \quad p^{27} = [p_0, p_0, (23)p_0, (23)p_0, (13)p_0]$$

$$p^{28} = [p_0, p_0, (123)p_0, (123)p_0, (12)p_0], \quad p^{29} = [p_0, p_0, (123)p_0, (123)p_0, (13)p_0]$$

$$p^{30} = [p_0, p_0, (132)p_0, (132)p_0, (13)p_0], \quad p^{31} = [p_0, p_0, (132)p_0, (132)p_0, (123)p_0]$$

Assume that there exist $(\varphi, \psi) \in G = S_5 \times S_3$ and $23 \leq j \leq 31$ such that $p^{22\,(\varphi,\psi)} = p^j$. Thus, we have

$$[\psi p_0, \psi p_0, \psi(12)p_0, \psi(12)p_0, \psi(13)p_0]^{(\varphi, id)} = p^j$$

and since we need p_0 appearing twice, we must have $\psi p_0 = p_0$ or $\psi(12)p_0 = p_0$. In the first case we have $\psi = id$ which cannot work because the p^j are distinct. In the second case we have instead $\psi = (12)$ which cannot work because that implies

$$p^{22\,(\varphi,\psi)} = p^{22\,(\varphi,(12))} = [(12)p_0, (12)p_0, p_0, p_0, (132)p_0]^{(\varphi, id)}$$

and $(132)p_0$ and $(12)p_0$ never appear together in p^j with $23 \leq j \leq 31$. The analysis can now proceed dealing with p^{23} and showing that its orbit cannot contain some p^j with $24 \leq j \leq 31$ and then going on with the same strategy up to p^{30}.

Once one faces a system of representatives, the computation of the set $C_\nu(p^j)$ for every $j \in \{1, \dots, 42\}$ and $\nu \in \{3, 4, 5\}$ is easy and mechanic. Again, for the sake of clarity, we explain how to compute $C_\nu(p^j)$ in one case. Let consider p^{20} and $\nu = 4$: we see that 1 is preferred to 2 four times and 1 is preferred to 3 five times, while 2 is preferred to 3 three times. Thus, the 4-majority condition applies only to 1 with respect to 2 and to 1 with respect to 3, imposing us to find the set of linear orders $q_0 \in \mathcal{L}(N)$ such that $(1,2), (1,3) \in q_0$. Thus, we get $C_4(p^{20}) = \{[1,2,3]^T, [1,3,2]^T\}$.

The following table describe the results we find. Note that, in agreement with Proposition 9, there exists $j \in \{1, \dots, 42\}$ such that $C_3(p^j) = \varnothing$ while, in agreement with Proposition 8, for every $j \in \{1, \dots, 42\}$, $C_4(p^j) \neq \varnothing$ and $C_5(p^j) \neq \varnothing$.

	C_3	C_4	C_5
p^1	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$
p^2	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$
p^3	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^4	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$
p^5	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$
p^6	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$
p^7	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$
p^8	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^9	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$
p^{10}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$
p^{11}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$
p^{12}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{13}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [3,2,1]^T, [1,3,2]^T\}$
p^{14}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$
p^{15}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{16}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{17}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{18}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{19}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{20}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$
p^{21}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{22}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{23}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\{[1,2,3]^T, [3,2,1]^T, [1,3,2]^T\}$
p^{24}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{25}	$\{[1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{26}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\{[1,2,3]^T, [3,2,1]^T, [1,3,2]^T\}$
p^{27}	$\{[1,3,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{28}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$
p^{29}	$\{[2,3,1]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{30}	$\{[3,1,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{31}	$\varnothing$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{32}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [3,2,1]^T, [1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{33}	$\{[2,1,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T, [2,3,1]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{34}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{35}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{36}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{37}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{38}	$\{[1,2,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{39}	$\{[1,3,2]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{40}	$\varnothing$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$
p^{41}	$\{[1,2,3]^T\}$	$\{[1,2,3]^T, [1,3,2]^T, [3,1,2]^T\}$	$\mathcal{L}(\{1,2,3\})$
p^{42}	$\{[2,1,3]^T\}$	$\mathcal{L}(\{1,2,3\})$	$\mathcal{L}(\{1,2,3\})$

