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Opportunity-based other regarding preferences in general equilibrium
Part I: existence and generic regularity

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Opportunity-Based Other Regarding Preferences in General Equilibrium
Part I: Existence and generic regularity

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Abstract

Following a seminal paper by Kranich (1988), we consider a pure exchange economy with households having other regarding preferences. The only differences with respect to the standard model are what follows: each household utility function depends not only on her own consumption but also on other households' welfare, measured by wealth; households are allowed to promise transfers to other households (and promised are bound to be honored).

We show existence of equilibria under the assumption of the presence of an upper bound on transfers, taking care of some minor details which are not addressed by the paper by Kranich.

We present a robust example of nonexistence of equilibria if the bound on transfers is not imposed.

We then introduce a variation of Kranich model in terms of relative wealth and in this model we show generic regularity of equilibria.

Keywords: General Equilibrium; exchange economies; other regarding preferences; existence, nonexistence and regularity of equilibria.

JEL classification: D50, D64.

Contents

1 Introduction

We consider a pure exchange economy with a finite number of goods and households. The only differences with respect to the standard model are what follows:

1. each household utility function depends not only on her own consumption but also on other households' welfare, measured by wealth;

2. households are allowed to promise transfers to other households (and promised are bound to be honored).

In Section 2, we present the set up of the model as it was introduced by Kranich (1988) and in Section 3, we show existence of equilibria taking care of some minor details which are not addressed by that paper.

In Section 4, we discuss two main modelling assumptions of Kranich's model. The first one amounts to perform a normalization of other household nominal wealth; as showed in Section 8.2, different normalizations lead to different allocation equilibria.

The second modelling assumption consists in imposing an ad hoc upper bound on promises of transfers. That bound allows to use a standard proof of existence which requires compactness of the choice set of each household.

To tackle the first problem, we follow a simple, widely accepted viewpoint and we assume that households may in general care about other households' "relative wealth". That choice seems to be more realistic and allows to make households' maximization problem homogeneous of degree zero in prices, a quite reasonable and convenient assumption.

The nonexistence problem seems harder to be dealt with and we provide a partial solution. First of all in Section 5, we present a simple Cobb-Douglas, two household, one good version of the model and we do show that there is indeed a "large" set of economies for which equilibria do not exists if upper bounds on promises of transfers are not imposed. Further research is needed on the non-existence problem, but we do get the following partial results: we verify that if an upper bound is added, indeed at least one of the households chooses the highest allowed level of transfer; some intuition on the nonexistence results are gathered; further conjectures on the set of equilibria are obtained. We introduce a strong assumptions on preferences which insure existence.

We then move to further analysis of equilibria. Since it is quite hard to describe properties of equilibria under the general assumptions under which existence was proved, then we make some stronger assumptions about households' preferences and transfers possibilities. Indeed, we assume that i. households' utility functions are sufficiently differentiable, ii. they take an additive form with respect to selfish preferences and other-regarding preferences and iii. promises of transfers can take place only in terms of a numeraire commodity. In this new set up, we cannot apply the existence result used in Section 3: if each household's utility is the sum of a selfish utility function and an other-regarding function, then the needed assumption of (quasi)-concavity of the overall function is lost if the given household dislikes "too much" some other household. An equivalence result between true equilibria and some fictitious equilibria allows to overcome the problem.

Given all the above preliminary work, we can then proceed as follows. In Section **??**, an indispensable, so-called generic regularity result is proved: typically, in the space of economies, equilibria are finite and depend smoothly upon parameters defining an economy. Moreover, extending the result obtained in the Cobb-Douglas economy case,we show that a. there exists a set of economies contained in a closed measure set, for which the number of equilibria is infinite, and in which all household choose a strictly positive transfer; b. typically, for any pair of households, at most one of them provides a transfer to the other one.

In a work in progress paper which is the continuation of the work presented here, after having discussed the notion of Pareto Optimal, we show there exists an open, non-empty set of economies for which at least one associated equilibrium allocation is not Pareto Optimal. Finally, we prove that there exists a open, nonempty set of economies, for which there exists an equilibrium it is possible to Pareto improve upon, through a redistribution of wealth and a change in other regarding attitude of a very small number of households.

2 Set up of the model

A good or commodity is denoted by $c \in \{1, ..., C\} := \mathcal{C}$. A household is denoted by $h \in \{1, ..., H\} := \mathcal{H}$ and she is described by a consumption set X_h, a transfer set T_h, an endowment vector e_h and a utility function u_h.

A generic element of X_h is denoted by $(x_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where x_h^c denotes the consumption of good c by household h. Define also $X = \times_{h \in \mathcal{H}} X_h$

A generic element of T_h is denoted by $t_h := (t_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)}$, where $t_{hh'} = (t_{hh'}^c)_{c \in \mathcal{C}}$ and $t_{hh'}^c$ denotes the transfer of good c from household h to household h'. We also define $T = \times_{h \in \mathcal{H}} T_h$ with generic element $t = (t_h)_{h \in \mathcal{H}}$ and $T_{\setminus h} = \times_{h' \in \mathcal{H} \setminus \{h\}} T_{h'}$ with generic element $t_{\setminus h} = (t_{h'})_{h' \in \mathcal{H} \setminus \{h\}}$. Further notation on transfers will be introduced when needed.

$e_h = (e_h^c)_{c \in \mathcal{C}}$ is the vector of goods owned by household h.

To describe the utility function we need some preliminary definitions. Commodities can be exchanged with other commodities at some exchange ratios described by a price vector $p \in \mathbb{R}^C$. θ_h is household h's wealth and $\theta := (\theta_h)_{h\in\mathcal{H}}$. Utility of each household depends upon her own consumption and anyone's wealth. Then, household h's utility function is defined has follows.

$$u_h : X_h \times \mathbb{R}^H \longrightarrow \mathbb{R}, \quad (x_h, \theta) \mapsto u_h(x_h, \theta).$$

We make the following assumptions.
For any $h \in \mathcal{H}$,
1. $X_h = \mathbb{R}^C_+$;
2. There exists $k_h = (k_{hh'})_{h'\in\mathcal{H}\backslash\{h\}} \in \mathbb{R}_+^{C(H-1)}$ such that $T_h = \left\{t_h \in \mathbb{R}^{C(H-1)} : 0 \le t_h \le k_h\right\}$.
3. u_h is continuous, quasi-concave, strictly increasing in x_h.
4. $e_h \in \mathbb{R}^C_{++}$.

Remark 1 *All thea above assumptions are quite standard. They are used to show existence.To get regularity of equilibria and other results, we are going to introduce stronger assumptions.*

An economy is $\mathcal{E} := (k_h, e_h, u_h)_{h\in\mathcal{H}}$ and we denote by $\mathbb{E}$ the set of economies satisfying the above assumptions.

Given the monotonicity assumption on utility, we can restrict prices to belong to $\mathbb{R}^C_+$. Households are assumed to choose consumption and transfer vectors in order to maximize utility under a budget constraint, as formalized below.

Definition 2 *The "budget set-valued function" or "budget correspondence" for household h is denoted and defined as follows.*

$$B_h : \mathbb{R}^C_+ \times T_{\backslash h} \longrightarrow\longrightarrow X_h \times T_h,$$

$$\left(p, t_{\backslash h}\right) \mapsto\mapsto \left\{(x_h, t_h) \in X_h \times T_h : px_h \le p\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} (t_{h'h} - t_{hh'})\right)\right\}.$$

Household h's wealth depends upon the value of her initial endowment and net transfer. We define household h's wealth function as follows.

$$w_h : \mathbb{R}^C_+ \times T \longrightarrow \mathbb{R}, \qquad (p,t) \mapsto p\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} (t_{h'h} - t_{hh'})\right),$$

and, with innocuous abuse of notation,

$$w_h : \mathbb{R}^C_+ \times T_h \times T_{\backslash h} \longrightarrow \mathbb{R}, \qquad \left(p, t_h, t_{\backslash h}\right) \mapsto w_h\left(p, t_h, t_{\backslash h}\right),$$

$$\begin{array}{l} \max_{(x_h,t_h)\in X_h\times T_h} \quad u_h\left(x_h, w\left(p^*, t_h, t^*_{\backslash h}\right)\right) \\ \text{s.t.} \\ p^* x_h \le p^*\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} (t^*_{h'h} - t_{hh'})\right) \\ \text{or} \\ (x_h, t_h) \in B_h\left(p, t_{\backslash h}\right). \end{array} \tag{1}$$

and

$$w : \mathbb{R}^C_+ \times T_h \times T_{\backslash h} \longrightarrow \mathbb{R}, \qquad \left(p, t_h, t_{\backslash h}\right) \mapsto \left(w_h\left(p, t_h, t_{\backslash h}\right)\right)_{h\in\mathcal{H}}.$$

Remark 3 *Problem (1) can be rewritten as*

$$\begin{array}{l} \max_{(x_h,t_h)\in X_h\times T_h} \quad u_h\left(x_h, \left(p^*\left(e_{h'} + t_{hh'} + \sum_{h''\neq h',h} t^*_{h''h'} - \sum_{h''\neq h'} t^*_{h'h''}\right)\right)_{h'\in\mathcal{H}\backslash\{h\}}\right) \\ \textit{s.t.} \\ p^*\left(x_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} t_{hh'}\right) \le p^*\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} t^*_{h'h}\right), \end{array}$$

and, defined

$$t^{net\ *}_{(\neg h),h'} = \sum_{h''\neq h',h} t^*_{h''h'} - \sum_{h''\neq h'} t^*_{h'h''}$$

$$t^*_{\longrightarrow h} = \sum_{h'\in\mathcal{H}\backslash\{h\}} t^*_{h'h}$$

$$t_{h\longrightarrow} = \sum_{h'\in\mathcal{H}\backslash\{h\}} t_{hh'},$$

also as follows

$$\begin{array}{l} \max_{(x_h,t_h)\in X_h\times T_h} \quad u_h\left(x_h,\left(p^*\left(e_{h'}+t_{hh'}+t^{net\ *}_{(\neg h),h'}\right)\right)_{h'\in\mathcal{H}\setminus\{h\}}\right) \\ s.t. \\ p^*\left(x_h+t_{h\longrightarrow}\right)\leq p^*\left(e_h+t^*_{\longrightarrow h}\right) \end{array}$$

Definition 4 *The vector $(x^*, t^*, p^*) \in X \times T \times \mathbb{R}^C_+$ is an **equilibrium** vector for the economy $(k_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathbb{E}$ if*

1.

For any $h \in \mathcal{H}$, household h maximizes, i.e., for given $(k_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathbb{E}$, $p^* \in \mathbb{R}^C_+, t^*_{\setminus h} \in T_{\setminus h}$, $(x^*_h, t^*_h) \in X_h \times T_h$ solves Problem (1)

2.

Markets clear, i.e.,

$$\sum_{h=1}^{H} x^*_h = \sum_{h=1}^{H} e_h.$$

Remark 5 *Wealth in the objective function in the above maximization problem can be rewritten in a more detailed manner in order to stress that variables which are not starred are chosen by household h:*

$$w\left(p^*, t_h, t^*_{\setminus h}\right) = \left(p^*\left(e^*_h - \sum_{h'\neq h} t_{hh'} + \sum_{h'\neq h} t^*_{h'h}\right), \left(p^*\left(e^*_{h'} + t_{hh'} + \sum_{h''\neq h',h} t^*_{h''h'} - \sum_{h''\neq h'} t^*_{h'h''}\right)\right)_{h'\in\mathcal{H}\setminus\{h\}}\right).$$

Remark 6 *In showing existence, we follow the strategy proposed by Kranich. We strongly believe that the paper by Kranich is the best paper we know on general equilibrium and other regarding preferences. In our opinion, his results are really pathbreaking both from an economic and a mathematical viewpoint. Below, we list some differences between his proof and the proof we provide below.*

a. Kranich uses a theorem which is not stated properly: a correct statement requires lower semicontinuity of the constraint set-valued functions.

b. Kranich does not show that equilibrium prices are strictly positive.

c. The boundedness of T has to be imposed to get existence, as we show in Section 5. Kranich require boundedness of X_h as well, a requirement which it is easy to dispense of.

d. the statement that Walras law holds is not proved in the presence of bounds on the consumption levels - see our Proposition 14 below.

We complete the description of the set up of the model adding some pieces of notation. We denote and define the vector of total resources as

$$r = \sum_{h\in\mathcal{H}} e_h.$$

We restrict our analysis to the set of normalized prices

$$S = \left\{p \in \mathbb{R}^C_+ : pr = 1\right\}.$$

Remark 7 *We adopt the above normalization just following Kranich. In Section 4, we discuss the above modelling assumption and we modify the set up of the model as a consequence of that discussion. We do keep Kranich's normalization to stress that Kranich result is quite brilliant: a simple result shows that existence of an equilibrium in the model with the normalization proposed by Kranich implies existence in our version of the model as well - see Proposition 32.*

The set of feasible allocations is

$$F := \left\{x \in \times_{h\in\mathcal{H}} X_h : \sum_{h\in\mathcal{H}} (x_h - e_h) \leq 0\right\}.$$

3 Existence of equilibria

To show existence of equilibria, we proceed as follows.

1. Define equilibria with an artificial bound on consumption sets;

2. Define generalized games and state a theorem which gives sufficient conditions to get existence of equilibria for generalized games;

3. Introduce a well chosen generalized game associated with each economy presented in the model under analysis and verify that game satisfies the sufficient conditions stated in the theorem;

4. Show any equilibrium of the generalized game for a given economy is an equilibrium with artificial bound for that economy.

5. Show that an equilibrium with artificial bound is an equilibrium (without the artificial bound).

3.1 An equilibrium with artificial bound

Remark 8 *Since $pr = 1$, then for any $c \in \mathcal{C}$, $p^c \leq \frac{1}{r^c}$ and then, defined $r^* = \max\{r^c : c \in \mathcal{C}\}$, for any $c \in \mathcal{C}$, $p^c \leq \frac{1}{r^*}$, or $p \leq \frac{1}{r^*} \cdot \mathbf{1}$ and $p \cdot \mathbf{1} \leq \frac{1}{r^*} \cdot (\mathbf{1} \cdot \mathbf{1})$ or $p \cdot \mathbf{1} \leq \frac{C}{r^*}$*

Suppose otherwise; then there exists $c' \in \mathcal{C}$ such tha $p^{c'} > \frac{1}{r^{c'}}$ and then $p^{c'} r^{c'} > 1$. Then $1 = pr = p^{c'} r^{c'} + \sum_{c \neq c'} p^c r^c \geq p^{c'} r^{c'} > 1$, which is the desired contradiction.

Definition 9

$$\widetilde{k}_x = \tfrac{1}{r^*} \cdot \mathbf{1} \cdot \sum_{h \in \mathcal{H}} \left(e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} k_{h'h}\right) + 1 \in \mathbb{R}_{++}.$$

$$\widetilde{k}_x^c = \max\left\{\widetilde{k}_x \widetilde{k}_x \cdot r^c\right\} \in \mathbb{R}_{++},$$

$$k_x := \left(\widetilde{k}_x^c\right)_{c \in \mathcal{C}} \in \mathbb{R}_{++}^C.$$

Consistently with what said above we make the following provisional assumption.

Assumption 1'.

For any $h \in \mathcal{H}$, the (artificially bounded) consumption set is

$$X_h^a = \left\{x_h \in \mathbb{R}_+^C : x_h \leq k_x\right\}.$$

Remark 10 *For any (p,t) and $h \in \mathcal{H}$, $w_h(p,t) < \widetilde{k}_x$. Indeed, using the facts $p \leq \frac{1}{r^*} \cdot \mathbf{1}$ and for any h, h' with $h' \neq h$, $t_{h'h} \leq k_{hh'}$ and $t_{hh'} \geq 0$, we have*

$$w_h(p,t) := pe_h + p\sum_{h' \in \mathcal{H} \setminus \{h\}} (t_{h'h} - t_{hh'}) \leq pe_h + p\sum_{h' \in \mathcal{H} \setminus \{h\}} k_{h'h} \leq$$

$$\leq p\left(e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} k_{h'h}\right) \leq \tfrac{1}{r^*} \cdot \mathbf{1}\left(e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} k_{h'h}\right) < \widetilde{k}_x,$$

where last inequality follows from the definition of $\widetilde{k}_x$.

An economy is then defined as a list $\left((k_h, e_h, u_h)_{h \in \mathcal{H}}\right) \in \mathbb{R}_+^{H(H-1)} \times \mathcal{U} \times \mathbb{R}_{++}^{CH}$. Define $X^a = \times_{h \in \mathcal{H}} X_h^a$. Then "the artificially bounded" budget constraint set valued function is defined as follows.

Definition 11

$$\begin{aligned} B_h^a(p, t_{\setminus h}) = \quad & \{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)} : \quad px_h - p\left(e_h + \textstyle\sum_{h' \in \mathcal{H} \setminus \{h\}} (t_{h'h} - t_{hh'})\right) \leq 0, \\ & \qquad x_h \geq 0 \\ & \qquad x_h \leq k_x \cdot \mathbf{1} \\ & \qquad t_h \geq 0 \\ & \qquad t_h \leq k_h \quad \} = \\ & \{(x_h, t_h) \in X_h^a \times T_h : \quad p\left(x_h + \textstyle\sum_{h' \in \mathcal{H} \setminus \{h\}} t_{hh'}\right) \leq p\left(e_h + \textstyle\sum_{h' \in \mathcal{H} \setminus \{h\}} t_{h'h}\right) \quad \} \end{aligned}$$

Definition 12 *The vector $(x^*, t^*, p^*) \in X^a \times T \times S$ is an* ***artificially bounded equilibrium*** *vector for the economy $\left((k_h, e_h, u_h)_{h \in \mathcal{H}}, k_x\right)$ if*

1.

For any $h \in \mathcal{H}$, household h maximizes, i.e., for given $\left((k_h, e_h, u_h)_{h \in \mathcal{H}}, k_x\right)$, $p^* \in \mathbb{R}_+^C, t_{\setminus h}^* \in T_{\setminus h}$, $(x_h^*, t_h^*) \in X_h^a \times T_h$ solves

$$\begin{aligned} & \max_{(x_h, t_h) \in X_h \times T_h} \quad u_h\left(x_h, w\left(p^*, t_h, t_{\setminus h}^*\right)\right) \\ & s.t. \\ & (x_h, t_h) \in B_h^a(p, t_{\setminus h}). \end{aligned} \tag{2}$$

2.

Markets clear, i.e.,

$$\sum_{h=1}^{H} x_h^* = \sum_{h=1}^{H} e_h.$$

Remark 13 *Observe that some or all components of k_h may be zero, allowing, for example, the case in which transfers may occur only using, say, commodity 1. Therefore, from this viewpoint, the proof presented below applies also to the case of the model presented in the second part of the paper.*

Below, we present some preliminary Remarks and prove that an appropriate Walras law for the present model does hold true.

Proposition 14 *Let an economy $\left((k_h, e_h, u_h)_{h\in\mathcal{H}}, k_x\right)$ be given. If for any $h \in \mathcal{H}$, (x_h^*, t_h^*) solves problem (2) at $\left(p^*, t_{\backslash h}^*\right)$, then*

1.

$$0 = \sum_{h\in\mathcal{H}} \sum_{h'\neq h} t_{h,h'}^* - \sum_{h\in\mathcal{H}} \sum_{h'\neq h} t_{h',h}^*. \quad (3)$$

2.

$$p^* \left(\sum_{h\in\mathcal{H}} (x_h^* - e_h) \right) = 0.$$

Proof. 1.

Defined

$$\mathcal{H}^* = \{(h, h') \in \mathcal{H} \times \mathcal{H} : h' \neq h\}, \text{ and}$$

$$\mathcal{H}^{**} = \{(h', h) \in \mathcal{H} \times \mathcal{H} : h' \neq h\}.$$

Since $\mathcal{H}^* = \mathcal{H}^{**}$, then the desired result follows.

2. Using the strict monotonicity of u_h with respect to x_h and the fact that $p^* \in S$, we can show that budget constraints hold as equalities. Suppose otherwise, i.e.,

$$p^* x_h^* < p^* \left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}} (t_{h'h}^* - t_{hh'}^*) \right) := w_h(p^*, t^*). \quad (4)$$

Since $p^* \in S$, there exists a set $\mathcal{C}^+$ such that $\varnothing \neq \mathcal{C}^+ \subseteq \mathcal{C}$ and such that for any $c \in \mathcal{C}^+$ we have $p^{*c^*} > 0$ and for any $c \in \mathcal{C}\backslash\mathcal{C}^+ := \mathcal{C}^0$, we have $p^{*c} = 0$. We distinguish the following two cases.

Case a. There exists $\widetilde{c} \in \mathcal{C}^+$ such that $x_h^{*\widetilde{c}} < \widetilde{k}_x^c$;

Case b. For any $c \in \mathcal{C}^+$, $x_h^{*c} = \widetilde{k}_x^c$.

Case a.

Define $x_h^{**} = (x_h^{**c})_{c\in\mathcal{C}}$ such that

$$x_h^{**c} = \begin{cases} x_h^{*c} & \text{if } c \in \mathcal{C}^+\backslash\{\widetilde{c}\} \\ x_h^{*\widetilde{c}} + \frac{1}{2}\min\left\{ \frac{w_h(p^*,t^*)-p^*x_h^*}{p^{*\widetilde{c}}}, \quad \widetilde{k}_x^c - x_h^{*\widetilde{c}} \right\} > x_h^{*\widetilde{c}} & \text{if } c = \widetilde{c} \\ x_h^{*c} & \text{if } c \in \mathcal{C}^0, \end{cases}$$

where the strictly inequality follows from the fact that $w_h(p^*, t^*) - p^* x_h^* \overset{(4)}{>} 0$ and $\widetilde{k}_x^c - x_h^{*\widetilde{c}} > 0$. It then suffices to show that (x_h^{**}, t_h^*) belongs to the budget set, as verified below.

i. $x_h^{**} \leq \widetilde{k}_x^c$: it is enough to verify that $x_h^{**\widetilde{c}} \leq \widetilde{k}_x^{\widetilde{c}}$.

$$x_h^{**\widetilde{c}} \leq x_h^{*\widetilde{c}} + \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}}\right) = \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} + x_h^{*\widetilde{c}}\right) < \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} + \widetilde{k}_x^{\widetilde{c}}\right) = \widetilde{k}_x^{\widetilde{c}}.$$

ii. affordability:

$$p^* x_h^{**} \leq p^* x_h^* + p^{*\widetilde{c}}\frac{1}{2}\min\left\{ \frac{w_h(p^*,t^*)-p^*x_h^*}{p^{*\widetilde{c}}}, \quad \widetilde{k}_x^c - x_h^{*\widetilde{c}} \right\} \leq p^* x_h^* + \frac{1}{2}\left(w_h(p^*, t^*) - p^* x_h^*,\right) =$$

$$= \frac{1}{2}\left(w_h(p^*, t^*) + p^* x_h^*\right) \overset{(4)}{<} \frac{1}{2}\left(w_h(p^*, t^*) + w_h(p^*, t^*)\right) = w_h(p^*, t^*).$$

Case b.

This case cannot hold. Assume it does. Then,

$$\widetilde{k}_x \overset{(10)}{>} w_h(p^*, t^*) \overset{\text{budget constraint}}{\geq} p^* x_h^* = \sum_{c\in\mathcal{C}} p^{*c}\widetilde{k}_x^c \geq \sum_{c\in\mathcal{C}} p^{*c}\cdot \widetilde{k}_x \cdot r^c = \widetilde{k}_x \cdot (p^* r) \overset{p^* r=1}{=} \widetilde{k}_x,$$

which is the desired contradiction. ■

Remark 15 *If the upper bound on consumption is not big enough, then Walras' law does not hold, because the consumption vector hits the corner of the box* $[0, \textit{upper bound vector}]$.

Kranich uses Walras law on page 377, last paragraph in the proof of Proposition 3.4., but he does not seem to consider that possibility.

3.2 Generalized games

For the definition below see for example Kreps (2013), page 537, and the simple discussion following the definition proposed there. See also Facchinei and Kanzow (2010).

Definition 16 *Given* $n \in \mathbb{N}$, *an* n*-player generalized game is* $G = \{A_i, C_i, u_i\}_{i=1}^n$ *where for each* $i \in \{1, ..., n\}$,

1. A_i *is a set of strategies or actions with generic element* a_i;
2. $C_i : A_{\setminus i} := \times_{j\in\{1,...,n\}\setminus\{i\}} A_j \longrightarrow\!\!\!\!\longrightarrow A_i, \quad a := (a_i)_{i=1}^n \mapsto C_i(a)$ *is a constraint set valued function;*
3. $u_i : A := \times_{i\in\{1,..,n\}} A_i \longrightarrow \mathbb{R}$, $a \mapsto u_i(a)$ *is a utility function.*

Definition 17 *A Nash equilibrium for the generalized game* $G = \{A_i, C_i, u_i\}_{i=1}^n$ *is* $a^* := (a_i^*)_{i=1}^n \in A$ *such that for any* $i \in \{1, ..., n\}$, a_i^* *solves the following problem. For given* $a_{\setminus i}^* := \left(a_j^*\right)_{j\in\{1,...,n\}\setminus\{i\}} \in A_{\setminus i}$,

$$\max_{a_i\in A_i} u_i\left(a_i, a_{\setminus i}^*\right) \qquad s.t. \qquad a_i \in C_i\left(a_{\setminus i}^*\right).$$

Theorem 18 *Let a generalized game* $G = \{A_i, C_i, u_i\}_{i=1}^n$ *be given. If for any* $i \in \{1, ..., n\}$,

1. *there exists* $n_i \in \mathbb{N}$ *such that* A_i *is a nonempty, compact, convex subset of* $\mathbb{R}^{n_i}$;

2. C_i *is a non-empty value, convex valued, lower semicontinuous and upper semicontinuous set-valued function;*

3. u_i *is continuous and for any* $a_{\setminus i} \in A_{\setminus i}$, *the function* $u_i\left(\cdot, a_{\setminus i}\right) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto u_i\left(a_i, a_{\setminus i}\right)$ *is quasi-concave,*

then G *has a Nash equilibrium.*

The standard reference for the above theorem is Debreu (1952). Indeed, exactly the same statement of the above theorem and a proof of it can be found in Kreps (2013), page 538 .

3.3 The generalized game associated with our economy

We now define the generalized game associated with our economy.

There are $n = 1+H$ players. For each player $h \in \{0, 1..., H\}$ we describe below the triple of a. set of actions, b. constraint set valued functions and c. utility functions.

a.

$$\begin{aligned} A_0 &= \Delta \\ A_h &= X_h \times T_h \qquad \text{for any } h \in \mathcal{H} \end{aligned}$$

b.

$$A_0 \qquad = \Delta$$

$$\begin{aligned} &C_0 : \times_{h\in\mathcal{H}} A_h \longrightarrow\!\!\!\!\longrightarrow A_0 \\ &C_0 : (\times_{h\in\mathcal{H}} (X_h \times T_h)) \longrightarrow \Delta \qquad (x,t) \mapsto\!\!\!\!\mapsto \Delta \end{aligned}$$

$$\begin{aligned} &C_h : A_0 \times \left(\times_{h'\in\mathcal{H}\setminus\{h\}} A_{h'}\right) \longrightarrow\!\!\!\!\longrightarrow A_h \\ &B_h^a : S \times T_{\setminus h} \longrightarrow\!\!\!\!\longrightarrow X_h \times T_h \qquad \left(p, t_{\setminus h}\right) \mapsto\!\!\!\!\mapsto B_h^a\left(p, t_{\setminus h}\right) \end{aligned}$$

c.

$$\begin{array}{lcl} u_0 : A_0 \times (\times_{h\in\mathcal{H}} A_h) \longrightarrow \mathbb{R} & = & X_h \times T_h \\ u_0 : \Delta \times (\times_{h\in\mathcal{H}} (X_h \times T_h)) \longrightarrow \mathbb{R} & & (p,x,t) \longrightarrow p \cdot \sum_{h\in\mathcal{H}} (x_h - e_h) . \end{array}$$

$$\begin{array}{ll} u_h : A_0 \times (\times_{h\in\mathcal{H}} A_h) \longrightarrow \mathbb{R} & \\ u_h : \Delta \times (\times_{h\in\mathcal{H}} (X_h \times T_h)) \longrightarrow \mathbb{R} & (p,x,t) \mapsto u_h \left(x_h, w\left(p, t_h, t_{\setminus h}\right)\right) = u \circ (id_{X_h}, w) . \end{array}$$

Proposition 19 *Under Assumptions 1', 2', 3 and 4', the generalized game*

$$\left((\Delta, X \times T) , (C_0, (B_h^a)_{h\in\mathcal{H}}) , (u_h)_{h\in\{0\}\times\mathcal{H}} \right)$$

presented above has a Nash equilibrium (p^*, x^*, t^*).

Proof. We have to verify that the Assumptions of Theorem 18 are verified.

1. there exists $n_i \in \mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$.

$A_0 = S := \left\{p \in \mathbb{R}^C_+ : pr = 1\right\}$ satisfies the needed assumptions.

For any $h \in \mathcal{H}$, $X_h \times T_h$ satisfies the needed assumptions from our Assumptions 1' and 2'.

3. u_i is continuous and for any $a_{\setminus i} \in A_{\setminus i}$, the function $u_i\left(\cdot, a_{\setminus i}\right) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto u_i\left(a_i, a_{\setminus i}\right)$ is quasi-concave.

The desired result follows immediately from Assumption 3.

2. C_i is a non-empty value, convex valued, lower semicontinuous and upper semicontinuous set-valued function.

By definition of S and since $C_0 : (\times_{h\in\mathcal{H}} (X_h \times T_h)) \longrightarrow \Delta, (x,t) \mapsto\mapsto S$, the desired results follow. Indeed, C_0 is the constant set valued function and S is a compact set.

Verification of the needed properties for B_h^a is the content of Lemma 20 below. ■

Lemma 20 *For any* $h \in \mathcal{H}$

1. B_h *is non-empty valued;*
2. B_h *is convex valued;*
3. B_h *is closed;*
4. B_h *is compact valued;*
5. B_h *is lower hemi-continuous;*
6. B_h *is upper hemi-continuous.*

Proof. Under our assumption we can write B_h as follows

$$B_h\left(p, t_{\setminus h}\right) = \left\{ (x_h, t_h) \in S \times \mathbb{R}^{C(H-1)} : \begin{array}{l} p x_h - p\left(e_h + \sum_{h'\in\mathcal{H}\setminus\{h\}} (t_{h'h} - t_{hh'})\right) \leq 0, \\ \\ x_h \geq 0 \\ x_h \leq k_x \\ t_h \geq 0 \\ t_h \leq k_h \end{array} \right\}$$

1.

Take $x_h = e_h$, $t_h = 0$.

2.

Obvious.

3.

We want to show that for every sequence $\left(p^n, t^n_{\setminus h}\right)_{n\in\mathbb{N}} \in \left(\Delta \times \mathbb{R}^{C(H-1)}\right)^\infty$ such that $\left(p^n, t^n_{\setminus h}\right)_{n\in\mathbb{N}} \rightarrow \left(p, t_{\setminus h}\right)$, and for every sequence $(x^n_h, t^n_h)_{n\in\mathbb{N}} \in (X_h \times T_h)^\infty$ such that $(x_h, t_h) \in B_h\left(p^n, t^n_{\setminus h}\right)$ and $(x^n_h, t^n_h) \rightarrow (x_h, t_h)$, it is the case that $(x_h, t_h) \in B_h\left(p, t_{\setminus h}\right)$.

By assumption,

$$p^n x^n_h - p^n\left(e_h + \sum_{h'\in\mathcal{H}\setminus\{h\}} (t^n_{h'h} - t^n_{hh'})\right) \leq 0,$$

$$\begin{array}{l} x^n_h \geq 0 \\ x^n_h \leq k_x \\ t^n_h \geq 0 \\ t^n_h \leq k_h \end{array}$$

Taking limits, using the continuity of the involved functions, we get the desired result.

4.

Since B_h is defined in terms of weak inequalities via continuous function, it is closed valued. Moreover, $\operatorname{Im}(B_h) \subseteq X_h \times T_h$ and $X_h \times T_h$ is a compact set. Since closed subset of compact sets are compact, the desired result follows.

5.

We first present a proof in which we make the additional assumption that $k \gg 0$. The proof in the general case is presented in Lemma 21.

Define

$$\widetilde{B}_h\left(p, t_{\backslash h}\right) = \{x_h, t_h) \in X_h^a \times \mathbb{R}^{C(H-1)} : \; p\left(x_h + \textstyle\sum_{h' \in \mathcal{H}\backslash\{h\}} t_{hh'}\right) - p\left(e_h + \textstyle\sum_{h' \in \mathcal{H}\backslash\{h\}} t_{h'h}\right) < 0,$$

$$x_h \gg 0$$
$$x_h \ll k_x$$
$$t_h \gg 0$$
$$t_h \ll k \qquad \}$$

Now to show that B_h is lower hemicontinuous, we go through 3 steps Step 0. $\widetilde{B}_h$ is nonempty valued. Step 1. B_h is the closure of $\widetilde{B}_h$. Step 2. $\widetilde{B}_h$ is lower hemicontinuous. Step 3. Desired result.

Step 0.

$$\widetilde{B}_h\left(p, t_{\backslash h}\right) \neq \varnothing. \tag{5}$$

Take

$$x_h = \frac{e_h}{2H} \gg 0, \qquad \text{for any } c \in \mathcal{C} \text{ and for any } h' \in \mathcal{H}\backslash\{h\}, \qquad t_{hh'}^c = \frac{1}{2}\min\left\{k_{hh'}^c, \frac{e_h^c}{2H}\right\} \in \left(0, k_{hh'}^c\right).$$

Then,

$$p\left(x_h + \sum_{h' \in \mathcal{H}\backslash\{h\}} t_{hh'}\right) \leq p\frac{e_h}{2H} + p\sum_{h' \in \mathcal{H}\backslash\{h\}} \frac{e_h}{2H} < \frac{1+H-1}{2H} pe_h < pe_h \leq p\left(e_h + \sum_{h' \in \mathcal{H}\backslash\{h\}} t_{h'h}\right).$$

Step 1. $B_h = \operatorname{Cl}\left(\widetilde{B}_h\right)$.

