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Opportunity-Based Other Regarding Preferences in General Equilibrium: Existence

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Abstract.

We consider a pure exchange economy with a finite number of goods and households. Following Kranich (1988), we introduce only two differences with respect to the standard model: i. each household utility function depends not only on her own consumption, but also on other households' welfare, measured by wealth; then, *other regarding preferences* are *based* on other households *opportunities*; ii. households are allowed to promise transfers to other households (and promised are bound to be honored).

We show existence of equilibria under the assumption of the presence of an upper bound on transfers, taking care of some problems which are not addressed by the paper by Kranich (1988).

We present a robust example of non existence of equilibria if the bound on transfers is not imposed, and we describe equilibria and their Pareto Optimality properties in a simple Cobb-Douglas economy.

Keywords: General Equilibrium; exchange economies; other regarding preferences; existence, nonexistence and regularity of equilibria.

JEL classification: D50, D64.

1 Introduction

We consider a pure exchange economy with a finite number of goods and households. Following Kranich (1988), we introduce only two differences with respect to the standard model: i. each household utility function depends not only on her own consumption, but also on other households' welfare, measured by wealth (then, other regarding preferences are based on other households opportunities); ii. households are allowed to promise transfers to other households (and promised are bound to be honored).

The main contributions of the paper are as follows.

First of all, we discuss some problems and flows contained in the proof of existence of equilbria presented in Kranich (1988). Similar problems are contained in Mercier Ythier (2000) which analyzes an analogous model. At the best of our knowledge, no other work available in the literature presents a contribution similar to the one presented here. Then, we provide (two) sets of assumptions which allow to get the existence result. Such assumptions are less general than those provided by Kranich (1988) and Mercier Ythier (2000).

Moreover, we consider the example of a one-good-two household Cobb Douglas economy and we analyze and discuss the non-existence problem under the reasonable assumption of absence of an artificial bound on the amount of transfers households can provide. In the case in which the bound is imposed, the example allows to get some conjectures about properties of equilibria in more general cases.

Finally, as a by-product of our analysis, we present a result which gives very easy to check conditions which are sufficient to guarantee crucial properties of constraint set-valued functions associated with commonly studied maximization problems in economics - see Proposition 107.[1]

The paper is organized as follows. In Section 1, we first present the set up of the model as it was introduced by Kranich (1988).[2] Indeed, the very definition of the maximization problem under analysis requires several observations related to the quasi-concavity of the utility function, the compactness of the set of admissible transfers, a consistent definition of the maximization problem and the related need of an extension of the utility function, the role of price normalizations. Each above problem is discussed and a solution is proposed.

In Section 2, starting from the Definition of equilibrium obtained as a consequence of the above discussion, we describe a further problem we face in showing existence: the possibility of empty constraint set faced by households. That problem can be addressed using some extension theorem for (quasi-)concave continuous utility function from a subset of an Euclidean space to the whole space. Using those theorems, we do present two existence results under different assumptions on the utility functions. As a consequence, we can say that the existence result claimed by Kranich (1988) can be shown to be true under assumptions which are stronger than those he proposes.

In Section 3, we discuss in detail the nonexistence problem in the model without upper bound on transfers. First of all, we present a simple Cobb-Douglas, two household, one good version of the model and we do show that there is indeed an open, nonempty and "interesting" set of economies for which equilibria do not exists if upper bounds on promises of transfers are not imposed. Intuition on the nonexistence results is presented. We then consider the Cobb-Douglas, two household, one good model, with an upper bound on transfer. In the case of endowments of both households equal to 1, economies can be represented as points in the positive orthant, and it is possible to compute equilibria for each economy. We verify that if an upper bound is added, the result of non-existence of equilibria is substituted by the existence of equilibria in which the upper bound on transfer is binding. Moreover, we analyze the Pareto Optimality of equilibria. The simple Cobb-Douglas economies allows to get some results on the equilibria structure which are used to get conjectures we study in a companion paper.Set up of the model

1.1 A first version

We describe an economy in which households exchange goods (or commodities) in order to maximize their well-being. A commodity is denoted by $c \in \{1, ..., C\} := \mathcal{C}$. A household is denoted by $h \in$

[1] A more detailed version of the present paper can be found in its working paper version at
https://www.disei.unifi.it/vp-110-working-papers-quantitative-methods-for-social-sciences.html
That version contains the proofs or precise references for the proofs of all results presented here.

[2] In a companion paper, we analyze a similar interesting model introduced by Mercier Ythier (2000).

$\{1, ..., H\} := \mathcal{H}$ and she is described by the following objects:[3]

a consumption set $X_h \subseteq \mathbb{R}^C$ with generic element $x_h = (x_h^c)_{c \in \mathcal{C}}$, where $x_h^c \in \mathbb{R}$ denotes the consumption of good c by household h;

an endowment vector $e_h = (e_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $e_h^c \in \mathbb{R}$ denotes the amount of good c owned by household h;

a transfer vector $t_h = (t_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)}$, where $t_{hh'} = (t_{hh'}^c)_{c \in \mathcal{C}}$ and $t_{hh'}^c$ denotes the transfer of good c from household h to household h'. We also define $t_{\setminus h} = (t_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{(H-1)C(H-1)}$.

Commodities can be exchanged with other commodities at exchange ratios described by a price vector p belonging to a price set $P \subseteq \mathbb{R}^C$.

To describe households' utility functions, we need some preliminary definitions. $\theta_h \in \Theta \subseteq \mathbb{R}$ is household h's wealth. We also define $\theta = (\theta_h)_{h \in \mathcal{H}}$ and $\theta_{\setminus h} := (\theta_{h'})_{h' \in \mathcal{H} \setminus \{h\}}$. Household h's wealth function, which depends upon the value of her initial endowment and net transfers, is denoted and defined as follows.[4]

$$w_h : P \times \mathbb{R}^{C(H-1)H} \longrightarrow \Theta, \qquad (p, t) \mapsto p \left(e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} (t_{h'h} - t_{hh'}) \right).$$

Household h's utility function depends upon her own consumption and anyone else's wealth and it is denoted and defined as follows.

$$u_h : X_h \times \Theta^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h \left(x_h, \theta_{\setminus h}\right).$$

For "physical/biological" reasons, we assume nonnegativity of consumption and endowment vectors; for institutional reasons (households are not allowed to "steal" goods), we assume nonnegativity of the transfer vectors. Moreover, since we are going to assume that households "like goods", prices are restricted to be nonnegative. Since wealth is going to be completely used to buy goods which are consumed, wealth as well is going to be nonnegative. Finally, defined the vector of total resources as $r = \sum_{h \in \mathcal{H}} e_h$, for the time being and following Kranich (1988), we consider the following set of "normalized" prices:

$$S = \left\{ p \in \mathbb{R}_+^C : pr = 1 \right\}.$$

Summarizing, we assume $X_h = \mathbb{R}_+^C$, $P = S$, $\Theta = \mathbb{R}_+$ and then the utility function is specified as follows.

$$u_h : \mathbb{R}_+^C \times \mathbb{R}_+^{H-1} \longrightarrow \mathbb{R}, \quad \left(x_h, \theta_{\setminus h}\right) \mapsto u_h \left(x_h, \theta_{\setminus h}\right).$$

A sort of naive version of household $h \in \mathcal{H}$ maximization problem is defined as follows.

Definition 1 *For given $e_h \in \mathbb{R}_{++}^C$, $p^* \in S$, $t_{\setminus h}^* \in \mathbb{R}_+^{C(H-1)(H-1)}$,*

$$\max_{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}} \quad u_h \left(x_h, \left(p^* \left(e_{h'} + \textstyle\sum_{h'' \in \mathcal{H} \setminus \{h'\}} (t_{h''h'} - t_{h'h''}) \right) \right)_{h' \in \mathcal{H} \setminus \{h\}} \right)$$

s.t.

$$p x_h \leq p \left(e_h + \textstyle\sum_{h' \in \mathcal{H} \setminus \{h\}} (t_{h'h} - t_{hh'}) \right)$$

$$x_h \geq 0$$

$$t_h \geq 0$$

Indeed, the Definition above requires careful discussion. In each of the three subsections below, we present a problem related to it and a proposal on how to address that problem.

[3] Economically meaningful restrictions on the sets defined below will be presented in the remainder of the section.

[4] Conditions will be imposed below to get a well given definition of the function w_h.

1.2 A discussion of the set-up of the model

1.2.1 Compactness of the choice set at the economy level

In many proof of existence of equilibria, it is provisionally assumed that the consumption vectors of each household is bounded above by (a vector bigger than) total resources. Using a standard trick, it is then shown the upper bound is never reached in equilibrium. That procedure does not work in the case of transfers, for which no natural, physical bound exists. Following Kranich (1988), we assume that there exists an *artificial* bound on transfer. In Section 3 below, a discussion on that assumption is presented. Formally, we assume that for any $h \in \mathcal{H}$, there exists $k_h = (k_{hh'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)}_{++}$ such that $t_h \leq k_h$.