We can finally conclude that when $n = 3$ and $h = 5$ we have

$$|\mathcal{F}| = 6^{7776}, \quad |\mathcal{F}_5| = 2^{6690}3^{7590}, \quad |\mathcal{F}_4| = 2^{4740}3^{5460}, \quad |\mathcal{F}_3| = 0, \quad |\mathcal{F}_{\min}| = 2^{360}3^{540},$$

$$|\mathcal{F}^{\mathrm{an}}| = 6^{42}, \quad |\mathcal{F}_5^{\mathrm{an}}| = 2^{31}3^{37}, \quad |\mathcal{F}_4^{\mathrm{an}}| = 2^{21}3^{20}, \quad |\mathcal{F}_3^{\mathrm{an}}| = 0, \quad |\mathcal{F}_{\min}^{\mathrm{an}}| = 18.$$

11 Concluding comments

In the paper, we deal with the problem to understand whether the members of a committee could strict rank a given family of alternatives obeying the principles of anonymity, neutrality and majority. After having noted that anonymity and neutrality requirements are consistent only with members expressing their own preferences via strict rankings, we restricted to that case and proved that anonymous and neutral ranking procedures can be designed if and only if a suitable relation between the number of members in the committee and the number of alternatives holds true (Theorem A). Moreover, we showed there are anonymous and neutral ranking procedures satisfying also a given majority principle if and only if a further arithmetic relation among the number of members, the number of alternatives and the chosen majority threshold holds true (Theorem B).

Those results are proved thanks to a preliminary theoretical analysis of the set of preference profiles developed with some tools from group theory. Indeed, principles of anonymity and neutrality are naturally associated with an action of the group $S_h \times S_n$ on the set of preference profiles. That action allows to get a great deal of information about the set of preference profiles and allow us to prove not only the above mentioned results but many other facts. For instance, we find out a procedure to build and count all anonymous and neutral majority rules as well as the rules that are anonymous and neutral only. We firmly believe that the way we apply group theory for dealing with anonymous and neutral rules could be developed into further interesting directions.

Indeed, in the paper we focused on situations where individuals and alternatives cannot be distinguished and thus we focus on the action of the group $S_h \times S_n$. However, suppose to deal with a collective decision problem where a committee is assumed to be divided into two or more sub-committees whose members are known and impartially treated within. In that case, the anonymity principle in its universal version may fail because people belonging to certain sub-committees may influence the final decision in a different manner than the others, so that only a weaker type of anonymity can reasonably be expected. On the other hand, we can also conceive situations where one or more alternatives are favourite or distinguished, so that full neutrality is not a suitable property to impose to any model describing these new contexts. Nevertheless, our approach allows to model also partial anonymity and partial neutrality by considering the action on the set of preference profiles of suitable subgroups of $S_h \times S_n$. In fact, Proposition 1 can be proved in that more general framework and most of the algebraic machinery still works.

Note also that in our paper individual and social preferences are modelled as linear orders. In the interesting case where indifference is allowed and preferences are modelled as orders, it is immediate to verify that fundamental Proposition 1 still holds true in that new framework. Then, even though new tools and methods seem to be required to fully treat this setting, some parts of the theory we developed can surely be used to carry on that analysis.

We finally observe that our paper has an essentially constructive flavour as evidenced by the analysis of the case with three alternatives and five individuals. That suggests the possibility to implement algorithmically our results in order to explicitly determine the various kinds of rules.

Appendix

We provide here the proofs of Propositions 2, 19, 20, 21 and equality (18).

Proof of Proposition 2. First note that $p \in \mathrm{Fix}_{\mathcal{P}}(\varphi, \psi)$ means that, for every $i \in H$, $\psi p_{\varphi^{-1}(i)} = p_i$, that is, $\psi p_i = p_{\varphi(i)}$. Moreover, let $g = (\varphi, \psi)$, $J = \{1, \ldots, r(\lambda)\}$ and let $\varphi = \gamma_1 \ldots \gamma_{r(\lambda)}$ be a decomposition of φ into disjoint cycles $\gamma_j \in S_h$ such that $\lambda = (|\gamma_1|, \ldots, |\gamma_{r(\lambda)}|)$.

We prove, at first, that $\mathrm{Fix}_{\mathcal{P}}(g) \neq \varnothing$ implies $|\psi| \mid \gcd(\lambda)$. If $p \in \mathrm{Fix}_{\mathcal{P}}(g) \neq \varnothing$, then $g \in \mathrm{Stab}_G(p)$, so that, for every $j \in J$, also $g^{\lambda_j} = (\varphi^{\lambda_j}, \psi^{\lambda_j}) \in \mathrm{Stab}_G(p)$. Then, for every $j \in J$ and $i \in H$, we have

$\psi^{\lambda_j} p_i = p_{\varphi^{\lambda_j}(i)}$. Since φ^{λ_j} has at least $\lambda_j \geq 1$ fix points, picking one of them, say i_0, we get $\psi^{\lambda_j} p_{i_0} = p_{i_0}$ and thus $\psi^{\lambda_j} = id$. It follows that, for every $j \in J$, $|\psi| \mid \lambda_j$, that is, $|\psi| \mid \gcd(\lambda)$.