We go through three substeps: i. for any $\left(p, t_{\backslash h}\right)$,

$$\operatorname{Int}\left(B_h\left(p, t_{\backslash h}\right)\right) = \widetilde{B}_h\left(p, t_{\backslash h}\right) \overset{(5)}{\neq} \varnothing; \tag{6}$$

ii. $\operatorname{Cl}(\operatorname{Int}(B_h\left(p, t_{\backslash h}\right))) = \operatorname{Cl}\left(B_h\left(p, t_{\backslash h}\right)\right)$;

iii. desired result.

i.

$\cdot$ $\widetilde{B}_h\left(p, t_{\backslash h}\right) \subseteq \operatorname{Int}(B_h)$

By definition, $\widetilde{B}_h\left(p, t_{\backslash h}\right) \subseteq B_h\left(p, t_{\backslash h}\right)$ and then $\operatorname{Int}(\widetilde{B}_h\left(p, t_{\backslash h}\right)) \subseteq \operatorname{Int}(B_h\left(p, t_{\backslash h}\right))$. Since $\widetilde{B}_h\left(p, t_{\backslash h}\right)$ is an open set, then $\widetilde{B}_h\left(p, t_{\backslash h}\right) = \operatorname{Int}(\widetilde{B}_h\left(p, t_{\backslash h}\right))$. Hence $\widetilde{B}_h\left(p, t_{\backslash h}\right) \subseteq \operatorname{Int}(B_h\left(p, t_{\backslash h}\right))$.

$\cdot\cdot$ $\operatorname{Int}(B_h) \subseteq \widetilde{B}_h$.

Take $\left(\widehat{x}_h, \widehat{t}_h\right) \in \operatorname{Int}(B_h\left(p, t_{\backslash h}\right))$. Then, there exists $\delta > 0$ such that $\mathcal{B}\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right) \subseteq B_h\left(p, t_{\backslash h}\right)$, where $B\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right)$ is the ball centered at $\left(\widehat{x}_h, \widehat{t}_h\right)$ with radius δ.

Suppose now our desired result does not hold true, i.e., $\left(\widehat{x}_h, \widehat{t}_h\right) \notin \widetilde{B}_h\left(p, t_{\backslash h}\right)$, i.e., since $\left(\widehat{x}_h, \widehat{t}_h\right) \in \operatorname{Int}(B_h\left(p, t_{\backslash h}\right)) \subseteq B_h\left(p, t_{\backslash h}\right))$, there is a strict inequality among those defining $\widetilde{B}_h\left(p, t_{\backslash h}\right)$ which holds as an equality. Below we show that any of those possibilities leads to a contradiction.

If there exists $c \in \mathcal{C}$ such that $\widehat{x}_h^c = 0$, then define $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right)$ as done below.

$$\text{for any } c' \in \mathcal{C}, \quad \widehat{\widehat{x}}_h^{c'} = \begin{cases} \widehat{x}_h^c & \text{if } \; c' \neq c \\ \widehat{x}_h^c - \frac{1}{n} = -\frac{1}{n} & \text{if } \; c' = c; \end{cases}$$

$$\widehat{\widehat{t}}_h = \widehat{t}_h.$$

Then, for $n \in \mathbb{N}$ large enough, $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right) \in \mathcal{B}\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right) \subseteq B_h\left(p, t_{\backslash h}\right)$ and $\widehat{\widehat{x}}_h^{c'} < 0$, a contradiction.

If there exists $(c,h') \in \mathcal{C} \times (\mathcal{H} \backslash \{h\})$ such that $\widehat{t}^c_{hh'} = 0$, then define $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right)$ as done below.

$$\widehat{\widehat{x}}_h = \widehat{x}_h$$

$$\widehat{\widehat{t}}^c_{hh'} = \widehat{t}^c_{hh'} - \tfrac{1}{n} = -\tfrac{1}{n}$$
$$\widehat{\widehat{t}}^{c'}_{hh''} = \widehat{t}^{c'}_{hh''} \text{ if } h'' \neq h', h \text{ and } c' \neq c.$$

Then, for $n \in \mathbb{N}$ large enough, $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right) \in \mathcal{B}\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right) \subseteq B_h\left(p, t_{\backslash h}\right)$ and $\widehat{\widehat{t}}_{hh'} < 0$, a contradiction.

If $p\widehat{x}_h - p\left(e_h + \sum_{h' \in \mathcal{H} \backslash \{h\}} \left(t_{h'h} - \widehat{t}_{hh'}\right)\right) = 0$, then define $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right)$ as done below:

$$\text{for any } c' \in \mathcal{C}, \quad \widehat{\widehat{x}}^{c'}_h = \begin{cases} \widehat{x}^c_h & \text{if} \quad c' \neq c \\ \widehat{x}^c_h + \frac{1}{n} & \text{if} \quad c' = c; \end{cases}$$

$$\widehat{\widehat{t}}_h = \widehat{t}_h.$$

Then, for $n \in N$ large enough, $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right) \in \mathcal{B}\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right) \subseteq B_h\left(p, t_{\backslash h}\right)$ and $p\widehat{x}_h - p\left(e_h + \sum_{h' \in \mathcal{H} \backslash \{h\}} \left(t_{h'h} - \widehat{t}_{hh'}\right)\right) > 0$, a contradiction.

If there exists $c' \in \mathcal{C}$ and h' such that $\widehat{t}^c_{h,h'} = k$, then define $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right)$ as done below.

$$\widehat{\widehat{x}}_h = \widehat{x}_h$$

$$\widehat{\widehat{t^{c'}}}_{hh'} = \widehat{t}^{c'}_{hh'} + \tfrac{1}{n} = k + \tfrac{1}{n}$$
$$\widehat{\widehat{t}}^c_{hh''} = \widehat{t}^c_{hh''} \text{ if } (h'', c) \neq (h', c')$$

Then, for $n \in N$ large enough, $\left(\widehat{\widehat{x}}_h, \widehat{\widehat{t}}_h\right) \in \mathcal{B}\left(\left(\widehat{x}_h, \widehat{t}_h\right), \delta\right) \subseteq B_h\left(p, t_{\backslash h}\right)$ and $\widehat{\widehat{t}}_{hh'} > k$, a contradiction.

ii.

It follows from Proposition 75 in the Appendix.

iii.

Observe that $B_h\left(p, t_{\backslash h}\right)$ is a closed set and that $\widetilde{B}_h\left(p, t_{\backslash h}\right)$ is an open set because defined in terms of strict inequalities. Then

$$\varnothing \overset{(6)}{\neq} \text{Int}B_h\left(p, t_{\backslash h}\right) = \widetilde{B}_h\left(p, t_{\backslash h}\right) = \text{Int}\left(\widetilde{B}_h\left(p, t_{\backslash h}\right)\right). \tag{7}$$

Then, we can apply again apply the result mentioned in ii. above to get

$$\text{Cl}\left(\widetilde{B}_h\left(p, t_{\backslash h}\right)\right) \overset{(7)}{=} \text{Cl}\left(\text{Int}B_h\left(p, t_{\backslash h}\right)\right) \overset{\text{Prop. 75}}{=} \text{Cl}\left(B_h\left(p, t_{\backslash h}\right)\right) \overset{(1)}{=} B_h\left(p, t_{\backslash h}\right),$$

where (1) follows from the fact that $B_h\left(p, t_{\backslash h}\right)$ is closed.

Step 2. $\widetilde{B}_h$ is lower hemicontinuous.

We present two different proofs. One is similar to the one in Ok (2007), page 224,. The other one is similar to the one in Werner (1985).

First proof of Step 2..

Recall the definition of lower hemicontinuity for set valued functions: $\varphi : X \to\to Y$ is Lower Hemicontinuous (LHC) at $x \in X$ if $\varphi(x) \neq \varnothing$ and for any open set V in Y such that $\varphi(x) \cap V \neq \varnothing$, there exists an open neighborhood U of x such that for every $x' \in U$, $\varphi(x') \cap V \neq \varnothing$.

$\widetilde{B}_h\left(p, t_{\backslash h}\right)$ is not empty from (6). Suppose our Claim is false. Then, by Definition of hemicontinuity, we have

there exists $\left(p, t_{\backslash h}\right) \in S \times T_{\backslash h}$ and an open set V in $X_h \times T_h$ such that $\widetilde{B}_h\left(p, t_{\backslash h}\right) \cap V \neq \varnothing$ and for any $m \in \mathbb{N}$ there exists $\left(p^m, t^m_{\backslash h}\right) \in \mathcal{B}\left(\left(p, t_{\backslash h}\right), \frac{1}{m}\right)$ such that $\widetilde{B}_h\left(p^m, t^m_{\backslash h}\right) \cap V = \varnothing$. (8)

Since $\widetilde{B}_h\left(p, t_{\setminus h}\right) \cap V \neq \varnothing$, we can take $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right) \cap V$. Moreover,

$$
\begin{array}{c}
0 < \overset{\in \mathbb{S}}{p}\left(\overset{\in \mathbb{R}^C_{++}}{x_h} + \sum_{h' \in \mathcal{H}\setminus\{h\}} \overset{\in \mathbb{R}_+}{t_{hh'}}\right) \overset{\text{Def. } \widetilde{B}_h}{<} p\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{h'h}\right) \\
\text{and for any } \lambda \in [0,1)\,,\ \lambda p\left(x_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{hh'}\right) < p\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{h'h}\right).
\end{array} \tag{9}
$$

Since V is an open set and $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right)$ - and therefore $(x_h, t_h) \gg 0$ - then there exists $\lambda \in (0,1)$, sufficiently close to 1 such that

$$
\lambda \cdot (x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right) \cap V. \tag{10}
$$

Then, from (8), we do have that $\left(p^m, t^m_{\setminus h}\right) \in \mathcal{B}\left(\left(p, t_{\setminus h}\right), \frac{1}{m}\right)$ and therefore

$$
\lambda p^m \cdot x_h + \lambda p^m \cdot \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{hh'} - p^m\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} t^m_{h'h}\right) \overset{m}{\longrightarrow} \lambda p \cdot x_h + \lambda p \cdot \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{hh'} - p\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} t_{h'h}\right) \overset{(9)}{<} 0,
$$

and, for m large enough,

$$
\lambda (x_h, t_h) \in \widetilde{B}_h\left(p^m, t^m_{\setminus h}\right) \overset{(10)}{\cap} V,
$$

contradicting the fact that $B\left((p, w), \frac{1}{m}\right) \cap V = \varnothing$ stated in (8) .

Second proof of Step 2. (Werner)

$\widetilde{B}_h\left(p, t_{\setminus h}\right)$ is not empty from (6). We now want to show that $\widetilde{B}_h$ is LHC at any $\left(p, t_{\setminus h}\right) \in S \times T_{\setminus h}$, i.e., for any sequence $\left(p^n, t^n_{\setminus h}\right)_{n \in \mathbb{N}} \in \left(S \times T_{\setminus h}\right)^\infty$ such that $\left(p^n, t^n_{\setminus h}\right) \to \left(p, t_{\setminus h}\right)$ and any $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right)$, there exists a sequence $(x^n_h, t^n_h)_{n \in \mathbb{N}} \in (X_h \times T_h)^\infty$ such that $\forall n \in \mathbb{N}, (x^n_h, t^n_h) \in \widetilde{B}_h\left(p^n, t^n_{\setminus h}\right)$ and $(x^n_h, t^n_h) \to (x_h, t_h)$.

Observe that

$$
p^n x_h - p^n\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \left(t_{h'h} - t^n_{hh'}\right)\right) \overset{n}{\longrightarrow} p x_h - p\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \left(t_{h'h} - t_{hh'}\right)\right) < 0
$$

where the last strict inequality follows from the fact that $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right)$. Then, there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$
p^n x_h - p^n\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \left(t_{h'h} - t^n_{hh'}\right)\right) < 0.
$$

For every $n > N$, there exists $\varepsilon_n > 0$ such that for any $(x^n_h, t^n_h) \in \mathcal{B}\left((x_h, t_h), \varepsilon_n\right)$,

$$
p^n x^n_h - p^n\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \left(t^n_{h'h} - t^n_{hh'}\right)\right) < 0.
$$

For any $n > N$, choose $(x^n_h, t^n_h) = (x_h, t_h) + \frac{1}{\sqrt{CH}} \cdot \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1}$. Then,

$$
d\left((x^n_h, t^n_h), (x_h, t_h)\right) = \left((CH)\left(\frac{1}{\sqrt{CH}} \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\}\right)^2\right)^{\frac{1}{2}} = \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} < \varepsilon_n,
$$

i.e.,

$$
(x^n_h, t^n_h) \in \mathcal{B}\left((x_h, t_h), \varepsilon_n\right).
$$

and then

$$
p^n x^n_h - p^n\left(e_h + \sum_{h' \in \mathcal{H}\setminus\{h\}} \left(t^n_{h'h} - t^n_{hh'}\right)\right) < 0.. \tag{11}
$$

Observe that since $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\setminus h}\right)$, we have

$$
\text{for any } n > N,\ (x^n_h, t^n_h) \gg (x_h, t_h) \gg 0. \tag{12}
$$

Moreover, since $(x_h, t_h) \in \widetilde{B}_h\left(p, t_{\backslash h}\right)$ by assumption, we do have $x_h \ll k_x$ and $t_h \ll k$. Since $(x_h^n, t_h^n) = (x_h, t_h) + \frac{1}{\sqrt{CH}} \cdot \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1}$, then there exists $N' \in \mathbb{N}$ such that for any $n > N'$, $x_h^n \ll k_x$ and $t_h^n \ll k$.

Therefore, from (11) and (12), we have that

$$\text{for any } n > \max\{N, N'\},\ (x_h^n, t_h^n) \in \widetilde{B}_h\left(p^n, t_{\backslash h}^n\right). \tag{13}$$

Finally,

$$\lim_{n \longrightarrow +\infty} (x_h^n, t_h^n) = \lim_{n \longrightarrow +\infty} \left((x_h, t_h) + \frac{1}{\sqrt{CH}} \cdot \min\left\{\frac{\varepsilon_n}{2}, \frac{1}{n}\right\} \cdot \mathbf{1}_{CH}\right) = (x_h, t_h),$$

as desired.

Step 3. B_h is lower hemicontinuous.

Since B_h is the closure of a lower hemicontinuous set valued function, then the desired result follows from Proposition 76 in the Appendix.

6. It follows from the four results above and Proposition 78 or 77 in the Appendix.

■

Lemma 21 *For any $h \in \mathcal{H}$ and any $k_h \in \mathbb{R}_+^{C(H-1)}$, B_h^a is lower semicontinuous.*

Proof. The proof goes through two steps.

Step 1. Given metric spaces X, Y and Z, if $\varphi : X \longrightarrow\!\!\!\!\longrightarrow Y$, $x \mapsto\!\!\!\!\mapsto \varphi(x)$ is lower semicontinuous, then $\psi : X \longrightarrow\!\!\!\!\longrightarrow Y \times Z$, $x \mapsto\!\!\!\!\mapsto \varphi(x) \times \{0_Z\}$ is lower semicontinuous.

Step 2. Desired result.

Step 1.

By assumption, for any $x \in X$ and any $y \in \varphi(x)$ and for any $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \longrightarrow x$, there exist $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that for any $n \in \mathbb{N}$, $y_n \in \varphi(x_n)$, and $y_n \longrightarrow y$. Then, for any $x \in X$ and any $(y, 0) \in \psi(x)$ and for any $(x_n)_{n\in\mathbb{N}} \in X^\infty$ such that $x_n \longrightarrow x$, there exist $(y_n, 0)_{n\in\mathbb{N}} \in (Y \times Z)^\infty$ such that for any $n \in \mathbb{N}$, $(y_n, 0) \in \varphi(x_n) \times \{0_Z\} = \psi(x_n)$, and$(y_n, 0) \longrightarrow (y, 0)$.

Step 2.

Define

$$\mathcal{Z}_h = \{(c, h') \in \mathcal{C} \times (\mathcal{H} \backslash \{h\}) : k_{hh'}^c = 0\}$$

$$\mathcal{P}_h = \{(c, h') \in \mathcal{C} \times (\mathcal{H} \backslash \{h\}) : k_{hh'}^c > 0\}$$

$$t_h^0 = (t_{hh'}^c)_{(c,h')\in\mathcal{Z}_h} \in \mathbb{R}^{\#\mathcal{Z}_h}$$

$$t_h^+ := (t_{hh'}^c)_{(c,h')\in\mathcal{P}_h} \in \mathbb{R}^{\#\mathcal{P}_h}$$

$$k_h^+ = \left\{(k_{hh'}^c)_{(c,h')\in\mathcal{P}_h} \in \mathbb{R}^{\#\mathcal{P}_h} : k_{hh'}^c > 0\right\}$$

$$T_h^0 = \left\{t_h^0 \in \mathbb{R}^{\#\mathcal{Z}_h}\right\}$$

$$T_h^+ = \left\{t_h^+ \in \mathbb{R}^{\#\mathcal{P}_h} : 0 \le t_{hh'}^c \le k_h^+\right\}$$

$$\widetilde{t}_{hh'} = \left(\widetilde{t}_{hh'}^c\right)_{c\in\mathcal{C}} \text{ such that } \widetilde{t}_{hh'}^c = \begin{cases} \widetilde{t}_{hh'}^c & \text{if } (c, h') \in \mathcal{P}_h \\ 0 & \text{if } (c, h') \in \mathcal{Z}_h \end{cases}$$

$$B_h\left(p, t_{\backslash h}\right) = \Big\{x_h, t_h^+, t_h^0) \in X_h \times T_h^+ \times T_h^0 : \ p\left(x_h + \textstyle\sum_{h'\in\mathcal{H}\backslash\{h\}} t_{hh'}\right) - p\left(e_h + \textstyle\sum_{h'\in\mathcal{H}\backslash\{h\}} t_{h'h}\right) \le 0,$$
$$x_h \ge 0$$
$$x_h \le k_x$$
$$0 \le t_h^+ \le k_h^+$$
$$t_h^0 = 0 \qquad \Big\}$$

$$B_h^+\left(p, t_{\backslash h}\right) = \Big\{x_h, t_h^+) \in X_h \times T_h^+ : \ p\left(x_h + \textstyle\sum_{h'\in\mathcal{H}\backslash\{h\}} t_{hh'}^+\right) - p\left(e_h + \textstyle\sum_{h'\in\mathcal{H}\backslash\{h\}} t_{h'h}^+\right) \le 0,$$
$$x_h \ge 0$$
$$x_h \le k_x$$
$$0 \le t_h^+ \le k_h^+ \qquad \Big\}$$

With innocuous abuse of notation

$$B_h\left(p,t_{\backslash h}\right)=B_h^{+}\left(p,t_{\backslash h}\right)\times\{0\}$$

with $0\in\mathbb{R}^{\#\mathcal{Z}_h}$. Then the desired result follows from Step 1 and Lemma 20.5. ■

3.4 Equilibria of the game and equilibria with artificial bound of the economy

Remark 22 *The proof of existence of equilibria in an exchange economy goes back to Debreu (1952).*

Proposition 23 *If (x^*,p^*,t^*) is a Nash equilibrium for the generalized game presented above, then it is an equilibrium and $p^*\gg 0$.*

Proof. By definition of Nash equilibrium, each player is maximizing. Therefore, for player $h=0$, we have that

$$\text{for any } p\in S,\quad p^*\cdot\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right)\geq p\cdot\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right). \tag{14}$$

For player $h\in\mathcal{H}$, we have

$$\begin{array}{ll}\left(x_h^*,t_h^*\right)\in B_h\left(p^*,t_{\backslash h}^*\right) & \text{and}\\ \text{for any }\left(x_h,t_h\right)\in B_h\left(p^*,t_{\backslash h}^*\right),\quad u_h\left(x_h^*,w\left(p^*,t_h^*,t_{\backslash h}^*\right)\right)\geq u_h\left(x_h,w\left(p^*,t_h,t_{\backslash h}^*\right)\right). & \end{array} \tag{15}$$

Then, (1) is satisfied. We are left with checking market clearing. From (1), we get that for any $h\in\mathcal{H}$

$$0\geq p^*x_h^*-p^*\left(e_h+\sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t_{h'h}^*-t_{hh'}^*\right)\right).$$

Summing up with respect to $h\in\mathcal{H}$, we get

$$0\geq\sum_{h\in\mathcal{H}}p^*\left(x_h^*-e_h\right)+\sum_{h\in\mathcal{H}}\sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t_{h'h}^*-t_{hh'}^*\right)=\sum_{h\in\mathcal{H}}p^*\left(x_h^*-e_h\right), \tag{16}$$

where last equality follows from Proposition 14. From (14) and (16), we then get

$$\text{for any } p\in S,\qquad 0\geq p^*\cdot\textstyle\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right)\geq p\cdot\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right). \tag{17}$$

For any $c\in\mathcal{C}$, define $p(c)=\left(p(c)^{c'}\right)_{c'\in\mathcal{C}}$ such that

$$p(c)^{c'}=\begin{cases}\frac{1}{r^{c'}} & \text{if}\quad c'=c\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $p(c)\in S$. Then from (17), we get

$$0\geq\frac{1}{r^c}\sum_{h\in\mathcal{H}}\left(x_h^{*c}-e_h^c\right),$$

and therefore,

$$\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right)\leq 0. \tag{18}$$

Let's now show that $p^*\gg 0$. Suppose our claim is false and without loss of generality, assume that $p^{*1}=0$. Then, by strict monotonicity of u_h in x_h, we would have

$$\text{for any } h\in\mathcal{H},\ x_h^{*1}=\widetilde{k}_x^1.$$

Then, we would have

$$\sum_{h\in\mathcal{H}}x_h^{*1}=H\widetilde{k}_x^1\overset{\text{Def. 9}}{>}Hr^1>r^1,$$

contradicting (18).

Finally, since $p^*>>0$, from (18) and Walras' law, i.e., Proposition 14, we also have $\sum_{h\in\mathcal{H}}\left(x_h^*-e_h\right)=0$. ■

3.5 Equilibria with artificial bound and equilibria

Definition 24 *Let a nonempty, convex subset X of $\mathbb{R}^n$ and a function $f : X \to \mathbb{R}$ be given. f is semistrictly quasiconcave if for any $x, y \in X$ and any $\lambda \in (0,1)$, $f(x) > f(y) \Rightarrow f((1-\lambda)x + \lambda y) > f(y)$.*

Proposition 25 *If X is a convex metric space and $u : X \to \mathbb{R}$ is continuous, then*

u is semistrictly quasiconcave and Non-Satiated $\Leftrightarrow$ u is quasiconcave and Locally NonSatiated.

Proof. See, for example, Villanacci (2022), Corollary 39, page 14. ■

Remark 26 *Since u_h strictly increasing in x_h, then u_h is Locally NonSatiated. Therefore, u_h is semistrictly quasiconcave.*

Proposition 27 *For any economy, an equilibrium with bound is an equilibrium.*

Proof. Consider household h's maximization problem with artificial constraint on consumption, i.e.,

$$\begin{array}{ll} \max_{(x_h,t_h)\in X_h\times T_h} & u_h\left(x_h, w\left(p^*, t_h, t^*_{\setminus h}\right)\right) \\ \text{s.t.} & \\ (x_h,t_h) \in B^A_h\left(p, t_{\setminus h}\right), & \\ \text{where} & \end{array}$$

$$\begin{array}{lll} B^A_h\left(p, t_{\setminus h}\right) = & \{x_h, t_h) \in X_h \times T_h : & px_h - p\left(e_h + \sum_{h'\in\mathcal{H}\setminus\{h\}}\left(t_{h'h} - t_{hh'}\right)\right) \le 0, \\ & & x_h \ge 0 \\ & & x_h \le k_x \\ & & t_h \ge 0 \\ & & t_h \le k_h \qquad \} \end{array}$$

Consider also the problem without the artificial constraint on consumption, i.e.,

$$\begin{array}{ll} \max_{(x_h,t_h)\in X_h\times T_h} & u_h\left(x_h, w\left(p^*, t_h, t^*_{\setminus h}\right)\right) \\ \text{s.t.} & \\ (x_h,t_h) \in B_h\left(p, t_{\setminus h}\right), & \\ \text{where} & \end{array}$$

$$\begin{array}{lll} B_h\left(p, t_{\setminus h}\right) = & \{x_h, t_h) \in X_h \times T_h : & px_h - p\left(e_h + \sum_{h'\in\mathcal{H}\setminus\{h\}}\left(t_{h'h} - t_{hh'}\right)\right) \le 0, \\ & & x_h \ge 0 \\ & & \\ & & t_h \ge 0 \\ & & t_h \le k \qquad \} \end{array}$$

Let (x^*, t^*, p^*) be an equilibrium with artificial bound on consumption. We want to show that if

$$\begin{array}{lll} (a) & (x^*_h, t^*_h) \in B^A_h\left(p^*, t^*_{\setminus h}\right), \text{ and} & \\ (b) & \text{for any } (x_h,t_h) \in B^A_h\left(p^*, t^*_{\setminus h}\right), & u_h\left(x^*_h, w\left(p^*, t^*_h, t^*_{\setminus h}\right)\right) \ge u_h\left(x_h, w\left(p^*, t_h, t^*_{\setminus h}\right)\right), \end{array} \tag{19}$$

then

$$\begin{array}{lll} (1) & (x^*_h, t^*_h) \in B_h\left(p^*, t^*_{\setminus h}\right), \text{ and} & \\ (2) & \text{for any } (x_h,t_h) \in B_h\left(p^*, t^*_{\setminus h}\right), & u_h\left(x^*_h, w\left(p^*, t^*_h, t^*_{\setminus h}\right)\right) \ge u_h\left(x_h, w\left(p^*, t_h, t^*_{\setminus h}\right)\right). \end{array} \tag{20}$$

Since $B^A_h\left(p^*, t^*_{\setminus h}\right) \subseteq B_h\left(p^*, t^*_{\setminus h}\right)$, conclusion (20.1) follows from assumption (19.a). Now suppose that conclusion (20.2) does not hold, i.e.,

$$\exists\ \left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h\left(p^*, t^*_{\setminus h}\right) \setminus B^A_h\left(p^*, t^*_{\setminus h}\right) \quad \text{such that} \quad u_h\left(\widetilde{x}_h, w\left(p^*, \widetilde{t}_h, t^*_{\setminus h}\right)\right) > u_h\left(x^*_h, w\left(p^*, t^*_h, t^*_{\setminus h}\right)\right). \tag{21}$$

Then

$$\left(\widetilde{x}_h, \widetilde{t}_h\right) \neq \left(x_h^*, t_h^*\right). \tag{22}$$

Since $\left(x_h^*, t_h^*\right), \left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h\left(p^*, t_{\setminus h}^*\right)$ and $B_h\left(p^*, t_{\setminus h}^*\right)$ is convex, then

$$\forall \lambda \in (0,1), \quad \left(\widehat{x}_h, \widehat{t}_h\right) := (1-\lambda)\left(x_h^*, t_h^*\right) + \lambda\left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h\left(p^*, t_{\setminus h}^*\right). \tag{23}$$

From semistrict quasiconcavity of u_h, (21), (22) and (23), we have

$$u_h\left(\widehat{x}_h, w\left(p^*, \widehat{t}_h, t_{\setminus h}^*\right)\right) > u_h\left(x_h^*, w\left(p^*, t_h^*, t_{\setminus h}^*\right)\right). \tag{24}$$

Now, for any $\delta > 0$, $\left\|\left(\widehat{x}_h, \widehat{t}_h\right) - \left(x_h^*, t_h^*\right)\right\| \overset{\text{Def. } \left(\widehat{x}_h, \widehat{t}_h\right)}{=} \lambda \cdot \left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\| < \delta$ if and only if $\lambda < \frac{\delta}{\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\|} \in \mathbb{R}_{++}$, where $\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\| > 0$ from the fact $\left(\widetilde{x}_h, \widetilde{t}_h\right) \neq \left(x_h^*, t_h^*\right)$. Then, chosen $\lambda < \frac{\delta}{\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\|}$, we have that

$$\left(\widehat{x}_h, \widehat{t}_h\right) \in \mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta\right) \overset{(23)}{\cap} B_h\left(p^*, t_{\setminus h}^*\right), \tag{25}$$

where $\mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta\right) := \left\{(x_h, t_h) \in X_h \times T_h : \left\|(x_h, t_h) - \left(x_h^*, t_h^*\right)\right\| < \delta\right\}$. Observe that

$$B_h^A\left(p^*, t_{\setminus h}^*\right) = B_h\left(p^*, t_{\setminus h}^*\right) \cap [0, k_x], \tag{26}$$

where $[0, k_x] := \left\{x \in \mathbb{R}^C : 0 \leq x \leq k_x\right\}$. Moreover, and *this is a crucial step*, from market clearing

$$x_h^* \leq \sum_{h \in \mathcal{H}} e_h = r \ll k_x.$$

Then, there exists $\delta > 0$ such that $\mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta\right) \subseteq [0, k_x]$. Moreover,

$$B_h\left(p^*, t_{\setminus h}^*\right) \cap \mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta\right) \subseteq B_h\left(p^*, t_{\setminus h}^*\right) \cap [0, k_x] \overset{(26)}{=} B_h^A\left(p^*, t_{\setminus h}^*\right). \tag{27}$$

From (25) and (27), we have

$$\left(\widehat{x}_h, \widehat{t}_h\right) \in B_h^A\left(p^*, t_{\setminus h}^*\right). \tag{28}$$

(28) and (24) contradict assumption (19.2). ■

Theorem 28 *For any economy an equilibrium exists.*

Proof. It a consequence of Propositions 27, 23 and 19. ■

4 The relative wealth model

4.1 Indeterminacy: price normalization matters

Following the paper by Kranich, in the model we presented in the previous sections, we assumed that the utility function of household h is

$$u_h : \mathbb{R}_+^C \times \mathbb{R}^H \longrightarrow \mathbb{R}, \quad (x_h, \theta) \mapsto u_h\left(x_h, \left(w_h\left(p, t_h, t_{\setminus h}\right)\right)_{h \in \mathcal{H}}\right).$$

where for any $h \in \mathcal{H}$, $w_h\left(p, t_h, t_{\setminus h}\right) = p\left(e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} \left(t_{h'h} - t_{hh'}\right)\right)$.

We want to observe that to "normalize prices" amounts to multiply prices by a strictly positive real number, a fact which does affect the value of u_h unless u_h is homogenous of degree zero in p. To be better understand the above observation, let's we write household h's maximization problem using prices expressed in units of account - as usually done in general equilibrium models. That problem is

$$\max_{(x_h, t_h) \in \mathbb{R}_{++}^C \times \mathbb{R}^{C(H-1)}} \quad u_h\left(x_h, \left(p\left(e_{h'} + t_{h,h'} + \tau_{(\neg h),h'}\right)\right)_{h \in \mathcal{H}}\right)$$

s.t.

$$p \cdot \left(x_h + \sum_{h' \neq h} t_{h,h'}\right) \leq p\left(e_h + \widetilde{t}_{\longrightarrow h}\right),$$

$$t_h \geq 0,$$

$$t_h \leq k_h.$$

Multiplying the price vector p by $\alpha > 0$ does not affect the budget constraint, but it does affect the utlity function - see the Appendix 8.2 for further discussion in a simplified version of the model.The above discussion says that the altruistic part of the utility function of each household in the Kranich's model is not well specified: different choice of normalizations of prices give rise to different equilibria and there is no natural choice of normalization. Indeed, it seems obvious to formulate the following conjecture

Conjecture 29 *Generically in the economy space, the set of equilibria in the nominal wealth model exhibits indeterminacy of degree 1, i.e., the set of equilibrium allocation contains the image of open interval via a $\mathcal{C}^1$ one-to-one function.*

To avoid the fact that equilibrium allocations are normalization dependent, we propose a simple change in the model: we substitute wealth of other households in the utlity function with "relative wealth". There is indeed a vast literature in partial equilibrium, game theory and behavioral economic analysis which follows this approach - see Dhami (2006), Chapter 6 and references quoted there.

As usual in standard general equilibrium models, we start our description of the set-up of the model using prices which are measured in units of account.

We can then write household h's maximization problem as follows. For any $h \in \mathcal{H}$, for given $(\mathcal{E}, p, t_{\backslash h}) \in \mathbb{E} \times \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)(H-1)}$, $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}$ solves problem

$$\max_{(x_h,t_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{C(H-1)}} \quad u_h\left(x_h, \left(\frac{p\left(e_{h'}+t_{h,h'}+\tau_{(\neg h),h'}\right)}{pr}\right)_{h\in\mathcal{H}}\right)$$

$$\text{s.t.}$$

$$p\cdot\left(x_h+\textstyle\sum_{h'\neq h} t_{h,h'}\right) \le p\left(e_h+\widetilde{t}_{\longrightarrow h}\right),$$

$$t_h \ge 0,$$

$$t_h \le k_h.$$

Remark 30 *$(\widetilde{x}_h, \widetilde{t}_h)$ solves Problem (51) at $(\mathcal{E}, p, t_{\backslash h})$ if and only if for any $\alpha > 0$, $(\widetilde{x}_h, \widetilde{t}_h)$ solves Problem (51) at $(\mathcal{E}, \alpha p, t_{\backslash h})$. That follows simply from the facts that*

$$\frac{p\left(e_{h'}+t_{h,h'}+\tau_{(\neg h),h'}\right)}{pr} = \frac{\alpha p\left(e_{h'}+t_{h,h'}+\tau_{(\neg h),h'}\right)}{\alpha pr}$$

$$p\cdot\left(x_h+\textstyle\sum_{h'\neq h} t_{h,h'}-e_h-\widetilde{t}_{\longrightarrow h}\right) = \alpha p\cdot\left(x_h+\textstyle\sum_{h'\neq h} t_{h,h'}-e_h-\widetilde{t}_{\longrightarrow h}\right)$$

Therefore we can normalize the price of good 1 to 1 and define

$$p^{\backslash} = (p^c)_{c\in\mathcal{C}\backslash\{1\}} \in \mathbb{R}^{C-1}_{++}, \qquad p = \left(1, p^{\backslash}\right).$$

The above is indeed an innocous normalization.