Remark 2 *The value of the bound k_h plays a role in the proof of existence of equilibria. If each household h has to satisfy the constraint*

$$k_h \leq e_h, \tag{1}$$

using the strategy proposed in Section 2, it is possible to prove that result under more general assumption than those presented inTheorem 34.[5] *On the other hand, the constraint $k_h \leq e_h$ is really strong and it is more natural to look for equilibria in which k_h is "as large as possible", say, pippo*

$$k_h \geq \sum_{h' \in \mathcal{H}} e_{h'}. \tag{2}$$

We do provide an existence result for arbitary positive values of k_h.

A natural conjecture is that any equilibrium with constraint (1) *is allocation equivalent to an equilibrium with constraint* (2)*. That conjecture turns out to be true in the one good, two household, Cobb-Douglas economy presented in Section 3.2. For the more general case presented in Section 2, the conjecture is easy to be proved if in equilibriuim the constraints hold with all strict inequalities, the proof being very similar to the one presented in Proposition 33. If that is not the case, the proof of the conjecture remains an open problem.*

1.2.2 The role of quasi-concavity

To discuss the role of quasi-concavity of the utility function, we proceed as follows.

1. We explain why quasi-concavity and envy are somehow inconsistent. 2. We propose a different set of assumptions on households' utility functions. 3. Given those assumptions, we show that there is no loss of generality in assuming away the part of the utility function related to households who are disliked. The basic idea of the presented argument is based on the following simple observation: "If household h is maximizing and she does not like household h', i.e., u_h is decreasing in $\theta_{h'}$, then she will transfer nothing to that household, i.e., $t_{hh'} = 0$".

Remark 3 *Kranich (1988) does assume quasi-concavity.*

1. Quasi-concavity, goods and bads Consider a simple "Econ 1" example, in which the utility function is as follows. For any $a \in \mathbb{R}_{++}$,

$$u : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}, \quad (x, y) \mapsto \ln(x) - a \ln(y).$$

Then u is a function of a good (whose quantity is x) and a bad (whose quantity is y). As a simple application of the Implicit Function Theorem, it is easy to verify that associated indifference curves are (increasing) and strictly concave if $a > 1$ and strictly convex if $a \in (0, 1)$, as verified below.

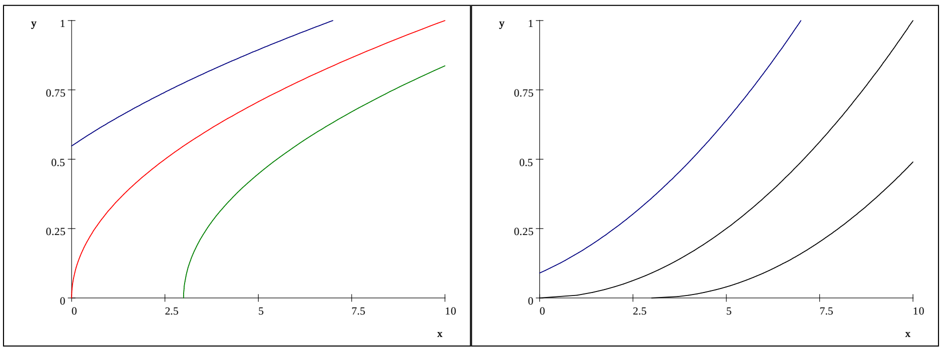
Defined as $g : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$, $x \mapsto g(x)$ the function whose graph is an indifference curve associated to an arbitrary level of the utility,we have

$$g'(x) = -\left.\frac{\frac{1}{x}}{-\frac{a}{y}}\right|_{y=g(x)} = \frac{g(x)}{ax},$$

[5] See Remark 36.

and

$$g''(x) = \frac{axg'(x) - ag(x)}{(ax)^2} = \frac{ax\frac{g(x)}{ax} - ag(x)}{(ax)^2} = \frac{g(x)}{(ax)^2}(1-a)$$



Then $g''(x) < 0$, i.e., the utility function is quasi-concave iff $a > 1$. Observe also that for any $(\overline{x}, \overline{y}) \in \mathbb{R}^2_{++}$,

$$\mathrm{Cl}_{\mathbb{R}^2}\left\{(x,y) \in \mathbb{R}^2_{++} : u(x,y) \geq u(\overline{x}, \overline{y})\right\}$$

is not contained in $\mathbb{R}^2_{++}$.

2. Different assumptions on the utility function We are now in the following situation.

a. the assumption of convex preferences, i.e., quasi-concavity of the utility function, is somehow unreasonable if households strongly dislike some other households;

b. most of the results used to prove existence of equilibria do assume quasi-concavity.

There are then two possibilities:

1. try to apply some known results in the case of arbitrary (not necessarily convex) preferences; this is the path we try to follow in a parallel research program;

2. show that, under some conditions, we can "restore quasi-concavity", which is the path we follow below.

For any $h \in \mathcal{H}$, let $\mathcal{B}_h \subseteq \mathcal{H} \backslash \{h\}$ be the set of households such that "household h conceives to transfer some wealth to", i.e., "the set of households that household h potentially likes". Define also $\mathcal{B} = (\mathcal{B}_h)_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} \mathcal{P}(\mathcal{H} \backslash \{h\})$, where $\mathcal{P}(\mathcal{H} \backslash \{h\})$ is the family of all subsets of $\mathcal{H} \backslash \{h\}$; $\mathcal{B}_h^{\backslash} = (\mathcal{H} \backslash \{h\}) \backslash \mathcal{B}_h$, $B_h = \#\mathcal{B}_h$, $B := \sum_{h \in \mathcal{H}} B_h$, $B_h^{\backslash} = \#\mathcal{B}_h^{\backslash}$.

We define household h utility function as follows.

$$u_h : \mathbb{R}^C_+ \times \mathbb{R}^{B_h}_+ \times \mathbb{R}^{B_h^{\backslash}}_+ \longrightarrow \mathbb{R}, \quad \left(x_h, (\theta_h)_{h \in \mathcal{B}_h}, (\theta_h)_{h \in \mathcal{B}_h^{\backslash}}\right) \mapsto u_{\mathcal{B}_h}\left(x_h, (\theta_h)_{h \in \mathcal{B}_h}\right) + v_h\left((\theta_h)_{h \in \mathcal{B}_h^{\backslash}}\right).$$

We make the following provisional assumption on u_h.

u1. $u_{\mathcal{B}_h}$ is continuous;

u2. $u_{\mathcal{B}_h}$ is strictly increasing in x_h; increasing in $\theta_{h'}$;

u3. $u_{\mathcal{B}_h}$ is quasi concave;

v4. v_h is continuous and decreasing.

For any $\mathcal{B}_h \subseteq \mathcal{H} \backslash \{h\}$, let $\mathcal{U}_{\mathcal{B}_h}$ and $\mathcal{V}_{\mathcal{B}_h}$ the sets of functions satisfying Assumptions u.1.,u.2., u.3. and v4, respectively.

3. Ignoring people we dislike

Definition 4 *For any $n \in \mathbb{N}$, and any $a, b \in \mathbb{R}^n$ such that $a \leq b$, we define $[a,b] = \{x \in \mathbb{R}^n : a \leq x \leq b\}$.*

An economy is

$$\mathcal{E}' := \left((\mathcal{B}_h, u_{\mathcal{B}_h}, v_{\mathcal{B}_h}, e_h, k'_h)_{h \in \mathcal{H}}\right) \in \times_{h \in \mathcal{H}} \left(\mathcal{P}(\mathcal{H} \backslash \{h\}) \times \times \mathcal{U}_{\mathcal{B}_h} \times \mathcal{V}_{\mathcal{B}_h} \times \mathbb{R}^C_{++} \times \mathbb{R}^{C(H-1)}_{++}\right) := \mathbb{E}'.$$

We can then write household h's maximization problem as follows. For any $h \in \mathcal{H}$, for given $\mathcal{E}' \in \mathbb{E}'$, $p \in S$ and $t_{\backslash h} \in [0, k'_h]$, $(x^*_h, t^*_h) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}$ solves problem

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{C(H-1)}}$$
$$u_{\mathcal{B}_h}\left(x_h,\left(p\left(e_{h'}+\sum_{h''\in\mathcal{H}\backslash\{h'\}}\left(t_{h''h'}-t_{\prime h''}\right)\right)\right)_{h'\in\mathcal{B}_h}\right)+$$
$$+v_h\left(\left(p\left(e_{h'}+\sum_{h''\in\mathcal{H}\backslash\{h'\}}\left(t_{h''h'}-t_{\prime h''}\right)\right)\right)_{h'\in\mathcal{B}_h^{\backslash}}\right)$$
$$\text{s.t.} \tag{3}$$
$$px_h\leq p\left(e_h+\sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t_{h'h}-t_{hh'}\right)\right)$$
$$x_h\geq 0$$
$$0\leq t_h\leq k_h'.$$

For any $h\in\mathcal{H}$, define $\mathcal{B}_{\longrightarrow h}=\{h'\in\mathcal{H}:h\in\mathcal{B}_{h'}\}$, i.e., the set of households who are potential donors to household h.