Next we show that if $|\psi| \mid \gcd(\lambda)$, then $|\mathrm{Fix}_{\mathcal{P}}(g)| = n!^{r(\lambda)}$. Note that $|\psi| \mid \gcd(\lambda)$ means that, for every $j \in J$, $\psi^{\lambda_j} = id$. Let $J_1 = \{j \in J : \lambda_j = 1\}$ and $J_2 = \{j \in J : \lambda_j \geq 2\}$. If $J_2 = \varnothing$ we have that $r(\lambda) = h$ and, for every $j \in J$, $\lambda_j = 1$, that is, $\varphi = id$. Then $\psi = id$ and $g = (id, id)$ which obviously fix any profile $p \in \mathcal{P}$, so that $|\mathrm{Fix}_{\mathcal{P}}(g)| = |\mathcal{P}| = n!^{r(\lambda)} = n!^h$.

Thus let assume that $J_2 \neq \varnothing$. Define, for every $j \in J_2$, the nonempty set $H_j = \{i \in H : \gamma_j(i) \neq i\}$. Define also the possibly empty set $\mathrm{Fix}_H(\varphi) = \{i \in H : \varphi(i) = i\}$. We get

$$H = \bigcup_{j \in J_2} H_j \cup \mathrm{Fix}_H(\varphi),$$

where the union above is a disjoint one, $|\mathrm{Fix}_H(\varphi)| = |J_1|$, $|H_j| = |\gamma_j| = \lambda_j \geq 2$, and

$$\varphi_{|H_j} = \gamma_{j|H_j}. \tag{22}$$

In order to describe the sets $\mathrm{Fix}_H(\varphi)$ and H_j, consider a bijection $f : \mathrm{Fix}_H(\varphi) \to J_1$ and fix, for every $j \in J_2$, an element $h_j \in H_j$. Then $\gamma_j = (h_j\ \gamma_j(h_j)\ \ldots \gamma_j^{\lambda_j - 1}(h_j))$ and

$$H_j = \{\gamma_j^s(h_j) : s \in \mathbb{Z}\}. \tag{23}$$

Our task is to define a subset $\mathcal{S}$ of $\mathrm{Fix}_{\mathcal{P}}(g)$ having order $n!^{r(\lambda)}$ and show that the profiles in $\mathcal{S}$ are the only ones in $\mathrm{Fix}_{\mathcal{P}}(g)$. In order to define $p \in \mathcal{S}$, we need to define, for every $i \in H$, its component p_i. First of all, choose, for every $j \in J$, $q_j \in \mathcal{L}(N)$. Then define

$$\text{for every } i \in \mathrm{Fix}_H(\varphi),\ p_i = q_{f(i)} \text{ and, for every } j \in J_2, s \in \mathbb{Z},\ p_{\gamma_j^s(h_j)} = \psi^s q_j. \tag{24}$$

Let us prove that, for every $j \in J_2$, (24) defines unambiguously the components of p whose index is in H_j. Consider $s_1, s_2 \in \mathbb{Z}$ such that $\gamma_j^{s_1}(h_j) = \gamma_j^{s_2}(h_j)$, that is, $\gamma_j^{s_1 - s_2}(h_j) = h_j$. Since $h_j \in H_j$, we have $\gamma_j(h_j) \neq h_j$ and, being γ_j a cycle of order λ_j, we have $\gamma_j^s(h_j) = h_j$ for some $s \in \mathbb{Z}$ if and only if $\lambda_j \mid s$. So λ_j divides $s_1 - s_2$ and by $|\psi| \mid \lambda_j$, we get that $|\psi|$ divides $s_1 - s_2$ and finally $\psi^{s_1} q_j = \psi^{s_2} q_j$. Clearly $|\mathcal{S}| = n!^{r(\lambda)}$.