Definition 31 *The vector $(\widetilde{x}, \widetilde{t}, \widetilde{p}^{\backslash}) \in \mathbb{R}^{CH}_{++} \times T \times \mathbb{R}^{C-1}_{++}$ is an* ***equilibrium*** *vector associated with an economy $\mathcal{E} := (u, v, e, \alpha, k) \in \mathbb{E}$ if*

1. *for given $(\mathcal{E}, p, t_{\backslash h}) \in \mathbb{E} \times \mathbb{R}^{C-1}_{++} \times \mathbb{R}^{C(H-1)(H-1)}$, $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}$ solves problem*

$$\max_{(x_h,t_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{C(H-1)}} \quad u_h\left(x_h, \left(\frac{p\left(e_{h'}+t_{h,h'}+\tau_{(\neg h),h'}\right)}{pr}\right)_{h\in\mathcal{H}}\right)$$

$$s.t.$$

$$p\cdot\left(x_h+\textstyle\sum_{h'\neq h} t_{h,h'}\right) \le p\left(e_h+\widetilde{t}_{\longrightarrow h}\right), \tag{29}$$

$$t_h \ge 0,$$

$$t_h \le k_h.$$

2. *markets clear, i.e.,*

$$\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h) = 0.$$

4.2 Existence

We are going to call the equilibrium studied in Section 3 as "equilibrium in the Kranick's model". We call the equilibrium presented above as "equilibrium in the Relative Wealth model".

Proposition 32 *If (x,t,p) is an equilibrium in Kranick's model, then $\left(x,t,\frac{p}{p^1}:=\widehat{p}\right)$ it is an equilibrium in the Relative Wealth model.*

Proof. Take $p \in S_{++} := \left\{z \in \mathbb{R}^C_{++} : zr = 1\right\}$. Drop h and denote by $\beta_K(p,t)$, $u_K(x,t,p)$ and $\beta_{RW}(p,t)$, $u_{RW}(x,t,p)$ the (budget set and the objective function) in Kranick and Relative Wealth models, respectively, i.e.,

$$\beta_K(p,t) = \{(x_h,t_h) : p(x_h + t_{h\longrightarrow}) \leq p(e_h + t_{\longrightarrow h})\} \qquad \text{with} \quad pr = 1$$

$$\beta_{RW}(\widehat{p},t) = \{(x_h,t_h) : \widetilde{p}(x_h + t_{h\longrightarrow}) \leq \widetilde{p}(e_h + t_{\longrightarrow h})\} \qquad \text{with} \quad \widetilde{p}^1 = 1$$

$$u_K(x,t,p) = u_h\left(x_h, \left(p\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\neq h}\right)$$

$$u_{RW}(x,t,\widetilde{p}) = u_h\left(x_h, \left(\tfrac{\widetilde{p}}{\widetilde{p}r}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\neq h}\right)$$

First of all observe that $\widehat{p}^1 = 1$, as required by the definition of RW model.

Step 1. $\beta_K(p,t) = \beta_{RW}\left(\frac{p}{p^1},t\right)$.

Indeed, the budget equations are the same. In Kranick's model, we have

$$p(x_h + t_{h\longrightarrow}) \leq p(e_h + t_{\longrightarrow h})$$

In the Relative Wealth model, we have

$$\frac{p}{p^1}(x_h + t_{h\longrightarrow}) \leq \frac{p}{p^1}(e_h + t_{\longrightarrow h})$$

Step 2. For any (x,t,p) such that $p \in S_{++}$, we have $u_{RW}\left(x,t,\frac{p}{p^1}\right) = u_K(x,t,p)$.

$$u_{RW}\left(x,t,\tfrac{p}{p^1}\right) = u_h\left(x_h, \left(\tfrac{\frac{p}{p^1}}{\frac{p}{p^1}r}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\neq h}\right) = u_h\left(x_h, \left(\tfrac{p}{pr}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\neq h}\right) \stackrel{pr=1}{=}$$

$$= u_h\left(x_h, \left(p\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\neq h}\right) = u_K(x,t,p).$$

Step 3. Desired result.

Let (x,t,p) be an equilibrium in Kranick's model. For any $(x'_1,t'_{12}) \in \beta_K(p,t_{21}) \stackrel{\text{Step 1}}{=} \beta_{RW}\left(\frac{p}{p^1},t_{21}\right)$, we have

$$u_{RW}\left(x,t,\frac{p}{p^1}\right) \stackrel{\text{Step 2}}{=} u_k(x,t,p) \stackrel{(x,t,p)\text{ is Kranick equilibrium}}{\geq} u_k(x',t',p) \stackrel{\text{Step 2}}{=} u_{RW}\left(x',t',\frac{p}{p^1}\right),$$

i.e, households maximize. Clearly markets clear. ■

Corollary 33 *For any economy, an equilibrium in the Relative Wealth model exists.*

Proof. It follows from Proposition 32 and Theorem 28. ■

4.3 Upper bound on promises of transfers and non-existence of equilibria

An easy conjecture is that without imposing an upper bound on the promises of transfers, equilibria may not exist. That conjecture is proved to be true in Section 5.1.

Imposing an upper bound is anyway quite an ad-hoc assumption: at the best of our knowledge, no limitation on charites or gifts are indeed used in "real life". There are then two possible ways out.

1. Find some conditions on economies for which existence is insured without imposing any upper bound on promises of transfers.

2. Describe the set of economies for which the upper bound is hit: explain why that set of economies is economically not relevant.

Consistently with statement 1. above, an easy condition which insures existence is presented below in Definition 34. In Proposition 35, we verify that indeed equilibria exist for any economy satifsying that condition. Below, we consider the case of promises of transfers in terms of the numeraire good.

In Section 5.1 below, we present a simple Cobb-Douglas economy case and we find a subset of the parameter space for which equilibria do not exist. Those parameters correspond to cases in which each household cares about some other household more than the latter households care about themselves. That situation may imply nonexistence if consumption of each household is assumed to be strictly positive. Probably, to model the behavior of "losing his own life to save a beloved one", we need to assume the possibility of zero consumption.

Further research on the nonexistence problem is needed.

Definition 34 *Household h is called $k_h \in \mathbb{R}_+^{H-1}$ moderately altruistic if*

for any $h' \neq h$,$\exists k_{hh'} \in \mathbb{R}_+$ and $\overline{v}_{hh'} \in \mathbb{R}$ such that for any $x_h \in \mathbb{R}_+^C$ and any $(\theta_{h''})_{h'' \in \mathcal{H} \setminus \{h,h'\}}$,

$$\text{if } k \geq k_{hh'}, \text{ then } u\left(x_h,\ k, (\theta_{h''})_{h'' \neq h}\right) = u\left(x_h,\ k_{hh'}, (\theta_{h''})_{h'' \neq h}\right).$$

Proposition 35 *If (x^*, t^*, p^*) is an equilibrium with upper bound k on promises of transfers, then (x^*, t^*, p^*) is an equilibrium in an economy in which households are k moderately altruistic.*

Proof. Market clearing conditions are clearly satisfied. Now, suppose otherwise and assume there exists $h \in \mathcal{H}$ such that there exists $\left(\widehat{x}_h, \widehat{t}_h\right)$ which is affordable and gives a higher utility than (x_h^*, t_h^*) in the k moderately altruistic model.

If $\widehat{t}_h \leq k_h$, then $\left(\widehat{x}_h, \widehat{t}_h\right)$ it is affordable and gives a higher utility than (x_h^*, t_h^*) in the k upper bound model, a contradiction.

Ift $\widehat{t}_h > k_h$, take

$$\left(x_h^* + \left(\widehat{t}_h - k_h\right), k_h\right)$$

which is clearly affordable:

$$p^* \left(x_h^* + t_h^* - k_h\right) - k_h = p^* \left(x_h^* + t_h^*\right)$$

and

$$\begin{aligned} u_h\left(x_h^* + \left(\widehat{t}_h - k_h\right)\right) &> u_h\left(x_h^*\right), \qquad \text{since } u_h \text{ is strictly increasing,} \\ v_h\left(pe_{h'} - t_{h'} + k_h\right) &= v_h\left(k_h\right) = \overline{v}_h, \end{aligned}$$

contradicting the definition of (x_h^*, t_h^*) as part of an equilibrium. ■

5 Discussion of the Equilibrium set properties

Before moving to further analysis of the model at somehow high level of generality, we want to address a quite reasonable question about equilibria. Could we get existence without imposing a bound on transfers? The answer is negative, as the analysis of the following Cobb-Douglas economy shows. Indeed, a first simple intuitive explanation is as follows.

Consider the following informally described game. There are two players: each player chooses one real number, i.e., her strategy set is $\mathbb{R}$. The player who chooses a bigger number than the other player wins 1 euro; player who chooses smaller number gets 0 euros. If both players choose the same number, they both get 0 euro. Since basically a best response against $x \in \mathbb{R}$ is $x + 1 \in \mathbb{R}$, then the game has no Nash equilibria - not even in mixed strategies. Then, we can argue that individuals simply do not play that game.

Conjecture: assuming some form of selfishness or the fact that for a sufficiently large level of transfers, the marginal increase in the other household utility becomes smaller than the marginal increase in her own utility should suffice for existence. For example, you may assume that the selfish utility is unbounded above and the altruistic utility is bounded above.

We formalize the intuition about nonexistence of equilibria providing an analysis in a Cobb-Douglas economy. That analysis provide also some further intuition about the nature of the equilibrium set.

The basic idea is to compare the model without and with a bound on transfers.

The main results we get in the Cobb-Douglas economy we analyze are what follows.

1. In the model without bounds on transfer,

a. there is a set $\mathcal{N}$ of economies such that $\mathcal{N}$ has nonempty interior and for which equilibria do not exist - see Proposition 40.5;

b. an infinite number of equilibria arise only for a closed and measure zero set $\mathcal{D}$ of economies - see Proposition 40.1;

c. there is a set $\mathcal{N}'$ of economies such that $\mathcal{N}'$ has nonempty interior and for which only one (or none) of the two households choose a strictly positive transfer - see Proposition 40.2 ,3 and 4;

2. in the model with bounds on transfers,

a. in the set $\mathcal{N}$, equilibria exists (transfer are equal to the upper bound for at least one household);

b. and c. holds true, keepinn bounds into consideration - see Proposition **??**.

5.1 A Cobb-Douglas economy with no bound on transfers

In this section we compute equilibria in a 2 household- one good - Cobb-Douglas economy. The utility function of household 1 is

$$u_1 : \mathbb{R}^2_{++} \times \mathbb{R}_+ \times (-e_2 - t_{21}, +\infty) \longrightarrow \mathbb{R}, \qquad (x_1, e_2, t_{21}, t_{12}) \mapsto \log x_1 + \beta_{12} \log \left(\frac{e_2 - t_{21} + t_{12}}{r} \right).$$

An economy is $(\beta_{12}\beta_{21}) \in \mathbb{R}^2_{++}$. The maximization problem is what follows. For given, $\beta_{12}, \beta_{21}, e_1, e_2 \in \mathbb{R}_{++}$ and $t_{\setminus h} \in \mathbb{R}_+$,

$$\begin{array}{lll} \max_{(x_1,t_{12}) \in \mathbb{R}_{++} \times (-e_2 - t_{21}, +\infty)} \quad \log x_1 + \beta_{12} \log \left(\frac{1}{r} (e_2 - t_{21} + t_{12}) \right) = & \text{s.t.} & -x_1 - t_{12} + e_1 + t_{21} = 0 \\ = \log x_1 + \beta_{12} \log \left((e_2 - t_{21} + t_{12}) \right) - \beta_{12} \log r & & t_{12} \geq 0. \end{array}$$

Then household 1's maximization problem is as follows. For given, $\beta_{12}, \beta_{21}, e_1, e_2 \in \mathbb{R}_{++}$ and $t_{\setminus h} \in \mathbb{R}_+$,

$$\max_{(x_1,t_{12}) \in \mathbb{R}_{++} \times (-e_2 - t_{21}, +\infty)} \quad \log x_1 + \beta_{12} \log (e_2 - t_{21} + t_{12}) \quad \text{s.t.} \quad \begin{array}{ll} -x_1 - t_{12} + e_1 + t_{21} & = 0 \\ t_{12} & \geq 0. \end{array}$$

In Proposition 57 in Section **??** below, we show that in a more general version of the above problem the set of maximizers is characterized by theassociated Kuhn-Tucker conditions. Therefore, we can then present the following Definition.

Definition 36 $((x_1^*, t_{12}^*, \lambda_1^*, \gamma_{12}^*), (x_2^*, t_{21}^*, \lambda_2^*, \gamma_{21}^*)) \in (\mathbb{R}_{++} \times \mathbb{R}^3)^2$ *is an equilibrium for the economy* $(\beta_{12}, \beta_{21}, e_1, e_2) \in \mathbb{R}^4_{++}$ *if it is a solution to the following system.*

$$\begin{array}{ll} \frac{1}{x_1} - \lambda_1 & = 0 \\ \beta_{12} \frac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} & = 0 \\ -x_1 - t_{12} + e_1 + t_{21} & = 0 \\ \min \{t_{12}, \gamma_{12}\} & = 0 \\ & \\ \frac{1}{x_2} - \lambda_2 & = 0 \\ \beta_{21} \frac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} & = 0 \\ -x_2 - t_{21} + e_2 + t_{12} & = 0 \\ \min \{t_{21}, \gamma_{21}\} & = 0 \\ & \\ \sum_{h \in \mathcal{H}} (x_h - e_h) & = 0 \\ & \\ t_{12} > -e_2 + t_{21} & \\ t_{21} > -e_1 + t_{12} & \end{array} \tag{30}$$

Remark 37 *Observe since* $e_2 - t_{21} + t_{12} > 0$*, by definition of* log*, then* $\beta_{12} \frac{1}{e_2 - t_{21} + t_{12}} = \lambda_1 - \gamma_{12} > 0$*. Moreover, from budget sets we get* $x_2 = e_2 - t_{21} + t_{12} > 0$ *and* $x_1 = e_1 - t_{12} + t_{21} > 0$*.*

Remark 38 *Consistently with the* min *equations in system* (30)*, we have the following exhaustive cases.*

$\gamma_{21} \longrightarrow$	$= 0$		> 0
$\gamma_{12} \downarrow$			
$= 0$	*1*		*3*
> 0	*4*		*2*

(31)

To be able to describe equilibria in the economy space $\mathbb{R}^2_{++}$*, the basic idea we follow is to work "logically backward": we make conjecture about existence of equilibria consistently with the four above described cases and we then check if that conjecture is not contradictory. Roughly speaking, "we assume some properties about endogenous variables and then we infer which values of endogenous variables are consistent with them". The Proposition below follows and formalizes that idea.*

Remark 39 *The main idea on how to get conjectures is what follows.*

1. If $\beta_{12}\beta_{21} > 1$, then at least one household cares really a lot about the other household and there is no equilibrium.

2. Otherwise, to get a conjcture we use Table 31. The following observations may also help.

$$\beta_{12}e_1 - e_2$$

is an indicator of how much household 1 care about household 2 (as described by β_{12}) keeping into account the relative endowment. Indeed, $\beta_{12}e_1 - e_2 > 0$ if and only if

$$\beta_{12} > \frac{e_2}{e_1},$$

i.e., if and only if the degree of empaty is bigger than the relative wealth. In that case, $t_{12} > 0$.

Proposition 40 *1. a. If $1 = \beta_{12}\beta_{21}$ and $\beta_{12}e_1 - e_2 \geq 0$, then for any $m \in \mathbb{R}_+$, equilibrium is* [1]

$$\begin{array}{l} x_1 = \frac{r}{1+\beta_{12}} \\ \lambda_1 = \frac{1+\beta_{12}}{r} \\ t_{12} = \frac{\beta_{12}\cdot r}{1+\beta_{12}} - e_2 + m = \frac{\beta_{12}e_1 - e_2}{1+\beta_{12}} + m \\ \gamma_{12} = 0 \\ \\ x_2 = \frac{\beta_{12}\cdot r}{1+\beta_{12}} \\ \lambda_2 = \frac{1+\beta_{12}}{\beta_{12}\cdot r} \\ t_{21} = m \\ \gamma_{21} = 0 \end{array}$$

1.b .If $1 = \beta_{12}\beta_{21}$ and $\beta_{21}e_2 - e_1 \geq 0$, then for any $m \in \mathbb{R}_+$, equilibrium is symmetric[2] *to the above ones.*

2.

If $1 - \beta_{12} > 0$ and $1 - \beta_{21} > 0$, then the unique equilibrium is

$$\begin{array}{l} x_1 = e_1 \\ \lambda_1 = \frac{1}{e_1} \\ t_{12} = 0 \\ \gamma_{12} = \frac{1-\beta_{12}}{e_1 e_2} > 0 \\ \\ x_2 = e_2 \\ \lambda_2 = \frac{1}{e_2} \\ t_{21} = 0 \\ \gamma_{21} = \frac{1-\beta_{21}}{e_1 e_2} > 0 \end{array}$$

3.

If $\beta_{12}\beta_{21} < 1$ and $\beta_{12}e_1 - e_2 \geq 0$, then the unique equilibrium is

$$\begin{array}{l} x_1 = \frac{r}{1+\beta_{12}} \\ \lambda_1 = \frac{1+\beta_{12}}{r} \\ t_{12} = \frac{\beta_{12}e_1 - e_2}{1+\beta_{12}} \geq 0 \\ \gamma_{12} = 0 \\ \\ x_2 = \frac{\beta_{12}r}{1+\beta_{12}} \\ \lambda_2 = \frac{1}{e_2} \\ t_{21} = 0 \\ \gamma_{21} = \frac{(1-\beta_{12}\beta_{21})(1+\beta_{12})}{\beta_{12}r} > 0 \end{array}$$

4.

If $1 - \beta_{12}\beta_{21} > 0$ and $\beta_{21}e_2 - e_1 \geq 0$, then the unique equilibrium is symmetric to the equilibrium presented in 3. above.

5. If $\beta_{12}\beta_{21} > 1$, then no equilibrium exists.

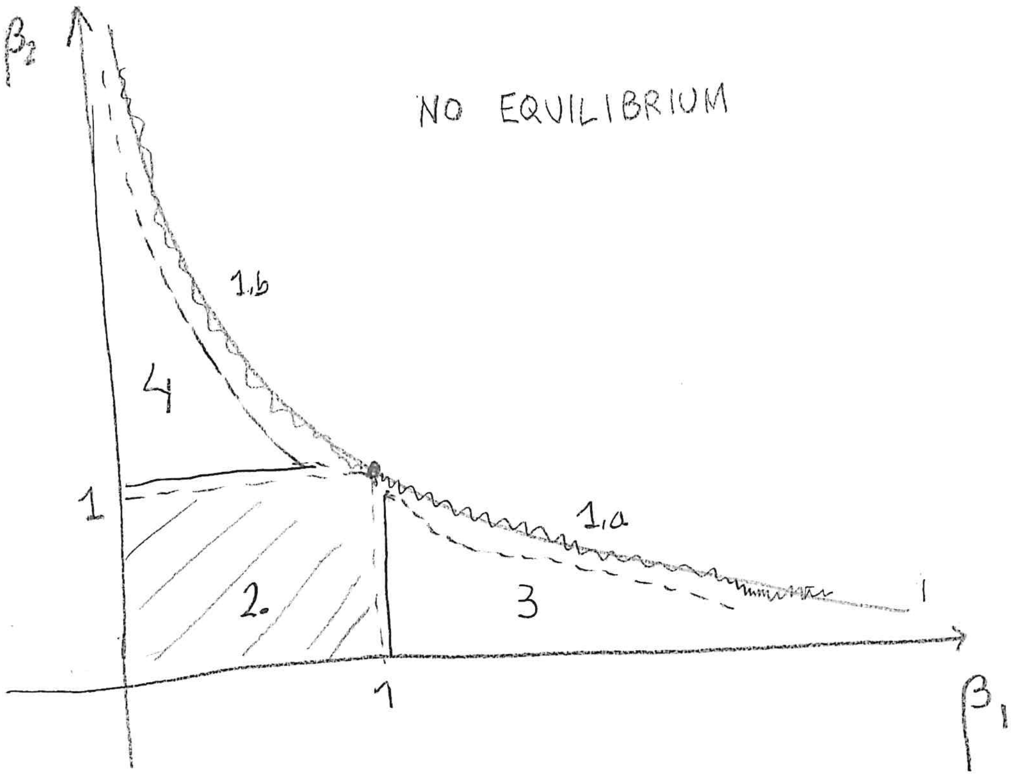
Remark 41 *Before presenting the proof, let"s summarize the above result in a graph in the case in which $e_1 = e_2 = 1$ and the economy is a point $(\beta_1, \beta_2) \in \mathbb{R}^2_{++}$.*

[1] What we show in the present Proposition is that in each analyzed case, the proposed vector is an equilibrium. We should show that the equilibrium is unique.

[2] Here symmetric means "interchange 1 and 2" .

NO BOUNDS ON TRANSFERS, and values of transfers

$e_1 = e_2 = 1$



1.a. $t_1 = * + m$ $\quad$ 1b. $t_{12} = m$ $\quad$] infinite number of equilibria

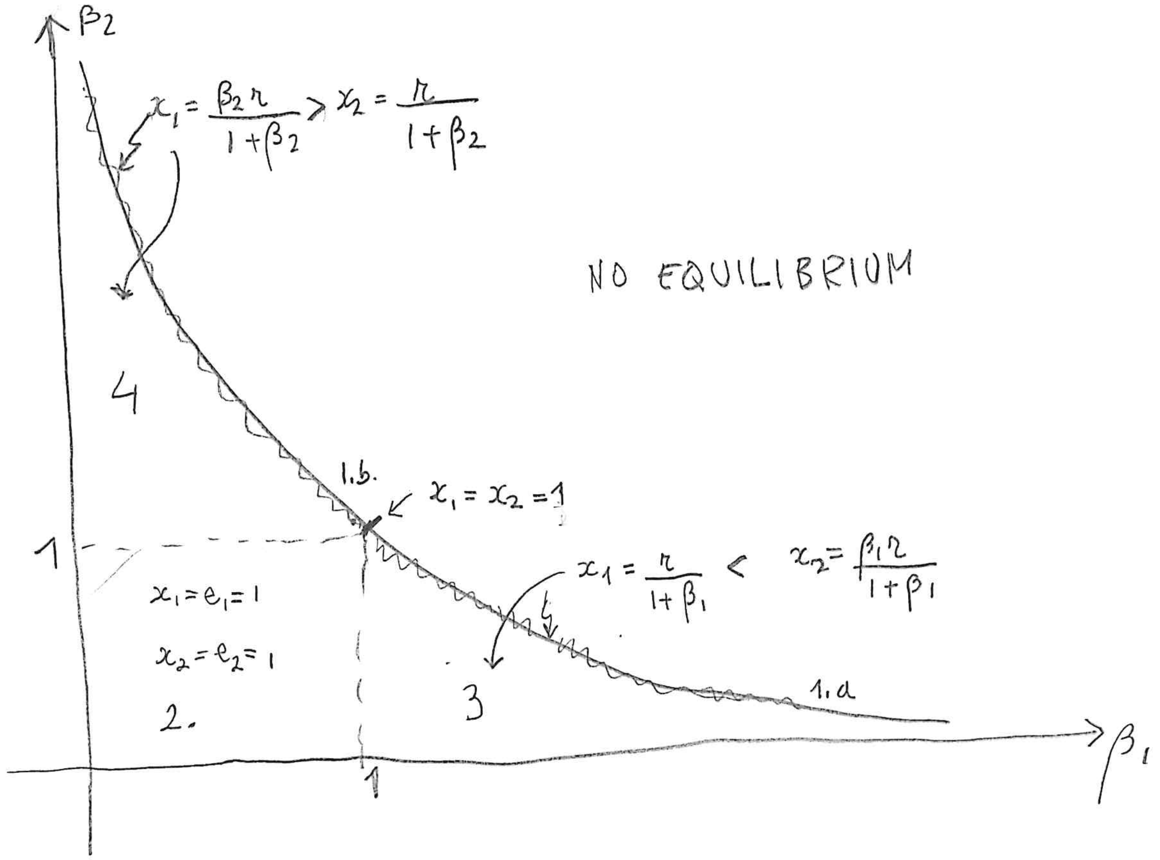
$t_2 = m$ $\quad$ $t_{21} = * + m$

2. $t_1 = t_2 = 0$ $\quad$] unique equilibrium

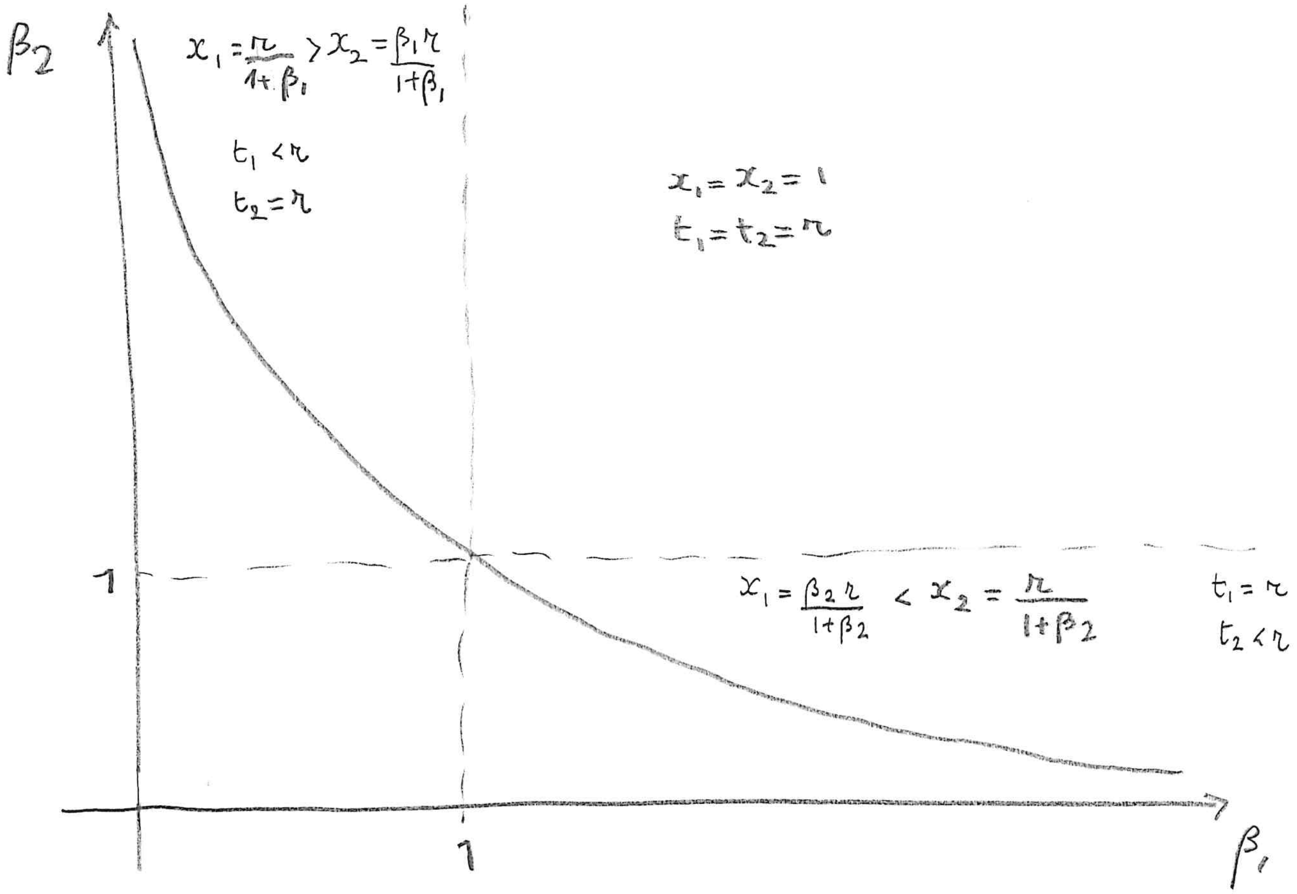
3. $t_1 > 0 \quad t_2 = 0$ $\quad$] "

4. $t_1 = 0 \quad t_2 > 0$

NO BOUNDS ON TRANSFERS
and value of consumption



BOUNDS ON TRANSFERS



Proof. of Proposition 40.

We first make some preliminary observations. From equations related to household 1, we get what follows.

$$x_1 = \frac{1}{\lambda_1} \tag{32}$$

$$\begin{array}{l} \beta_{12}\frac{1}{e_2 - t^*_{21} + t_{12}} = \lambda_1 - \gamma_{12} \\ \frac{\beta_{12}}{\lambda_1 - \gamma_{12}} = e_2 - t^*_{21} + t_{12} \\ t_{12} = \frac{\beta_{12}}{\lambda_1 - \gamma_{12}} - e_2 + t^*_{21} \end{array} \tag{33}$$

Defined

$$r := e_1 + e_2,$$

and substituting the expressions found in (32) and (33) in the budget equation, we get

$$\begin{array}{l} 0 = -\frac{1}{\lambda_1} - \left(\frac{\beta_{12}}{\lambda_1 - \gamma_{12}} - e_2 + t_{21}\right) + e_1 + t_{21} \\ 0 = -\frac{1}{\lambda_1} - \frac{\beta_{12}}{\lambda_1 - \gamma_{12}} + (e_1 + e_2) \\ \frac{1}{\lambda_1} + \frac{\beta_{12}}{\lambda_1 - \gamma_{12}} = r \end{array} \tag{34}$$

1.

Let's first simply observe that our claim can be proved just by inserting the proposed values of the endogenous variables in system (30) and check that all equations do become identities.

Consider again the equilibrium system.

$$\begin{array}{ll} \frac{1}{x_1} - \lambda_1 & = 0 \\ \beta_{12}\frac{1}{e_2 - t^*_{21} + t_{12}} - \lambda_1 + \gamma_{12} & = 0 \\ -x_1 - t_{12} + e_1 + t^*_{21} & = 0 \\ \min\{t_{12}, \gamma_{12}\} & = 0 \\ & \\ \frac{1}{x_2} - \lambda_2 & = 0 \\ \beta_{21}\frac{1}{e_1 + t_{21} - t^*_{12}} - \lambda_2 + \gamma_{21} & = 0 \\ -x_2 - t_{21} + e_2 + t^*_{12} & = 0 \\ \min\{t_{21}, \gamma_{21}\} & = 0 \\ & \\ \sum_{h\in\mathcal{H}}(x_h - e_h) & = 0 \end{array} \tag{35}$$

Let's plug in the proposed solution

$$\begin{array}{l} x_1 = \frac{1\cdot r}{1+\beta_{12}} \\ \lambda_1 = \frac{1+\beta_{12}}{r} \\ t_{12} = \frac{\beta_{12}\cdot r}{1+\beta_{12}} - e_2 + k = \frac{\beta_{12}e_1 - 1e_2}{1+\beta_{12}} + k \\ \gamma_{12} = 0 \\ \\ x_2 = \frac{\beta_{12}\cdot r}{1+\beta_{12}} \\ \lambda_2 = \frac{1\ \beta_{12}\cdot r}{1+\beta_{12}} \\ t_{21} = k \\ \gamma_{21} = 0 \end{array}$$

Below, we provide a way to get the proposed conjecture: as said in Remark 38, we proceed "logically backward": from endogenous to exogenous variables.

Observe that in the present case, the fact that the multipliers are zero signals that the maximization problems with or without the nonnegativity constraints on t_{12} and t_{21} have the same solution.

Claim. If $\gamma_{12} = \gamma_{21} = 0$, then $1 = \beta_{12}\beta_{21}$.

Proof of the Claim.

From (44), we have

$$\frac{1}{\lambda_1} + \frac{\beta_{12}}{\lambda_1} = r$$

$$\lambda_1 = \frac{1+\beta_{12}}{r}$$

From (32),

$$x_1 = \frac{1}{\lambda_1} = \frac{r}{1+\beta_{12}}.$$

From (33),

$$t_{12} = \frac{\beta_{12}}{\lambda_1} - e_2 + t_{21} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 + t_{21}$$
or
$$t_{12} - t_{21} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 = \frac{\beta_{12}\ e_1 + \beta_{12}\ e_2 - 1e_2 - \beta_{12} e_2}{1+\beta_{12}} = \frac{\beta_{12}\ e_1 - 1e_2}{1+\beta_{12}} \tag{36}$$

Writing (36) symmetrically for household 2, we have

$$t_{21} - t_{12} = \frac{\beta_{21} \cdot r}{1+\beta_{21}} - e_1. \tag{37}$$

Observe that

$$t_{12} - t_{21} \geq 0 \quad \text{if} \quad \beta_{12} e_1 - e_2 \geq 0$$

and

$$t_{21} - t_{12} \geq 0 \quad \text{if} \quad \beta_{21} e_2 - e_1 \geq 0$$

Then, from (36) , (37), we have

$$0 = t_{12} - t_{21} + t_{21} - t_{12} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 + \frac{\beta_{21} \cdot r}{1+\beta_{21}} - e_1 \tag{38}$$

and

$$0 = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 + \frac{\beta_{21} \cdot r}{1+\beta_{21}} - e_1 = r\left(\frac{\beta_{12}}{1+\beta_{12}} + \frac{\beta_{21}}{1+\beta_{21}} - 1\right) =$$
$$= \frac{1}{(1+\beta_{12}) \cdot (1+\beta_{21})} \left(\beta_{12} \cdot (1+\beta_{21}) + \beta_{21} \cdot (1+\beta_{12}) - (1+\beta_{12})(1+\beta_{21})\right) =$$
$$= \frac{1}{(1+\beta_{12}) \cdot (1+\beta_{21})} \left(\beta_{12}\beta_{21} - 1\right)$$

End of the proof of the Claim.