An economy is

$$\mathcal{E}:=\left(\mathcal{B}_h,u_{\mathcal{B}_h},e_h,k_h\right)_{h\in\mathcal{H}}\in\left(\times_{h\in\mathcal{H}}\left(\mathcal{P}\left(\mathcal{H}\backslash\{h\}\right)\times\mathcal{U}_{\mathcal{B}_h}\times\mathbb{R}^C_{++}\times\mathbb{R}^{CB_h}_{++}\right)\right):=\mathbb{E}.$$

Define for any $h\in\mathcal{H}$,

$$T_h=[0,k_h]\subseteq\mathbb{R}^{CB_h},\qquad T=\times_{h\in\mathcal{H}}T_h\ \ \text{and}\ T_{\backslash h}=\times_{h'\in\mathcal{H}\backslash\{h\}}T_{h'}.$$

We can then define the $\mathcal{B}$-problem for household $h\in\mathcal{H}$ as follows. For given $\mathcal{E}\in\mathbb{E}$, $p\in S$ and $t_{\backslash h}\in T_{\backslash h}$, $(x_h^*,t_h^*)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}$ solves

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}}\quad u_{\mathcal{B}_h}\left(x_h,\left(p^*\left(e_{h'}+\sum_{h''\in\mathcal{B}_{\longrightarrow h'}}t_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}t_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}\right)$$
$$\text{s.t.} \tag{4}$$
$$(x_h,t_h)\in B_h^*\left(p^*,t^*_{\backslash h}\right),$$

where

$$B_h^*:S\times T_{\backslash h}\longrightarrow\longrightarrow\mathbb{R}^C\times\mathbb{R}^{CB_h},$$
$$\left(p^*,t^*_{\backslash h}\right)\mapsto\mapsto\ \{\ (x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}:\ p^*x_h\leq p^*\left(e_h+\sum_{h'\in\mathcal{B}_{\longrightarrow h}}t^*_{h'h}-\sum_{h'\in\mathcal{B}_h}t_{hh'}\right)$$
$$x_h\geq 0$$
$$t_h\geq 0$$
$$t_h\leq k_h\ \}$$

We now present an informal statement of the desired result of the section; a precise statement and its proof are provided in Appendix 4.4.

Proposition 5 *There is no loss of generality in proving existence of equilibria for the case in which households solve problem* (4) *and markets clear.*

1.2.3 The need of an extension of the utility function

An economically sound assumption we made about the economy under analysis is the non-negativity of wealth of household $h'\in\mathcal{B}_h$ as an argument of the utility function of household h. Then, it is immediate to observe that household h 's maximization problem presented in the definition of equilibrium (without or with upper bound on consumption) may have no solution for arbitrary values of variables which are taken for given by household h. That is the case if the value of the transfers from household h' to other households are very large and household h budget constraint does not allow her to choose a transfer large enough in order to compensate that fact. Take for example $C = 3$ $H = 3$, $e_1 = e_2 = e_3 = (1, 1)$, $p \cdot (3, 3) = 1$, $k_h = (3, 3)$, $t_{21} = t_{31} = t_{32} = 0$ and $t_{23} = (2.5, 2.5)$. Consider household 1. Her budget constraint is

$$p (x_1 + t_{12} + t_{13}) \leq pe_1 \text{ and then } pt_{12} \leq p (1, 1) . \tag{5}$$

Non-negativity of household 2's wealth requires

$$0 \leq p (e_2 + t_{12} + t_{32} - t_{21} - t_{23}) = pt_{12} - p (1.5.1.5) \text{ and then } pt_{12} \geq p (1.5.1.5) \tag{6}$$

(5) and (6) cannot both hold. Kranich (1988) does not acknowledge the problem described above. Some ways out are possible.

1. Assume that "by law", households cannot transfer more than the value of their endowment. That approach has been followed in a previous version of the paper, but, first of all, it uses an assumption which is not consistent with basic competitive behavior, which requires that households can behave freely as long as they satisfy a budget constraint. Moreover, that approach does have the same problem related to the need of extending the utility function as described below.

2. Observe that to show existence of an equilibrium does not require to show existence of a solution to households' maximization problem for any value of the variables which are taken for given by that household - as often done in existing general equilibrium models. Indeed, it is enough to show existence of a solution to the *equilibrium values* of the variables taken for given by households. On the other hand, as shown above, assuming that other households can choose transfers in an exogenously given compact set leads to the possibility of negative values of some households' wealths, a fact which is inconsistent with the very definition of the utility function and therefore with a correct formulation of the maximization problem. Since dispensing the compactness assumption prevents the use of the main tool in showing existence - i.e., Debreu's Theorem 19 below - we decided to introduce a fictitious utility function which extends the true one, in order to allow negative wealth (and consumption). We then construct a game associated with "the original economy with the extended utility function" ; we show that game has an equilibrium and finally that equilibrium is an equilibrium of the "true" economy.

Remark 6 *If the upper bound k_h is equal to the endowment e_h, then the possibility of negative wealth does not arise and there is no need to extend the original utility function.*

1.2.4 Indeterminacy: price normalization matters

Following the paper by Kranich (1988), in the model we presented in the previous sections, we assumed that the utility function of household h depends upon other households' *nominal* wealth. Then, "normalizing prices" , or multiplying prices by a strictly positive real number, does affect the value of u_h (unless u_h is homogenous of degree zero in prices). In other words, different choices of normalizations of prices give rise to different equilibria and there is no natural choice of normalization.

To avoid the fact that equilibrium allocations are normalization dependent, we propose a simple change in the model: we substitute wealth of other households in the utility function with "relative wealth". There is indeed a vast literature in partial equilibrium, game theory and behavioral economic analysis which follows this approach - see Dhami (2006), Chapter 6 and references quoted there. Indeed, what is important in the analysis of other regarding preferences which are opportunity based is not (the value of) the amount of goods other households own, but those amounts compared with what is generally available in the economy.

We can then write household h's utility function as

$$u_{\mathcal{B}_h} \left(x_h, \left(\frac{p \left(e_{h'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'}} t_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''} \right)}{p \cdot r} \right)_{h' \in \mathcal{B}_h} \right).$$

Under the above specification of the utility function, price normalizations do not affect households maximizing choices.

Remark 7 *To show existence of equilibrium in the relative wealth model, we proceed as follows.*

a. We show existence of an "equilibrium in Kranich (1988) model" in the following sections;

b. we use that result to show existence an "equilibrium in the relative wealth model" in Section 2.4.

2 Existence of equilibria

2.1 The definition of equilibrium

Definition 8 *The vector* $(x^*, t^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{BC} \times S$ *is an* ***equilibrium*** *for the economy* $\mathcal{E} := (\mathcal{B}_h, u_{\mathcal{B}_h}, e_h, k_h)_{h\in\mathcal{H}} \in \left(\times_{h\in\mathcal{H}} \left(\mathcal{P}(\mathcal{H}\backslash\{h\}) \times \mathcal{U}_{\mathcal{B}_h} \times \mathbb{R}_{++}^C \times \mathbb{R}_{++}^{CB_h}\right)\right) := \mathbb{E}$. *if*

1. for any $h \in \mathcal{H}$, *household* h *maximizes, i.e., for given* $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $t^*_{\backslash h} \in T_{\backslash h}$, $(x_h^*, t_h^*) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ *solves*

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \quad u_{\mathcal{B}_h}\left(x_h, \left(p^*\left(e_{h'} + t_{hh'} + \textstyle\sum_{h''\in\mathcal{B}_{\longrightarrow h'}\backslash\{h\}} t^*_{h''h'} - \textstyle\sum_{h''\in\mathcal{B}_{h'}} t^*_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}\right)$$

s.t.

$$(x_h, t_h) \in B_h^*\left(p^*, t^*_{\backslash h}\right),$$

where B^* *is defined in* (35).

2.

Markets clear, i.e.,

$$\sum_{h\in\mathcal{H}} (x_h^* - e_h) = 0.$$

Remark 9 *The following Proposition simply says that "if households maximize, supply and demand of transfer are equal".*

Proposition 10 *If the vector* $(x^*, t^*, p^*) \in \mathbb{R}_+^{CH} \times \mathbb{R}_+^{BC} \times S$ *is such that for any* $h \in \mathcal{H}$, *household* h *solves the maximization problem in Definition 8, then*

$$\sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} t^*_{hh'} = \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\longrightarrow h}} t^*_{h'h}. \tag{7}$$

Proof. Recall that $\mathcal{B}_{\longrightarrow h} := \{h' \in \mathcal{H}\backslash\{h\} : h \in \mathcal{B}_{h'}\}$ and then also $\mathcal{B}_{\longrightarrow h'} := \{h \in \mathcal{H}\backslash\{h'\} : h' \in \mathcal{B}_h\}$. Then, by definition of $\mathcal{B}_{\longrightarrow h'}$, we have

$$h \in \mathcal{B}_{\longrightarrow h'} \Leftrightarrow h' \in \mathcal{B}_h. \tag{8}$$

Defined

$$\mathcal{S} = \left\{(h, h') \in \mathcal{H}^2 : h' \in \mathcal{B}_h\right\},$$

$$\mathcal{T} = \left\{(h', h) \in \mathcal{H}^2 : h' \in \mathcal{B}_{\longrightarrow h}\right\},$$

we want to show that $\mathcal{S} = \mathcal{T}$. Indeed,

$$\mathcal{S} := \left\{(h, h') \in \mathcal{H}^2 : h' \in \mathcal{B}_h\right\} \overset{(8)}{=} \left\{(h, h') \in \mathcal{H}^2 : h \in \mathcal{B}_{\longrightarrow h'}\right\} = \left\{(h', h) \in \mathcal{H}^2 : h' \in \mathcal{B}_{\longrightarrow h}\right\} := \mathcal{T},$$

as desired. ■

We can also give the definition of **equilibrium with upper bound on consumption**, which is the same as the one above apart from the fact that the constraint set is

$$\begin{aligned} &B_h : S \times T_{\backslash h} \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h}, \\ &\left(p^*, t^*_{\backslash h}\right) \mapsto\!\!\!\!\mapsto B_h^*\left(p^*, t^*_{\backslash h}\right) \cap \left\{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : x_h \le k_x\right\} \end{aligned} \tag{9}$$

Our (standard) strategy of proof to show existence of equilibria is to use the definition and a main result associated with a so-called generalized game. We present both definition and result in the section below.