Let us show now that $\mathcal{S} \subseteq \mathrm{Fix}_{\mathcal{P}}(g)$. Then we have to show that, for every $p \in \mathcal{S}$ and $i \in H$, $p_{\varphi(i)} = \psi p_i$. Let $p \in \mathcal{S}$ and let first $i \in H_j$ for some $j \in J_2$: then, by (23) we can write $i = \gamma_j^s(h_j)$ for some $s \in \mathbb{Z}$ and thus, by (24), (22) and (23), we get

$$p_{\varphi(i)} = p_{\varphi(\gamma_j^s(h_j))} = p_{\gamma_j^{s+1}(h_j)} = \psi^{s+1} q_j = \psi \psi^s q_j = \psi p_{\gamma_j^s(h_j)} = \psi p_i.$$

If $\mathrm{Fix}_H(\varphi) = \varnothing$ our check is finished. On the other hand, if $\mathrm{Fix}_H(\varphi) \neq \varnothing$ there exists $j \in J$ with $\lambda_j = 1$ and therefore, recalling that by assumption, for every $j \in J$, $\psi^{\lambda_j} = id$, we get $\psi = id$. It follows that $g = (\varphi, id)$ and the equality which remain to control is $p_{\varphi(i)} = p_i$ for all $i \in \mathrm{Fix}_H(\varphi)$, which is trivially true.

Now we see that $\mathrm{Fix}_{\mathcal{P}}(g) \subseteq \mathcal{S}$. Let $p \in \mathrm{Fix}_{\mathcal{P}}(g)$, then, for every $s \in \mathbb{Z}$, $p \in \mathrm{Fix}_{\mathcal{P}}(g^s)$, which means that, for every $i \in H$ and $s \in \mathbb{Z}$

$$p_{\varphi^s(i)} = \psi^s p_i. \tag{25}$$

For a profile in $\mathcal{S}$ the components corresponding to the individuals in $\mathrm{Fix}_H(\varphi)$, if any, are completely free and so we need only to check that for each $j \in J_2$ there exists $q_j \in \mathcal{L}(N)$ such that $p_{\gamma_j^s(h_j)} = \psi^s q_j$ for each $s \in \mathbb{Z}$. This is easily done defining $q_j = p_{h_j}$ and observing that by (22) and (25) $p_{\gamma_j^s(h_j)} = p_{\varphi^s(h_j)} = \psi^s p_{h_j} = \psi^s q_j$. □

Proof of Proposition 19. Let $((B_1, \ldots, B_r), (q_1, \ldots, q_r)) \in \mathfrak{R}$ be the block representation of p, where $r = r(b) \leq n!$. This means that

$$p_i = q_k \quad \text{if and only if} \quad i \in B_k. \tag{26}$$

Consider the subgroup Σ of S_h which leaves fixed each component B_k of the partition $B = (B_1, \ldots, B_r)$ of H, that is

$$\Sigma = \{\varphi \in S_h : \forall k \in \{1, \ldots, r\}, \varphi(B_k) = B_k\}. \tag{27}$$

Clearly Σ is isomorph to $\times_{k=1}^r \mathrm{Sym}(B_k)$ and then also to $\times_{j=1}^h S_j^{a_j(b)}$.

We show that $\mathrm{Stab}_G(p) = \Sigma \times \{id\}$. Let $(\varphi, id) \in \Sigma \times \{id\}$ and see that $p^{(\varphi,id)} = p$ that is $p_{\varphi(i)} = p_i$, for all $i \in H$. Since B is a partition of H, this can test through $p_{\varphi(i)} = p_i$, for all $i \in B_k$, $k \in \{1, \ldots, r\}$. But if $i \in B_k$ also $\varphi(i) \in B_k$, by definition of Σ, and thus $p_i = q_k = p_{\varphi(i)}$, by (26). Next let $g \in \mathrm{Stab}_G(p)$. Then, by Proposition 3, we have that $g = (\varphi, id)$ for some $\varphi \in S_h$ and thus $p_{\varphi(i)} = p_i$ for all $i \in H$. We need to see that $\varphi \in \Sigma$, that is $\varphi(B_k) \subseteq B_k$ for all $k \in \{1, \ldots, r\}$. But if $i \in B_k$ we have $p_i = q_k$ and thus also $p_{\varphi(i)} = q_k$, which by (26) gives $\varphi(i) \in B_k$.