Then from the budget equation

$$x_2 = e_2 + t_{12} - t_{21} = e_2 + \left(\frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2\right) = \frac{\beta_{12} \cdot r}{1+\beta_{12}},$$

or, from market clearing

$$x_2 = r - x_1 = r - \frac{r}{1+\beta_{12}} = \frac{\beta_{12} \cdot r}{1+\beta_{12}}$$

And since $\lambda_2 = \frac{1}{x_2}$, we have

$$\lambda_2 = \frac{1+\beta_{12}}{\beta_{12} \cdot r}.$$

Finally, we conjecure

$$t_{21} = 0$$

and then, from (37), we have

$$t_{12} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 = \frac{\beta_{12}\ e_1 - e_2}{1+\beta_{12}} \geq 0.$$

Finally, since $t_{12} \geq t_{21} \geq 0$, then the desired result follows from Lemma 85 in the appendix.

2.

We are in Case 2 of Table 31. Since $\gamma_{12} > 0$ and $\gamma_{21} > 0$, then from the buget equation we have that $x_1 = e_1$, $x_2 = e_2$ and $\lambda_1 = \frac{1}{e_1}, \lambda_2 = \frac{1}{e_2}$. $e_2 - \beta_{12} e_1 > 0$ and $e_1 - \beta_{21} e_2 > 0$. Then,

$$(2) \qquad \frac{\beta_{12}}{\lambda_1 - \gamma_{12}} - e_2 = 0$$

$$\beta_{12} - \lambda_1 e_2 + \gamma_{12} e_2 = 0$$
$$\gamma_{12} = \frac{\lambda_1 e_2 - \beta_{12}}{e_2} = \frac{\frac{1}{e_1} e_2 - \beta_{12}}{e_2} = \frac{e_2 - \beta_{12} e_1}{e_1 e_2} > 0$$

Then,

$$e_2 - \beta_{12} e_2 > 0 \text{ and } e_1 - \beta_{21} e_1 > 0.$$

Then the unique equilibrium is

$$\begin{aligned}
&x_1 = e_1\\
&\lambda_1 = \tfrac{1}{e_1}\\
&t_{12} = 0\\
&\gamma_{12} = \tfrac{1e_2 - \beta_{12} e_2}{e_1 e_2} > 0
\end{aligned}$$

$$\begin{aligned}
&x_2 = e_2\\
&\lambda_2 = \tfrac{1}{e_2}\\
&t_{21} = 0\\
&\gamma_{21} = \tfrac{1e_1 - \beta_{21} e_1}{e_1 e_2} > 0
\end{aligned}$$

3.

We first observe that our conjecture is $\gamma_{12} = 0$ and $\gamma_{21} > 0$; then $t_{21} = 0$. Moreover, $\beta_{12}e_1 - e_2 \geq 0$ and $1 - \beta_{12}\beta_{21} > 0$. From (44), we have

$$\tfrac{1}{\lambda_1} + \tfrac{\beta_{12}}{\lambda_1} = r$$

$$\lambda_1 = \tfrac{1+\beta_{12}}{r} > 0$$

From (32),

$$x_1 = \frac{1}{\lambda_1} = \frac{1 \cdot r}{1+\beta_{12}}.$$

From (33),

$$t_{12} = \frac{\beta_{12}}{\lambda_1} - e_2 + t_{21} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 + t_{21} \tag{39}$$

Then,

$$t_{12} - t_{12} = t_{12} = \frac{\beta_{12} \cdot r}{1+\beta_{12}} - e_2 = \frac{\beta_{12}e_1 - e_2}{1+\beta_{12}} \geq 0 \quad \text{iff} \quad \beta_{12}e_1 - 1e_2 \geq 0.$$

Household 2's Kuhn-Tucker conditions are what follows.

$$\begin{aligned}
&\tfrac{1}{x_2} - \lambda_2 &&= 0\\
&\beta_{21}\tfrac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} &&= 0\\
&-x_2 - t_{21} + e_2 + t_{12} &&= 0\\
&\min\{t_{21}, \gamma_{21}\} &&= 0
\end{aligned}$$

In the present case, they become.

$$\begin{aligned}
&\tfrac{1}{x_2} - \lambda_2 &&= 0\\
&\beta_{21}\tfrac{1}{e_1 - t_{12}} - \lambda_2 + \gamma_{21} &&= 0\\
&-x_2 + e_2 + t_{12} &&= 0\\
&t_{21} &&= 0
\end{aligned}$$

and

$$\begin{aligned}
&x_2 &&= \tfrac{1}{\lambda_2}\\
&\beta_{21}\tfrac{1}{e_1 - \left(\frac{\beta_{12}e_1 - 1e_2}{1+\beta_{12}}\right)} - \lambda_2 + \gamma_{21} &&= 0\\
&-\tfrac{1}{\lambda_2} + e_2 + \tfrac{\beta_{12}e_1 - 1e_2}{1+\beta_{12}} &&= 0\\
&t_{21} &&= 0
\end{aligned}$$

$$\textstyle\sum_{h\in\mathcal{H}}(x_h - e_h) = 0$$

$$0 = -\tfrac{1}{\lambda_2} + e_2 + \tfrac{\beta_{12}e_1 - 1e_2}{1+\beta_{12}} = \tfrac{-1(1+\beta_{12}) + \lambda_2(1+\beta_{12})e_2 + \lambda_2(\beta_{12}e_1 - 1e_2)}{\lambda_2(1+\beta_{12})} =$$

$$= \tfrac{-1(1+\beta_{12}) + \lambda_2(1+\beta_{12})e_2 + \lambda_2(\beta_{12}e_1 - 1e_2)}{\lambda_2(1+\beta_{12})} = \tfrac{-1(1+\beta_{12}) + \lambda_2(1e_2 + \beta_{12}e_2 + \beta_{12}e_1 - 1e_2)}{\lambda_2(1+\beta_{12})}$$

$$\lambda_2 = \tfrac{1(1+\beta_{12})}{\beta_{12}e_2 + \beta_{12}e_1} > 0$$

$$x_2 = \frac{\beta_{12}e_2 + \beta_{12}e_1}{1\,(1+\beta_{12})}$$

$$\gamma_{21} = \lambda_2 - \beta_{21} \frac{1}{e_1 - \left(\frac{\beta_{12} e_1 - 1 e_2}{1+\beta_{12}}\right)} = \frac{1+\beta_{12}}{\beta_{12} e_2 + \beta_{12} e_1} - \beta_{21} \frac{1}{\frac{e_1 + \beta_{12} e_1 - \beta_{12} e_1 + 1 e_2}{1+\beta_{12}}} =$$

$$= \frac{1(1+\beta_{12})}{\beta_{12}(e_1+e_2)} - \beta_{21} \frac{1+\beta_{12}}{1(e_1+e_2)} = \frac{1(1+\beta_{12}) - \beta_{12}\beta_{21}(1+\beta_{12})}{\beta_{12} r} =$$

$$= \frac{(1-\beta_{12}\beta_{21})(1+\beta_{12})}{\beta_{12} r} > 0$$

Then the unique equilibrium is

$$\begin{aligned} x_1 &= \frac{r}{1+\beta_{12}} \\ \lambda_1 &= \frac{1+\beta_{12}}{r} \\ t_{12} &= \frac{\beta_{12} e_1 - e_2}{1+\beta_{12}} \geq 0 \\ \gamma_{12} &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\beta_{12} r}{1+\beta_{12}} \\ \lambda_2 &= \frac{1}{e_2} \\ t_{21} &= 0 \\ \gamma_{21} &= \frac{(1-\beta_{12}\beta_{21})(1+\beta_{12})}{\beta_{12} r} > 0 \end{aligned}$$

and then we have $1 - \beta_{12}\beta_{21} > 0$.

4.

It is perfectly symmetric to point 3 above.

5.

From equilibrium conditions, we have

$$\begin{aligned} &\tfrac{1}{x_1} - \lambda_1 &&= 0 \\ &\beta_{12} \tfrac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} &&= 0 \\ &-x_1 - t_{12} + e_1 + t_{21} &&= 0 \\ &\min\{t_{12}, \gamma_{12}\} &&= 0 \\ \\ &\tfrac{1}{x_2} - \lambda_2 &&= 0 \\ &\beta_{21} \tfrac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} &&= 0 \\ &-x_2 - t_{21} + e_2 + t_{12} &&= 0 \\ &\min\{t_{21}, \gamma_{21}\} &&= 0 \\ \\ &\textstyle\sum_{h \in \mathcal{H}} (x_h - e_h) &&= 0 \end{aligned}$$

Observe that $x_2 = -t_{21} + e_2 + t_{12} = e_2 + t_{12} - t_{21}$. Then, $\beta_{12} \frac{1}{e_2 - t_{21} + t_{12}} = \lambda_1 - \gamma_{12}$, $\beta_{12} \frac{1}{x_2} = \lambda_1 - \gamma_{12}$ and $\beta_{12}\lambda_2 = \lambda_1 - \gamma_{12}$ or $\lambda_1 - \beta_{12}\lambda_2 = \gamma_{12}$. Then,

$$\begin{cases} \lambda_1 - \beta_{12}\lambda_2 - \gamma_{12} = 0 \\ \\ \lambda_2 - \beta_{21}\lambda_1 - \gamma_{21} = 0 \end{cases}$$

or

$$\begin{cases} \lambda_1 - \beta_{12}\lambda_2 = \gamma_{12} \\ \\ -\beta_{21}\lambda_1 + \lambda_2 = \gamma_{21} \end{cases} \tag{40}$$

Let's show the above system has no solutions, using Rouchè-Capelli Theorem. Indeed.

$$\det \begin{bmatrix} 1 & -\beta_{12} \\ -\beta_{21} & 1 \end{bmatrix} = 1 - \beta_{12}\beta_{21} < 0, \text{ by assumption.}$$

Therefore the system has a unique solution. Using Cramer's rule, we compute the following determinants.

$$\det \begin{bmatrix} \gamma_{12} & -\beta_{12} \\ \gamma_{21} & 1 \end{bmatrix} = \gamma_{12} + \beta_{12}\gamma_{21} \qquad \text{and} \qquad \det \begin{bmatrix} 1 & \gamma_{12} \\ -\beta_{21} & \gamma_{21} \end{bmatrix} = \gamma_{21} + \beta_{21}\gamma_{12}.$$

and

$$\lambda_1 = \frac{\gamma_{12} + \beta_{12}\gamma_{21}}{1 - \beta_{12}\beta_{21}}, \qquad \lambda_2 = \frac{\gamma_{21} + \beta_{21}\gamma_{12}}{1 - \beta_{12}\beta_{21}}.$$

We are left with showing that at least one of the above determinants is positive or zero and therefore either $\lambda_1 \leq 0$ or $\lambda_2 \leq$

0, which is a desired contradiction.

Case 1. either $\gamma_{12} > 0$ or $\gamma_{21} > 0$.

In this case either $\gamma_{12} + \beta_{12}\gamma_{21} > 0$ or $\gamma_{21} + \beta_{21}\gamma_{12} > 0$,as desired.

Case 2. $\gamma_{12} = \gamma_{21} = 0$.

Then $\lambda_1 = \lambda_2 = 0$, again a contradiction. ■

5.2 A Cobb Douglas economy with an artificial bound on transfers

The main result in the present section is Proposition **??**. The basic idea of the (proof of that) Proposition is as follows. Assume that the upper bound on trasfer is $k_1 = k_2 := k = r$. If $(x^*, t^*, \lambda^*, \gamma^*)$ is an equilibrium for an economy (e, β) in the model with no bounds and $t^* \leq r \cdot \mathbf{1}$, then $(x^*, t^*, \lambda^*, \gamma^*, \delta^* = 0)$ is an equilibrium for the economy (e, β) in the model with bound. To show that result is enough to compare the equilibrium systems in the models without and with bound on trasfers. In the subset of economies for which there is no equilibrium in the model without bound, to construct an equilibrium in the model with bound prooceed as follows. Assume that one or both household "hit the bound" and then work backward.

In this economy, household 1 maximization problem is as follows. For given economy $\beta_{12}, \beta_{21} \in \mathbb{R}^2_{++}$, $t_{21} \in \mathbb{R}_+$ and price of the good normalized to 1:

$$\max_{(x_1,t_{12})} 1 \log x_1 + \beta_{12} \log (e_2 - t_{21} + t_{12}) \qquad \text{s.t.} \qquad \begin{aligned} -x_1 - t_{12} + e_1 + t_{21} &= 0 \\ t_{12} &\geq 0. \\ k - t_{12} &\geq 0 \end{aligned}$$

To fix ideas we take $k = r$. Other choices could be made.

Similarly, we can write maximization problem for household 2. We can then give the definition of equilibrium as follows.

Definition 42 $((x_1^*, t_{12}^*, \lambda_1^*, \gamma_{12}^*, \delta_{12}^*), (x_2^*, t_{21}^*, \lambda_2^*, \gamma_{21}^*, \delta_{21}^*))$ *is an equilibrium for the economy* (β_{12}, β_{21}) *if it is a solution to the following system.*

$$\begin{aligned}
&\tfrac{1}{x_1} - \lambda_1 &&= 0 \\
&\beta_{12}\tfrac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} - \delta_{12} &&= 0 \\
&-x_1 - t_{12} + e_1 + t_{21} &&= 0 \\
&\min\{t_{12}, \gamma_{12}\} &&= 0 \\
&\min\{\overline{k} - t_{12}, \delta_{12}\} && \\
\\
&\tfrac{1}{x_2} - \lambda_2 &&= 0 \\
&\beta_{21}\tfrac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} - \delta_{21} &&= 0 \\
&-x_2 - t_{21} + e_2 + t_{12} &&= 0 \\
&\min\{t_{21}, \gamma_{21}\} &&= 0 \\
&\min\{\overline{k} - t_{21}, \delta_{21}\} && \\
\\
&\textstyle\sum_{h \in \mathcal{H}} (x_h - e_h) &&= 0
\end{aligned} \tag{41}$$

Remark 43 *Observe that* $e_2 - t_{21} + t_{12} > 0$*, by definition of* log*, and then* $\beta_{12}\frac{1}{e_2 - t_{21} + t_{12}} = \lambda_1 - \gamma_{12} + \delta_{12} > 0$.

From equations related to household 1, we get what follows.

$$x_1 = \frac{1}{\lambda_1} \tag{42}$$

$$\begin{aligned}
&\beta_{12}\tfrac{1}{e_2 - t_{21} + t_{12}} = \lambda_1 - \gamma_{12} + \delta_{12} \\
&\tfrac{\beta_{12}}{\lambda_1 - \gamma_{12} + \delta_{12}} = e_2 - t_{21} + t_{12} \\
&t_{12} = \tfrac{\beta_{12}}{\lambda_1 - \gamma_{12} + \delta_{12}} - e_2 + t_{21}
\end{aligned} \tag{43}$$

Defined

$$r := e_1 + e_2,$$

and substituting the expressions found in (32) and (33) in the budget equation, we get

$$\begin{aligned}
0 &= -\tfrac{1}{\lambda_1} - \left(\tfrac{\beta_{12}}{\lambda_1-\gamma_{12}+\delta_{12}} - e_2 + t_{21}\right) + e_1 + t_{21} \\
0 &= -\tfrac{1}{\lambda_1} - \tfrac{\beta_{12}}{\lambda_1-\gamma_{12}+\delta_{12}} + (e_1+e_2) \\
\tfrac{1}{\lambda_1} &+ \tfrac{\beta_{12}}{\lambda_1-\gamma_{12}+\delta_{12}} = r
\end{aligned} \tag{44}$$

Similarly, for household 2, we have what follows.

$$x_2 = \tfrac{1}{\lambda_2} \tag{45}$$

$$\begin{aligned}
\beta_{21}\tfrac{1}{e_1-t_{12}+t_{21}} &= \lambda_2 - \gamma_{21} + \delta_{21} \\
\tfrac{\beta_{21}}{\lambda_2-\gamma_{21}+\delta_{21}} &= e_1 - t^*_{12} + t_{21} \\
t_{21} &= \tfrac{\beta_{21}}{\lambda_2-\gamma_{21}+\delta_{21}} - e_1 + t_{12}
\end{aligned} \tag{46}$$

$$\begin{aligned}
0 &= -\tfrac{1}{\lambda_2} - \left(\tfrac{\beta_{21}}{\lambda_2-\gamma_{21}+\delta_{21}} - e_1 + t_{12}\right) + e_2 + t_{12} \\
0 &= -\tfrac{1}{\lambda_2} - \tfrac{\beta_{21}}{\lambda_2-\gamma_{21}+\delta_{21}} + (e_2+e_1) \\
\tfrac{1}{\lambda_2} &+ \tfrac{\beta_{21}}{\lambda_2-\gamma_{21}+\delta_{21}} = r
\end{aligned} \tag{47}$$

Proposition 44 *1. a. If* $1 = \beta_{12}\beta_{21}$ *and* $\beta_{12}e_1 - e_2 \geq 0$*, then for any* $m \in \left[0, r - \frac{\beta_{12}e_1-e_2}{1+\beta_{12}}\right]$*,*[3] *equilibria are*

$$\begin{aligned}
x_1 &= \tfrac{r}{1+\beta_{12}} \\
\lambda_1 &= \tfrac{1+\beta_{12}}{r} \\
t_{12} &= \tfrac{\beta_{12}\cdot r}{1+\beta_{12}} - e_2 + m = \tfrac{\beta_{12}e_1-e_2}{1+\beta_{12}} + m \\
\gamma_{12} &= 0
\end{aligned}$$

$$\begin{aligned}
x_2 &= \tfrac{\beta_{12}\cdot r}{1+\beta_{12}} \\
\lambda_2 &= \tfrac{\beta_{12}\cdot r}{1+\beta_{12}} \\
t_{21} &= m \\
\gamma_{21} &= 0
\end{aligned}$$

1.b .If $1 = \beta_{12}\beta_{21}$ *and* $\beta_{21}e_2 - e_1 \geq 0$*, then for any* $m \in \mathbb{R}_+$*, equilibria are symmetric*[4] *to the above ones.*
2.
If $e_2 - \beta_{12}e_2 > 0$ *and* $e_1 - \beta_{21}e_1 > 0$*, then the unique equilibrium is*

$$\begin{aligned}
x_1 &= e_1 \\
\lambda_1 &= \tfrac{1}{e_1} \\
t_{12} &= 0 \\
\gamma_{12} &= \tfrac{e_2-\beta_{12}e_2}{e_1e_2} > 0
\end{aligned}$$

$$\begin{aligned}
x_2 &= e_2 \\
\lambda_2 &= \tfrac{1}{e_2} \\
t_{21} &= 0 \\
\gamma_{21} &= \tfrac{1e_1-\beta_{21}e_1}{e_1e_2} > 0
\end{aligned}$$

3.
If $1 - \beta_{12}\beta_{21} > 0$ *and* $\beta_{12}e_1 - e_2 \geq 0$*, then the unique equilibrium is*

$$\begin{aligned}
x_1 &= \tfrac{r}{1+\beta_{12}} \\
\lambda_1 &= \tfrac{1+\beta_{12}}{r} \\
t_{12} &= \tfrac{\beta_{12}e_1-e_2}{1+\beta_{12}} \geq 0 \\
\gamma_{12} &= 0
\end{aligned}$$

$$\begin{aligned}
x_2 &= \tfrac{\beta_{12}r}{1+\beta_{12}} \\
\lambda_2 &= \tfrac{1}{e_2} \\
t_{21} &= 0 \\
\gamma_{21} &= \tfrac{(1-\beta_{12}\beta_{21})(1+\beta_{12})}{\beta_{12}r} > 0
\end{aligned}$$

[3] You want that transfers are not bigger than artificially imposed bounded $k = r$.
[4] Here symmetric means "interchange 1 and 2" .

4.

If $1-\beta_{12}\beta_{21}>0$ *and* $\beta_{21}e_2-e_1\geq 0$*, then the unique equilibrium is symmetric to the equilibrium presented in 3. above.*

5. If $\beta_1\beta_2>1$*, then there are the following subcases .*

	$\beta_2 e_2 - e_1 \geq 0$		$\beta_2 e_2 - e_1 < 0$
$\beta_{12}e_1 - e_2 \geq 0$	*1*		*2*
$\beta_{12}e_1 - e_2 < 0$	*3*		*4*

Subcase 4. cannot hold.

In Subcase 1, equilibrium is

$$x_1 = e_1$$
$$\lambda_1 = \frac{1}{e_1}$$
$$t_{12} = r$$
$$\gamma_{12} = 0$$
$$\delta_{12} = \frac{\beta_1}{e_2} - \frac{1}{e_1} \geq 0$$

$$x_2 = e_2$$
$$\lambda_2 = \frac{1}{e_2}$$
$$t_{21} = r$$
$$\gamma_{21} = 0$$
$$\delta_{21} = \frac{\beta_2}{e_1} - \frac{1}{e_2} \geq 0$$

In Subcase 2, equilibrium is

$$x_1 = \frac{\beta_2 r}{1+\beta_2}$$
$$\lambda_1 = ...$$
$$t_{12} = r$$
$$\gamma_{12} = 0$$
$$\delta_{12} = \frac{1}{\beta_2 r}\left(\beta_1\beta_2 - 1\right)\left(1+\beta_2\right) > 0$$

$$x_2 = \frac{r}{1+\beta_2}$$
$$\lambda_2 = ...$$
$$t_{21} = \frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2}$$
$$\gamma_{21} = 0$$
$$\delta_{21} = 0$$

In Subcase 3, equilibrium is symmetric to that one in Subcase 2.

Conjecture. *If* $e_1 = e_2 (= 1)$ *there are two (!) equilibria.*

Proof. 1.

Basically the idea is that: all equilibria in the model without bound which satisy the bound requirements are indeed equilibria in the model with bounds.

$x_1 = \frac{1}{\lambda_1}$

$t_{12} = \frac{\beta_{12}}{\lambda_1} - e_2 + t_{21}$

$\frac{1}{\lambda_1} + \frac{\beta_{12}}{\lambda_1} = r$

$\lambda_1 = \frac{1+\beta_{12}}{r}, \qquad x_1 = \frac{1r}{1+\beta_{12}}.$

Similarly, for household 2, we have what follows.

$x_2 = \frac{1}{\lambda_2}$

$t_{21} = \frac{\beta_{21}}{\lambda_2} - e_1 + t_{12}^*$

$\frac{1}{\lambda_2} + \frac{\beta_{21}}{\lambda_2} = r$

$\lambda_2 = \frac{1+\beta_{21}}{r}, \qquad x_2 = \frac{1r}{1+\beta_{21}}$

To get market clearing, we must have

$$\begin{aligned} 0 = \frac{r}{1+\beta_{21}} + \frac{r}{1+\beta_{12}} - r = r\frac{1}{1+\beta_{12}} - r + r\frac{1}{1+\beta_{21}} = \\ r\frac{1}{1+\beta_{12}} - r + r\frac{1}{1+\beta_{21}} = \left(1+\beta_{12}\right)^{-1}\left(1+\beta_{21}\right)^{-1}\left(1-\beta_{12}\beta_{21}\right)r, \end{aligned} \tag{48}$$

i.e.,

$$1-\beta_{12}\beta_{21}=0.$$

We are left with finding t_{12} and t_{21}.

$$\begin{cases} t_{12}-t_{21} &= \frac{\beta_{12}}{\lambda_1}-e_2 \\ \\ -t_{12}+t_{21} &= \frac{\beta_{21}}{\lambda_2}-e_1 \end{cases}$$

The augmented matrix associated with the above system is

$$\begin{matrix} 1 & -1 & | & \frac{\beta_{12}}{\lambda_1}-e_2 \\ & & | & \\ -1 & 1 & | & \frac{\beta_{21}}{\lambda_2}-e_1 \end{matrix}$$

Since $\det\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}=0$, then the system has solutions if (summing up rows)

$$0=\frac{\beta_{12}}{\frac{1+\beta_{12}}{r}}-e_2+\frac{\beta_{21}}{\frac{1+\beta_{21}}{r}}-e_1=\frac{\beta_{12}r}{1+\beta_{12}}+\frac{\beta_{21}r}{1+\beta_{21}}-r$$

which is exactly (**??**). Then,

$$t_{12}=t_{21}+\frac{\beta_{12}}{\frac{1+\beta_{12}}{r}}-e_2=t_{21}+\frac{\beta_{12}r}{1+\beta_{12}}-e_2=t_{21}+\frac{\beta_{12}e_1+\beta_{12}e_2-1e_2-\beta_{12}e_2}{1+\beta_{12}}=t_{21}+\frac{\beta_{12}e_1-1e_2}{1+\beta_{12}}.$$

2. 3. and 4.

The equilibria are the same because all the equilibrium values of the transfer are smaller or equal than r. To be chcked again.

5.

This is the only substantially different case.

Since in the model with no bound we have no equilibria, then in the model with bound at least one of the households has to have a transfer equal to the bound. In case 1, both transfers are equal to k; in the present case, it seems that we should have $t_{12}=k=r$ and household 1 would like to violate the bound because $\beta_1<\beta_2$.

Let's present an informal discussion which motivates the analysis below.

Conjecture $t_1=t_2=r>0$.Then $\gamma_{12}=\gamma_{21}=0$ and from the budget equations $x_1=e_1$ and $x_2=e_2$.From the second equation in the kt conditions we have

$$\delta_{12}=\frac{\beta_1}{e_2}-\frac{1}{e_1}\geq 0 \quad\Leftrightarrow\quad \beta_{12}e_1-e_2\geq 0$$

and similarly

$$\delta_{21}=\frac{\beta_2}{e_1}-\frac{1}{e_2}\geq 0 \quad\Leftrightarrow\quad \beta_2e_2-e_1\geq 0$$

Then, we have to distinguish the following cases.

	$\beta_2e_2-e_1\geq 0$		$\beta_2e_2-e_1<0$
$\beta_{12}e_1-e_2\geq 0$	1		2
$\beta_{12}e_1-e_2<0$	3		4

Case 1. $\beta_{12}e_1-e_2\geq 0$ and $\beta_2e_2-e_1\geq 0$.

As a consequence of what said above in the informal discussion, we have that in this case the equilibrium is

$$\begin{aligned} &x_1=e_1\\ &\lambda_1=\tfrac{1}{e_1}\\ &t_{12}=r\\ &\gamma_{12}=0\\ &\delta_{12}=\tfrac{\beta_1}{e_2}-\tfrac{1}{e_1}\geq 0\\ \\ &x_2=e_2\\ &\lambda_2=\tfrac{1}{e_2}\\ &t_{21}=r\\ &\gamma_{21}=0\\ &\delta_{21}=\tfrac{\beta_2}{e_1}-\tfrac{1}{e_2}\geq 0 \end{aligned}$$

Case 4. $\beta_{12}e_1 - e_2 < 0$and $\beta_2 e_2 - e_1 < 0$.

We are going to show that this case cannot hold because we are assuming that $\beta_1\beta_2 > 1$. Indeed, in this case we have $\beta_1 < \frac{e_2}{e_1}$ and $\beta_2 < \frac{e_1}{e_2}$ and then

$$\beta_1\beta_2 < 1.$$

Case 2. $\beta_{12}e_1 - e_2 \geq 0$and $\beta_2 e_2 - e_1 < 0$.

We conjecture $t_{12} = r > 0$.Then $\gamma_{12} = 0$ and

$$x_1 = -r + e_1 + t_{21} = t_{21} - e_2$$

$$x_2 = -t_{21} + e_2 + r = e_1 + 2e_2 - t_{21}$$

Moreovoer fom the first equation for each household we get

$$\beta_{12}\frac{1}{x_2} - \frac{1}{x_1} = \delta_{12}$$

$$\beta_{21}\frac{1}{x_1} - \frac{1}{x_2} = 0 \quad \text{or} \quad x_1 = \beta_2 x_2$$

Then,

$$t_{21} - e_2 = \beta_2\left(e_1 + 2e_2 - t_{21}\right)$$

$$\left(1+\beta_2\right)t_2 = e_2 + \beta_2\left(e_1 + 2e_2\right) = e_2 + \beta_2\left(e_2 + r\right) = \left(1+\beta_2\right)e_2 + \beta_2 r$$

$$t_2 = \frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2}$$

Observe

$$\frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2} < r \quad \Leftrightarrow$$

$$\left(1+\beta_2\right)e_2 + \beta_2 r - \left(1+\beta_2\right)r = \left(1+\beta_2\right)e_2 + \beta_2 r - \left(1+\beta_2\right)e_1 - \left(1+\beta_2\right)e_2 =$$
$$= \beta_2 e_1 + \beta_2 e_2 - e_1 + \beta_2 e_1 = \beta_2 e_2 - e_1 < 0$$

which is true in the present Case 2. Then $\delta_{21} = 0$. Then,

$$x_1 = \frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2} - e_2 = \frac{\beta_2 r}{1+\beta_2}$$

$$x_2 = -\frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2} + e_2 + r = \frac{r}{1+\beta_2}$$

$$0 = \beta_{12}\frac{1}{e_2 - t_{21}^* + t_{12}} - \lambda_1 + \gamma_{12} - \delta_{12} = \beta_1\frac{r}{1+\beta_2} - \frac{1+\beta_2}{\beta_2 r} - \delta_{12}$$

$$\delta_{12} = \beta_1\frac{1+\beta_2}{r} - \frac{1+\beta_2}{\beta_2 r} = \frac{1}{\beta_2 r}\left(\beta_1\beta_2 - 1\right)\left(1+\beta_2\right) > 0$$

$$\beta_{21}\frac{1}{e_1 + t_{21} - t_{12}^*} - \lambda_2 + \gamma_{21} - \delta_{21} = \beta_2\frac{1+\beta_2}{\beta_2 r} - \frac{1+\beta_2}{r} = 0.$$

Summarizing the equilibrium is

$$\begin{aligned}
&x_1 = \frac{\beta_2 r}{1+\beta_2}\\
&\lambda_1 = \ldots\\
&t_{12} = r\\
&\gamma_{12} = 0\\
&\delta_{12} = \frac{1}{\beta_2 r}\left(\beta_1\beta_2 - 1\right)\left(1+\beta_2\right) > 0\\
\\
&x_2 = \frac{r}{1+\beta_2}\\
&\lambda_2 = \ldots\\
&t_{21} = \frac{(1+\beta_2)e_2 + \beta_2 r}{1+\beta_2} \overset{\beta_2 e_2 - e_1 < 0}{<} r\\
&\gamma_{21} = 0\\
&\delta_{21} = 0.
\end{aligned}$$

■

5.3 The intuition about nonexistence provided by the Cobb-Douglas economy

The above analysis says that there is no equilibrium if and only if

$$\begin{aligned} &\beta_{12}\beta_{21} > 1, && \text{i.e..} \\ &\beta_{12} > \tfrac{1}{\beta_{21}} && \text{i.e.,} \\ &\beta_{21} > \tfrac{1}{\beta_{12}}. \end{aligned} \tag{49}$$

Since the objective functions of households 1 and 2 are respectively

$$u_1 + \beta_{12} v_{12}$$

and

$$u_2 + \beta_{21} v_{21} \quad \underline{\text{or}} \quad \frac{1}{\beta_{21}} u_2 + v_{21},$$

condition (49) seems to suggest that nonexistence of equilibria occurs if

household 1 cares about household 2 more than household 2 cares about herself,	or	$\beta_1 > \frac{1}{\beta_2}$
and		
household 2 cares about household 1 more than household 1 cares about herself,	or	$\beta_2 > \frac{1}{\beta_1}$

In that situation, three possibilities may arise:

a.

household 1 cares about household 2 more than household 1 cares about household 1,	or	$\beta_1 > 1$
and		
household 2 cares about household 1 more than household 2 cares about household 2,	or	$\beta_2 > 1$

In this case, each promise of transfer by each household is upbeaten by a higher promise of the other household.

b.

household 1 cares about household 2 more than household 1 cares about household 1,	or	$\beta_1 > 1$
and		
household 2 cares about household 1 less than household 2 cares about household 2,	or	$\beta_2 < 1$,

In this case, each promise of transfer by household 2 is upbeaten by a higher promise of household 1.

c.

Symmetric situation with respect to b.

To overcome the nonexistence problem, we can introduce an assumption on the relationship between "care of herself and care of the other household".

This is still an open problem for future research.

6 Generic regularity for a simpler model

6.1 The simpler model - "α case"

Showing regularity is a necessary step to show the other results, but it is quite hard under the general assumptions presented in the model described in the Section about existence. Indeed, in Remark 63 below, we present a detail explanation of the encountered difficulties.

In the remainder of the paper, we present a less general version of the model in which generic regularity (and the other results mention in the title of the section) are indeed proved to hold true. The basic idea is to allow transfer only in the numerair good and to write the utility function is a separable manner.

Assumption 1. For any $h, h' \in \mathcal{H}$ with $h \neq h'$, transfers from household h to household h' are done in terms of the the numeraire good.

With some abuse of notation, we are using the same symbol used in the previous sections.

We also assume that each household utility functions has the following additive, or separable, specification.

$$U_h : \mathbb{R}^C_{++} \times \mathbb{R}^{CH-1H}_{+} \times \mathbb{R}^{C-1}_{++} \to \mathbb{R},\ \left(x_h, t, p^{\backslash}\right) \mapsto\ u_h\left(x_h\right) + \sum_{h' \neq h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \sum_{h'' \neq h'}\left(t_{h'',h'} - t_{h',h''}\right)}{pr}\right), \tag{50}$$

where for any $h, h' \in \mathcal{H}$ with $h \neq h'$, $\alpha_{hh'} \in \mathbb{R}$ and the function $v_{hh'}$is defined as follows.

$$v_{hh'} : \mathbb{R}_{++} \longrightarrow \mathbb{R}, \qquad s \mapsto v_{hh'}\left(s\right)$$

Assumption 2. Selfish preferences of each household $h \in \mathcal{H}$ are described by the utility function u_h which satisfies the following standard "smooth assumptions".

u1. $u_h\ : \mathbb{R}^C_{++} \to \mathbb{R}$ is a twice continuously differentiable function;

u2. u_h is differentiably strictly increasing, i.e., for every $x_h \in \mathbb{R}^C_{++},\ Du_h\left(x_h\right) \gg 0$;

u3. u_h is differentiably strictly concave,i.e., for every $x_h \in \mathbb{R}^C_{++}$ and $D^2 u_h\left(x_h\right)$ is negative definite;

u4. for any $\underline{u} \in \mathbb{R}$, $\left\{x \in \mathbb{R}^C_{++} : u_h\left(x\right) \geq \underline{u}\right\}$ is closed in the Euclidean topology of $\mathbb{R}^C$.