More precisely, following also what said in Section 1.2, our strategy of proof goes through the following steps - each of which is the content of a subsection of Section 2.2.

1. We prove some preliminary results;
2. We present conditions which insure existence of the needed extension of the utility function.
3. We describe a generalized game associated with the economy under analysis and verify that game satisfies the sufficient conditions stated in Theorem 19 and therefore has an equilibrium.
4. We show equilibria of the generalized game are such that the associated wealth is positive and therefore they are equilibria with upper bound on consumption of the economic model under analysis.
5. Using a standard trick, we show that an equilibrium with upper bound on consumption is an equilibrium.

2.2 Existence of equilibria for concave, Lipschitz utility functions

2.2.1 Preliminary results

For reasons explained in the proofs of Lemma 12, Proposition 27 and Lemma 24, we are going to introduce the upper bounds k_x on consumption.

Definition 11

	where it is used
$\mathbf{1} = (1, ..., 1),$	
$r^m = \min\{r^c : c \in \mathcal{C}\}, \qquad r^M = \max\{r^c : c \in \mathcal{C}\}$	
$\widetilde{k}_x = \frac{1}{r^m} \cdot \sum_{\mathbf{c}\in\mathcal{C}} \sum_{h\in\mathcal{H}} \left(e_h^c + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} k_{h'h}^c\right) + 1 \in \mathbb{R}_{++}.$	*Remark* 13
$\widetilde{k}_x^c = \max\left\{\widetilde{k}_x,\ r^c + 1, \widetilde{k}_x \cdot r^c\right\} \in \mathbb{R}_{++},$	*Proposition 33 and 29; Proposition* 27
$k_x := \left(\widetilde{k}_x^c\right)_{c\in\mathcal{C}} \in \mathbb{R}_{++}^C.$	*Proposition* 27

Lemma 12 *For any $p \in S$,*

1. *for any $c \in \mathcal{C}$, $p^c \leq \frac{1}{r^m}$,*
2. *there exists $c \in \mathcal{C}$ such that $p^c \geq \frac{1}{C \cdot r^M}$, and*
3. *for any $e_h \in \mathbb{R}_{++}^C$, $pe_h \geq \frac{e_h^m}{C \cdot r^M}$, where $e_h^m := \min\{e_h^c : c \in \mathcal{C}\}$.*

Proof. 1.

Since $pr = 1$, then for any $c \in \mathcal{C}$, $p^c r^c + \sum_{c'\in\mathcal{C}\setminus\{c\}} p^c r^c \leq 1$, $p^c r^c \leq 1 - \sum_{c'\in\mathcal{C}\setminus\{c\}} p^c r^c \leq 1$ and $p^c \leq \frac{1}{r^c} \leq \frac{1}{r^m}$.

2.

Suppose otherwise, i.e., for any $c \in \mathcal{C}$, $p^c < \frac{1}{C \cdot r^M}$. Then,

$$1 = \sum_{c\in\mathcal{C}} p^c r^c < \sum_{c\in\mathcal{C}} \frac{1}{C} \frac{r^c}{r^M} \leq \sum_{c\in\mathcal{C}} \frac{1}{C} = 1,$$

which is the desired contradiction.

3.

From 2. above, we have that for any $p \in S$ and any $e_h \in \mathbb{R}_{++}^C$

$$\text{there exists } c \in \mathcal{C} \text{ such that } pe_h = p^c e_h^c + \overbrace{\sum_{c'\neq c} p^{c'} e_h^{c'}}^{(\geq 0)} \geq p^c e_h^c \geq \frac{e_h^c}{C \cdot r^M} \geq \frac{e_h^m}{C \cdot r^M}.$$

■

Remark 13 *For any* $p \in S$, *any* $h \in \mathcal{H}$, *any* $h' \in \mathcal{B}_h$ *any* $t = \left((t_{hh'})_{h' \in \mathcal{B}_h} \right)_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} \left(\times_{h' \in \mathcal{B}_h} [0, k_{hh'}] \right) \subseteq \mathbb{R}^{C \cdot \Sigma_{h \in \mathcal{H}} B_h}$, $h \in \mathcal{H}$, $w_h(p,t) < \widetilde{k}_x$. *Indeed, using the facts* $p \leq \frac{1}{r^{m.}} \cdot \mathbf{1}$ *and for any* h, h' *with* $h' \neq h$, $t_{h'h} \leq k_{hh'}$ *and* $t_{hh'} \geq 0$, *we have*

$$w_h(p,t) := p \left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h} - \sum_{h' \in \mathcal{B}_h} t_{hh'} \right) \leq p \left(e_h + \sum_{h' \in \mathcal{B}_{\rightarrow h}} k_{h'h} \right) \overset{Lemma\ 12.1}{\leq}$$

$$\leq \frac{1}{r^{m.}} \cdot \mathbf{1} \left(e_h + \sum_{h' \in \mathcal{B}_{\rightarrow h}} k_{h'h} \right) = \frac{1}{r^{m.}} \cdot \sum_{\mathbf{c} \in \mathcal{C}} \left(e_h^c + \sum_{h' \in \mathcal{B}_{\rightarrow h}} k_{h'h}^c \right) <$$

$$< \frac{1}{r^{m.}} \cdot \sum_{h \in \mathcal{H}} \sum_{\mathbf{c} \in \mathcal{C}} \left(e_h^c + \sum_{h' \in \mathcal{B}_{\rightarrow h}} k_{h'h}^c \right) < \widetilde{k}_x.$$

2.2.2 Extension of the utility function

It is usually assumed that for any $h \in \mathcal{H}$,

$$u_{\mathcal{B}_h} : \mathbb{R}_+^C \times \mathbb{R}_+^{B_h} \longrightarrow \mathbb{R}, \quad (x_h, \theta_{\mathcal{B}_h}) \mapsto u_h(x_h, \theta_{\mathcal{B}_h})$$

is continuous and quasi-concave. We would like to extend $u_{\mathcal{B}_h}$ to a function $U_h : \mathbb{R}^C \times \mathbb{R}^{H-1} \longrightarrow \mathbb{R}$, which is still continuous and quasi-concave. From what said in Appendix 4.9.1 with respect to Conjecture 120, that extension is not guaranteed to exist. Indeed, stronger assumptions on $u_{\mathcal{B}_h}$ have to be added. We repeat below the main result presented in the appendix to reach the desired goal.

Proposition 14 *Let* A *be a convex subset of a normed space* X. *If* $g : A \to \mathbb{R}$ *is an L-Lipschitz concave function then it admits an L-Lipschitz concave extension* G *to the whole* X*; moreover, such an extension* G *can be defined by the formula*

$$G(x) = \sup_{y \in A} \left[g(y) - L\|x - y\| \right], \qquad x \in X.$$

We can apply the above result to our case identifying X, A, g with $\mathbb{R}^C$,$\mathbb{R}_+^C$ and u respectively. Observe that L-Lipschitz implies continuity.

Therefore, our existence result is going to be proved under the following assumption on the utility function.

For any $h \in \mathcal{H}$,

$$u_{\mathcal{B}_h} : \mathbb{R}_+^C \times \mathbb{R}_+^{B_h} \longrightarrow \mathbb{R}, \quad (x_h, \theta_{\mathcal{B}_h}) \mapsto u_{\mathcal{B}_h}(x_h, \theta_{\mathcal{B}_h})$$

is concave and Lipschitz.

Remark 15 *The above assumption are quite strong. Definition and sufficient conditions for a function to be Lipschitz are presented in Corollary 15. Roughly speaking, we need to assume that the utility function has slopes in points close to zero which are bounded above.*

Using Proposition 14 and the above Assumption, we can give the following definition.

Definition 16 U_h *is a concave Lipschitz extension of* $u_{\mathcal{B}_h}$ *on the whole Euclidean space, i.e.,*

$$U_h : \mathbb{R}^C \times \mathbb{R}^{B_h} \longrightarrow \mathbb{R}$$

is a concave Lipschitz function and for any $\left(x_h, \theta_{\setminus h} \right) \in \mathbb{R}_+^C \times \mathbb{R}_+^{B_h}$ *we have* $U_h \left(x_h, \theta_{\setminus h} \right) = u_{\mathcal{B}_h} \left(x_h, \theta_{\setminus h} \right)$.

2.2.3 The generalized game associated with the economy

For the definition below see for example Kreps (2013), page 339, and the simple discussion following the definition proposed there.