It follows that

$$|\mathrm{Stab}_G(p)| = |\Sigma| = \left| \times_{j=1}^h S_j^{a_j(b)} \right| = \prod_{j=1}^h j!^{\,a_j(b)}$$

which, by (10), gives immediately (16). □

Proof of Proposition 20. By Proposition 19, we know that $\mathcal{P}(b)$ is union of orbits of equal known order and therefore we can get the desired result just computing $|\mathcal{P}(b)|$. For simplicity, set $r = r(b)$, $a_j = a_j(b)$ for $j \in \{1, \ldots, h\}$, and write $\overline{B} = (\overline{B}_1, \ldots, \overline{B}_r)$ instead of $\overline{B}(b) = (\overline{B}_1(b), \ldots, \overline{B}_r(b))$. Define $A = S_h \times \mathcal{L}(N)_*^r$ and consider the function

$$t : A \to \mathfrak{B}_r(H) \times \mathcal{L}(N)_*^r, \quad (\varphi, q) = (\varphi, (q_1, \ldots, q_r)) \mapsto \left((\varphi(\overline{B}_1), \ldots, \varphi(\overline{B}_r)), q\right).$$

Let now $f : \mathfrak{R} \to \mathcal{P}$ be the function defined in (14) and $\Theta : \mathcal{P} \to \mathfrak{R}$ its right inverse defined in (15). Since $\mathfrak{B}_r(H) \times \mathcal{L}(N)_*^r \subseteq \mathfrak{R}$, we can consider the function $ft : A \to \mathcal{P}$.

We claim first that $(ft)(A) = \mathcal{P}(b)$. In fact, if $(\varphi, q) \in A$ and $p = f(t(\varphi, q))$, then the partition of h associated to p is given by $b(p) = (|\varphi(\overline{B}_1)|, \ldots, |\varphi(\overline{B}_r)|) = (|\overline{B}_1|, \ldots, |\overline{B}_r|) = b$, so that $(ft)(A) \subseteq \mathcal{P}(b)$. On the other hand, let $p \in \mathcal{P}(b)$ and consider its block representation $\Theta(p) = ((B_1, \ldots, B_r), q)$. Of course, for every $j \in \{1, \ldots, r\}$, we have $|B_j| = b_j = |\overline{B}_j|$ and then, since both $(\overline{B}_1, \ldots, \overline{B}_r)$ and $(B_1, \ldots, B_r)$ are partitions of H, there exists a bijection $\varphi : H \to H$ such that, for every $j \in \{1, \ldots, r\}$, $\varphi(\overline{B}_j) = B_j$. Then taking $(\varphi, q) \in A$ we get $ft(\varphi, q) = f((\varphi(\overline{B}_1), \ldots, \varphi(\overline{B}_r)), q) = f((B_1, \ldots, B_r), q) = f\Theta(p) = p$.

So if we set $v = ft$, we have a surjective map $v : A \to \mathcal{P}(b)$. We consider the equivalence classes naturally defined in A by v, that is the sets $[(\varphi^0, q^0)]_v = v^{-1}(v(\varphi^0, q^0))$ where $(\varphi^0, q^0) \in A$ and we show that their order is $\left|[(\varphi^0, q^0)]_v\right| = \prod_{j=1}^h j!^{a_j} a_j!$, which is the same for any choice of $(\varphi^0, q^0) \in A$. First of all, we observe that, for every $\varphi \in S_h$ and $q \in \mathcal{L}(N)_*^r$,

$$v(\varphi, q) = v(id, q)^{(\varphi, id)}. \tag{28}$$

In fact, if $p = v(\varphi, q)$ we have $p = f((\varphi(\overline{B}_1), \ldots, \varphi(\overline{B}_r)), q)$ and so, for every $j \in \{1, \ldots, r\}$ and $i \in \varphi(\overline{B}_j)$, $p_i = q_j$, that is, for every $j \in \{1, \ldots, r\}$ and $i \in \overline{B}_j$, $p_{\varphi(i)} = q_j$. Now let $p' = v(id, q)^{(\varphi, id)}$: by definition of action (4) we have that, for every $j \in \{1, \ldots, r\}$ and $i \in \overline{B}_j$, $p'_{\varphi(i)} = v(id, q)_i = f((\overline{B}_1, \ldots, \overline{B}_r), q)_i = q_j = p_i$. Thus $p = p'$. This property implies that there is a bijection between $[(\varphi^0, q^0)]_v$ and $[(id, q^0)]_v$. Indeed, $(\varphi, q) \in [(\varphi^0, q^0)]_v$ means $v(\varphi, q) = v(\varphi^0, q^0)$. By (9) and (28), we have $v(id, q)^{(\varphi(\varphi^0)^{-1}, id)} = v(id, q^0)$, that is, $v(\varphi(\varphi^0)^{-1}, q) = v(id, q^0)$ equivalent to $(\varphi(\varphi^0)^{-1}, q) \in [(id, q^0)]_v$. Thus we get the map $\delta : [(\varphi^0, q^0)]_v \to [(id, q^0)]_v$ defined by $\delta(\varphi, q) = (\varphi(\varphi^0)^{-1}, q)$ which is clearly a bijection. As a consequence, given $(\varphi^0, q^0) \in A$, in order to compute the order of $[(\varphi^0, q^0)]_v$ it is enough to compute that of $[(id, q^0)]_v$.