The set of all utility functions u_h satisfying the above Assumptions is denoted by $\widehat{\mathcal{U}}$. Moreover $\mathcal{U} := \widehat{\mathcal{U}}^H$ with generic element $u \equiv \left(u_h\right)_{h=1}^H$.

Assumption 3. For any $h, h' \in \mathcal{H}$ with $h \neq h'$, altruistic preferences are described by the function

$$v_{hh'} : \mathbb{R}_{++} \longrightarrow \mathbb{R}, \qquad s \mapsto v_{hh'}\left(s\right)$$

which satisfies the following properties

v1. $v_{hh'}$ is twice continuously differentiable;

v2. $v_{hh'}$ is differentiably strictly increasing, i.e., for every $s \in \mathbb{R}_{++},\ v'_{hh'}\left(s\right) > 0$;

v3. $v_{hh'}$ is differentiably strictly concave, i.e., for every $s \in \mathbb{R}_{++}$, $v''_{hh'}\left(s\right) < 0$.

The set of all functions v_h satisfying the above Assumptions is denoted by $\widehat{\mathcal{V}}$. Moreover $\mathcal{V} := \widehat{\mathcal{V}}^H$ with generic element $v := \left(v_h\right)_{h=1}^H$.

Moreover,

$$\alpha_h := \left(\alpha_{hh'}\right)_{h' \in \mathcal{H} \backslash \{h\}} \in \mathbb{R}^{H-1} \text{ and } \alpha := \left(\alpha_h\right)_{h \in \mathcal{H}} \in \mathbb{R}^{H(H-1).}$$

An economy is $\mathcal{E} := (u, v, e, \alpha, k) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H(H-1)} \times \mathbb{R}^{H(H-1)}_{++} := \mathbb{E}$.

We can then write household h's maximization problem as follows. For any $h \in \mathcal{H}$, for given $(\mathcal{E}, p^{\backslash}, t_{\backslash h}) \in \mathbb{E} \times \mathbb{R}^{C-1}_{++} \times \mathbb{R}^{(H-1)(H-1)}$, $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}$ solves problem

$$\begin{array}{ll} \max_{(x_h, t_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}} & u_h\left(x_h\right) + \sum_{h' \neq h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \left(t_{h,h'} + \tau_{(\neg h),h'}\right)}{pr}\right) \\ \text{s.t.} & \\ p \cdot x_h + \sum_{h' \neq h} t_{h,h'} \leq pe_h + t_{\longrightarrow h}, & \\ t_h \geq 0, & \\ t_h \leq k_h. & \end{array} \tag{51}$$

Definition 45 *The vector* $(\widetilde{x}, \widetilde{t}, \widetilde{p}^{\backslash}) \in \mathbb{R}^{CH}_{++} \times T \times \mathbb{R}^{C-1}_{++}$ *is an* ***equilibrium*** *vector associated with an economy* $\mathcal{E} := (u, v, e, \alpha, k) \in\ \mathbb{E}$ *if*

1. *for given* $(\mathcal{E}, p, t_{\backslash h}) \in \mathbb{E} \times \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)(H-1)}$, $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}$ *solves Problem* (51)
2. *markets clear.*

Definition 46 *The "budget set-valued function" or "budget correspondence" for household* $h \in \mathcal{H}$ *is denoted and defined as follows.*

$$B_h : \mathbb{R}^{C-1}_{++} \times \mathbb{R}^{C(H-1)(H-1)} \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)},$$

$$(p^{\backslash}, t_{\backslash h}) \mapsto\!\!\!\!\mapsto \left\{(x_h, t_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)} : p \cdot x_h + \sum_{h' \neq h} t_{h,h'} \leq pe_h + t_{\longrightarrow h}, \qquad t_h \geq 0, \qquad t_h \leq k_h\right\}.$$

Remark 47 *An unpleasant property of the chosen utility function is that it is not necessarily strictly concave because some of the $\alpha_{hh'}$ may be negative. What we try to do below is to get rid of the negative $\alpha_{hh'}$, without loss of generality.*

Definition and Proposition below formalize the following simple observation "If household h is maximizing and she does not like household h', i.e., $\alpha_{h,h'} \leq 0$, then she will not transfer anything to that household, i.e., $t_{h,h'} = 0$".

Definition 48 *For any $\alpha_h = (\alpha_{h,h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{H-1}$, define*

$$\mathcal{G}_h(\alpha_h) = \{h' \in (\mathcal{H} \setminus \{h\}) : \alpha_{h,h'} > 0\} \qquad \text{and } \mathcal{G}_h^{\setminus}(\alpha_h) = (\mathcal{H} \setminus \{h\}) \setminus \mathcal{G}_h = \{h' \in (\mathcal{H} \setminus \{h\}) : \alpha_{h,h'} \leq 0\};$$

Remark 49 *Observe that $\mathcal{G}_h(\alpha_h)$ is the set of households that household h likes, consistently with α_h. We then also introduce the following definition.*

$$\mathcal{G}_{\longrightarrow h}(\alpha) = \{h' \in \mathcal{H} \setminus \{h\} : h \in \mathcal{G}_{h'}(\alpha_{h'})\} = \{h' \in (\mathcal{H} \setminus \{h\}) : \alpha_{h'h} > 0\},$$

i.e., the set of households who like household h.

We are going to construct a fictitious economy in which for any $h \in \mathcal{H}$, $\mathcal{B}_h$ is the set of households such that "household h conceives to transfer some wealth". More precisely, if $h' \notin \mathcal{B}_h$, then h' does not even appear either in the utility function or the budget constraint of household h, while if $h' \in \mathcal{B}_h$, then household h''s welfare has a strictly positive weight in household h's utility function and household h' may receive a transfer from household h. The above informal description is formalized below.

Definition 50 *For any $h \in \mathcal{H}$, let $\mathcal{B}_h$ be a subset of $\mathcal{H} \setminus \{h\}$ and $\mathcal{B} := \times_{h \in \mathcal{H}} \mathcal{B}_h$. Define also $\beta_h = (\beta_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}_{++}^{\#\mathcal{B}_h}$ and $\beta = (\beta_h)_{h \in \mathcal{H}} \in \mathbb{R}_{++}^{\sum_{h \in \mathcal{H}}(\#\mathcal{B}_h)}$.*

Consistently with the above definition, we have to use a different notation for transfers and related object in this new fictitious environment, as done below (superscript f used below stands for "fictitious").

Definition 51 *Given $\mathcal{B}$,*

$$\begin{array}{lll} \tau_{h,h'} \in \mathbb{R} & & \text{is the transfer from household } h \text{ to household } h', \\ \tau_h & := & (\tau_{h,h'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{\#\mathcal{B}_h}, \\ \tau & := & (\tau_h)_{h \in \mathcal{H}} \in \mathbb{R}^{\sum_{h \in \mathcal{H}} \#\mathcal{B}_h} \\ \tau_{\setminus h} & := & (\tau_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{\sum_{h' \in \mathcal{H} \setminus \{h\}} \#\mathcal{B}_{h'}} \\ \mathcal{G}^f_{\longrightarrow h} & := & \{h' \in \mathcal{H} \setminus \{h\} : h \in \mathcal{B}_{h'}\} \\ \tau_{\longrightarrow h} & := & \sum_{h' \in \mathcal{G}^f_{\longrightarrow h}} \tau_{h',h}\ , \end{array}$$

$$\tau^{net}_{(\neg h),h'} := \sum_{h'' \in \mathcal{G}^f_{\longrightarrow h'}} \tau_{h'',h'} - \sum_{h'' \in \mathcal{B}_{h'}} \tau_{h',h''}$$

$$U_h^f : \mathbb{R}_{++}^{CH} \times \mathbb{R}^{C\left(\sum_{h \in \mathcal{H}} \#\mathcal{B}_h\right)} \to \mathbb{R}, \qquad (x_h, \tau) \mapsto u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \tau_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right)$$

Definition 52 *For any $\mathcal{B}_h \subseteq \mathcal{H} \setminus \{h\}$,$\beta_h = \left(\beta_{h,h'}\right)_{h' \in \mathcal{B}_h} \in \mathbb{R}_{++}^{\#\mathcal{B}_h}$, $\tau^{net}_{(\neg h),h'} \in \mathbb{R}^C$, $\widetilde{\tau}_{\longrightarrow h} \in \mathbb{R}$, the following maximization problem is called $\mathcal{B}_h$-problem for household $h \in \mathcal{H}$.*

$$\begin{array}{ll} \max_{(x_h, \tau_h) \in \mathbb{R}_{++}^C \times \mathbb{R}^{C \cdot (\#\mathcal{B}_h)}} & u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \tau_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) \\ s.t. & \\ p \cdot x_h + \sum_{h' \in \mathcal{B}_h} \tau_{h,h'} - pe + \tau_{\longrightarrow h} \leq 0 & \\ (\tau_{h,h'})_{h' \in \mathcal{B}_h} \geq 0. & \end{array} \tag{52}$$

Definition 53 *The "budget set valued function" or "budget correspondence" for household $h \in \mathcal{H}$ associated to the above problem is denoted and defined as follows.*

$$B_h^f : \mathbb{R}^C_{++} \times \mathbb{R}^{\sum_{h' \in \mathcal{H} \setminus \{h\}} \# \mathcal{B}_{h'}} \longrightarrow \mathbb{R}^C_{++} \times \mathbb{R}^{\# \mathcal{B}_h}$$

$$\left(p, \tau_{\setminus h}\right) \mapsto\mapsto \left\{ (x_h, \tau_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{\sum_{h' \in \mathcal{H} \setminus \{h\}} \# \mathcal{B}_{h'}} : p \cdot x_h + \textstyle\sum_{h' \in \mathcal{B}_h} \tau_{h,h'} \leq pe + \widetilde{\tau}_{\longrightarrow h}, \qquad \tau_h \geq 0 \right\}.$$

Proposition 54 *Let $p \in \mathbb{R}^C_{++}$ be given.*

1. If $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}$ solves problem (51) at $\alpha_h \in \mathbb{R}^{H-1}$ and $\widetilde{t}_{\setminus h} \in \mathbb{R}^{(H-1)(H-1)}$, then $\left(\widetilde{t}_{h,h'}\right)_{h' \in \mathcal{G}_h^{\setminus}(\alpha_h)} = 0$.

2. If $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}$ solves problem (51) at $\alpha_h \in \mathbb{R}^{H-1}$, then $(\widetilde{x}_h, (\widetilde{\tau}_{hh'})_{h' \in \mathcal{G}_h(\alpha_h)}) \in \mathbb{R}^C_{++} \times \mathbb{R}^{\# \mathcal{G}_h(\alpha_h)}$, with $(\widetilde{\tau}_{hh'})_{h' \in \mathcal{G}_h(\alpha_h)} = \left(\widetilde{t}_{hh'}\right)_{h' \in \mathcal{G}_h(\alpha_h)}$, solves $\mathcal{G}_h(\alpha_h)$-problem (52) at $\beta_h := (\alpha_{hh'})_{h' \in \mathcal{G}(\alpha_h)}$ and $\tau^{net}_{(\neg h),h'} = t^{net}_{(\neg h),h'} \in \mathbb{R}$, $\tau_{\longrightarrow h} = t_{\longrightarrow h} \in \mathbb{R}$.

3. If $\left(\widetilde{x}_h, \left(\widetilde{\widetilde{\tau}}_h\right)_{h \in \mathcal{B}_h}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}^{\# \mathcal{B}_h}$ solves problem $\mathcal{B}_h$-(52) at $\beta_h \in \mathbb{R}^{\# \mathcal{B}_h}_{++}$ and $\tau^{net}_{(\neg h),h'} \in \mathbb{R}$, $\tau_{\longrightarrow h} \in \mathbb{R}$, then $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C_{++} \times \mathbb{R}^{H-1}$, with

$$\widetilde{t}_{h,h'} = \begin{cases} \widetilde{\widetilde{\tau}}_{h,h'} & \text{if} \quad h' \in \mathcal{B}_h, \\ 0 & \text{if} \quad h' \notin \mathcal{B}_h, \end{cases}$$

solves problem (51) at $\alpha_h \in \mathbb{R}^{H-1}$ and $t^{net}_{(\neg h),h'} = \tau^{net}_{(\neg h),h'} \in \mathbb{R}$, $t_{\longrightarrow h} = \tau_{\longrightarrow h} \in \mathbb{R}$ with

$$\alpha_{h,h'} = \begin{cases} \beta_{h,h'} & \text{if} \quad h' \in \mathcal{B}_h, \\ \in (-\infty, 0] & \text{if} \quad h' \notin \mathcal{B}_h, \end{cases}$$

Proof. 1. Suppose otherwise, i.e.,[5]

$$\left(\widetilde{t}_{h,h'}\right)_{h' \in \mathcal{G}_h^{\setminus}(\alpha_h)} > 0. \tag{53}$$

Then, the simple idea of the proof is to use these transfer as consumption of household h to get the desired contradiction.

We are going to show that $(\widetilde{x}_h, \widetilde{t}_h)$ does not solve problem (**??**) at $\alpha_h \in \mathbb{R}^{H-1}$, verifying that

$$\exists \, (x_h^*, t_h^*) \in \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)} \text{ such that a. } U_h\left(x_h^*, t_h^*\right) > U_h(\widetilde{x}_h, \widetilde{t}_h) \text{ and b. } \left(x_h^*, t_h^*\right) \in B_h\left(p, t_{\setminus h}\right).$$

Take $x_h^* = \widetilde{x}_h + \sum_{h' \in \mathcal{G}_h^{\setminus}} \widetilde{t}_{h,h'} \overset{(53)}{>} \widetilde{x}_h$ and $t_h^* = \left(t^*_{h,h'}\right)_{h' \neq h}$ such that

$$t^*_{h,h'} = \begin{cases} \widetilde{t}_{h,h'} & \text{if} \quad h' \in \mathcal{G}_h(\alpha_h) \\ 0 & \quad h' \in \mathcal{G}_h^{\setminus}(\alpha_h). \end{cases} \tag{54}$$

Then,

$$0 \overset{(54)}{\leq} \left(t^*_{h,h'}\right)_{h' \neq h} \overset{(53)}{<} \left(\widetilde{t}_{h,h'}\right)_{h' \neq h}.$$

a.

$$U_h\left((x_h^*, t_h^*)\right) =$$

$$= U_h\left(\widetilde{x}_h + \textstyle\sum_{h' \in \mathcal{G}_h^{\setminus}} \widetilde{t}_{h,h'}\right) + \sum_{h' \in \mathcal{G}_h} \overset{(+)}{\alpha_{hh'}} \cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \sum_{h' \in \mathcal{G}_h^{\setminus}} \overset{(-)}{\alpha_{hh'}} \cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau^{net}_{(\neg h),h'}}{pr}\right) >$$

$$> u_h\left(\widetilde{x}_h\right) + \textstyle\sum_{h' \in \mathcal{G}_h} \overset{(+)}{\alpha_{hh'}} \cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \sum_{h' \in \mathcal{G}_h^{\setminus}} \overset{(-)}{\alpha_{hh'}} \cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) =$$

$$= U_h(\widetilde{x}_h, \widetilde{t}_h).$$

b.

[5] We use the "standard" definitions for $\geq, >, >>$ between vectors in $\mathbb{R}^n$.

Observe that by definition of t_h^* and since $\left\{\mathcal{G}_h(\alpha_h), \mathcal{G}_h^{\setminus}(\alpha_h)\right\}$ is a partition of $\mathcal{H}\setminus\{h\}$, then

$$\sum_{h'\neq h} t^*_{h,h'} = \sum_{h'\in\mathcal{G}_h(\alpha_h)} t^*_{h,h'} = \sum_{h'\neq h} \widetilde{t}_{h,h'} - \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \widetilde{t}_{h,h'}.$$

Therefore,

$$p\cdot x_h^* + \textstyle\sum_{h'\neq h} t^*_{h,h'} - pe + \widetilde{t}_{\longrightarrow h} =$$

$$= p\cdot \widetilde{x}_h + \textstyle\sum_{h'\in\mathcal{G}_h^{\setminus}} \widetilde{t}_{h,h'} + \sum_{h'\neq h} \widetilde{t}_{h,h'} - \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \widetilde{t}_{h,h'} - pe + \widetilde{t}_{\longrightarrow h} =$$

$$= p\cdot \widetilde{x}_h + \textstyle\sum_{h'\neq h} \widetilde{t}_{h,h'} - pe + \widetilde{t}_{\longrightarrow h} \leq 0$$

$$t_h^* \geq 0.$$

2.

Suppose our claim is false, i.e., there exists $\left(x_h^*, \left(\tau^*_{h,h'}\right)_{h'\in\mathcal{B}}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}^{\#\mathcal{G}_h(\alpha_h)}$ such that

$$\left(x_h^*, \left(\tau^*_{h,h'}\right)_{h'\in\mathcal{G}_h(\alpha_h)}\right) \in B_h^f\left(p, \tau_{\setminus h}\right) \tag{55}$$

and

$$u_h\left(x_h^*\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \tau^*_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) > u_h\left(\widetilde{x}_h\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{\tau}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right),$$

or

$$\begin{array}{c} u_h\left(x_h^*\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + t^*_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau^{net}_{(\neg h),h'}}{pr}\right) > \\ > u_h\left(\widetilde{x}_h\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{e_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau^{net}_{(\neg h),h'}}{pr}\right) \end{array}, \tag{56}$$

Now choose $\widehat{x}_h = x_h^*$ and

$$\widehat{t}_{h,h'} = \begin{cases} t^*_{h,h'} & \text{if} \quad h'\in\mathcal{G}_h(\alpha_h), \\ 0 & \quad h'\in\mathcal{G}_h^{\setminus}(\alpha_h). \end{cases}$$

Observe that

$$\begin{array}{l} u_h\left(\widehat{x}_h\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \alpha_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widehat{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \\ + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \alpha_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widehat{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) = \\ = u_h\left(x_h^*\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + t^*_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \\ + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau^{net}_{(\neg h),h'}}{pr}\right) \overset{(56)}{>} \\ > u_h\left(\widetilde{x}_h\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \\ + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau^{net}_{(\neg h),h'}}{pr}\right) \overset{\text{Conclusion 1 above}}{=} \\ = u_h\left(\widetilde{x}_h\right) + \sum_{h'\in\mathcal{G}_h(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right) + \\ + \sum_{h'\in\mathcal{G}_h^{\setminus}(\alpha_h)} \beta_{hh'}\cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{t}_{h,h'} + \tau^{net}_{(\neg h),h'}}{pr}\right). \end{array} \tag{57}$$

From (55), we have

$$0 \geq p \cdot x_h^* + \sum_{h' \in \mathcal{G}_h(\alpha_h)} t_{h,h'}^* - pe + \widetilde{t}_{\longrightarrow h} = p \cdot x_h^* + \sum_{h' \in \mathcal{G}_h(\alpha_h)} t_{h,h'}^* + \sum_{h' \in \mathcal{G}_h^{\setminus}(\alpha_h)} 0 - pe - \widetilde{t}_{\longrightarrow h}$$

$$= p \cdot x_h^* + \sum_{h' \in \mathcal{G}_h(\alpha_h)} \widehat{t}_{h,h'} + \sum_{h' \in \mathcal{G}_h^{\setminus}(\alpha_h)} \widehat{t}_{h,h'} - pe - \widetilde{t}_{\longrightarrow h}$$

$$t_h^* \geq 0,$$

i.e.,

$$\left(\widehat{x}_h, \left(\widehat{t}_{h,h'}\right)_{h' \neq h}\right) \in B_h\left(p, t_{\setminus h}\right) \tag{58}$$

(57) and (58) contradicts the fact that $(\widetilde{x}_h, \widetilde{t}_h)$ solves problem (**??**) at $\alpha_h \in \mathbb{R}^{H-1}$.
3.
By assumption, for any $\left(x_h, (\tau_h)_{h \in \mathcal{B}_h}\right) \in B_h^f\left(e_h, \tau_{\setminus h}\right)$

$$u_h\left(\widetilde{x}_h\right) + \sum_{h' \in \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{\widetilde{\tau}}_{h,h'} + \tau_{(\neg h),h'}^{net}}{pr}\right) \geq u_h\left(x_h\right) + \sum_{h' \in \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \tau_{h,h'} + \tau_{(\neg h),h'}^{net}}{pr}\right).$$

Since $\alpha_{hh'} \leq 0$ for any $h' \in \mathcal{H} \setminus \mathcal{B}_h$, taking into account Assumption v2, we have

$$\sum_{h' \in \mathcal{H} \setminus \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau_{(\neg h),h'}^{net}}{pr}\right) \geq \sum_{\mathcal{H} \setminus \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + t_{h,h'} + \tau_{(\neg h),h'}^{net}}{pr}\right).$$

Therefore,

for any $\left(x_h, (\tau_h)_{h \in \mathcal{B}_h}\right) \in B_h^f\left(e_h, \tau_{\setminus h}\right)$

$$u_h\left(\widetilde{x}_h\right) + \sum_{h' \in \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \widetilde{\widetilde{\tau}}_{h,h'} + \tau_{(\neg h),h'}^{net}}{pr}\right) + \sum_{h' \in \mathcal{H} \setminus \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + 0 + \tau_{(\neg h),h'}^{net}}{pr}\right) \geq \tag{59}$$

$$u_h\left(x_h\right) + \sum_{h' \in \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \tau_{hh'} + \tau_{(\neg h),h'}^{net}}{pr}\right) + \sum_{\mathcal{H} \setminus \mathcal{B}_h} \alpha_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'} + \tau_{h,h'} + \tau_{(\neg h),h'}^{net}}{pr}\right).$$

Moreover,

$$\left(\widetilde{x}_h, \left(\widetilde{t}_h\right)_{h \in \mathcal{B}_h}\right) \in B_h\left(e_h, t_{\setminus h}\right), \tag{60}$$

as verified below. By assumption, $\left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h^f\left(p, t_{\setminus h}\right)$ and therefore

$$p \cdot \widetilde{x}_h + \sum_{h' \in \mathcal{B}_h} \widetilde{\widetilde{t}}_{h,h'} \leq pe + \widetilde{t}_{\longrightarrow h}, \qquad t_h \geq 0$$

and then

$$p \cdot \widetilde{x}_h + \sum_{h' \in \mathcal{B}_h} \widetilde{\widetilde{t}}_{h,h'} + \sum_{h' \in \mathcal{B}_h} \overset{(=0)}{\widetilde{\widetilde{t}}_{h,h'}} \leq pe + \widetilde{t}_{\longrightarrow h}, \qquad t_h \geq 0$$

(59) and (60) mean that $(\widetilde{x}_h, \widetilde{t}_h)$ solves problem (**??**) at α_h, as desired. ■

Corollary 55 *For any* $t_{(\neg h),h'}^{net} \in \mathbb{R}$ *and* $t_{\longrightarrow h} \in \mathbb{R}$,
$\left(x_h, (t_{h,h'})_{h' \in \mathcal{H} \setminus \{h\}}\right) = \left(x_h, (t_{h,h'})_{h' \in \mathcal{G}_h(\alpha_h)}, (t_{h,h'} = 0)_{h' \notin \mathcal{G}_h(\alpha_h)}\right)$ *is a solution to (the true) maximization problem* (51) *at* $\alpha_h \in \mathbb{R}^{H-1}$
$\Leftrightarrow$
$\left(x_h, (\tau_{h,h'})_{h' \in \mathcal{G}_h(\alpha_h)}\right)$ *is a solution to (the fictitious) maximization problem* (52) *at* $(\alpha_{h,h'})_{h' \in \mathcal{G}_h(\alpha_h)} \in \mathbb{R}_{++}^{\#\mathcal{G}_h(\alpha_h)}$.

6.2 The simpler model - "β case"

Using the analysis of the previous subsection, we can then rewrite the model as follows.

An economy is a list of the following objects. First of all,endowments $e \in \mathbb{R}^{CH}_{++}$ and selfish utility functions $u \in \mathcal{U}$ are part of the definition of an economy. Moreover, for any $h \in \mathcal{H}$, $\mathcal{B}_h \subseteq \mathcal{H} \backslash \{h\}$ is the set of households which household h may be willing to make a transfer to; $\mathcal{B}_h$ may be empty; we define $B_h = \#\mathcal{B}_h$and $B = \sum_{h \in \mathcal{H}} B_h$. Then, an economy is also described by

altruistic utility functions, i.e., for any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, $(v_{hh'})_{h \in \mathcal{H},\ h' \in \mathcal{B}_h} := v$;

altruistic weights, i.e., for any $h \in \mathcal{H}$, $\beta_h := (\beta_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{B_h}_{++}$ and $\beta = (\beta_h)_{h \in \mathcal{H}} \in \mathbb{R}^{B}_{++}$;

upper bounds on transfers, for any $h \in \mathcal{H}$, $k_h := (k_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{B_h}_{++}$ and $k = (k_h)_{h \in \mathcal{H}} \in \mathbb{R}^{B}_{++}$.

Then an economy is an element $(e, u, v, \beta, k) \in \mathbb{R}^{CH}_{++} \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{B}_{++} \times \mathbb{R}^{B}_{++}$. For easier notation, we denote transfers still by t.

Definition 56 *(x^*, t^*) is an equilibrium associated to an economy $\mathcal{E}$, if*

for any $h \in \mathcal{H}$, household h maximizes, i.e., for given (u, v, e, β, k), $p^* \in \mathbb{R}^{C}_{++}, t^*_{\backslash h} \in T_{\backslash h}$, $(x^*_h, t^*_h) \in X_h \times \mathbb{R}^{C(H-1)}$ solves

$$\begin{array}{ll}\max_{(x_h, t_h) \in X_h \times T_h} & u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(w_{h'}\left(p^*, t_h, t^*_{\backslash h}\right)\right) \\ s.t. & \\ p^* x_h \leq p^* e_h + t^*_{\longrightarrow h} - t_{h \longrightarrow} & \\ t_h \geq 0. & \\ t_h \leq k_h. & \end{array}$$

and market clear, i.e.,

$$\sum_{h \in \mathcal{H}} \left(x^{\backslash *}_h - e^{\backslash}_h\right) = 0.$$

Proposition 57 *1. A solution to the maximization problem* (52) *exists. 2. Solution is unique. 3. It is characterized by Kuhn-Tucker conditions.*

Proof. 1.

To show existence, we use a standard trick of adding the constraint $U_h(x_h, t_h) \geq U_h(e_h, 0)$,where $U_h(x_h, t_h) := u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(w_{h'}\left(p, t_h, t_{\backslash h}\right)\right)$: clearly the solution sets to the problems with and without the additional constraint coincide - see Subsection "The constraint set is not compact" in Math 2 Notes. Then, as a consequence of the Extreme Value Theorem, it suffices to show the constraint set defined below is compact.

$$B^{f*}_h\left(e_h, t_{\backslash h}\right) :=$$

$$\left\{(x_h, t_h) \in \mathbb{R}^{C}_{++} \times \mathbb{R}^m : p^* x_h \leq p^* e_h + t^*_{\longrightarrow h} - t_{h \longrightarrow}, \quad t_h \geq 0, \quad t_h \leq k_h, \quad U_h(x_h, t_h) \geq U_h(e_h, 0)\right\}$$

t_h does belong to the compact set $[0, k_h]$. x_h is bounded below by *zero*. Boundedness above follows from the inequalities below.

$$\text{for any } c \in \mathcal{C},\ x^c_h \leq \frac{1}{p^c}\left(pe_h + \sum_{h' \in \mathcal{D}_h} t_{h'h} - \sum_{h' \in \mathcal{B}_h} \overset{(\geq 0)}{t_{h,h'}} - \sum_{c' \neq c} p^{c'} \cdot \overset{(>0)}{x^{c'}_h}\right) \leq \frac{1}{p^c}\left(pe_h + \sum_{h' \in \mathcal{H} \backslash \{h\}} k_{h'h}\right).$$

To show that the budget set is $\mathbb{R}^C \times \mathbb{R}^{C(H-1)}$-closed, it suffices to show that

$$\left\{(x_h, t_h) \in \mathbb{R}^{C}_{++} \times \mathbb{R}^m : \quad U_h(x_h, t_h) \geq U_h(e_h, 0)\right\}$$

satisfies that properties, a fact which is shown in Lemma 58 below. Since

$$\begin{array}{l} B^{f*}_h\left(e_h, \tau_{\backslash h}\right) = \\ \left\{(x_h, t_h) \in \mathbb{R}^{C}_{++} \times \mathbb{R}^m : \quad U_h(x_h, \tau_h) \geq U_h(e_h, 0)\right\} \\ \cap \\ \left\{(x_h, \tau_h) \in \overset{\downarrow}{\mathbb{R}^C} \times \mathbb{R}^m : px_h \leq p\left(e_h + \sum_{h' \in \mathcal{H} \backslash \{h\}} (t_{h'h} - t_{hh'})\right), \quad t_h \geq 0, \quad t_h \leq k_h, \right\}, \end{array}$$

then $B^{f*}\left(e_h, \tau_{\backslash h}\right)$ is $\mathbb{R}^C \times \mathbb{R}^m$-closed because it is intersection of $\mathbb{R}^C \times \mathbb{R}^m$-closed sets.

2.

Observe that the objective function is strictly concave; the constraint functions are affine. Therefore, uniqueness follows.

3.

Observe that

$$\left(x_h^{++}, t_h^{++}\right) := \left(\frac{e_h}{4}, \frac{pe_h}{4} \cdot \frac{1}{\#\mathcal{B}_h}\right) >> 0$$

is such that

$$-p \cdot \left(x_h^{++} - e_h\right) - \left(\sum_{h' \in \mathcal{B}_h} t_{h,h'}^{++} - \widetilde{t}_{\longrightarrow h}\right) =$$

$$= -p \cdot \left(\tfrac{e_h}{4} - e_h\right) - \left(\sum_{h' \in \mathcal{B}_h} \tfrac{pe_h}{4} \cdot \tfrac{1}{\#\mathcal{B}_h} - \widetilde{t}_{\longrightarrow h}\right) \geq$$

$$\geq -p \cdot \left(\tfrac{e_h}{4} - e_h\right) - p\tfrac{e_h}{4} = p\tfrac{e_h}{2} > 0.$$

All other assumptions for Kuhn-Tucker theorems are satisfied. ■

Lemma 58 *Take* $\underline{V} \in \mathbb{R}$.

Defined $B = \left\{(x_1, \theta_2) \in \mathbb{R}_{++}^C \times \mathbb{R} : \quad V(x_1, \theta_2) \geq \underline{V}\right\}$, *then,*

1. $\mathrm{Cl}_{\mathbb{R}^C \times \mathbb{R}^C} B \subseteq \mathbb{R}_{++}^C \times \mathbb{R}$, *and*

2. B *is* $\mathbb{R}^C \times \mathbb{R}$*-closed.*

Defined $B^* = \left\{(x_1, t_{12}) \in \mathbb{R}_{++}^C \times \mathbb{R} : \quad U_1(x_1, t_{12}) \geq \underline{V}\right\}$, *then,*

3. $\mathrm{Cl}_{\mathbb{R}^C \times \mathbb{R}^C} B^* \subseteq \mathbb{R}_{++}^C \times \mathbb{R}$, *and*

4. B^* *is* $\mathbb{R}^C \times \mathbb{R}$*-closed.*

Proof. We prove conclusions 1 and 2 as follows.

`Notation for the proof of part 1 and 2 to be fixed.`

We want to us Proposition 69 in the Appendix, identifying

$$X, \qquad B, \qquad Y,$$

there with

$$\mathbb{R}^C \times \mathbb{R}^H, \qquad B, \qquad \mathbb{R}_{++}^C \times \mathbb{R}^H,$$

here. Observe that since w is a continuous function, then B is $\mathbb{R}_{++}^C \times \mathbb{R}^m$-closed. Therefore, we are left with showing that for any $\underline{w} \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C \times \mathbb{R}^C} \left\{(x, \theta) \in \mathbb{R}_{++}^C \times \mathbb{R}^H : w(x, \theta) \geq \underline{w}\right\} \subseteq \mathbb{R}_{++}^C \times \mathbb{R}^H$. Then from Proposition 69, also desired conclusion 2 holds.

Let's carefully state what we want to show.

Let the functions $u, v : \mathbb{R}_{++}^C \longrightarrow \mathbb{R}$, $x \mapsto u(x)$ and $\theta \mapsto v(\theta)$ be given.

Assumption.

p. For any $\underline{u} \in \mathbb{R}$, $\quad \mathrm{Cl}_{\mathbb{R}^C} \left\{x' \in \mathbb{R}_{++}^C : u(x') \geq \underline{u}\right\} \subseteq \mathbb{R}_{++}^C$.

Desired conclusion.

q. For any $\underline{w} \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C \times \mathbb{R}^C} \left\{(x, \theta) \in \mathbb{R}_{++}^C \times \mathbb{R}^C : w(x, t) \geq \underline{w}\right\} \subseteq \mathbb{R}_{++}^C \times \mathbb{R}^C$.