Definition 17 *Given* $n \in \mathbb{N}$, *an n-player generalized game is a triple* $G = \{A_i, C_i, u_i\}_{i=1}^n$, *where for any* $i \in \{1, ..., n\}$,

1. A_i *is a set of strategies or actions with generic element* a_i*;*

2. $C_i : A_{\setminus i} := \times_{j \in \{1,...,n\} \setminus \{i\}} A_j \longrightarrow\!\!\!\!\longrightarrow A_i, \quad a := (a_i)_{i=1}^n \mapsto\!\!\!\!\mapsto C_i(a)$ *is a constraint set-valued function;*

3. $u_i : A := \times_{i \in \{1,..,n\}} A_i \longrightarrow \mathbb{R}$, $a \mapsto u_i(a)$ *is a utility or payoff function.*

Definition 18 *A Nash equilibrium for the generalized game* $G = \{A_i, C_i, u_i\}_{i=1}^n$ *is an n-tuple of actions* $a^* := (a_i^*)_{i=1}^n \in A$ *such that for any* $i \in \{1, ..., n\}$, a_i^* *solves the following problem. For given* $a_{\setminus i}^* := \left(a_j^*\right)_{j \in \{1,...,n\} \setminus \{i\}} \in A_{\setminus i}$,

$$\max_{a_i \in A_i} \; u_i\left(a_i, a_{\setminus i}^*\right) \qquad s.t. \qquad a_i \in C_i\left(a_{\setminus i}^*\right).$$

Theorem 19 *Let a generalized game* $G = \{A_i, C_i, u_i\}_{i=1}^n$ *be given. If for any* $i \in \{1, ..., n\}$,

1. there exists $m_i \in \mathbb{N}$ *such that* A_i *is a nonempty, compact, convex subset of* $\mathbb{R}^{m_i}$*;*

2. C_i *is a non-empty valued, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function;*

3. u_i *is a continuous function and for any* $a_{\setminus i} \in A_{\setminus i}$, *the function* $u_i\left(\cdot, a_{\setminus i}\right) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto u_i\left(a_i, a_{\setminus i}\right)$ *is quasi-concave,*

then G *has a Nash equilibrium.*

The standard reference for the above theorem is Debreu (1952). Indeed, exactly the same statement and a proof of the above theorem can be found in Kreps (2013), page 340 .

We now define the generalized game associated with an economy $\mathcal{E}$ we are going to use.

Definition 20 *There are* $n = 1 + H$ *players. For each player* $h \in \{0, 1..., H\}$, *we describe below the appropriate definition of the triple of 1. set of actions, 2. constraint set-valued functions and 3. utility functions.*

1.

$$\begin{aligned} A_0 &= S \subseteq \mathbb{R}^C \\ A_h &= X_h \times T_h \subseteq \mathbb{R}^{C+B_h \cdot C} \qquad \text{for any } h \in \mathcal{H} \end{aligned}$$

where for any $h \in \mathcal{H}$,

$$X_h = \left\{x_h \in \mathbb{R}^C : 0 \le x \le k_x\right\} \text{ and } T_h = \left\{t_h \in \mathbb{R}^{B_h \cdot C} : 0 \le t_h \le k_h\right\}.$$

2.

$$C_0 : \times_{h \in \mathcal{H}} A_h \longrightarrow\!\!\!\!\longrightarrow A_0$$
$$C_0 : \left(\times_{h \in \mathcal{H}} \left(X_h \times T_h\right)\right) \longrightarrow\!\!\!\!\longrightarrow S \qquad (x, t) \mapsto\!\!\!\!\mapsto S$$

$$C_h : A_0 \times \left(\times_{h' \in \mathcal{H} \setminus \{h\}} A_{h'}\right) \longrightarrow\!\!\!\!\longrightarrow A_h$$
$$\widehat{B}_h : S \times \left(\times_{h' \in \mathcal{H} \setminus \{h\}} \left(X_{h'} \times T_{h'}\right)\right) \longrightarrow\!\!\!\!\longrightarrow X_h \times T_h \qquad \left(p, \left(x_{h'}, t_{h'}\right)_{h' \in \mathcal{H} \setminus \{h\}}\right) \mapsto B_h\left(p, t_{\setminus h}\right)$$

3.

$$u_0 : A_0 \times \left(\times_{h \in \mathcal{H}} A_h\right) \longrightarrow \mathbb{R}$$
$$u_0 : S \times \left(\times_{h \in \mathcal{H}} \left(X_h \times T_h\right)\right) \longrightarrow \mathbb{R}$$
$$\left(p, \left(x_h, t_h\right)_{h \in \mathcal{H}}\right) \longrightarrow p \cdot \sum_{h \in \mathcal{H}} \left(x_h - e_h\right)$$

$$u_h : A_0 \times \left(\times_{h \in \mathcal{H}} A_h\right) \longrightarrow \mathbb{R}$$
$$u_h : S \times \left(\times_{h \in \mathcal{H}} \left(X_h \times T_h\right)\right) \longrightarrow \mathbb{R}$$
$$\left(p, \left(x_h, t_h\right)_{h \in \mathcal{H}}\right) \mapsto U_h\left(x_h, \left(p\left(e_{h'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'}} t_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''}\right)\right)_{h' \in \mathcal{B}_h}\right)$$

Definition 21 *A Nash equilibrium for the Generalized Game associated with an economy* $\mathcal{E} := \left(\mathcal{B}, u_{\mathcal{B}_h}, e_h, k_h\right)_{h \in \mathcal{H}} \in \mathbb{E}$, *as presented in Definition 20, is a vector* $\left(p^*, \left(x_h^*, t_h^*\right)_{h \in \mathcal{H}}\right) \in S \times \left(\times_{h \in \mathcal{H}} \left(X_h \times T_h\right)\right)$ *such that*

for given $\left(x_h^*, t_h^*\right)_{h \in \mathcal{H}}$,

p^* *solves*

$$\max_{p \in S} \quad p \cdot \sum_{h \in \mathcal{H}} \left(x_h^* - e_h\right),$$

and for any $h \in \mathcal{H}$,

for given p^*, $(x^*_{h'}, t^*_{h'})_{h \in \mathcal{H}}$,

(x^*_h, t^*_h) *solves*

$$\max_{(x_h,t_h)\in(X_h\times T_h)} \quad U_h\left(x_h, \left(p^*\left(e_{h'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}} t^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}\right)$$
s.t.
$$(x_h, t_h) \in \widehat{B}_h\left(p^*, (x^*_{h'}, t^*_{h'})_{h'\in\mathcal{H}\setminus\{h\}}\right) = B_h\left(p^*, t^*_h\right),$$

Proposition 22 *For any economy $\mathcal{E} := (\mathcal{B}, u_{\mathcal{B}_h}, e_h, k_h)_{h\in\mathcal{H}} \in \mathbb{E}$, the generalized game*

$$\left(\left(S, (X_h \times T_h)\right), \left(C_0, (B_h)_{h\in\mathcal{H}}\right), \left(u_0, (U_h)_{h\in\mathcal{H}}\right)\right)$$

presented above has a Nash equilibrium (p^*, x^*, t^*).

Proof. We show that the Assumptions of Theorem 19 are verified.

1. there exists $n_i \in \mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$.

$A_0 = S := \left\{p \in \mathbb{R}^C_+ : pr = 1\right\}$ satisfies the needed assumptions.

For any $h \in \mathcal{H}$, $X_h \times T_h$ satisfies the needed assumptions by definition.

3. u_i is continuous and for any $a_{\setminus i} \in A_{\setminus i}$, the function $u_i\left(\cdot, a_{\setminus i}\right) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto u_i\left(a_i, a_{\setminus i}\right)$ is quasi-concave.

For given $(x_h)_{h\in\mathcal{H}}$, u_0 is linear in p and therefore concave and quasi-concave. For $h \in \mathcal{H}$, observe what follows. Defined

$$\varphi_h : S \times \left(\times_{h\in\mathcal{H}}(X_h \times T_h)\right) \longrightarrow \mathbb{R}^C_+ \times \mathbb{R}^{B_h},$$

$$\left(p, (x_h, t_h)_{h\in\mathcal{H}}\right) \mapsto \left(x_h, \left(p\left[e_{h'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\setminus\{h\}} t_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}\right] + pt_{hh'}\right)_{h'\in\mathcal{B}_h}\right) := \tag{10}$$

$$:= \left(x_h, \left(a_{h'} + pt_{hh'}\right)_{h'\in\mathcal{B}_h}\right),$$

we have

$$.u_{\mathcal{B}_h} = U_h \circ \varphi_h.$$

Then, $.u_{\mathcal{B}_h}$ is continuous because U_h is Lipschitz continuous and φ is affine. We are going to show that

$$v_h := .u_{\mathcal{B}_h}\left(.; \left(p, (x_{h'}, t_{h'})_{h'\in\mathcal{H}\setminus\{h\}}\right)\right) : X_h \times T_h \longrightarrow \mathbb{R},$$

$$(x_h, t_h) \mapsto (U_h \circ \varphi_h)\left(p, (x_h, t_h)_{h\in\mathcal{H}}\right)$$

is concave and therefore quasi-concave. Indeed, for any $(x'_h, t'_h), (x''_h, t''_h)$ and any $\lambda \in [0,1]$, we have

$$v_h\left((1-\lambda)(x'_h, t'_h) + \lambda(x''_h, t''_h)\right) \overset{\text{Def. } v_h}{=}$$

$$U_h\left((1-\lambda)x'_h + \lambda x''_h, \quad \left(a_{h'} + p\left((1-\lambda)t'_{hh'} + \lambda t''_{hh'}\right)\right)_{h'\in\mathcal{B}_h}\right) =$$

$$U_h\left((1-\lambda)\left(x'_h, \quad (a_{h'} + pt'_{hh'})_{h'\in\mathcal{B}_h}\right) + \lambda\left(x''_h, \quad (a_{h'} + pt''_{hh'})_{h'\in\mathcal{B}_h}\right)\right) \overset{U_h \text{ concave}}{\geq}$$

$$(1-\lambda)U_h\left(\left(x'_h, \quad (a_{h'} + pt'_{hh'})_{h'\in\mathcal{B}_h}\right)\right) + \lambda U_h\left(x''_h, \quad (a_{h'} + pt''_{hh'})_{h'\in\mathcal{B}_h}\right) \overset{\text{Def. } v_h}{=}$$

$$(1-\lambda)v_h(x'_h, t'_h) + \lambda v_h(x''_h, t''_h),$$

as desired.