Consider the set

$$\Phi(\overline{B}) = \left\{\varphi \in S_h : \forall k \in \{1, \ldots, r\}, \varphi\left(\overline{B}_k\right) \in \{\overline{B}_j\}_{j=1}^r\right\}$$

and observe that it consists of those permutations of S_h permuting among themselves the components of $\overline{B}$ of the same order. Moreover, for each $\varphi \in \Phi(\overline{B})$, there exists a unique element $\widehat{\varphi} \in S_r$ such that, for every $j \in \{1, \ldots, r\}$, $\varphi\left(\overline{B}_j\right) = B_{\widehat{\varphi}(j)}$.

We claim that, for every $q^0 \in \mathcal{L}(N)_*^r$, $\left|[(id, q^0)]_v\right| = \left|\Phi(\overline{B})\right|$. Define the function

$$\Gamma : \Phi(\overline{B}) \to A, \quad \varphi \mapsto (\varphi, (q^0_{\widehat{\varphi}(1)}, \ldots, q^0_{\widehat{\varphi}(r)})),$$

and note that Γ is trivially injective. We show that the image of Γ is $[(id, q^0)]_v$. First of all for every $\varphi \in \Phi(\overline{B})$, we have that

$$v(\Gamma(\varphi)) = f((\varphi(\overline{B}_1), \ldots, \varphi(\overline{B}_r)), (q^0_{\widehat{\varphi}(1)}, \ldots, q^0_{\widehat{\varphi}(r)})) = f((\overline{B}_{\widehat{\varphi}(1)}, \ldots, \overline{B}_{\widehat{\varphi}(r)}), (q^0_{\widehat{\varphi}(1)}, \ldots, q^0_{\widehat{\varphi}(r)}))$$

$$= f((\overline{B}_1,\dots,\overline{B}_r),(q_1^0,\dots,q_r^0)) = v(id,q^0),$$

which says that $\Gamma(\varphi)\in[(id,q^0)]_v$. We are left to prove $[(id,q^0)]_v\subseteq\Gamma(\Phi(\overline{B}))$. Given $(\varphi,q)\in[(id,q^0)]_v$, we have that $p=p'$, where

$$p = f((\varphi(\overline{B}_1),\dots,\varphi(\overline{B}_r)),(q_1,\dots,q_r)) \quad \text{and} \quad p' = f((\overline{B}_1,\dots,\overline{B}_r),(q_1^0,\dots,q_r^0)).$$

Consider then $k\in\{1,\dots,r\}$ and assume by contradiction that there exists $i_1,i_2\in\varphi(\overline{B}_k)$ such that $i_1\in\overline{B}_{j_1}$ and $i_2\in\overline{B}_{j_2}$, where $j_1,j_2\in\{1,\dots,r\}$ and $j_1\neq j_2$. Then $q_k=p_{i_1}=p'_{i_1}=q^0_{j_1}$ and $q_k=p_{i_2}=p'_{i_2}=q^0_{j_2}$ so that $q^0_{j_1}=q^0_{j_2}$, against $q^0\in\mathcal{L}(N)^r_*$. Then $\varphi(\overline{B}_k)\in\{\overline{B}_j\}_{j=1}^r$ and consequently, for every $j\in\{1,\dots,r\}$, $q_j=q^0_{\widehat{\varphi}(j)}$. In other words, $\varphi\in\Phi(\overline{B})$ and $\Gamma(\varphi)=(\varphi,q)$. Since we have proved that $[(id,q^0)]_v=\Gamma(\Phi(\overline{B}))$ and we know that Γ is injective, we get also $\left|[(id,q^0)]_v\right|=\left|\Phi(\overline{B})\right|$.