Define

$$N(\underline{u}) = \left\{x \in \mathbb{R}_{++}^C : u(x) \geq \underline{u}\right\},$$

$$M(\underline{w}) = \left\{(x, \theta) \in \mathbb{R}_{++}^C \times \mathbb{R}^C : w(x, t) \geq \underline{w}\right\} = \left\{(x, \theta) \in \mathbb{R}_{++}^C \times \mathbb{R}^C : u(x) + \alpha \cdot v(\theta) \geq \underline{w}\right\}$$

We are going to show that $(\neg q) \Rightarrow (\neg p)$.

$$\begin{array}{l}
\neg q \\
\equiv \\
\exists\ \underline{w} \in \mathbb{R},\ \exists\ (x, \theta) \in \mathbb{R}^C \times \mathbb{R}^C \text{ s.th. } (x, \theta) \in \mathrm{Cl}_{\mathbb{R}^C \times \mathbb{R}^C} M(\underline{w}) \overset{\text{Proposition 72}}{=} \mathrm{Ad}_{\mathbb{R}^C \times \mathbb{R}^C} M(\underline{w}) \text{ and } (x, \theta) \notin \mathbb{R}_{++}^C \times \mathbb{R}^C \\
\equiv \\
\exists\ \underline{w} \in \mathbb{R},\ \exists\ (x, \theta) \in \mathbb{R}^C \times \mathbb{R}^C \text{ s.th.}^{6} \forall \varepsilon > 0,\ (B_{\mathbb{R}^C}(x, \varepsilon) \times B_{\mathbb{R}^C}(\theta, \varepsilon)) \cap M(\underline{w}) \neq \varnothing \text{ and } x \notin \mathbb{R}_{++}^C \\
\equiv \\
\exists\ \underline{w} \in \mathbb{R},\ \exists\ (x, \theta) \in \mathbb{R}^C \times \mathbb{R}^C \text{ s.th.} \\
\forall \varepsilon > 0,\ \exists \left(x', \theta'\right) \text{ s.th. } \left(x', \theta'\right) \in (B_{\mathbb{R}^C}(x, \varepsilon) \times B_{\mathbb{R}^C}(\theta, \varepsilon)),\ u(x') + \alpha \cdot v\left(\theta'\right) \geq \underline{w} \text{ and } x \notin \mathbb{R}_{++}^C \\
\Rightarrow \\
\exists\ \underline{w} \in \mathbb{R},\ \exists\ x \in \mathbb{R}^C \text{ s.th. } \forall \varepsilon > 0,\ \exists x' \in B_{\mathbb{R}^C}(x, \varepsilon) \text{ and } \theta' \in \mathbb{R} \text{ s.th. } u(x') \geq \underline{w} - \alpha \cdot v\left(\theta'\right) \text{ and } x \notin \mathbb{R}_{++}^C
\end{array} \tag{61}$$

[6] We are using Proposition 67.

$$
\begin{aligned}
&\neg p \\
&\equiv \\
&\exists\ \underline{u} \in \mathbb{R} \text{ and } \exists\ \overline{x} \in \mathbb{R}^C \text{ such that } \overline{x} \in \mathrm{Cl}_{\mathbb{R}^C} N\left(\underline{u}\right) \text{ and } \overline{x} \notin\ \mathbb{R}^C_{++} \\
&\equiv \\
&\exists\ \underline{u} \in \mathbb{R} \text{ and } \exists\ \overline{x} \in \mathbb{R}^C \text{ such that } \forall \varepsilon > 0,\ B_{\mathbb{R}^C}\left(\overline{x}, \varepsilon\right) \cap N\left(\underline{u}\right) \neq \varnothing \text{ and } \overline{x} \notin\ \mathbb{R}^C_{++} \\
&\equiv \\
&\exists\ \underline{u} \in \mathbb{R} \text{ and } \exists\ \overline{x} \in \mathbb{R}^C \text{ such that } \forall \varepsilon > 0,\ \ \exists \overline{x}' \in B_{\mathbb{R}^C}\left(\overline{x}, \varepsilon\right) \text{ such } u\left(\overline{x}'\right) \geq \underline{u} \text{ and } \overline{x} \notin\ \mathbb{R}^C_{++}
\end{aligned} \tag{62}
$$

Then, to get desired conclusion $\neg p$, it is enough to take $\underline{u} = \underline{w} - \alpha \cdot v\left(\theta'\right)$, $\overline{x} = x$ and $\overline{x}' = x'$ and use (61).

3.

I am dropping the subscripts.By Conclusion 1 above, we know that

$\forall\left(x, \theta\right) \in \mathbb{R}^C \times \mathbb{R}$ such that $\forall \varepsilon > 0$, $\mathcal{B}\left(\left(x, \theta\right), \varepsilon\right) \cap B \neq \varnothing$, we have $\left(x, \theta\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$, or

$$
\begin{aligned}
&\forall\left(x, \theta\right) \in \mathbb{R}^C \times \mathbb{R} \text{ such that } \forall \varepsilon > 0, \text{ there exists } \left(x', \theta'\right) \in \mathcal{B}\left(\left(x, \theta\right), \varepsilon\right) \text{ such that } V_1\left(x', \theta'\right) \geq \underline{V}, \\
&\text{we have } \left(x, \theta\right) \in \mathbb{R}^C_{++} \times \mathbb{R}.
\end{aligned} \tag{63}
$$

We want to show that

$\forall\left(x, t\right) \in \mathbb{R}^C \times \mathbb{R}$ such that $\forall \varepsilon > 0$, $\mathcal{B}\left(\left(x, t\right), \varepsilon\right) \cap B^* \neq \varnothing$, we have $\left(x, t\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$, or

$\forall\left(x, t\right) \in \mathbb{R}^C \times \mathbb{R}$ such that $\forall \varepsilon > 0$, there exists $\left(x'', t''\right) \in \mathcal{B}\left(\left(x, t\right), \varepsilon\right)$ such that $U_1\left(x'', t''\right) \geq \underline{V}$, we have $\left(x, t\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$.

Recall that

$$
U_1\left(x_1, t_{12}\right) = V_1\left(x_1, t_{12} + \left(pe_2 - t_{21}\right)\ \right). \tag{64}
$$

We want to show that

$\forall\left(x, t\right) \in \mathbb{R}^C \times \mathbb{R}$ such that $\forall \varepsilon > 0$, there exists $\left(x'', t''\right) \in \mathcal{B}\left(\left(x, t\right), \varepsilon\right)$ such that $V\left(x''_1, t''_{12} + \left(pe_2 - t_{21}\right)\ \right) \geq \underline{V}$, we have $\left(x, t\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$.

For lighter notation, define $pe_2 - t_{21} = a$.

Take $\left(x, \theta\right) = \left(x, t_{12} + a\right)$. Then $\forall \varepsilon > 0$,

$$
\left\|\left(x, t_{12} + a\right) - \left(x'', t''_{12} + a\right)\right\| = \left\|\left(x, t_{12}\right) - \left(x'', t''_{12}\right)\right\| \overset{\left(x'', t''\right) \in \mathcal{B}\left(\left(x, t\right), \varepsilon\right)}{<} \varepsilon
$$

and

$$
V\left(x''_1, t''_{12} + a\ \right) \geq \underline{V}.
$$

Then, by assumption (63), $\left(x, \theta\right) = \left(x, t_{12} + a\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$ and $\left(x, t_{12}\right) \in \mathbb{R}^C_{++} \times \mathbb{R}$, as desired.

4.

Take $\left(x_n, t_n\right)_{n \in \mathbb{N}} \in \left(B^*\right)^\infty$ such that $\left(x_n, t_n\right) \longrightarrow \left(\overline{x}, \overline{t}\right) \in \mathbb{R}^C \times \mathbb{R}$. We want to show that $\left(\overline{x}, \overline{t}\right) \in B^*$.

Observe that $\left(x_n, t_n\right)_{n \in \mathbb{N}} \in \left(B^*\right)^\infty$ means that for any $n \in \mathbb{N}$, $U_1\left(x^n_1, t^n_{12}\right) \geq \underline{V}$ and, from (64), $V_1\left(x^n_1, t^n_{12} + a\right) \geq \underline{V}$, i.e., $\left(x_n, t_n + a\right)_{n \in \mathbb{N}} \in \left(B\right)^\infty$. From Conclusion 2 above, since B is closed, we have that $\left(x_n, t_n + a\right) \longrightarrow \left(\overline{x}, \overline{t} + a\right) \in B$, i.e., by definition of $B = \left\{\left(x_1, \theta_2\right) \in \mathbb{R}^C_{++} \times \mathbb{R} : \quad V\left(x_1, \theta_2\right) \geq \underline{V}\right\}$, we have $V\left(\overline{x}, \overline{t} + a\right) \geq \underline{V}$, and from (64), $U_1\left(\overline{x}, \overline{t}\right) \geq \underline{V}$, i.e., $\left(\overline{x}, \overline{t}\right) \in B^*$. ■

7 Regularity with $H \in \mathbb{N}$ households in the β-model

Using the analysis of the previous subsection, we can then rewrite the model as follows. Recall that transfers are made in units of the numeraire good.

An economy is a list of the following objects:

endowments: $\left(e_h\right)_{h \in \mathcal{H}} := e \in \mathbb{R}^{CH}_{++}$;

selfish utility functions: $\left(u_h\right)_{h \in \mathcal{H}} := u \in \mathcal{U}$;

set of households household h may be willing to make a transfer to: for any $h \in \mathcal{H}$, $\mathcal{B}_h \subseteq \mathcal{H} \backslash \left\{h\right\}$; $\mathcal{B}_h$ may be empty; we define $B_h = \#\mathcal{B}_h$and $B = \sum_{h \in \mathcal{H}} B_h$;

set of households may be willing to make transfers to household h: $\mathcal{B}_{\longrightarrow h} = \left\{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}\right\}$;

transfer from household $h \in \mathcal{H}$ to household $h' \in \mathcal{B}_h$: $t_{hh'} \in \mathbb{R}$;

vector of transfers of household h: for any $h \in \mathcal{H}$, $t_h := \left(t_{hh'}\right)_{h' \in \mathcal{B}_h} \in \mathbb{R}^{B_h}$;

total transfer from household h: $t_{h \longrightarrow} := \sum_{h' \in \mathcal{B}_h} t_{hh'}$;

total transfer to household h: $t_{\longrightarrow h} := \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}$;

total transfer to household h without considering the transfer from $h'' \in \mathcal{B}_{\longrightarrow h}$: $t_{\left(\neg h''\right) \longrightarrow h} := \sum_{h' \in \mathcal{B}_{\longrightarrow h} \backslash \left\{h''\right\}} t_{h'h}$

altruistic utility functions: for any $h \in \mathcal{H}$ and $h' \in \mathcal{B}_h$, $\left(v_{hh'}\right)_{h \in \mathcal{H},\ h' \in \mathcal{B}_h} := v$;

altruistic weights: for any $h \in \mathcal{H}$, $\ \beta_h := \left(\beta_{hh'}\right)_{h' \in \mathcal{B}_h} \in \mathbb{R}^{B_h}_{++}$ and $\beta = \left(\beta_h\right)_{h \in \mathcal{H}} \in \mathbb{R}^B_{++}$;

upper bounds on transfers, for any $h \in \mathcal{H}$, $k_h := (k_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}_+^{B_h}$ and $k = (k_h)_{h \in \mathcal{H}} \in \mathbb{R}_+^B$;
wealth of household h: $pe_h + t_{\longrightarrow h} - t_{h\longrightarrow}$;
wealth of household h': $pe_{h'} + t_{hh'} + t_{(\neg h)\longrightarrow h'} - t_{h'\longrightarrow}$.
Then an economy is an element $\mathcal{E} := (e, u, v, \beta, k) \in \mathbb{R}_{++}^{CH} \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}_+^B \times \mathbb{R}_+^B$.

Definition 59 $(x^*, t^*.p^*) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^B \times \mathbb{R}_{++}^{C-1}$ *is an equilibrium associated to an economy* (e, u, v, β, k), *if for any* $h \in \mathcal{H}$, *household* h *maximizes, i.e., for given* (u, v, e, β, k), $p^* \in \mathbb{R}_{++}^C, t^*_{\backslash h} \in T_{\backslash h}$, $(x_h^*, t_h^*) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}^{B_h}$ *solves*

$$\begin{array}{l} \max_{(x_h,t_h)\in\mathbb{R}_{++}^{CH}\times\mathbb{R}^B} \quad u_h(x_h) + \sum_{h'\in\mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'}+t_{hh'}+t^*_{(\neg h)\longrightarrow h'}-t^*_{h'\longrightarrow}}{pr}\right) \\ s.t. \\ p^* x_h + t_{h\longrightarrow} \leq p^* e_h + t^*_{\longrightarrow h} \\ \\ t_h \geq 0. \\ \\ t_h \leq k_h. \end{array}$$

and market clear, i.e.,

$$\sum_{h\in\mathcal{H}} \left(x_h^{\backslash *} - e_h^{\backslash}\right) = 0.$$

Using Proposition 57, we can write the so called extended system associated with the above definition of equilibrium.

$$\begin{array}{ll} & Du_h(x_h) - \lambda_h p \\ h' \in \mathcal{B}_h & \frac{1}{pr}\beta_{hh'} \cdot v_{hh'}\left(\frac{pe_{h'}+t_{hh'}+t_{(\neg h)\longrightarrow h'}-t_{h'\longrightarrow}}{pr}\right) - \lambda_h + \gamma_{hh'} - \delta_{hh'} \\ & px_h + \sum_{h'\in\mathcal{B}_h} t_{hh'} \leq pe_h + t_{\longrightarrow h} \\ h' \in \mathcal{B}_h & \min\{t_{hh'}, \gamma_{hh'}\} \\ h' \in \mathcal{B}_h & \min\{\overline{k} - t_{hh'}, \delta_{hh'}\} \\ \\ & Du_{h'}(x_{h'}) - \lambda_{h'} p \\ h'' \in \mathcal{B}_{h'} & \frac{1}{pr}\beta_{h'h''} \cdot v_{h'h''}\left(\frac{pe_{h''}+t_{h'h''}+t_{(\neg h')\longrightarrow h''}-t_{h''\longrightarrow}}{pr}\right) - \lambda_{h'} + \gamma_{h'h''} - \delta_{h'h''} \\ & px_{h'} + \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''} \leq pe_{h'} + t_{\longrightarrow h'} \\ h'' \in \mathcal{B}_{h'} & \min\{t_{h'h''}, \gamma_{h'h''}\} \\ h'' \in \mathcal{B}_{h'} & \min\{\overline{k} - t_{h'h''}, \delta_{h'h''}\} \\ \\ & \sum_{h\in\mathcal{H}} \left(x_h^{\backslash} - e_h^{\backslash}\right) \end{array}$$

In Appendix 8.4, we describe what we have to prove to get the desired generic regularity result. For the reader's convenience, we repeat below the main steps to be followed.
1. For every $\mathcal{E}$, $F_{\mathcal{E}}^{-1}(0) \neq \emptyset$.
2. For every $\mathcal{E}$, $F_{\mathcal{E}}^{-1}(0)$ is compact.
3. $pr : F^{-1}(0) \to \mathbb{E}$, $\quad (\xi, \mathcal{E}) \mapsto \mathcal{E}$ is proper.
4. Border line cases are rare;
5. Final regularity result holds true, which is basically an easier version of the proof done in part of 4 above.
Since we have already shown existence in Corollary 33, we are going to formalize and prove results 2., 3., 4. and 5. in the subsections below.
We also present a sort of generalization of Proposition 40.1 in Subsection 7.0.5 below.

7.0.1 Compactness of $(F_{\mathcal{E}})^{-1}(0)$

Proposition 60 *If*

$$\text{there exists } h \in \mathcal{H} \text{ such that for any } h' \in \mathcal{B}_h, \text{ we have } \mathcal{B}_{h'} \subseteq \{h\}, \tag{65}$$

then for any $\mathcal{E} = (e, \beta) \in \mathbb{R}_{++}^{CH} \times \mathbb{R}_{++}^B$, $\quad F_{(e,\beta)}^{-1}(0)$ *is compact.*

Proof. Taken a sequence $\left(x^n, t^n, \lambda^n, \gamma^n, \delta^n, p^{\backslash n}\right) \in \left((F_{\mathcal{E}})^{-1}(0)\right)^\infty$, we want to show that it admits a subsequence converging to $\left(\overline{x}, \overline{t}, \overline{\lambda}, \overline{\gamma}, \overline{\delta}, \overline{p}^{\backslash}\right) \in (F_{\mathcal{E}})^{-1}(0)$.

1. $(x^n)_{n\in\mathbb{N}}$ admits a subsequence converging to $\overline{x}\in\mathbb{R}^{CH}_{++}$.

It suffices to show that $\{x^n : n\in\mathbb{N}\}$ is contained in a compact subset of $\mathbb{R}^{CH}_{++}$.

α. Boundedness from below. Since $\{x^n : n\in\mathbb{N}\}\subseteq\mathbb{R}^{CH}_{++}$ is bounded from below by zero.

β. Boundedness from above. By assumption, we have that for any $h'=1,...,H$, $x^n_{h'}=\sum_{h=1}^{H}e_h-\sum_{h\neq h'}x^n_h$. Therefore, since $\{x^n : n\in\mathbb{N}\}$ is bounded from below, the result follows.

γ. Closedness.

Case 1. $\mathcal{B}_h=\varnothing$.

Then, for any $n\in\mathbb{N}$, x^n_h solves $\max_{x_h\in\mathbb{R}^C_+}u_h(x_h)$ s.t. $p^n x^n_h\leq p^n e_h$ - i.e., household h is a Walrasian consumer. Then, for any $n\in\mathbb{N}$, $u_h(x^n_h)\geq u_h(e_h)$. From Assumption u4, $\{x^n_h : n\in\mathbb{N}\}$ is contained in the $\mathbb{R}^C$-closed set $\left\{x_h\in\mathbb{R}^C_{++} : u_h(x_h)\geq u_h(e_h)\right\}$.

Case 2. $\mathcal{B}_h\neq\varnothing$.

In this case we have that for any $n\in\mathbb{N}$, $(x^n_h,t^n_h)\in\mathbb{R}^C_{++}\times\mathbb{R}^{B_h}_+$ solves

$$\begin{array}{l}
\max_{(x_h,t_h)\in\mathbb{R}^{CH}_{++}\times\mathbb{R}^{B_h}} \quad u_h(x_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}\cdot v_{hh'}\left(\frac{p^n e_{h'}+t_{hh'}-t^n_{h'h}}{pr}\right)\\
\text{s.t.}\\
p^n x_h+\sum_{h'\in\mathcal{B}_h}t_{hh'}\leq p^n e_h+\sum_{h'\in\mathcal{B}_h}t_{h'h}\\
\\
t_h\geq 0.\\
\\
t_h\leq k_h,
\end{array}$$

where we use the convention that if $\mathcal{B}_{h'}=\varnothing$, then $t_{h'h}=0$.

For any $n\in\mathbb{N}$ and for any $\left(x^n,t^n,\lambda^n,\gamma^n,\delta^n,p^{\setminus n}\right)\in\left((F_{\mathcal{E}})^{-1}(0)\right)^{\infty}$, using assumption (65), we have

$$\begin{array}{ll}
u_h(x^n_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\frac{p^n e_{h'}+t^n_{hh'}-t^n_{h'h}}{p^n r}\right) & \overset{(1)}{\geq}\\
\\
u_1(e_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\frac{p^n e_{h'}}{p^n r}\right), &
\end{array}\tag{66}$$

where (1) follows from the fact $\left(e_h,(t^n_{h'h})_{h'\in_h}\right)\in B_h\left(p^n,(t^n_{h'h})_{h'\in_h}\right)$. Then,

$$u_h(x^n_h)\geq u_h(e_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\frac{p^n e_{h'}}{p^n r}\right)-\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\frac{p^n e_{h'}+t^n_{hh'}-t^n_{h'h}}{p^n r}\right)\tag{67}$$

First of all, observe that

$$\begin{array}{l}
v_{hh'}\left(\frac{p^n e_{h'}+t^n_{hh'}-t^n_{h'h}}{p^n r}\right)=v_{hh'}\left(\frac{p^n x_{h'}}{p^n r}\right)\overset{\text{mkt clearing}}{\leq}v_{hh'}(1)\\
\text{and therefore}\\
-\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}\left(\frac{p^n e_{h'}+t^n_{hh'}-t^n_{h'h}}{p^n r}\right)\geq-\sum_{h'\in\mathcal{B}_h}\beta_{hh'}v_{hh'}(1).
\end{array}\tag{68}$$

We now claim that

$$\text{there exist } b\in\mathbb{R}_{++} \text{ such that for any } n\in\mathbb{N},\ \beta_{hh'}v_{h'}\left(\frac{p^n e_{h'}}{p^n r}\right)\geq\beta_{hh'}v_{hh'}(b),\tag{69}$$

as verified below. Taken $b=\min\left\{\frac{e^c_{h'}}{r^c} : c\in\mathcal{C},\ h'\in\mathcal{B}_h\right\}$, then for any $c\in\mathcal{C}$ and $h'\in\mathcal{B}_h$,

$$b\leq\frac{e^c_{h'}}{r^c}\quad\text{or}\quad br^c\leq e^c_{h'}\quad\text{or}\quad e^c_{h'}-br^c\geq 0.$$

Therefore for any $h'\in\mathcal{B}_h$, we have $p^n e_{h'}-bp^n r=\sum_{c\in\mathcal{C}}p^{nc}\left(e^c_{h'}-br^c\right)\geq 0$ and therefore $p^n e_{h'}\geq bp^n r$ or $\frac{p^n e_{h'}}{p^n r}\geq b>0$ and since $v_{hh'}$ is increasing, we have

$$v_{hh'}\left(\frac{p^n e_{h'}}{p^n r}\right)\geq v_{hh'}(b),$$

as desired.

Then from (67), (68) and (69), we have

$$u_h(x^n_h)\geq u_h(e_h)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}\left(v_{hh'}(b)-v_{hh'}(1)\right).$$

Therefore,

$$\text{for any } n\in\mathbb{N},\ u_h\left(x_h^n\right)\geq u_h\left(e_h\right)+\sum_{h'\in\mathcal{B}_h}\beta_{hh'}\left(v_{hh'}\left(b\right)-v_{hh'}\left(1\right)\right):=\underline{l}\in\mathbb{R}. \tag{70}$$

From (70) and Assumption u4, $\{x_h^n : n\in\mathbb{N}\}$ is contained in the $\mathbb{R}^C$-closed set $\left\{x_h\in\mathbb{R}^C_{++} : u_h\left(x_h\right)\geq \underline{l}\right\}$.

Claim 2. $\left(\lambda_h^n\right)_{n\in\mathbb{N}}$ admits a subsequence converging to $\overline{\lambda}\in\mathbb{R}^H_{++}$.

Since $p^1=1$, for $h=1,...,H$, from equation $Du_h\left(x_h^n\right)-\lambda_h^n p^n=0$, we have

$$D_{x_h^1}u_h\left(x_h^n\right)=\lambda_h^n.$$

Taking $\lim_{n\to+\infty}$ of both sides, we get

$$\lim_{n\to+\infty}\lambda_h^n=\lim_{n\to+\infty}D_{x_h^1}u_h\left(x_h^n\right)=D_{x_h^1}u_h\left(\overline{x}_h\right):=\overline{\lambda}_h>0,$$

where last strict inequality comes from Assumption u2.

Claim 3. $\left(p^{\setminus n}\right)_{n\in\mathbb{N}}$ admits a subsequence converging to $\overline{p}^{\setminus}\in\mathbb{R}^{C-1}_{++}$.

Again from equation $D_{x_h^{\setminus}}u_h\left(x_h^n\right)-\lambda_h^n p^{\setminus}=0$, we get

$$\lim_{n\to+\infty}p^{\setminus n}=\lim_{n\to+\infty}\frac{D_{x_h^{\setminus}}u_h\left(x_h^n\right)}{\lambda_h^n}=\frac{D_{x_h^{\setminus}}u_h\left(\overline{x}_h\right)}{\overline{\lambda}_h}\equiv\overline{p}^{\setminus}>>0,$$

where again the last strict inequality comes from Assumption u2.

Claim 4. $\left(t_h^n\right)_{n\in\mathbb{N}}$ admits a subsequence converging to $\overline{t}_h\in[0,k_1]$.

This is obvious because $[0,k_1]$ is a compact set.

To show convergence of the remaining multipliers $\gamma_{hh'}$ and $\delta_{hh'}$, we distinguish three exhaustive cases with respect to the values of $\overline{t}_{hh'}$.

Case 1. $\overline{t}_{hh'}\in(0,k_1)$.

Then, taking the convergent subsequence to be the sequence itself, we have that there exists $N\in\mathbb{N}$ such that for any $n>N$, $t_{hh'}^n\in(0,k_1)$ and then

$$\text{for any } n>N,\ \ \gamma_{hh'}^n=0 \quad \text{ and } \delta_{hh'}^n=0,$$

i.e., $\gamma_{hh'}^n\longrightarrow 0$ and $\delta_{hh'}^n\longrightarrow 0$.

Case 2. $\overline{t}_{hh'}=0<k_1$.

Then, there exists $N\in\mathbb{N}$ such that for any $n>N$, $t_{hh'}^n<k_1$ and then

$$\text{for any } n>N,\ \delta_{hh'}^n=0,\quad \text{i.e.}\qquad \delta_{hh'}^n\longrightarrow 0.$$

From equation $\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(p^n e_{h'}+t_{hh'}^n-t_{h'h}^n\right)-\lambda_1^n+\gamma_{hh'}^n-\delta_{hh'}^n=0$, we have

$$0\leq\gamma_{hh'}^n=-\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{p^n e_{h'}+t_{hh'}^n-t_{h'h}^n}{p^n r}\right)+\lambda_1^n+\delta_{hh'}^n\longrightarrow-\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{\overline{p}e_{h'}+\overset{=0}{\overline{t}_{hh'}}-\overline{t}_{h'h}}{\overline{p}r}\right)+\overline{\lambda}_1+\overset{=0}{\overline{\delta}_{hh'}}:=\overline{\gamma}_{hh'}.$$

Case 3. $\overline{t}_{hh'}=k_1>0$.

Then, there exists $N\in\mathbb{N}$ such that for any $n>N$, $t_{hh'}^n>0$ and then

$$\text{for any } n>N,\ \gamma_{hh'}^n=0,\quad \text{i.e.}\qquad \gamma_{hh'}^n\longrightarrow 0.$$

From equation $\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(p^n e_{h'}+t_{hh'}^n-t_{h'h}^n\right)-\lambda_1^n+\gamma_{hh'}^n-\delta_{hh'}^n=0$, we have

$$0\leq\delta_{hh'}^n=\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{p^n e_{h'}+t_{hh'}^n-t_{h'h}^n}{p^n r}\right)-\lambda_1^n+\gamma_{hh'}^n\longrightarrow-\frac{1}{pr}\beta_{hh'}v'_{hh'}\left(\frac{\overline{p}e_{h'}+\overline{t}_{hh'}-\overline{t}_{h'h}}{\overline{p}r}\right)+\overline{\lambda}_1+\overset{=0}{\overline{\gamma}_{hh'}}:=\overline{\delta}_{hh'}.$$

Finally since for any $n\in\mathbb{N}$,

$$F_{\mathcal{E}}\left(x^n,t^n,\lambda^n,\gamma^n,\delta^n,p^{\setminus n}\right)=0,$$

taking then limit for $n\longrightarrow+\infty$ and using the continuity of all component functions of F, we do have

$$F_{\mathcal{E}}\left(\overline{x},\overline{t},\overline{\lambda},\overline{\gamma},\overline{\delta},\overline{p}^{\setminus}\right)=0.$$

■

7.0.2 Properness of the projection

Lemma 61

$$pr : F^{-1}(0) \to \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H}_{++} \times \mathbb{R}^{2}_{++} \times \mathbb{R}^{2}_{++} := \mathbb{R}^{s}_{++}, \quad \left(x, t, \lambda, \gamma, \delta, p^{\setminus}; e, \alpha, \beta, k\right) \mapsto (e, \alpha, \beta, k)$$

is proper .

Proof. Take a sequence $(e^n, \beta^n, k^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^{\cdots}_{++}$. By assumption, it admits a convergent subsequence, say to $\left(\overline{e}, \overline{\beta}, \overline{k}\right) \in \mathbb{R}^{\cdots}_{++}$. We want to show that the sequence $\left(\left(x^n, t^n, \lambda^n, \gamma^n, \delta^n, p^{\setminus n}\right), e^n, \beta^n, k^n\right)_{n\in\mathbb{N}}$ in $F^{-1}(0)$ does converge to an element of $F^{-1}(0)$. The proof follows the same steps as the proof of Lemma 60. ■

7.0.3 Border line cases are rare

Proposition 62 *For any $(v, u) \in \mathcal{U} \times \mathcal{V}$ and any $(\mathcal{B}_h)_{h\in\mathcal{H}}$, there exists an open full measure subset $\mathcal{D}_{nb}$ of $\mathbb{R}^{CH}_{++} \times \mathbb{R}^{B}_{++}$ such that any associated equilibrium $\left(x, t, \gamma, \delta, p^{\setminus}\right)$ are such that "border line cases" do not exist, i.e.,*

$$\forall h, h' \in \mathcal{H} \text{ such that } h \neq h', \quad (\gamma_{hh'} > 0 \text{ or } t_{hh'} > 0) \text{ and } (\delta_{hh'} > 0 \text{ or } k_{hh'} - t_{hh'} > 0).$$

Proof. Define the sets

$$C_1 := \left\{(\xi, e, \beta) \in F^{-1}(0) : \exists\, h \in \mathcal{H} \text{ and } \exists h' \in \mathcal{B}_h \text{ such that } \gamma_{hh'} = 0 \text{ and } t_{hh'} = 0\right\},$$

$$C_2 := \left\{(\xi, e, \beta) \in F^{-1}(0) : \exists\, h \in \mathcal{H} \text{ and } \exists h'' \in \mathcal{B}_h \text{ such that } \delta_{hh''} = 0 \text{ and } k_{hh''} - t_{hh''} = 0\right\},$$

$$C := C_1 \cup C_2.$$

Observe that $\mathcal{D}_{nb} = \left(\mathbb{R}^{CH}_{++} \times \mathbb{R}^{B}_{++}\right) \setminus pr\,(C)$. Since C is a closed set, openness of D^* follows from properness of pr. The proof of full measure needs some work. The possible cases are the following ones

	$t_{hh'} = 0$	$\gamma_{hh'} > 0$	$\delta_{hh'} = 0$
border line case 1	$\longrightarrow t_{hh'} = 0$	$\longrightarrow \gamma_{hh'} = 0$	$\delta_{hh'} = 0$
	$t_{hh'} \in (0, k)$	$\gamma_{hh'} = 0$	$\delta_{hh'} = 0$
border line case 2	$\longrightarrow t_{hh'} = k$	$\gamma_{hh'} = 0$	$\longrightarrow \delta_{hh'} = 0$
	$t_{hh'} = k$	$\gamma_{hh'} = 0$	$\delta_{hh'} > 0$

1. Let $\mathcal{P}$ be the family of all partitions of $\mathcal{H}$ into two subsets $\mathcal{H}_1, \mathcal{H}_2$. In the equilibrium system **??**, in place of min equations substitute the three no border line case; for $h \in \mathcal{H}_2$, substitue one of the possible border line case and the three border line case. (see the first column of Table **??** below for demonstration of how this is done).

2. Define

$$F^{i}_{\mathcal{H}_1,\mathcal{H}_2} : \Xi \times \left(\mathbb{R}^{CH}_{++} \times \mathbb{R}^{H}_{++}\right) \to \mathbb{R}^{\dim \Xi + 1}$$

which associates the left hand side of the equilibrium system **??** modified as explained above to each (ξ, e, β) in the domain, where $i \in \{1, 2\}$ refers to one of the two border line cases.

3. Define the set

$$B_{\mathcal{H}_1,\mathcal{H}_2} \equiv \{e \in \mathbb{R}^{CH}_{++} : \exists \xi \text{ such that } F_{\mathcal{H}_1,\mathcal{H}_2}(\xi, e, \beta) = 0\}$$

and observe that[7]

$$\mathcal{D}_{nb} \supseteq \left(\mathbb{R}^{CH}_{++} \times \mathbb{R}^{B}_{++}\right) \setminus \bigcup_{\{\mathcal{H}_1,\mathcal{H}_2\}\in\mathcal{P}} B_{\mathcal{H}_1,\mathcal{H}_2} \tag{71}$$

That $B_{\mathcal{H}_1,\mathcal{H}_2}$ is of measure zero follows from the parametric transversality theorem - see, for example, Hirsch (1976), Theorem 2.7, page 79 or from the appendix- and from the fact that zero is a regular value for $F_{\mathcal{H}_1,\mathcal{H}_2}$, which is shown below.

In the following Table, the components of $F_{\mathcal{H}_1,\mathcal{H}_2}$ are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the corresponding partial Jacobian of $D_{(\xi,e,\beta)} F_{\mathcal{H}_1,\mathcal{H}_2}(\xi, e, \beta)$ is displayed. Note also that, to simplify the exposition of the proof, only one household from each set is presented in the Table. We present the computation for the border line case 1, the other case being quite similar. The order in which rank preserving elementary column operations are performed is displayed in the last column.