2. C_i is a non-empty value, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function.

By definition of S and since $C_0 : (\times_{h\in\mathcal{H}} (X_h \times T_h)) \longrightarrow S, (x,t) \mapsto\mapsto S$, the desired results follow. Indeed, C_0 is the constant set valued function and S is a compact nonempty set.

Verification of the needed properties for $\widehat{B}_h$ goes through two steps: 1. If B_h has the desired properties, then $\widehat{B}_h$ has the desired properties - see Lemma 23; 2. B_h has the desired properties - see Lemma 24. ■

Lemma 23 *Let X_1, X_2 and Y be metric spaces and $\varphi : X_1 \to\to Y$, $x_1 \longmapsto\longmapsto \varphi(x_1)$ and $\psi : X_1 \times X_2 \to\to Y$, $(x_1,x_2) \longmapsto\longmapsto \varphi(x_1)$ be set valued functions.*

Then if φ satisfies any of the properties listed below, then ψ does as well: 1. non-empty valued; 2. convex valued; 3. closed; 4. compact valued; 5. lower hemi-continuous; 6. upper hemi-continuous.

Proof. Statements about properties 1. 2. and 4. are obvious. The proof of the other results are of the type: "write what you assume and what you have to prove".

3.

By assumption, for every sequence $(x_{1n})_{n\in\mathbb{N}} \in (X_1)^\infty$ such that $x_{1n} \to x_1$, and for every sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y_n \in \varphi(x_{1n})$ and $y_n \to y$, it is the case that $y \in \varphi(x_1)$.

We want to show that for every sequence $(x_{1n}, x_{2n})_{n\in\mathbb{N}} \in (X_1 \times X_2)^\infty$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$, and for every sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y_n \in \psi(x_{1n}, x_{2n})$ and $y_n \to y$, it is the case that $y \in \psi(x_1, x_2)$.

Take a sequence $(x_{1n}, x_{2n})_{n\in\mathbb{N}} \in (X_1 \times X_2)^\infty$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$, and a sequence $(y_n)_{n\in\mathbb{N}} \in Y^\infty$ such that $y_n \in \psi(x_{1n}, x_{2n}) = \varphi(x_{1n})$ and $y_n \to y$. Then, by assumption, $y \in \varphi(x_1) = \psi(x_1, x_2)$, as desired.

5.

By assumption, for any $x_1 \in X_1$ and for any open set V in Y such that $\varphi(x_1) \cap V \neq \varnothing$, there exists an open neighborhood U of x_1 such that for every $x_1' \in U$, $\varphi(x_1') \cap V \neq \varnothing$.

We want to show that for any $(x_1, x_2) \in X_1 \times X_2$ and for any open set V in Y such that $\psi(x_1, x_2) \cap V \neq \varnothing$, there exists an open neighborhood W of (x_1, x_2) such that for every for every $(x_1', x_2') \in W$, $\psi(x_1', x_2') \cap V \neq \varnothing$.

Take $(x_1, x_2) \in X_1 \times X_2$ and an open set V in Y such that $\psi(x_1, x_2) \cap V = \varphi(x_1) \cap V \neq \varnothing$. Then, by assumption, there exists an open neighborhood U of x_1 such that for every $x_1' \in U$, $\varphi(x_1') \cap V \neq \varnothing$. Take $W = U \times X_2$. Then, for any $(x_1', x_2') \in W$, $\psi(x_1', x_2') \cap V = \varphi(x_1') \cap V \neq \varnothing$, as desired.

6.

By assumption, for any $x_1 \in X_1$ and for every open neighborhood V of $\varphi(x_1)$, there exists an open neighborhood U of x_1 such that for every $x_1' \in U$, $\varphi(x_1') \subseteq V$.

We want to show that for any $(x_1, x_2) \in X_1 \times X_2$ and for every open neighborhood V of $\psi(x_1, x_2)$, there exists an open neighborhood W of (x_1, x_2) such that for every $(x_1', x_2') \in W$, $\psi(x_1', x_2') \subseteq V$.

Take $(x_1, x_2) \in X_1 \times X_2$ and an open neighborhood V of $\psi(x_1, x_2) := \varphi(x_1)$. Then, by assumption,there exists an open neighborhood U of x_1 such that for every $x_1' \in U$, $\varphi(x_1') \subseteq V$. Take $W = U \times X_2$. Then, for every $(x_1', x_2') \in W$, $\psi(x_1', x_2') := \varphi(x_1') \subseteq V$, as desired. ■

Lemma 24 *For any $h \in \mathcal{H}$*

1. *B_h is non-empty valued;*
2. *B_h is convex valued;*
3. *B_h is closed;*
4. *B_h is compact valued and* Im $B_h \subseteq X_h \times T_h$;
5. *B_h is lower hemi-continuous;*
6. *B_h is upper hemi-continuous.*

Proof. Define

$$\widetilde{B}_h : S \times T_{\setminus h} \longrightarrow\longrightarrow \mathbb{R}^C \times \mathbb{R}^{C\cdot B_h}$$

$$\widetilde{B}_h(p, t_{\setminus h}) = \{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{C\cdot B_h} : \begin{array}{l} -p\left(x_h + \sum_{h'\in\mathcal{B}_h} t_{hh'}\right) + p\left(e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} t_{h'h}\right) > 0, \\ \\ x_h >> 0 \\ k_x - x_h >> 0 \\ t_h >> 0 \\ k_h - t_h >> 0 \end{array} \} \tag{11}$$

and

$$f : S \times S \times T_{\backslash h} \longrightarrow \mathbb{R}^C \times \mathbb{R}^{C \cdot B_h}$$

$$f : \left((x_h, t_h), (p, t_{\backslash h})\right) \longmapsto \text{ Left Hand Side of inequalities in (11).}$$

To get the desired result, we have to check that the Assumptions of Proposition 107 about the function f used to define $\widetilde{B}_h$ and B_h are indeed satisfied. More precisely, we have to check that

1. $\widetilde{B}_h$ is nonempty valued,

2. f is continuous and for any $j = 1, ...m$ and for any $\pi \in \Pi$, $f_{j|\{\pi\}}$ is Locally NonSatiated and quasi-concave,

3. B_h is compact valued or b. $\operatorname{Im} B_h$ is contained in a compact set. Indeed, both results hold true.

1.

Take

$$\widetilde{x}_h = \frac{e_h}{2H} \gg 0 \text{ and } \qquad \text{for any } c \in \mathcal{C}, \text{ for any } h' \in \mathcal{B}_h, \qquad \widetilde{t}^c_{hh'} = \min\left\{\frac{k^c_{hh'}}{2}, \frac{e^c_h}{2H}\right\} \in (0, k^c_{hh'})\,.$$

Then,

$$p\left(\widetilde{x}_h + \textstyle\sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'}\right) \le p\frac{e_h}{2H} + p\sum_{h' \in \mathcal{B}_h} \frac{e_h}{2H} \overset{B_h \le H-1}{\le}$$

$$\le \tfrac{1+H-1}{2H} pe_h = \tfrac{1}{2} pe_h < pe_h \le p\left(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}\right).$$

2.

f is clearly continuous and any component function of f for fixed $\left(p, t_{\backslash h}\right)$ is affine and not constant a fact which implies the desired assumptions. Indeed, for example,

$$g : \mathbb{R}^C \times \mathbb{R}^{CB_h} \longrightarrow \mathbb{R}, \qquad (x_h, t_h) \mapsto -p\left(x_h + \textstyle\sum_{h' \in \mathcal{B}_h} t_{hh'}\right) + p\left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}\right)$$

can be written as

$$g\left(x_h, t_h\right) = \left(-\left(p, \overset{1}{p}, ..., \overset{B_h}{p}\right)\left(x_h, \left(t_{hh'}\right)_{h' \in \mathcal{B}_h}\right) + p\left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}\right)\right).$$

3.