Now we need to compute the order of $\Phi(\overline{B})$. Given $x,y\in\mathbb{N}$, it is well known that the set of the permutations in S_{xy} permuting among themselves y disjoint sets of order x, is a subgroup of S_{xy}. That subgroup is called the *wreath product* of S_x and S_y and it is denoted by $S_x\wr S_y$. Its order is $|S_x\wr S_y|=x!^y y\,!$. Thus, setting $J=\{j\in\{1,\dots,h\}:a_j\neq 0\}$, we have that

$$\left|\Phi(\overline{B})\right| = \left|\times_{j\in J}\left[S_j\wr S_{a_j}\right]\right| = \prod_{j\in J} j!^{a_j}a_j\,! = \prod_{j=1}^h j!^{a_j}a_j\,!.$$

It follows that

$$|\mathcal{P}(b)| = \frac{|A|}{\prod_{j=1}^h j!^{a_j}a_j\,!} = \frac{n!(n!-1)\cdots(n!-r+1)\,h!}{\prod_{j=1}^h j!^{a_j}a_j\,!}$$

and therefore, by Proposition 19, the number of orbits is

$$\frac{|\mathcal{P}(b)|}{|p^G|} = \frac{n!(n!-1)\cdots(n!-r+1)}{n!\prod_{j=1}^h a_j!} = \frac{\binom{n!}{r}r!}{n!\prod_{j=1}^h a_j!}.$$

□

Proof of Proposition 21. i) If $r(b)=1$, then $b=(h)$ so that, for every $j\in\{1,\dots,h-1\}$, $a_j(b)=0$ while $a_h(b)=1$. Thus, by Proposition 20 and Proposition 19, we get that $\mathcal{P}(b)$ is made up of one orbit of order $n!$.

ii) Let $r=r(b)$ and $|p^G|=n!$, for some profile $p\in\mathcal{P}(b)$. Let $((B_1,\dots,B_r),(q_1,\dots,q_r))\in\mathfrak{R}$ be the block representation of p. Consider the subgroup Σ of S_h formed by the permutations which leave each component of $B=(B_1,\dots,B_r)$ fixed and note that, since Σ is isomorph to $\times_{j=1}^h S_j^{a_j(b)}$, we have $|\Sigma|=\prod_{j=1}^h j!^{\,a_j(b)}$. By Proposition 19, $|p^G|=n!$ gives $\prod_{j=1}^h j!^{\,a_j(b)}=h!$. This means that $|\Sigma|=|S_h|$ and thus $\Sigma=S_h$. It follows that $r(b)=1$. Namely assume $r\geq 2$ and fix $i\in B_1$ and $j\in B_2$: since surely there exists a permutation $\varphi\in S_h$ with $\varphi(i)=j$ and so not belonging to Σ, we contradict $\Sigma=S_h$. □

Proof of Proposition 22. i) If $r(b)=h$, then $b_i=1$ for all $i\in\{1,\dots,h\}$ and thus, for every $j\geq 2$, we have $a_j(b)=0$ while $a_1(b)=h$. Thus, for each $p\in\mathcal{P}(b)$, Proposition 19 gives $|p^G|=n!h!$, that is p^G is a regular orbit. Moreover, by Proposition 20, $\mathcal{P}(b)$ is made up of $\frac{1}{n!}\binom{n!}{h}$ orbits.

ii) Let $p\in\mathcal{P}(b)$, with $|p^G|=n!h!$. Then, by Proposition 19, we get $\prod_{j=1}^h j!^{\,a_j(b)}=1$ and thus, for every $j\geq 2$, $a_j(b)=0$. It follows that $r(b)=h$. In particular $h\leq n!$ and, since $\gcd(h,n!)=1$, we get $h<n!$. Thus, if a regular orbit exists we necessarily have $h<n!$. On the other hand, if $h<n!$, then the unique partition b of h with $r(b)=h$, belongs to $\Pi_{n!}(h)$ and thus at least a regular orbit exists because, by i), each orbit in $\mathcal{P}(b)$ is regular.

Finally, observe that $\binom{n!}{h}=n!$ if and only if $h=1$ or $h=n!-1$ and the first case is excluded because, by assumption $h\geq 2$. □

Proof of equality (18). It is well known (see, for instance, Feller (1957)) that the number $W(m,k)$ of ways of distributing $m\in\mathbb{N}$ indistinguishable balls into $k\in\mathbb{N}$ distinguishable boxes is given by

$$W(m,k)=\binom{m+k-1}{k-1}. \tag{29}$$

We show that also

$$W(m,k)=\sum_{\substack{k\\ b\vdash m}}\frac{\binom{k}{r(b)}r(b)!}{\prod_{j=1}^m a_j(b)\,!} \tag{30}$$

Let $r\in\mathbb{N}$ with $r\le k$ be fixed. We associate to each partition $b\vdash m$ with $r(b)=r$ some distributions of the m balls into the k boxes: form bunches of b_1 balls, b_2 balls, up to b_r balls exhausting the m available balls. Note that there is only a way to do that because the balls are indistinguishable and recall that, by definition of $b\vdash m$ with $r(b)=r$, we have $b_1\ge b_2\ge\cdots\ge b_r\ge 1$ and $\sum_{j=1}^r b_j=h$.