[7] We cannot use the equality sign in (71), because $B_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}$ contains economies e such that $F_{\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3}(\xi, e) = 0$ and for which some components g_h and μ_h of ξ may be negative.

	x_{h_1}	$t_{h_1 h_1^3}$	$t_{h_1 h_1^4}$	$t_{h_1 h_1^5}$	λ_{h_1}	$\gamma_{h_1 h_1^3}$	$\gamma_{h_1 h_1^4}$	$\gamma_{h_1 h_1^5}$	$\delta_{h_1 h_1^3}$	$\delta_{h_1 h_1^4}$	$\delta_{h_1 h_1^5}$	x_{h_2}	$t_{h_2 h_2^1}$	$t_{h_2 h_2^3}$	$t_{h_2 h_2^4}$	$t_{h_2 h_2^5}$	λ_{h_2}	$\gamma_{h_2 h_2^1}$	$\gamma_{h_2 h_2^3}$	$\gamma_{h_2 h_2^4}$	$\gamma_{h_2 h_2^5}$	$\delta_{h_2 h_2^1}$	$\delta_{h_2 h_2^3}$	$\delta_{h_2 h_2^4}$	$\delta_{h_2 h_2^5}$	$\beta_{h_1 h_1^3}$ $\beta_{h_1 h_1^4}$ $\beta_{h_1 h5}$	$\beta_{h_2 h_2^1}$ $\beta_{h_2 h_2^3}$ $\beta_{h_2 h_2^4}$ $\beta_{h_2 h_2^5}$	$e^\backslash_{h_1}$	$e^1_{h_1}$	$e^1_{h_2}$	
$Du_{h_1} - \lambda_{h_1} p$	$D^2_{h_1}$				$-p$																										4
…		*	*	*	*	*	*	*	*	*	*		*	*	*	*										I				*	1
		−1	−1	−1																								$p^\backslash$	1		2
$t_{h_1 h_1^3}$		1																				*									5
$\delta_{h_1 h_1^3}$									1																						6
$\gamma_{h_1 h_1^4}$							1																								7
$\delta_{h_1 h_1^4}$										1																					8
$k - t_{h_1 h_1^5}$				−1																											9
$\gamma_{h_1 h_1^5}$								1																							10
$Du_{h_2} - \lambda_{h_2} p$											$D^2_{h_2}$																				4
…		*	*	*								*	*	*	*	*	*	*	*	*	*	*	*	*	*		I	*	*		1
…		*	*	*							$-p$	−1	−1	−1	−1															1	2
$\gamma_{h_2 h_1^1}$																		1				1									11
$t_{h_2 h_2^1}$													1																		12
$\delta_{h_2 h_2^1}$																						1									13
$t_{h_2 h_2^3}$														1																	14
$\delta_{h_2 h_2^3}$																							1								15
$\gamma_{h_2 h_2^4}$																				1											16
$\delta_{h_2 h_2^4}$																								1							17
$k - t_{h_2 h_2^5}$																−1															18
$\gamma_{h_2 h_2^5}$																					1										19
$\Sigma (x^\backslash_h - e^\backslash_h)$	$I \backslash 0$											$I \backslash 0$																I			3

■

Remark 63 *Let's say why the proof above becomes basically impossible if you consider a model with transfers in terms of vectors. In that case, household 1's maximization problem is as follows.*

$$\max_{(x_1,t_{12})\in\mathbb{R}^C_{++}\times\mathbb{R}} \quad u_1(x_1)+\beta_{12}v_{12}(p(e_2+t_{12}-t_{21})) \qquad s.t. \qquad \begin{array}{lcll} -p\cdot x_1+p(e_1-t_{12}+t_{21}) & \geq & 0 & \lambda_1\in\mathbb{R} \\ t_{12} & \geq & 0 & \gamma_{12}\in\mathbb{R}^C \\ k_1-t_{12} & \geq & 0 & \delta_{12}\in\mathbb{R}^C \end{array}$$

The associated extended system is then

$$\begin{array}{ll} Du_1(x_1)-\lambda_1 p & =0 \\ \beta_{12}v'_{12}(p(e_2+t_{12}-t_{21}))\cdot p-\lambda_1 p+\gamma_{12}-\delta_{12} & =0 \\ -p\cdot x_1+p(e_1-t_{21}+t_{12}) & =0 \\ \min\{t^c_{12},\gamma^c_{12}\} & =0 \\ \min\{k^c_1-t_{12},\delta^c_{12}\} & =0 \\ \\ .... \\ \\ \sum_{h\in\mathcal{H}}(x_h-e_h) & =0 \end{array}$$

and the associated Jacobian in the easy case $t=\gamma=\delta=0$ becomes:

		C	1	C	1	$C^\backslash$	1	1	$C^\backslash$	1	$C^\backslash$	1
		x_1	λ_1	x_2	λ_2	$p^\backslash$	β_{12}	β_{21}	$e_1^\backslash$	e_1^C	$e_2^\backslash$	e_2^C
C	$Du_1(x_1)$ $-\lambda_1 p$	D_1^2	$-p$			$\lambda_1\cdot I$ 0						
C	β_{12} $v'_{12}(pe_2)\cdot$ p $-\lambda_1 p$		$-p$			$\beta_{12}\cdot$ $v'_{12}(pe_2)\cdot$ I 0	$v'_{12}(pe_2)\cdot$ p				$\beta_{12}\cdot$ $v''(pe_2)\cdot$ D^p 0	0 β_{12} $v''(pe)\cdot$ $\left(p^C\right)^2$
1	$-p\cdot x_1+pe_1$					$-x_1+e_1$			$p^\backslash$	1		
C	$Du_2(x_2)$ $-\lambda_2 p$			D_2^2	$-p$	$\lambda_2 I$						
C	β_{21} $v'_{21}(pe_1)\cdot$ p $-\lambda_2 p$				$-p$	$\beta_{21}\cdot$ $v'_{21}(pe_1)\cdot$ I 0		$v'_{21}(pe_1)\cdot p$	$\beta_{21}\cdot$ $v''(pe_1)\cdot$ D^p 0	0 β_{21} $v''(pe_1)\cdot$ $\left(p^C\right)^2$		
1	$-px_2+pe_2$					$-x_2-e_2$					$-p^\backslash$	1
$C^\backslash$	$\Sigma\left(x_h^\backslash-e_h^\backslash\right)$	$I0$		$I0$		I			$-I$		$-I$	

where $D^p := diag\left((p^c)^2\right)_{c\neq C}$. Then β_{12},which is a scalar, is not enough to take care of the C equations $\beta_{12}Dv'_{12}(p(e_2))-\lambda_1 p$.

7.0.4 Regularity

Proposition 64 *For any $(v,u)\in\mathcal{U}\times\mathcal{V}$ and any $(\mathcal{B}_h)_{h\in\mathcal{H}}$, there exists an open full measure subset $\mathcal{D}_r$ of $\mathbb{R}^{CH}_{++}\times\mathbb{R}^H_{++}$ such that any associated equilibrium $\left(x,t,\gamma,\delta,p^\backslash\right)$ is locally unique and smoothly depending upon elements id $\mathcal{D}_r$, i.e.,[8] if $(e,\beta)\in\mathcal{D}_r$, then*

1. there exists $r\in\mathbb{N}$ such that

$$F^{-1}_{(e,\beta)}(0)=\left\{\xi^i\right\}_{i=1}^r$$

2. there exists an open set $Y\subseteq\mathbb{R}^{CH}_{++}$, and for each i, an open set $V_i\subseteq\mathbb{R}^{\dim\Xi}$ and a unique function $g_i:Y\to V_i$ such that

a. $\left(\xi^i,(e,\beta)\right)\in V_i\times Y$,

b. g_i is C^1,

c. $g_i((e,\beta))=\xi^i$,

d. for every $(e,\beta)'\in Y$, $F\left(g_i\left((e,\beta)'\right),(e,\beta)'\right)=0$,

e. $\{(\xi,(e,\beta)):(e,\beta)\in Y, F(\xi,(e,\beta))=0\}=\cup_{i=1}^r\text{graph } g_i$.

Proof. The proof is similar, indeed easier, than the one presented in Proposition 62. ■

[8] Below I just copying Theorem 99.

7.0.5 Equilibria with both households providing a strictly positive transfer

The proposition below is a generalization of Proposition 44.1.

Proposition 65 *Let (u, v) be given.*

1. If $\xi = \left(x, t_{12}, t_{21}, \lambda, \gamma, \delta, p^{\setminus}\right)$ is an equilibrium (associated with the economy (e, β)) such that $t_{12} \in (0, k_1)$ and $t_{21} \in (0, k_2)$, then for any $k \in \mathbb{R}$ such that $t_{12}(k) := t_{12} + k \in (0, k_1)$ and $t_{21}(k) := t_{21} + k \in (0, k_2)$, we have $\xi(k) := \left(x, t_{12}(k), t_{21}(k), \lambda, \gamma, \delta, p^{\setminus}\right)$ is an equilibrium (associated with the economy (e, β)).

2. The set of economies for which associated equilibria are such that $t_{12} \in (0, k_1]$ and $t_{21} \in (0, k_2)$ is contained in a closed and measure zero in the set of all economies.

Proof. 1.

Equilibria of the type described in the statement satisfy the following system of equations.

$Du_1(x_1) - \lambda_1 p$	$= 0$
$\frac{1}{pr}\beta_{12} v_{12}'\left(\frac{pe_2 + t_{12} - t_{21}}{pr}\right) - \lambda_1 + \gamma_{12} - \delta_{12}$	$= 0$
$-p \cdot x_1 + pe_1 - t_{12} + t_{21}$	$= 0$
$\min\{\gamma_{12}, t_{12}\} = \gamma_{12}$	$= 0$
$\min\{\delta_{12}, k_1 - t_{12}\} = \delta_{12}$	$= 0$
$Du_2(x_2) - \lambda_2 p$	$= 0$
$\frac{1}{pr}\beta_{21} v_{21}'\left(\frac{pe_1 + t_{21} - t_{12}}{pr}\right) - \lambda_2 + \gamma_{21} - \delta_{21}$	$= 0$
$-p \cdot x_2 + pe_2 - t_{21} + t_{12}$	$= 0$
$\min\{\gamma_{21}, t_{21}\} = \gamma_{21}$	$= 0$
$\min\{\delta_{21}, k_2 - t_{21}\} = \delta_{21}$	$= 0$
$\sum_h \left(x_h^{\setminus} - e_h^{\setminus}\right)$	$= 0$

Substituting $t_{12} + k \in (0, k_1)$, $t_{21} + k \in (0, k_2)$ in the place of t_{12}, t_{21}, respectively, the system is still satisfied.

2.

From Proposition 64, we know there exists an open and full measure subset Ω^* of Ω such that for any $\omega^* \in \Omega^*$, we have that 0 is a regular value for F_{ω^*}, i.e., for any $\omega \in \Omega \backslash \Omega^*$ there exists $\xi \in F_{\omega}^{-1}(0)$ such that $\operatorname{rank} D_\xi F_\omega(\xi) < \dim \Xi$. Define $F^0 = (F, \gamma, \delta)$ and $\Omega^0 = \left\{\omega \in \Omega : F^0(\xi, \omega) = 0\right\}$. If $\Omega^0 = \varnothing$, we are done. If $\Omega^0 \neq \varnothing$, then take $\omega^0 \in \Omega^0$. Then, $\xi \in \left(F_\omega^0\right)^{-1}(0) \Rightarrow \xi \in (F_\omega)^{-1}(0)$. Below, we are going to show that if $\omega^0 \in \Omega^0$, then

$$\operatorname{rank} D_\xi F_\omega(\xi) = \operatorname{rank} D_\xi F_\omega^0(\xi), \tag{72}$$

and

$$\operatorname{rank} D_\xi F_\omega^0(\xi) < \dim \Xi. \tag{73}$$

We then get that $\omega \in \Omega^0$ implies that $\omega \in \Omega \setminus \Omega^*$, i.e., Ω^0 is contained in a closed, measure zero subset of the economy space, which is our desired result.

Observe that $\omega \in \Omega^0$ if and only if

$Du_1(x_1) - \lambda_1 p$	$= 0$
$\frac{1}{pr}\beta_{12} v_{12}'\left(\frac{pe_2 + t_{12} - t_{21}}{pr}\right) - \lambda_1 + \gamma_{12} - \delta_{12}$	$= 0$
$-p \cdot x_1 + pe_1 - t_{12} + t_{21}$	$= 0$
$\min\{\gamma_{12}, t_{12}\} = \gamma_{12}$	$= 0$
$\min\{\delta_{12}, k_1 - t_{12}\} = \delta_{12}$	$= 0$
$Du_2(x_2) - \lambda_2 p$	$= 0$
$\frac{1}{pr}\beta_{21} v_{21}'\left(\frac{pe_1 + t_{21} - t_{12}}{pr}\right) - \lambda_2 + \gamma_{21} - \delta_{21}$	$= 0$
$-p \cdot x_2 + pe_2 - t_{21} + t_{12}$	$= 0$
$\min\{\gamma_{21}, t_{21}\} = \gamma_{21}$	$= 0$
$\min\{\delta_{21}, k_2 - t_{21}\} = \delta_{21}$	$= 0$
$\sum_h \left(x_h^{\setminus} - e_h^{\setminus}\right)$	$= 0$
γ_{12}	
δ_{12}	
γ_{21}	
δ_{21}	

Then $D_\xi F^0(\xi, \omega)$ is

	x_1	t_{12}	λ_1	x_2	t_{21}	λ_2	$p^{\setminus}$	β_{12}	β_{21}	$e_1^{\setminus}$	e_1^C	e_2^C	γ, δ
$Du_1(x_1) - \lambda_1 p$	D_1^2		$-p$			$-p$	$\lambda_1 I_{C-1}$ 0						
$\frac{1}{pr}$ β_{12} $v'_{12}\left(\frac{pe_2+t_{12}-t_{21}}{pr}\right)$ $-\lambda_1$		$\left(\frac{1}{pr}\right)^2$ $\beta_{12} v''_{12}$	-1		$-$ $\left(\frac{1}{pr}\right)^2$ $\beta_{12} v''_{12}$	-1	$*$	v'_{12}				$\beta_{12} v'_{12}$	
$-p \cdot x_1 + pe_1$ $-t_{12} + t_{21}$	$-p$	-1			1		$-x_1 + e_1^{\setminus}$			$p^{\setminus}$	1		
$Du_2(x_2) - \lambda_2 p$				D_2^2		$-p$	$\lambda_2 I_{C-1}$ 0						
$\frac{1}{pr}$ β_{21} $v'_{21}\left(\frac{pe_1+t_{21}-t_{12}}{pr}\right)$ $-\lambda_2$		$-\left(\frac{1}{pr}\right)^2$ $\beta_{21} v''_{21}$			$\left(\frac{1}{pr}\right)^2$ $\beta_{21} v''_{21}$	-1	$*$		v'_{21}	$*$	$*$		
$-p \cdot x_2 + pe_2$ $-t_{21} + t_{12}$		1		$-p$	-1		$-x_2 + e_2^{\setminus}$					1	
$\sum_h \left(x_h^{\setminus} - e_h^{\setminus}\right)$	$I_{C-1} 0$			$I_{C-1} 0$						$-I_{C-1} 0$			
γ, δ													I

Then, clearly statement (72) holds and since the column $D_{t_{12}} F(\xi, e, \beta)$ is (-1) times the column $D_{t_{21}} F(\xi, e, \beta)$, then statement (73) holds as well. ■

Remark 66 *The not full rank result presented above still holds true if you consider the case of transfers as vectors instead of scalars. The problem is that there we are not able to prove regularity.*

8 Appendices

8.1 Some basic facts on set valued functions, convex analysis, topology and measure theory

Definition 67 *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$, then*

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Z\}$$

is a basis for a topology on $X \times Y$, called the box or product topology $\mathcal{T}_{X \times Z}$.

Remark 68 *Therefore if $(x, z) \in S \in \mathcal{T}_{X \times Z}$, then there exist $U_x \in \mathcal{T}_X$ and $V_z \in \mathcal{T}_Z$ such that $(x, z) \in U_x \times V_z \subseteq S$.*

Proposition 69 *Given a topological space $(X, \mathcal{T}_X)$ and sets B, Y, if*

1. *$B \subseteq Y \subseteq X$,*
2. *B is Y-closed, and*
3. *$\mathrm{Cl}_{(X, \mathcal{T}_X)}(B) \subseteq Y$,*

then B is X-closed.

Proof. See page 7 in my handwritten notes "basic product-relative topologies.pdf". ■

Definition 70 *Given a topological space $(X, \mathcal{T}_X)$ and $x \in X$, the family of neighborhoods of x is denoted and defined as follows.*

$$\mathcal{N}_{(X, \mathcal{T}_X)}(x) = \{U \subseteq X : x \in U \text{ and } U \in \mathcal{T}_X\}.$$

Definition 71 *Given a set $A \subseteq (X, \mathcal{T}_X)$, the set of adherent points to A is denoted and defined as follows.*

$$\mathrm{Ad}_{(X, \mathcal{T}_X)}(A) = \left\{x \in X : \text{ for any } U \in \mathcal{N}_{(X, \mathcal{T}_X)}(x),\ U \cap A \neq \varnothing\right\}.$$

Proposition 72

$$\mathrm{Ad}_{(X, \mathcal{T}_X)}(A) = \mathrm{Cl}_{(X, \mathcal{T}_X)}(A).$$

Proof. See Math 2 notes. The proof is presented there for metric space, but it does generalize. ■

Proposition 73 *The intersection of a finite number of open and dense sets is open and dense.*

Proposition 74 *The intersection of a finite (in fact countable) number of full measure sets has full measure.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a family of sets with full measure, i.e., such that for any $n \in \mathbb{N}$, $\mu\left(A_n^C\right) = 0$. We want to show that $\cap_{n\in\mathbb{N}} A_n$ has full measure, i.e., $\mu\left(\left(\cap_{n\in\mathbb{N}} A_n\right)^C\right) = 0$. Indeed, $\mu\left(\left(\cap_{n\in\mathbb{N}} A_n\right)^C\right) = \mu\left(\cup_{n\in\mathbb{N}} A_n^C\right) \leq \sum_{n\in\mathbb{N}} \mu\left(A_n^C\right) = 0$, where the weak inequality follows from countable subadditivity of measure. ■

Proposition 75 *(Corollary 2.3.9, page 64, in Webster (1984)[12])If K is a convex subset of $\mathbb{R}^n$ such that $\text{Int}(K) \neq \varnothing$, then $\text{Cl}(\text{Int } K) = \text{Cl } K$.*

Proposition 76 *(Proposition 2.38, page 50 in Hu and Papageorgiou (1997) [5]) If a set-valued function $\varphi : X \rightrightarrows Y$ is lower semicontinuous if and only if $\text{Cl}(\varphi)$ is.*

Proof.

Proposition 77 *([4], bottom page 23) If $\varphi : X \rightrightarrows Y$ is a set valued function which is closed graph and if $\varphi(X)$ is contained in a compact set, then φ is upper semicontinuous.*

■

Proposition 78 *(Lemma 1, page 33, in Hildebrand (1974) [4]) If a set-valued function φ of a metric space in $\mathbb{R}^n$ is non-empty valued, compact valued, convex valued, closed and lower semicontinuous. Then φ is upper semicontinuous.*

8.2 Price normalization in the case of separable utility functions

The main goal of the present section is to show what follows. For any $\alpha \in \mathbb{R}_{++} \setminus \{1\}$, and any $\theta \in \mathbb{R}_+$

$$v(\alpha\theta) = v(\theta)$$

means that v is homogenous of degree zero. In that case, it obvious that (x, t, p) is an equilibrium if and only if $(x, t, \alpha p)$ is an equilibrium, simply because both v and the budget constraint functions are homogenous of degree zero. If v does not have property, then the equilibrium allocation is in general affected by normalizations, which is an undesirable property.

If you consider the extended system characterizing equilibria, then, as explained in detail in the proof of the Proposition below, a sufficient condition under which normalizations do not matter is that

$$\text{for any } \alpha > 0,\ \alpha v'(a\theta) = v'(\theta),$$

which does not hold true for many standardly used utility function (even though it is true for log function - see proof below).

To discuss the role of the price normalization proposed by Kranich, we write the definition of equilibrium using prices expressed in units of account - as usually done in general equilibrium models.

Definition 79 $(x_1^*, t_{12}^*, x_2^*, , t_{21}^*, p^*) \in \left(\mathbb{R}_{++}^C \times \mathbb{R}\right)^H \times \mathbb{R}_{++}^C$ *is an equilibrium associated with the economy (u, v, e, β, k) if*

1. *for given (u, v, e, β, k), p^*, t_{21}^*, (x_1^*, t_{12}^*) solves*

$$\max_{(x_1,t_{12})\in\mathbb{R}_{++}^C\times\mathbb{R}} \quad u_1(x_1) + \beta_{12} v_{12}\left(p^*\left(e_2 + t_{12} - t_{21}^*\right)\right) \quad s.t. \quad \begin{array}{rcl} -p^* \cdot x_1 + p^*\left(e_1 + t_{21}^* - t_{12}\right) & \geq & 0, \\ t_{12} & \geq & 0, \\ k_1 - t_{12} & \geq & 0, \end{array}$$

2. *for given (u, v, e, β, k), $p^{\backslash *}$, t_{12}^*, (x_2^*, t_{21}^*) solves*

$$\max_{(x_2,t_{21})\in\mathbb{R}_{++}^C\times\mathbb{R}} \quad u_2(x_2) + \beta_{21} v_{21}\left(p^*\left(e_1 + t_{21} - t_{12}^*\right)\right) \quad s.t. \quad \begin{array}{rcl} -p^* \cdot x_2 + p^*\left(e_2 + t_{12}^* - t_{21}\right) & \geq & 0 \\ t_{21} & \geq & 0 \\ k_2 - t_{21} & \geq & 0 \end{array}$$

3.

$$x_1^{*\backslash} + x_2^{*\backslash} = e_1^{\backslash} + e_2^{\backslash}.$$

Definition 80 $(x_1^*, t_{12}^*, x_2^*, , t_{21}^*, p^*) \in \left(\mathbb{R}_{++}^C \times \mathbb{R}\right)^H \times \mathbb{R}_{++}^C$ *is a Kranich equilibrium associated with the economy* (u, v, e, β, k) *if it is an equilibrium and* $p^* r = 1$.

Using Proposition 57, we can write the so called extended system associated with the above definition of equilibrium. Indeed the above proposition does not apply to the case we are going to analyze below since

$$v : \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \mapsto v(x) = -e^{cx}.$$

is a standard constant risk aversion utility function - see the appendix on quasiconcavity. On the other hand, the only problem may arise in the application of Proposition 57 is relative to the proof of existence of solutions to the maximization problem, but the proof is easier for the chosen v. Indeed, we can add the following innocuous constraint to make the constraint set compact

$$u_1(x_1) + \beta_{12} v_{12}\left(p^*\left(e_2 + t_{12} - t_{21}^*\right)\right) \geq u_1(e_1) + \beta_{12} \cdot v_{12}\left(p^* e_2\right)$$

and then

$$u_1(x_1) \geq u_1(e_1) + \beta_{12} \cdot v_{12}\left(p^* e_2\right) - \beta_{12} v_{12}\left(p^*\left(e_2 + t_{12} - t_{21}^*\right)\right) \geq u_1(e_1) + \beta_{12} \cdot v_{12}(0) - \beta_{12} v_{12}\left(p^*(r + k)\right).$$

Define $k^* := \max\{k^c : c \in \mathcal{C}\}$. Since $pr^* = 1$ and $pk \leq k^* p \cdot \mathbf{1} \leq k^* \frac{1}{r^{c*}} := \overline{k}$. Then

$$u_1(x_1) \geq u_1(e_1) + \beta_{12} \cdot v_{12}(0) - \beta_{12} v_{12}\left(1 + \overline{k}\right).$$

Definition 81 $\xi^* := \left(\left(x_1^*, t_{12}^*, \lambda_1^*, \gamma_{12}^*\right), \left(x_2^*, t_{21}^*, \lambda_2^*, \gamma_{21}^*\right), p^*\right) \in \Xi := \left(\mathbb{R}_{++}^C \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\right)^H \times \mathbb{R}_{++}^C$ *is an extended equilibrium for the economy* $\mathcal{E} = (\alpha_1, \beta_{12}, 1, \beta_{21}, k_1, k_2) \in \mathbb{E}$ *if it is a solution to the following system.*

$$\begin{array}{ll}
Du_1(x_1) - \lambda_1 p & = 0 \\
\beta_{12} v_{12}'\left(p\left(e_2 + t_{12} - t_{21}\right)\right) p - \lambda_1 p + \gamma_{12} - \delta_{12} & = 0 \\
-p \cdot x_1 + p\left(e_1 + t_{21} - t_{12}\right) & = 0 \\
\min\{t_{12}, \gamma_{12}\} & = 0 \\
\min\left\{\overline{k} - t_{12}, \delta_{12}\right\} & = 0 \\
 & \\
Du_2(x_1) - \lambda_2 p & = 0 \\
\beta_{21} v_{21}'\left(p\left(e_1 + t_{21} - t_{12}\right)\right) p - \lambda_2 p + \gamma_{21} - \delta_{21} & = 0 \\
-p \cdot x_2 + p\left(e_2 + t_{12} - t_{21}\right) & = 0 \\
\min\{t_{21}, \gamma_{21}\} & = 0 \\
\min\left\{\overline{k} - t_{21}, \delta_{21}\right\} & = 0 \\
 & \\
\sum_{h \in \mathcal{H}}\left(x_h^{\setminus} - e_h^{\setminus}\right) & = 0
\end{array} \tag{74}$$

Definition 82 $\xi^* := \left(\left(x_1^*, t_{12}^*, \lambda_1^*, \gamma_{12}^*\right), \left(x_2^*, t_{21}^*, \lambda_2^*, \gamma_{21}^*\right), p^*\right) \in \Xi := \left(\mathbb{R}_{++}^C \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\right)^H \times \mathbb{R}_{++}^C$ *is a Kranich extended equilibrium for the economy* $\mathcal{E} = (\alpha_1, \beta_{12}, 1, \beta_{21}, k_1, k_2) \in \mathbb{E}$ *if it is a an extended equilibrium and* $p^* r = 1$.

Proposition 83 *Normalizations matter, i.e.,*

if $\xi^* := \left(\left(x_1^*, t_{12}^*, \lambda_1^*, \gamma_{12}^*\right), \left(x_2^*, t_{21}^*, \lambda_2^*, \gamma_{21}^*\right), p^*\right)$ *is a Kranich extended equilibrium for any economy for which* $v_{hh'}$ *are of the constant risk aversion type, then there exists* $\widehat{\alpha} \in \mathbb{R}_{++} \setminus \{1\}$ *such that for any* $\alpha \in \mathbb{R}_{++} \setminus \{1, \widehat{\alpha}\}$ *and any* $(t, \lambda, \gamma, \delta) \in \mathbb{R}^{\cdot\cdot}$, $\quad (x^*, t, \lambda, \gamma, \delta, \alpha p^*)$ *is not a Kranich extended equilibrium for that given economy.*

Proof. Let's rewrite system (74) with αp in the place of p.

$$\begin{array}{ll}
Du_1(x_1) - \lambda_1(\alpha p) & = 0 \\
\beta_{12} v_{12}'\left(\alpha p\left(e_2 + t_{12} - t_{21}\right)\right)(\alpha p) - \lambda_1(\alpha p) + \gamma_{12} - \delta_{12} & = 0 \\
-\alpha p \cdot x_1 + \alpha p\left(e_1 + t_{21} - t_{12}\right) & = 0 \\
\min\{t_{12}, \gamma_{12}\} & = 0 \\
\min\left\{\overline{k} - t_{12}, \delta_{12}\right\} & = 0 \\
 & \\
Du_2(x_1) - \lambda_2(\alpha p) & = 0 \\
\beta_{21} v_{21}'\left(\alpha p\left(e_1 + t_{21} - t_{12}\right)\right)(\alpha p) - \lambda_2(\alpha p) + \gamma_{21} - \delta_{21} & = 0 \\
-\alpha p \cdot x_2 + \alpha p\left(e_2 + t_{12} - t_{21}\right) & = 0 \\
\min\{t_{21}, \gamma_{21}\} & = 0 \\
\min\left\{\overline{k} - t_{21}, \delta_{21}\right\} & = 0 \\
 & \\
\sum_{h \in \mathcal{H}}\left(x_h^{\setminus} - e_h^{\setminus}\right) & = 0
\end{array}$$

Then, all the equations but the ones below are indeed obviously satisfied.

$$
\begin{array}{lll}
(1.1) & Du_1(x_1) - \lambda_1(\alpha p) & = 0 \\
(1.2) & \beta_{12} v_{12}'(\alpha p(e_2 + t_{12} - t_{21}))(\alpha p) - \lambda_1(\alpha p) + \gamma_{12} - \delta_{12} & = 0 \\
 & & \\
(2.1) & Du_2(x_1) - \lambda_2(\alpha p) & = 0 \\
(2.2) & \beta_{21} v_{21}'(\alpha p(e_1 + t_{21} - t_{12}))(\alpha p) - \lambda_2(\alpha p) + \gamma_{21} - \delta_{21} & = 0
\end{array}
$$

Choosing $\lambda_h' = \frac{\lambda_h}{\alpha}$, then equations (1.1) and (1.2) are clearly satisfied. We are then left with the following equations.

$$
\begin{array}{lll}
(1.2) & \beta_{12} v_{12}'(\alpha p(e_2 + t_{12} - t_{21}))(\alpha p) - \frac{\lambda_1}{\alpha}(\alpha p) + \gamma_{12} - \delta_{12} & = 0 \\
 & & \\
(2.2) & \beta_{21} v_{21}'(\alpha p(e_1 + t_{21} - t_{12}))(\alpha p) - \frac{\lambda_2}{\alpha}(\alpha p) + \gamma_{21} - \delta_{21} & = 0
\end{array}
$$

Consider the open nonempty set of economies for which $t_{12}^* > 0$ and $\gamma_{12}^* = \delta_{12}^* = 0$ and $t_{21}^* = 0$. Then

$$\beta_{12} v_{12}'(\alpha p^*(e_2 + t_{12}^*))(\alpha p^*) - \lambda_1^* p^* = 0,$$

or

$$\beta_{12} v_{12}'(\alpha p^*(e_2 + t_{12}^*))(\alpha p^*) - \lambda_1^* = 0$$

and from system(74) we have

$$\beta_{12} v_{12}'(\alpha p^*(e_2 + t_{12}^*))(\alpha p^*) - \lambda_1^* = \beta_{12} v_{12}'(\alpha p^*(e_2 + t_{12}^*))(\alpha p^*) - \beta_{12} v_{12}'(p^*(e_2 + t_{12}^*))\, p^* = 0$$

Therefore, we must have

$$v_{12}'(\alpha p^*(e_2 + t_{12}^*))(\alpha p^*) = v_{12}'(p^*(e_2 + t_{12}^*)). \tag{75}$$

Indeed, the above equality is true for any $\alpha > 0$ if $v_{12} = \log$, simply because in that case $v_{12}'(z) = \frac{1}{z}$ and then $v_{12}'(p(e_2 + t_{12})) = \frac{1}{p(e_2+t_{12})} = \alpha \frac{1}{\alpha p(e_2+t_{12})} = \alpha v_{12}'(\alpha p(e_2 + t_{12}))$. On the other hand, condition (75) is not satisfied if $v(z) = -e^{-cx}$ with $c > 0$. Indeed, (75) becomes with $p^*(e_2 + t_{12}^*) := w$

$$\alpha c e^{-\alpha c w} = c e^{-cw}$$

or

$$\frac{\alpha}{e^{\alpha c w}} = \frac{1}{e^{cw}} \quad \text{or} \quad \alpha e^{cw} = e^{\alpha c w} \qquad \text{or} \qquad \log(\alpha) + cw = \alpha c w$$

$$\text{or} \quad \log(\alpha) = (cw) \cdot \alpha - (cw) \qquad \text{or} \qquad \log(\alpha) = (cw)(\alpha - 1),$$

which is satisfied for $\alpha = 1$ and for any value of $\alpha \in \mathbb{R}_{++}$ such that

$$\frac{\log \alpha}{\alpha - 1} = cw \tag{76}$$

Drawing a graph with α on the horizontal axis of $f(\alpha) = \frac{\log \alpha}{\alpha - 1}$, we get the following picture.Then, called $\alpha(cw)$ the unique solution to equation (76) ,then for any $\alpha \notin \{1, \alpha(cw)\}$, (76) is not satisfied. ■

8.3 On the case of an infinite number of equilibria

To make our argument simpler, we consider the case in which $H = 2$. Of course, we have to generalize the model to the case of arbitrary $H \in \mathbb{N}$

Observe that our fictitious $B-$ economies are such $B_1 \times B_2 \in \{\varnothing, \{2\}\} \times \{\varnothing, \{1\}\}$. Then, in those economies, household 1 and household 2 have, at most, one choice of transfers: t_{12} and t_{21}, respectively.

The equilibrium system with two households who potentially care about each other, i.e., $B_1 = \{2\}$ and $B_2 = \{1\}$ is as follows.

$Du_1(x_1) - \lambda_1 p$
$\frac{1}{pr}\beta_{12} v_{12}'\left(\frac{pe_2 + t_{12} - t_{21}}{pr}\right) - \lambda_1 + \gamma_{12}$
$-p \cdot x_1 + pe_1 - t_{12} + t_{21}$
$\min\{\gamma_{12}, t_{12}\}$
$Du_2(x_2) - \lambda_2 p$
$\beta_{21} v_{21}'\left(\frac{pe_1 + t_{21} - t_{12}}{pr}\right) - \lambda_2 + \gamma_{21}$
$-p \cdot x_2 + pe_2 - t_{21} + t_{12}$
$\min\{\gamma_{21}, t_{21}\}$
$\sum_h \left(x_h^{\setminus} - e_h^{\setminus}\right)$

(77)

Define

$$F : \Xi \times \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H}_{++} \longrightarrow \mathbb{R}^{\dim \Xi},\ (\xi, u, v, e, \beta) \mapsto \text{ Left Hand Side of } (77)$$

Remark 84 *The Lemma below says that any equilibrium is "equivalent" to an equilibrium in which at least one of the household chooses her transfer to be equal to zero.*

Lemma 85 *Assume that $B_1 = \{2\}$ and $B_2 = \{1\}$.*

Assume that $((x_1, t_{12}, \lambda_1, \gamma_{12}), (x_2, t_{21}, \lambda_2, \gamma_{21}), p)$ is an equilibrium associated with the economy (e, β, u, v).

a. if $t_{12} > t_{21}$, then $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21}, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := 0, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium associated with the economy (e, β, u, v).

b. if $t_{12} = t_{21}$, then $((x_1, 0, \lambda_1, \gamma_{12}), (x_2, 0, \lambda_2, \gamma_{21}), p)$ is an equilibrium associated with the economy (e, β, u, v).

c. (in a symmetric manner with respect to statement a.) if $t_{21} > t_{12}$, then $\left(\left(x_1, \widetilde{t}_{12} := 0, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t_{21} - t_{12}, \lambda_2, \gamma\right.\right.$ is an equilibrium associated with the economy (e, β, u, v).

Proof. The basic idea of the proof is what follows. To fix ideas assume that in the true equilibrium, we have $t_{12} > t_{21}$. If in that equilibrium $\gamma_{21} > 0$ and $t_{21} = 0$, then in the associated equilibrium you have to choose $\widetilde{t}_{21} = 0$. If in that equilibrium $\gamma_{21} = 0$ and $t_{21} \geq 0$, then in the associated equilibrium you may choose $\widetilde{t}_{21} \geq 0$.

a.

We are going to distinguish some cases as follows.

Case 1. $t_{12} > t_{21} = 0$ and $\gamma_{21} > 0$. Then $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21} = t_{12}, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t_{21} = 0, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium.