Since B_h is defined in terms of weak inequalities via continuous function, it is closed valued. Moreover, $\operatorname{Im}(B_h) \subseteq X_h \times T_h$where

$$X_h := \left\{x_h \in \mathbb{R}^C : 0 \le x_h \le k_x\right\}$$

$$T_h := \left\{t_h \in \mathbb{R}^{C \cdot B_h} : 0 \le t_h \le k_h\right\}$$

and $X_h \times T_h$ is a compact set. Since closed subsets of compact sets are compact, the desired result follows. ■

2.2.4 Equilibria of the game and equilibria with upper bound on consumption

Using Definition 10, we introduce a related function.

Definition 25 *For any* $\left(\left(x^*_{h'}\right)_{h' \in \mathcal{H} \backslash \{h\}}, p^*, t^*_{\backslash h}\right) \in \mathbb{R}^{C(H-1)}_+ \times S \times T_{\backslash h}$,

$$\varphi_{h|\left(p^*, t^*_{\backslash h}\right)} := \varphi_h\left(.; \left(x^*_{h'}\right)_{h' \in \mathcal{H} \backslash \{h\}}, p^*, t^*_{\backslash h}\right) : X_h \times T_h \longrightarrow \mathbb{R}^C_+ \times \mathbb{R}^{B_h},$$

$$(x_h, t_h) \mapsto \varphi_h\left(\left(x_h, t_h\right), \left(x^*_{h'}\right)_{h' \in \mathcal{H} \backslash \{h\}}, p^*, t^*_{\backslash h}\right)$$

Lemma 26 *If (x^*, t^*, p^*) is a Nash equilibrium as presented in Definition 21, then for any $h \in \mathcal{H}$,*

1. for any $h' \in \mathcal{B}_h$,

$$w_{h'}\left(p^*, t_h^*, t_{\setminus h}^*\right) := p^*\left(e_{h'} + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}^* - \sum_{h' \in \mathcal{B}_h} t_{hh'}^*\right) \geq 0.$$

2.

$$\left(U_h \circ \varphi_{h|\left(p^*, t_{\setminus h}^*\right)}\right)_{|B_h\left(p^*, t_{\setminus h}^*\right)} = \left(u_{\mathcal{B}_h} \circ \varphi_{h|\left(p^*, t_{\setminus h}^*\right)}\right)_{|B_h\left(p^*, t_{\setminus h}^*\right)}.$$

Proof. 1.

Observe that for any $h \in \mathcal{H}$, by definition of B_h, we have $x_h^* \geq 0$; by definition of S, we have $p^* \geq 0$. Then,

$$0 \leq p^* x_h^* \leq p^*\left(e_{h'} - \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''}^* + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'}} t_{h''h'}^*\right),$$

where the second inequality follows again from the definition of B_h.

2.

Observe that

$$U_{h|\mathbb{R}_+^C \times \mathbb{R}_+^{B_h}} = u_{\mathcal{B}_h},$$

and from 1. above, we have

$$\varphi_{h|\left(p^*, t_{\setminus h}^*\right)}\left(B_h\left(p^*, t_{\setminus h}^*\right)\right) \subseteq \mathbb{R}_+^C \times \mathbb{R}_+^{CB_h},$$

and then

$$\left(U_h \circ \varphi_{h|\left(p^*, t_{\setminus h}^*\right)}\right)_{|B_h\left(p^*, t_{\setminus h}^*\right)} = \left(u_{\mathcal{B}_h} \circ \varphi_{h|\left(p^*, t_{\setminus h}^*\right)}\right)_{|B_h\left(p^*, t_{\setminus h}^*\right)}.$$

■

Proposition 27 *If (x^*, p^*, t^*) is a Nash equilibrium for the generalized game presented above, then*

$$p^*\left(\sum_{h \in \mathcal{H}} (x_h^* - e_h)\right) = 0.$$

Proof. Using the strict monotonicity of $.u_{\mathcal{B}_h}$ with respect to x_h and the fact that $p^* \in S$, it is easy to claim that budget constraint holds as equalities. Then summing up with respect to households and using Proposition 10, we get the desired result. Suppose our claim does not hold, i.e., that budget constraint holds with a strict inquality:

$$p^* x_h^* < p^*\left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}^* - \sum_{h' \in \mathcal{B}_h} t_{hh'}^*\right) := w_h\left(p^*, t^*\right). \tag{12}$$

Since $p^* \in S$, then we can define $\mathcal{C}^+ = \{c \in \mathcal{C} : p^c > 0\} \neq \varnothing$. We then distinguish the following two cases.

Case a. There exists $\widetilde{c} \in \mathcal{C}^+$ such that $x_h^{*\widetilde{c}} < \widetilde{k}_x^{\widetilde{c}}$;

Case b. For any $c \in \mathcal{C}^+$, $x_h^{*c} = \widetilde{k}_x^c$.

Case a.

Define $x_h^{**} = \left(x_h^{**c}\right)_{c \in \mathcal{C}}$ such that

$$x_h^{**c} = \begin{cases} x_h^{*c} & \text{if} \quad c \in \mathcal{C}^+ \setminus \{\widetilde{c}\} \\ x_h^{*\widetilde{c}} + \frac{1}{2}\min\left\{ \frac{w_h(p^*, t^*) - p^* x_h^*}{p^{*\widetilde{c}}}, \quad \widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}} \right\} > x_h^{*\widetilde{c}} & \text{if} \quad c = \widetilde{c} \\ x_h^{*c} & \text{if} \quad c \in \mathcal{C}^0, \end{cases}$$

where the strictly inequality follows from the fact that $w_h(p^*, t^*) - p^* x_h^* \overset{(12)}{>} 0$ and $\widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}} > 0$. Below we show that $(x_h^{**}, t_h^*) \in B_h\left(p^*, t_{\backslash h}^*\right)$, a fact that contradicts the assumption that (x_h^*, t_h^*) is a solution to household h maximization problem.

i. $0 \leq x_h^{**} \leq \widetilde{k}_x$:

it is enough to verify that $x_h^{**\widetilde{c}} \leq \widetilde{k}_x^{\widetilde{c}}$.

$$x_h^{**\widetilde{c}} \leq x_h^{*\widetilde{c}} + \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}}\right) = \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} + x_h^{*\widetilde{c}}\right) < \frac{1}{2}\left(\widetilde{k}_x^{\widetilde{c}} + \widetilde{k}_x^{\widetilde{c}}\right) = \widetilde{k}_x^{\widetilde{c}}.$$

ii. affordability:

Recalling the definition of $w_h(p,t)$ presented in Remark 13, we get

$$p^* x_h^{**} = p^* x_h^* + p^{*\widetilde{c}} \tfrac{1}{2} \min\left\{\frac{w_h(p^*, t^*) - p^* x_h^*}{p^{*\widetilde{c}}}, \quad \widetilde{k}_x^{\widetilde{c}} - x_h^{*\widetilde{c}}\right\} \leq p^* x_h^* + \tfrac{1}{2}\left(w_h(p^*, t^*) - p^* x_h^*,\right) =$$

$$= \tfrac{1}{2}\left(w_h(p^*, t^*) + p^* x_h^*\right) \overset{(12)}{<} \tfrac{1}{2}\left(w_h(p^*, t^*) + w_h(p^*, t^*)\right) = w_h(p^*, t^*).$$

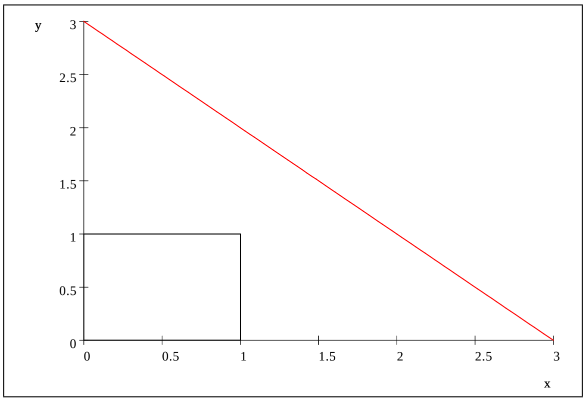
Case b.

This case cannot hold. Assume it does. Then,

$$\widetilde{k}_x \overset{\text{Remark 13}}{>} w_h(p^*, t^*) \overset{(12)}{\geq} p^* x_h^* = \sum_{c \in \mathcal{C}} p^{*c} \widetilde{k}_x^c \overset{\text{Def. 11}}{\geq} \sum_{c \in \mathcal{C}} p^{*c} \cdot \widetilde{k}_x \cdot r^c = \widetilde{k}_x \cdot (p^* r) \overset{p^* r = 1}{=} \widetilde{k}_x,$$

which is the desired contradiction. ■

Remark 28 *If the upper bound on consumption is not big enough, then Walras' law does not hold, because the consumption vector hits the corner of the box* $[0, \textit{upper bound vector}]$.



Kranich (1988) uses Walras law on page 377, last paragraph in the proof of Proposition 3.4., but he does not seem to consider that possibility.

Proposition 29 *If* (x^*, p^*, t^*) *is a Nash equilibrium for the generalized game presented above, then it is an equilibrium with upper bound on consumption and* $p^* \gg 0$.