Now choose r boxes among the k available to distribute the r bunches of balls each in a different box: since the boxes are distinguishable you have k choices for the first box in which you put the b_1 balls, $k-1$ choices for the second box in which you put the b_2 balls, up to $k-r+1$ choices for the box in which you put the last bunch of b_r balls. Let $a_j(b)\in\mathbb{N}\cup\{0\}$ be, as usual, the number of bunches of order j for $j\in\{1,\dots,m\}$: since the balls are indistinguishable the $a_j(b)$ bunches of order j may be interchanged among themselves producing the same final result. Thus we reach $\frac{\binom{k}{r}r!}{\prod_{j=1}^m a_j(b)\,!}$ different distributions associated to $b\vdash m$ with $r(b)=r$. It is clear that, with r fixed, different choices of the partition b produce different distributions. It is also clear that the varying of r leads to different distributions, because r is the number of boxes involved in the distribution of the balls. In this way we have in total reached

$$\sum_{r=1}^{k}\sum_{b\vdash m, r(b)=r}\frac{\binom{k}{r}r!}{\prod_{j=1}^m a_j(b)\,!}=\sum_{\substack{k\\ b\vdash m}}\frac{\binom{k}{r(b)}r(b)!}{\prod_{j=1}^m a_j(b)\,!}$$

ways of distributing the m indistinguishable balls into the k distinguishable boxes. On the other hand if any distribution is given, we may think that it arises from our procedure: simply look into each box, count the balls inside and extract them, consider the number r of boxes containing at least one ball and put in a non-decreasing order the number of balls found in the boxes reaching a sequence $b_1\ge b_2\ge\cdots\ge b_r\ge 1$, that is a partition $b\vdash m$. Now among the possibilities of our procedure starting from $b\vdash m$, there is the one consisting in putting the bunches in the boxes where they originally were. Thus we have shown (30). □

References

Asan, G., Sanver, M.R., 2002. Another characterization of the majority rule. Economics Letter 75, 409-413.

Asan, G., Sanver, M.R., 2006. Maskin monotonic aggregation rules. Economics Letters 75, 179-183.

Can, B., Storcken, T., 2012. Update monotone preference rules. Mathematical Social Sciences, doi:10.1016/j.mathsocsci.2012.10.004.

Chichilnisky, G., 1980. Social choice and the topology of spaces of preferences. Advances in Mathematics 37, 165-176.

Daugherty, Z., Eustin, A.K., Gregory, M., Orrison, M.E., 2009. Voting, the symmetric group, and representation theory. The American Mathematical Monthly 116, 667-687.

Eğecioğlu, O., 2009. Uniform generation of anonymous and neutral preference profiles for social choice rules. Monte Carlo Methods and Applications 15, 241-255.

Feller, W., 1957. *An introduction to Probability Theory and its Applications*, 3rd Edition, Wiley, New York.

Greenberg, J., 1979. Consistent majority rules over compact sets of alternatives. Econometrica 47, 627-636.

Kelly, J.S., 1991. Symmetry groups. Social Choice and Welfare 8, 89-95.

Maskin, E.S., 1995. Majority rule, social welfare functions, and game forms. In: Basu, K., Pattanaik, P.K., Suzumura, K. (Eds.), Choice, Welfare, and Development. The Clarendon Press, Oxford.

May, K.O., 1952. A set of independent necessary and sufficient conditions for simple majority decision. Econometrica 20, 680-684.

Miroiu, A., 2004. Characterizing majority rule: from profiles to societies. Economics Letters 85, 359-363.

Moulin, H., 1983. *The strategy of social choice*, Advanced Textbooks in Economics, North Holland Publishing Company.

Perry, J., Powers, R.C., 2008. Aggregation rules that satisfy anonymity and neutrality. Economics Letters 100, 108-110.

Rose , J.S., 1978. *A course on group theory*, Cambridge University Press, Cambridge.

Sanver, M.R., 2009. Characterizations of majoritarianism: a unified approach. Social Choice and Welfare 33, 159-171.

Wielandt, H., 1964. *Finite permutation groups*, Academic Press, New York.

Woeginger, G.J., 2003. A new characterization of the majority rule. Economics Letter 81, 89-94.