Case 2. $t_{12} > t_{21} = 0$ and $\gamma_{21} = 0$. Then, for any $t \geq 0$, $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21} + t, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium.

Case 3. $t_{12} > t_{21} > 0$ (and then $\gamma_{21} = 0$). Then for any $t \geq 0$, $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21} + t, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium.

Case 1. $t_{12} > t_{21} = 0$ and $\gamma_{21} > 0$.

We want to show that if $((x_1, t_{12}, \lambda_1, \gamma_{12}), (x_2, t_{21}, \lambda_2, \gamma_{21}), p)$ is an equilibrium associated with the economy (e, β, u, v), then $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21}, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := 0, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium associated with the economy (e, β, u, v). That is true simply because by the assumption of the case, we have

$$((x_1, t_{12}, \lambda_1, \gamma_{12}), (x_2, t_{21}, \lambda_2, \gamma_{21}), p) \overset{t_{12} > t_{21} = 0}{=} ((x_1, t_{12} - t_{21}, \lambda_1, \gamma_{12}), (x_2, t_{21} = 0, \lambda_2, \gamma_{21}), p),$$

Case 2. $t_{12} > t_{21} = 0$ and $\gamma_{21} = 0$.

Then the following system is satisfied.

$Du_1(x_1) - \lambda_1 p$
$\beta_{12} v'_{12}(pe_2 + t_{12} - 0) - \lambda_1 + \gamma_{12}$
$-p \cdot x_1 + pe_1 - t_{12} + 0$
$0 = \min\{\gamma_{12}, t_{12}\} \overset{t_{12}>0}{=} \gamma_{12}$
$Du_2(x_2) - \lambda_2 p$
$\beta_{21} v'_{21}(pe_1 + 0 - t_{12}) - \lambda_2 + \gamma_{21}$
$-p \cdot x_2 + pe_2 - 0 + t_{12}$
$\min\{\gamma_{21}, 0\} = 0$
$\sum_{h\in}\left(x_h^{\backslash} - e_h^{\backslash}\right)$

(78)

and by assumption of Case 2,

$$\gamma_{21} = -\beta_{21} v'_{21}(pe_1 + 0 - t_{12}) + \lambda_2 = 0. \tag{79}$$

We want to show that for any $t \geq 0$, $\left(\left(x_1, \widetilde{t}_{12} := t_{12} - t_{21} + t, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium, i.e., it solves system (77). Plugging in those values, we get

$Du_1(x_1)-\lambda_1 p$
$\beta_{12}v_{12}'(pe_2+(t_{12}-t_{21}+t)-t)-\lambda_1+\gamma_{12}$
$-p\cdot x_1+pe_1-(t_{12}-t_{21}+t)+t$
$\min\{\gamma_{12},(t_{12}-t_{21}+t)\}$
$Du_2(x_2)-\lambda_2 p$
$\beta_{21}v_{21}'(pe_1+t-(t_{12}-t_{21}+t))-\lambda_2+\gamma_{21}$
$-p\cdot x_2+pe_2-t+(t_{12}-t_{21}+t)$
$\min\{\gamma_{21},t\}$
$\sum_{h\in}\left(x_h^{\backslash}-e_h^{\backslash}\right)$

or

$Du_1(x_1)-\lambda_1 p$
$\frac{1}{pr}\beta_{12}v_{12}'\left(\frac{pe_2+t_{12}-t_{21}}{pr}\right)-\lambda_1+\gamma_{12}$
$-p\cdot x_1+pe_1-t_{12}+t_{21}$
$0=\min\{\gamma_{12},(t_{12}-t_{21}+t)\}\overset{t_{12}-t_{21}+t>0}{=}\gamma_{12}$
$Du_2(x_2)-\lambda_2 p$
$\beta_{21}v_{21}'(pe_1-t_{12}+t_{21})-\lambda_2+\gamma_{21}$
$-p\cdot x_2+pe_2+t_{12}-t_{21}$
$0=\min\{\gamma_{21},t\}\overset{(79)}{=}\gamma_{21}$
$\sum_{h\in}\left(x_h^{\backslash}-e_h^{\backslash}\right)$

which is exactly system (78).

Case 3. $t_{12}>t_{21}>0$ (and then $\gamma_{21}=0$).

By assumption, the following system is satisfied.

$Du_1(x_1)-\lambda_1 p$
$\frac{1}{pr}\beta_{12}v_{12}'\left(\frac{pe_2+t_{12}-t_{21}}{pr}\right)-\lambda_1+\gamma_{12}$
$-p\cdot x_1+pe_1-t_{12}+t_{21}$
$0=\min\{\gamma_{12},t_{12}\}=\gamma_{12}$
$Du_2(x_2)-\lambda_2 p$
$\beta_{21}v_{21}'\left(\frac{pe_1+t_{21}-t_{12}}{pr}\right)-\lambda_2+\gamma_{21}$
$-p\cdot x_2+pe_2-t_{21}+t_{12}$
$0=\min\{\gamma_{21},t_{21}\}=\gamma_{21}$
$\sum_{h}\left(x_h^{\backslash}-e_h^{\backslash}\right)$

(80)

and also

$$\gamma_{21}=-\beta_{21}v_{21}'\left(\frac{pe_1+t_{21}-t_{12}}{pr}\right)+\lambda_2=0. \tag{81}$$

We want to show that for any $t\geq 0$, $\left(\left(x_1,\widetilde{t}_{12}:=t_{12}-t_{21}+t,\lambda_1,\gamma_{12}\right),\left(x_2,\widetilde{t}_{21}:=t,\lambda_2,\gamma_{21}\right),p\right)$ is an equilibrium, i.e., it solves system (77). Plugging in those values, we get

$$
\begin{array}{|l|}
\hline
Du_1(x_1) - \lambda_1 p \\ \hline
\beta_{12} v_{12}' \left(pe_2 + (t_{12} - t_{21} + t) - t \right) - \lambda_1 + \gamma_{12} \\ \hline
-p \cdot x_1 + pe_1 - (t_{12} - t_{21} + t) + t \\ \hline
\min \{\gamma_{12}, (t_{12} - t_{21} + t)\} \\ \hline
\\ \hline
Du_2(x_2) - \lambda_2 p \\ \hline
\beta_{21} v_{21}' \left(pe_1 + t - (t_{12} - t_{21} + t) \right) - \lambda_2 + \gamma_{21} \\ \hline
-p \cdot x_2 + pe_2 - t + (t_{12} - t_{21} + t) \\ \hline
\min \{\gamma_{21}, t\} \\ \hline
\\ \hline
\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash} \right) \\ \hline
\end{array}
$$

or

$$
\begin{array}{|l|}
\hline
Du_1(x_1) - \lambda_1 p \\ \hline
\frac{1}{pr} \beta_{12} v_{12}' \left(\frac{pe_2 + t_{12} - t_{21}}{pr} \right) - \lambda_1 + \gamma_{12} \\ \hline
-p \cdot x_1 + pe_1 - t_{12} + t_{21} \\ \hline
0 = \min \{\gamma_{12}, (t_{12} - t_{21} + t)\} \overset{t_{12} - t_{21} + t > 0}{=} \gamma_{12} \\ \hline
\\ \hline
Du_2(x_2) - \lambda_2 p \\ \hline
\beta_{21} v_{21}' \left(pe_1 - t_{12} + t_{21} \right) - \lambda_2 + \gamma_{21} \\ \hline
-p \cdot x_2 + pe_2 + t_{12} - t_{21} \\ \hline
0 = \min \{\gamma_{21}, t\} \overset{(81)}{=} \gamma_{21} \\ \hline
\\ \hline
\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash} \right) \\ \hline
\end{array}
$$

which is exactly system (80).

b.

We are going to distinguish some cases as follows

Case 1. $t_{12} = t_{21} = 0$, $\gamma_{12} = 0$ and $\gamma_{21} > 0$. Then $\left((x_1, \widetilde{t}_{12} = 0, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} = 0, \lambda_2, \gamma_{21}), p \right)$ is an equilibrium.

Case 2. $t_{12} = t_{21} = 0$, $\gamma_{12} > 0$ and $\gamma_{21} = 0$. Then $\left((x_1, \widetilde{t}_{12} = 0, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} = 0, \lambda_2, \gamma_{21}), p \right)$ is an equilibrium..

Case 3. $t_{12} = t_{21} = 0$, $\gamma_{12} > 0$ and $\gamma_{21} > 0$. Then $\left((x_1, \widetilde{t}_{12} = 0, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} = 0, \lambda_2, \gamma_{21}), p \right)$ is an equilibrium.

Case 4. $t_{12} = t_{21} = 0$, $\gamma_{12} = 0$ and $\gamma_{21} = 0$. and $\gamma_{21} = 0$. Then, for any $t \geq 0$, $\left((x_1, \widetilde{t}_{12} := t, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}), \right.$ is an equilibrium.

Case 5. $t_{12} = t_{21} > 0$ (and then $\gamma_{12} = \gamma_{21} = 0$). Then for any $t \geq 0$, $\left((x_1, \widetilde{t}_{12} := t, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}), p \right)$ is an equilibrium.

For Cases 1,2 and 3, there is nothing to be proved.

Case 4.

By assumption, the following system is satisfied.

$$
\begin{array}{|l|}
\hline
Du_1(x_1) - \lambda_1 p \\ \hline
\beta_{12} v_{12}' (pe_2) - \lambda_1 + \gamma_{12} \\ \hline
-p \cdot x_1 + pe_1 \\ \hline
\\ \hline
\min \{\gamma_{12}, t_{12}\} = \gamma_{12} = t_{12} = 0 \\ \hline
\\ \hline
Du_2(x_2) - \lambda_2 p \\ \hline
\beta_{21} v_{21}' (pe_1) - \lambda_2 + \gamma_{21} \\ \hline
-p \cdot x_2 + pe_2 \\ \hline
\min \{\gamma_{21}, t_{21}\} = \gamma_{21} = t_{21} = 0 \\ \hline
\\ \hline
\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash} \right). \\ \hline
\end{array}
\tag{82}
$$

We want to show that for any $t \geq 0$, $\left((x_1, \widetilde{t}_{12} := t, \lambda_1, \gamma_{12}), (x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}), p \right)$ is an equilibrium, i.e., it solves system (77). Plugging in those values, we get

$Du_1(x_1) - \lambda_1 p$
$\beta_{12} v'_{12}(pe_2 + t - t) - \lambda_1 + \gamma_{12}$
$-p \cdot x_1 + pe_1 - t + t$
$\min\{\gamma_{12}, t\} = \gamma_{12} = 0$
$Du_2(x_2) - \lambda_2 p$
$\beta_{21} v'_{21}(pe_1 + t - t) - \lambda_2 + \gamma_{21}$
$-p \cdot x_2 + pe_2 - t + t$
$\min\{\gamma_{21}, t\} = \gamma_{21} = 0$
$\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash}\right)$

which is satisfied because of (82).

Case 5.

By assumption, the following system is satisfied.

$Du_1(x_1) - \lambda_1 p$
$\beta_{12} v'_{12}(pe_2) - \lambda_1 + \gamma_{12}$
$-p \cdot x_1 + pe_1$
$\min\{\gamma_{12}, \widetilde{t}_{12}\} = \gamma_{12} = 0$
$Du_2(x_2) - \lambda_2 p$
$\beta_{21} v'_{21}(pe_1) - \lambda_2 + \gamma_{21}$
$-p \cdot x_2 + pe_2$
$\min\{\gamma_{21}, \widetilde{t}_{21}\} = \gamma_{21} = 0$
$\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash}\right)$

(83)

We want to show that for any $t \geq 0$, $\left(\left(x_1, \widetilde{t}_{12} := t, \lambda_1, \gamma_{12}\right), \left(x_2, \widetilde{t}_{21} := t, \lambda_2, \gamma_{21}\right), p\right)$ is an equilibrium, i.e., it solves system (77). Plugging in those values, we get

$Du_1(x_1) - \lambda_1 p$
$\beta_{12} v'_{12}(pe_2 + t - t) - \lambda_1 + \gamma_{12}$
$-p \cdot x_1 + pe_1 - t + t$
$\min\{\gamma_{12}, t\} = \gamma_{12} = 0$
$Du_2(x_2) - \lambda_2 p$
$\beta_{21} v'_{21}(pe_1 + t - t) - \lambda_2 + \gamma_{21}$
$-p \cdot x_2 + pe_2 - t + t$
$\min\{\gamma_{21}, t\} = \gamma_{21} = 0$
$\sum_{h \in} \left(x_h^{\backslash} - e_h^{\backslash}\right)$

which is satisfied because of (83). ∎

Remark 86 *Before proceeding, let's comment on the meaning of the multiplier* γ_{12}*. Consistently with Kuhn-Tucker conditions, we have that* $\frac{1}{pr}\beta_{12}v'_{12}\left(\frac{pe_2+t_{12}-t_{21}}{pr}\right) - \lambda_1 + \gamma_{12} = 0$*, i.e.,*

$$\gamma_{12} = \lambda_1 - \frac{1}{pr}\beta_{12}v'_{12}\left(\frac{pe_2 + t_{12} - t_{21}}{pr}\right) \geq 0 \text{ and } \min\{\gamma_{12}, t_{12}\} = 0$$

Now, since

$\lambda_1 =$
$=$ *(marginal utility of one extra unit of the numeraire good in the household's wealth)* $:=$
$:=$ *(MU income)*
and
$\frac{1}{pr}\beta_{12}v'_{12}\left(\frac{pe_2+t_{12}-t_{21}}{pr}\right) =$
=*(marginal utility of one extra unit of the numeraire good in transfer to other household)* $:=$
:=*(MU transfer)*,

we then have what follows

$\gamma_{12} = 0$	$\Leftrightarrow$	*(MU income)* $=$ *(MU transfer)*	$\Leftrightarrow$	*Ms.1 does not want to change the amount of transfer*	*and*	$t_{12} = 0$	*means*	*Ms.1 chooses* $t_{12} = 0$ *and she is happy with that*
$\gamma_{12} = 0$	$\Leftrightarrow$	*(MU income)* $=$ *(MU transfer)*	$\Leftrightarrow$	*Ms.1 does not want to change the amount of transfer*	*and*	$t_{12} > 0$	*means*	*Ms.1 chooses* $t_{12} > 0$ *and she is happy with that*
$\gamma_{12} > 0$	$\Leftrightarrow$	*(MU income)* $>$ *(MU transfer)*	$\Leftrightarrow$	*Ms.1 would like to decrease transfer to increase income*	*and*	$t_{12} = 0$	*means*	*Ms.1 chooses* $t_{12} = 0$ *and she would like to decrease it.*

Now, consider the following maximization problem for household 1: she cannot choose t_{12}. *For given* $(e, \beta, u, v), t_{21}, e_2, p,$

$$\max_{x_1 \in \mathbb{R}^C_{++}} u_1(x_1) + \beta_{12}v_{12}(pe_2 - t_{21}) \qquad s.t. \qquad -px_1 + pe_1 + t_{21} \geq 0. \tag{84}$$

Given our assumptions, the Kuhn-Tucker conditions characterizing the above problem are the following ones.

$$\begin{array}{|l|}\hline Du_1(x_1) - \lambda_1 p = 0 \\ \hline -p\cdot x_1 + pe_1 + t_{21} = 0 \\ \hline \end{array} \tag{85}$$

We define some "fictitious Kuhn-Tucker conditions" associated to problem (84) *as follows*

$Du_1(x_1) - \lambda_1 p$
$\gamma_{12} = -\beta_{12}v'_{12}(pe_2 - t_{21}) + \lambda_1$
$-p\cdot x_1 + pe_1 + t_{21}$
$t_{12} = 0$

Observe that if $(x_1, t_{12} = 0, \lambda_1, \gamma_{12})$ *is a solution to the above system and* $\gamma_{12} \geq 0$, *then* $(x_1, t_{12} = 0, \lambda_1, \gamma_{12})$ *characterize the solution to the true household' 1 maximization problem; in the above case, we can also have*

$\gamma_{12} < 0$	$\Leftrightarrow$	*(MU income)* $<$ *(MU transfer)*	$\Leftrightarrow$	*Ms.1 would like to increase transfer to decrease income*	*but*	$t_{12} = 0$	*and*	*Ms.1 would not be maximizing, if she could choose* $t_{12} \geq 0$

8.4 Some facts on differential topology

We now introduce a definition of regular economies and show that they have nice properties (Theorems 99 and 100) and constitute a "very large" subset of the set of all economies (Theorem 101).

Needed mathematical results.

Proper functions

Theorem 87 *Let (X,d) and (Y,d') be metric spaces, and let $f : X \to Y$ be a continuous function. The following conditions are equivalent:*

1. *f is closed and $f^{-1}(y)$ is compact for each $y \in Y$;*
2. *$f^{-1}(K)$ is a compact subset of X for each compact subset K of Y;*
3. *every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(f(x_n))_{n\in\mathbb{N}}$ converges has a converging subsequence $(x_{n_k})_{k\in\mathbb{N}}$.*

Definition 88 *Let (X,d) and (Y,d') be metric spaces. A function $f : X \to Y$ is* ***proper****, if it is continuous and one among conditions 1-3 of Theorem 87 holds.*

Transversality results.

The implicit function theorem asserts some regularity of the set X of solutions to an equation

$$f(x) = y \tag{86}$$

when $f : D_f \subseteq \mathbb{R}^{n+k} \to \mathbb{R}^n$ is a sufficiently regular function. Namely, if $x \in X$ is such a solution, f is of class C^1 in a neighborhood of x, and the differential of f at x is surjective, then there exists a neighborhood U of x such that $X \cap U$ is the graph of a uniquely determined C^1 function.

The following theorem has a central role in the study of manifolds, and extends the implicit function theorem to functions defined between manifolds. According to the theorem, whenever y is a regular value for f, the set of solutions to equation (86) is either empty or a C^1 manifold.

Theorem 89 (Regular value theorem) *Let M and N be C^r manifolds of dimensions m and n respectively, let $f : M \to N$ be a C^r function, and let y be a regular value for f. If $y \in \mathrm{Im} f$, then $f^{-1}(y)$ is a C^r manifold of dimension $m - n$.*

Theorem 90 *Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. If y is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, y is a regular value for f_ω (where, $f_\omega : M \to N$, $x \mapsto f(x,\omega)$).*

Corollary 91 *(I modify the statement and proof of this theorem of our book)*

Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. Let y be a regular value for f. Let Ω^* be a full measure subset of Ω such that, for any $\omega \in \Omega^*$, y is a regular value of f_ω. Then

1. *if $\dim M < \dim N$, then $f_\omega^{-1}(y) = \emptyset$ for any $\omega \in \Omega^*$,*
2. *if $\dim M > \dim N$, then, for any $\omega \in \Omega^*$, either $f_\omega^{-1}(y) = \emptyset$ or $f_\omega^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M.*
3. *if $\dim M = \dim N$, then, for any $\omega \in \Omega^*$, either $f_\omega^{-1}(y) = \emptyset$ or $f_\omega^{-1}(y)$ is a set of isolated points.*
4. *if the projection $\pi : f^{-1}(y) \to \Omega$ is proper, then Ω^* is open in Ω.*

Some other results on regularity

Corollary 92 *Let M and N be C^r manifolds of the same dimension, respectively, let $f : M \to N$ be a C^r function, and let y be a regular value for f such that $f^{-1}(y)$ is compact. Then $f^{-1}(y)$ is a finite subset of M.*

Corollary 93 *Let $X \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^m$ be open sets; let $f : X \times Z \to \mathbb{R}^n$, and let y be an element of $\mathbb{R}^n$ and z be any element of Z. Let pr be defined as follows*

$$pr : f^{-1}(y) \to Z, \quad (x,z) \mapsto z$$

If

1. *y is a regular value for f;*

2. *y is a regular value for the partial function f_z;*

3. *pr is surjective;*

4. *pr is proper;*

then

1. *z is a regular value for pr;*

2. *$pr^{-1}(z)$ is a finite set, i.e., $pr^{-1}(z) = \left\{ \left(x^i, z\right) \right\}_{i=1}^r$ for some $r \in \mathbb{N}$;*

3. *there exist an open neighborhood V of z in Z, and for each i an open neighborhood U_i of $\left(x^i, z\right)$ in $f^{-1}(y)$, such that*

   (a) *$U_j \cap U_k = \emptyset$ if $j \neq k$;*
   (b) *$pr^{-1}(V) = \cup_{i=1}^r U_i$;*
   (c) *$pr_{|U_i} : U_i \to V$ is a diffeomorphism.*

Needed economic results.

Lemma 94 *For every e, $F_e^{-1}(0) \neq \emptyset$.*

Lemma 95 *For every e, $F_e^{-1}(0)$ is compact*

Lemma 96

$$pr : F^{-1}(0) \to \mathbb{R}_{++}^{CH}, \quad (\xi, e) \mapsto e \tag{87}$$

is proper.

Lemma 97 *0 is a regular value for F.*

Definition 98 *$e \in \mathbb{R}_{++}^{CH}$ is a regular economy if 0 is a regular value for F_e. Let $\mathcal{R}$ be the set of regular economies.*

Theorem 99 *If $e \in \mathcal{R}$, then*

1. *there exists $r \in \mathbb{N}$ such that*
$$F_e^{-1}(0) = \left\{\xi^i\right\}_{i=1}^r$$

2. *there exists an open set $Y \subseteq \mathbb{R}_{++}^{CH}$, and for each i, an open set $V_i \subseteq \mathbb{R}^{\dim \Xi}$ and a unique function $g_i : Y \to V_i$ such that*

   (a) *$\left(\xi^i, e\right) \in V_i \times Y$,*
   (b) *g_i is C^1,*
   (c) *$g_i(e) = \xi^i$,*
   (d) *for every $e' \in Y$, $F(g_i(e'), e') = 0$,*
   (e) *$\{(\xi, e) : e \in Y, F(\xi, e) = 0\} = \cup_{i=1}^r \text{graph } g_i$.*

Proof. 1. From the Lemma 94, for every e, $F_e^{-1}(0) \neq \emptyset$. Moreover, from Lemma 95, $F_e^{-1}(0)$ is compact. Then the result follows from Corollary 92.

2. From the implicit function theorem, we have that for each i, there exist an open sets $Y_i \subseteq \mathbb{R}_{++}^{CH}$, an open set $V_i \subseteq \mathbb{R}^{\dim \Xi}$ and a unique function $\widehat{g}_i : Y_i \to V_i$ such that $\left(\xi^i, e\right) \in V_i \times Y_i$ and statements 2.b, 2.c and 2.d hold. Then defined $Y \equiv \cap_{i=1}^r Y_i$, an open neighborhood of e, for each i, it is enough to take $g_i \equiv \widehat{g}_{i|Y}$. ■

Theorem 100 *Given $e \in \mathcal{R}$. Then*

1. *There exists $r \in \mathbb{N}$ such that $pr^{-1}(e) = \left\{\left(\xi^i, e\right)\right\}_{i=1}^r$;*

2. *there exist an open neighborhood Y of e in $\mathbb{R}_{++}^{CH}$, and for each i an open neighborhood U_i of $\left(\xi^i, e\right)$ in $F^{-1}(0)$, such that:*

   (a) *$U_j \cap U_k = \emptyset$ if $j \neq k$;*

(b) $pr^{-1}(Y) = \cup_{i=1}^{r} U_i$;

(c) $pr_{|U_i} : U_i \to Y$ is a diffeomorphism.

Proof. It follows from Corollary 93, identifying the function f and pr there with the functions F and pr here, and the variables x and z there with ξ and e here. In particular, Assumption 1 of that Corollary follows from Theorem 97; Assumption 2 from the assumption that $e \in \mathcal{R}$; Assumption 3 from Theorem 94; Assumption 4 from Lemma 96. ■

Theorem 101 *The set $\mathcal{R}$ has the following properties.*

1. *$\mathcal{R}$ is of full Lebesgue measure in $\mathbb{R}_{++}^{CH}$,*
2. *$\mathcal{R}$ is open in $\mathbb{R}_{++}^{CH}$.*

The proof of the above result contained in the chapter on Exchange economies is based on the fact that the equilibrium set is indeed an equilibrium manifold. In our model, that is not the case due to the presence of min functions in the equilibrium system. We can then use the analysis provided in the chapter on restricted participation, which I summarize below. (Indeed what follows is the beginning of the section 3. Regular economies in the Chapter on Restricted Participation).

As explained in detail at the beginning of Section on regular economies in the Chapter on numeraire assets of our book, besides existence of equilibria, the two crucial steps to get the desired results are

1. 0 is a regular value for F, and therefore $F^{-1}(0)$ is a manifold, and there exists an open and full measure subset $\mathcal{R}$ of $\mathbb{R}_{++}^{GH}$ such that

$$\forall\, e^* \in \mathcal{R},\ \forall \xi^* \in F_{e^*}^{-1}(0),\ \text{rank}\, DF_{e^*}(\xi^*) \text{ is full} \tag{88}$$

2. $pr : F^{-1}(0) \to \mathbb{R}_{++}^{GH}$, $(\xi, e) \mapsto e$ is proper.

A preliminary step is to show the Jacobian matrix of the left hand side of the first order conditions of the household maximization problem has full row rank. In the case of inequality constraints, one of the equations of Kuhn-Tucker conditions is expressed in terms of min between each constraint and the associated multiplier. The obvious problem is that the min function is not even differential where both constraint and multiplier are equal to zero, a sort of "border line" case. To be able to show result (88), we have therefore to show that border line cases occur outside an open and full measure subset Π^* of the economy space. As usual that result is obtained showing the properness of the projection from the equilibrium set to the economy space and then verifying a rank condition crucial as an application of the transversality theorem (see Theorem 90 and Corollary 91). The further step is to show that in an open and full measure subset $\mathcal{R}$ of Π^* condition (88) does hold. That step turns out to be an easy consequence of the first one.

Summarizing, to prove the desired regularity result, we need to show the following results.

Let $\mathcal{E}$ be an economy, $\mathbb{E}$ be the space of economies and $F(\xi, \mathcal{E}) = 0$ be the equilibrium system.

1. For every $\mathcal{E}$, $F_{\mathcal{E}}^{-1}(0) \neq \emptyset$.
2. For every $\mathcal{E}$, $F_{\mathcal{E}}^{-1}(0)$ is compact.
3. $pr : F^{-1}(0) \to \mathbb{E}$, $(\xi, \mathcal{E}) \mapsto \mathcal{E}$ is proper.
4. Border line cases are rare;
5. Regularity result holds.

8.5 On quasiconcavity of the utility function

We want to find conditions under which the utility function is quasiconcave in both x_h and $\theta_{h'}$. That for sure it is the case in which $\beta > 0$. In that case, we utility function is strictly concave, since the Hessian matrix is

$$\begin{matrix} u'' & 0 \\ 0 & \beta v'' \end{matrix}$$

and $u'' < 0, v'' < 0$ and $\beta > 0$.

If $\beta < 0$, we have the following proposition.

Proposition 102 *Consider a one-good economies* (β, u, v) *with* $\beta < 0$. *Define*

$$V : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}, \ (x, y) \mapsto \ u(x) + \beta v(y).$$

1. *A sufficient condition for quasiconcavity of* V *is*

$$\overset{(-)}{v''} \cdot \overset{(+)}{(u')}{}^2 + \overset{(-)}{\beta} \cdot \overset{(-)}{u''} \cdot \overset{(+)}{v'} \ > 0,$$

or, denoted the coefficient of absolute risk aversion associated with function f *by* $R_{A(f)}$,

$$(-\beta)\, R_A(u) < u' \cdot R_A(v)$$

2. *There exist economies for which* $\beta < 0$ *and* V *is quasi-concave.*

Proof. Preliminary observation. Below, we present two approaches to find sufficient conditions for quasi-concavity.

After the preliminary observation, we present the proof of the desired results.

Approach 1. Implicit Function Theorem.

Let the following equation be given.

$$V(x, y) := u(x) + \beta v(y) - k = 0,$$

with

$$u' > 0, \quad u'' < 0, \quad \beta < 0, \quad v' > 0, \quad v'' < 0.$$

From the Implicit Function Theorem, we have

$$\frac{dx}{dy} := g'(y) = -\frac{\frac{\partial V(x,y}{\partial y}}{\frac{\partial V(x,y}{\partial x}} = -\frac{\beta v'(y)}{u'(x)} > 0.$$

We now want to give conditions under which $g''(y) > 0$.

$$g''(y) = \frac{d\left(-\frac{\beta v'(y)}{u'(g(y))}\right)}{dx} = -\frac{1}{(u'(g(y)))^2} \left(\beta \cdot v''(y) \cdot u'(g(y)) - \beta \cdot v'(y) \cdot u''(g(y)) \cdot g'(y)\right) \ =$$

$$-\frac{1}{(u'(g(y)))^2} \overset{(-)}{\beta} \left(v'' \cdot u' + v' \cdot u'' \cdot \beta \cdot v' \cdot \frac{1}{u'}\right)$$

Then

$$sign\ g''(y) = sign\ \left(\overset{(-)}{v''} \cdot \overset{(+)}{u'} + \overset{(+)}{v'} \cdot \overset{(-)}{u''} \cdot \overset{(-)}{\beta} \cdot \overset{(+)}{v'} \cdot \overset{(+)}{\frac{1}{u'}}\right) \tag{89}$$

For example for large $|\beta|$, the indifference curve is convex and "therefore" V is quasiconcave.

Approach 2. The bordered Hessian.

We are using the following result. If $n \geq 2$ and $\forall x \in X$, for any $k \in \{3, ..., n+1\}$,

$$sign\,(k - \text{leading principal minor of } Bf(x)) = sign\ (-1)^{k-1},$$

then f is pseudo concave and, therefore, quasi-concave.

$$DV = [u'(x), \beta v'(y)]$$

$$\begin{array}{lll} & x & y \\ u'(x) & u'' & 0 \\ \beta v'(y) & 0 & \beta v'' \end{array}$$

$$\begin{array}{lll} 0 & u' & \beta v' \\ u' & u'' & 0 \\ \beta v' & 0 & b\beta v'' \end{array}$$

row and column 1

$\det \begin{bmatrix} u'' & 0 \\ 0 & \beta v'' \end{bmatrix} = \beta u'' v'' < 0$

r c 2

$\det\begin{bmatrix} 0 & \beta v' \\ \beta v' & \beta v'' \end{bmatrix} = -\beta^2 (v')^2 < 0$

r c 3

$\det\begin{bmatrix} 0 & u' \\ u' & u'' \end{bmatrix} = -(u')^2 < 0.$

$$\det\begin{bmatrix} 0 & u' & \beta v' \\ u' & u'' & 0 \\ \beta v' & 0 & \beta v'' \end{bmatrix} = -\beta(u')^2 v'' - \beta^2 u''(v')^2 = (-\beta)\cdot\left(\overset{(-)}{v''}\cdot\overset{(+)}{(u')}^2 + \overset{(-)}{\beta}\cdot\overset{(-)}{u''}\cdot\overset{(+)}{v'}\right) > 0 \tag{90}$$

if

$$\overset{(-)}{v''}\cdot\overset{(+)}{(u')}^2 + \overset{(-)}{\beta}\cdot\overset{(-)}{u''}\cdot\overset{(+)}{v'} < 0, \qquad \frac{\overset{(-)}{v''}}{v'}\cdot\overset{(+)}{u'} + \overset{(-)}{\beta}\cdot\frac{\overset{(-)}{u''}}{u'} < 0, \qquad (-\beta)\left(-\frac{u''}{u'}\right) < u'\left(-\frac{v''}{v'}\right) \tag{91}$$

Observe that (90) has a structure similar to the expression in (89).

2.

Consider $f(x) = -e^{-ax}$;

Observe that the above assumption does satisfy our existence maintained assumptions, but not our regularity maintained assumption: closure of the upper level set is not closed in $\mathbb{R}$: take $k > 0$.

$$\left\{x \in \mathbb{R}_{++} : -e^{-ax} \geq -k\right\} = \left(\frac{e^k}{a}, +\infty\right)$$

$-e^{-x}$

Observe also that[9] $R_A(x) := -\frac{f''(x)}{f'(x)} = -\frac{-a^2 e^{-ax}}{a e^{-ax}} = a.$

The higher a, the more concave the function is (graphs below correspond to $a \in \left\{\frac{1}{2}, 1, 2\right\}$).

Then assuming $u(x) = -e^{-ax}$ and $v(x) = -e^{-bx}$, condition (91) becomes

$$-\beta a < a e^{-ax} b \quad \text{or} \quad -\beta < e^{-ax} b$$

Now, observe that $v_{12}(pe_2 + t_{12} - t_{21})$ and $pe_2 + t_{12} - t_{21} \leq pe_2 + k_1 \leq pr + k_1 = 1 + k_1$. Then we can express condition (91) as

$$-\beta < e^{-a(1+k_1)} b,$$

i.e., in terms of exogenous variables. That show the nonemptyness statement. ■

Remark 103 *As discussed below, it is not easy to say something as in the above proposition is the utility function are* log.

Assume $v = u = \log$. *Then,*

$$\left(\overset{(-)}{v''}\cdot\overset{(+)}{(u')}^2 + \overset{(-)}{\beta}\cdot\overset{(-)}{u''}\cdot\overset{(+)}{v'}\right) = \left(-\frac{1}{y^2}\right)\left(\frac{1}{x}\right)^2 + \beta\left(-\frac{1}{x^2}\right)\frac{1}{y} > 0$$

$1 + \beta y < 0$

$\beta y < -1$

$(-\beta) > \frac{1}{y}$

$y > \frac{1}{(-\beta)}$

$\frac{1}{(-\beta)} < y$ *indeed* $y = pe_2 + t_{12} - t_{21} \overset{\beta \leq 0}{=} pe_2 - t_{21}$

$\leq pr + k = 1 + k.$ *Then, we must have*

$$\frac{1}{(-\beta)} < 1 + k$$

In the relative wealth model, we have

$$y = \frac{pe_2 + t_{12} - t_{21}}{pr} = \frac{px_2}{pr} \overset{in\ equilibrium}{\leq} \frac{pr}{pr} = 1$$

[9] See Mas Colell, page 191.

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