Proof. By definition of Nash equilibrium, each player is maximizing. Therefore, for player $h = 0$, we have that

$$\text{for any } p \in S, \quad p^* \cdot \sum_{h \in \mathcal{H}} (x_h^* - e_h) \geq p \cdot \sum_{h \in \mathcal{H}} (x_h^* - e_h). \tag{13}$$

We want to show that

for given $p^* \in S$ and $t_{\backslash h}^* \in \times_{h' \in \mathcal{H} \backslash \{h\}} [0, k_{h'}]$,

(x_h^*, t_h^*) solves

$\max_{(x_h, t_h) \in (X_h \times T_h)} \left(u_{\mathcal{B}_h} \circ \varphi_{h|\left(p^*, t_{\backslash h}^*\right)}\right)(x_h, t_h)$

s.t.

$(x_h, t_h) \in B_h\left(p^*, t_{\backslash h}^*\right)$.

By assumption, for any $h \in \mathcal{H}$, we have that

$$\begin{array}{l} \text{for given } p^* \in S \quad \text{and} \quad (x^*_{h'}, t^*_{h'}) \in \times_{h' \in \mathcal{H} \setminus \{h\}} (X_{h'} \times T_{h'}) \\ (x^*_h, t^*_h) \quad \text{solves} \\ \max_{(x_h, t_h) \in (X_h \times T_h)} \quad \left(U_h \circ \varphi_{h|\left(p^*, t^*_{\setminus h}\right)} \right) (x_h, t_h) \\ \text{s.t.} \\ (x_h, t_h) \in B_h (p^*, t^*_h), \end{array} \tag{14}$$

Observe that from Lemma 26.2 we have

$$\left(U_h \circ \varphi_{h|\left(p^*, t^*_{\setminus h}\right)} \right)_{|B_h\left(p^*, t^*_{\setminus h}\right)} = \left(u_{\mathcal{B}_h} \circ \varphi_{h|\left(p^*, t^*_{\setminus h}\right)} \right)_{|B_h\left(p^*, t^*_{\setminus h}\right)}.$$

Then, the desired result holds true simply because constraint sets and objective functions restricted to the constraint sets of the two problems are the same. Then, households maximize. We are left with checking market clearing. From (14), we get that for any $h \in \mathcal{H}$

$$0 \geq p^* x^*_h - p^* \left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t^*_{h'h} - \sum_{h' \in \mathcal{B}_h} t^*_{hh'} \right).$$

Summing up with respect to $h \in \mathcal{H}$, we get

$$0 \geq \sum_{h \in \mathcal{H}} p^* (x^*_h - e_h) + \sum_{h \in \mathcal{H}} \left(\sum_{h' \in \mathcal{B}_{\longrightarrow h}} t^*_{h'h} - \sum_{h' \in \mathcal{B}_h} t^*_{hh'} \right) = \sum_{h \in \mathcal{H}} p^* (x^*_h - e_h), \tag{15}$$

where last equality follows from Proposition 10. From (15) and (13), we then get

$$\text{for any } p \in S, \qquad 0 \geq p^* \cdot \textstyle\sum_{h \in \mathcal{H}} (x^*_h - e_h) \geq p \cdot \sum_{h \in \mathcal{H}} (x^*_h - e_h). \tag{16}$$

For any $c \in \mathcal{C}$, define $p(c) = \left(p(c)^{c'} \right)_{c' \in \mathcal{C}}$ such that

$$p(c)^{c'} = \begin{cases} \frac{1}{r^{c'}} & \text{if} \quad c' = c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $p(c) \in S$. Then from (16), we get

$$0 \geq \frac{1}{r^c} \sum_{h \in \mathcal{H}} (x^{*c}_h - e^c_h),$$

and therefore,

$$\sum_{h \in \mathcal{H}} (x^*_h - e_h) \leq 0. \tag{17}$$

Let's now show that $p^* \gg 0$. Suppose our claim is false and without loss of generality, assume that $p^{*1} = 0$. Then, from strict monotonicity of $u_{\mathcal{B}_h}$ in x_h (and since $x^*_h \in \text{dom } u_{\mathcal{B}_h}$), we would have

$$\text{for any } h \in \mathcal{H}, \;\; x^{*1}_h = \widetilde{k}^1_x.$$

Then,

$$\sum_{h \in \mathcal{H}} x^{*1}_h = H \widetilde{k}^1_x \overset{\text{Def. 11}}{>} H r^1 > r^1,$$

contradicting (17).

From Proposition 27.2, we have $p^* \sum_{h \in \mathcal{H}} (x^*_h - e_h) = 0$. Since $p^* >> 0$, from (17), we also have $\sum_{h \in \mathcal{H}} (x^*_h - e_h) = 0$. ■

2.2.5 Equilibria with upper bound on consumption and equilibria

Definition 30 *Let a nonempty, convex subset X of $\mathbb{R}^n$ and a function $f : X \to \mathbb{R}$ be given. f is semistrictly quasi-concave if for any $x, y \in X$ and any $\lambda \in (0,1)$, $f(x) > f(y) \Rightarrow f((1-\lambda)x + \lambda y) > f(y)$.*

Proposition 31 *If X is a convex metric space and $u : X \to \mathbb{R}$ is continuous, then*

u is semistrictly quasi-concave and Non-Satiated $\Leftrightarrow$ u is quasi-concave and Locally NonSatiated.

Proof. See, for example, Villanacci (2022), Corollary 42, page 15. ■

Remark 32 *Since $.u_{\mathcal{B}_h}$ strictly increasing in x_h, then $.u_{\mathcal{B}_h}$ is Locally NonSatiated. Therefore, $.u_{\mathcal{B}_h}$ is semistrictly quasi-concave.*

Proposition 33 *For any economy, an equilibrium with upper bound on consumption is an equilibrium (without the upper bound on consumption).*

Proof. The proof is quite standard. See for example, Donato and Villanacci (2023), Theorem 49.
Let (x^*, t^*, p^*) be an equilibrium with upper bound on consumption. We want to show that if

(a) $(x_h^*, t_h^*) \in B_h\left(p^*, t_{\setminus h}^*\right)$, and

$$(b) \quad \text{for any } (x_h, t_h) \in B_h\left(p^*, t_{\setminus h}^*\right), \quad u_{\mathcal{B}_h}\left(x_h^*, w\left(p^*, t_h^*, t_{\setminus h}^*\right)\right) \geq .u_{\mathcal{B}_h}\left(x_h, w\left(p^*, t_h, t_{\setminus h}^*\right)\right), \tag{18}$$

then

(1) $(x_h^*, t_h^*) \in B_h^*\left(p^*, t_{\setminus h}^*\right)$, and

$$(2) \quad \text{for any } (x_h, t_h) \in B_h^*\left(p^*, t_{\setminus h}^*\right), \quad u_{\mathcal{B}_h}\left(x_h^*, w\left(p^*, t_h^*, t_{\setminus h}^*\right)\right) \geq .u_{\mathcal{B}_h}\left(x_h, w\left(p^*, t_h, t_{\setminus h}^*\right)\right). \tag{19}$$

Observe that

$$\text{for any } h \in \mathcal{H}, \quad 0 \leq x_h^* \overset{(1)}{\leq} \sum_{h' \in \mathcal{H}} e_{h'} \overset{(2)}{<<} k_x, \tag{20}$$

where (1) follows from market clearing and (2) from the definition of k_x. Since

$$\left\{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : x_h << k_x\right\}$$

is an open set which contains (x_h^*, t_h^*), then there exists $\delta^* \in \mathbb{R}_{++}$ such that

$$\begin{array}{l} \mathcal{B}\left((x_h^*, t_h^*), \delta^*\right) := \left\{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : d\left((x_h^*, t_h^*), (x_h, t_h)\right) < \delta^*\right\} \\ \subseteq \\ \left\{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : x_h << k_x\right\}. \end{array}$$

Then,

$$\mathcal{B}\left((x_h^*, t_h^*), \delta^*\right) \cap \left(\mathbb{R}_+^C \times [0, k_h]\right) \subseteq [0, k_x] \times [0, k_h] \tag{21}$$

Since $B_h\left(p^*, t_{\setminus h}^*\right) \subseteq B_h^*\left(p^*, t_{\setminus h}^*\right)$, conclusion (19.1) follows from assumption (18.a). Now suppose conclusion (19.2) does not hold, i.e.,

$$\exists \ (\widetilde{x}_h, \widetilde{t}_h) \in B_h^*\left(p^*, t_{\setminus h}^*\right) \setminus B_h\left(p^*, t_{\setminus h}^*\right) \text{ such that } .u_{\mathcal{B}_h}\left(\widetilde{x}_h, w\left(p^*, \widetilde{t}_h, t_{\setminus h}^*\right)\right) > .u_{\mathcal{B}_h}\left(x_h^*, w\left(p^*, t_h^*, t_{\setminus h}^*\right)\right). \tag{22}$$

Then

$$(\widetilde{x}_h, \widetilde{t}_h) \neq (x_h^*, t_h^*). \tag{23}$$

Since $\left(x_h^*, t_h^*\right), \left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h^*\left(p^*, t_{\backslash h}^*\right)$ and $B_h^*\left(p^*, t_{\backslash h}^*\right)$ is convex, then

$$\forall \lambda \in (0,1), \quad \left(\widehat{x}_h, \widehat{t}_h\right)(\lambda) := (1-\lambda)\left(x_h^*, t_h^*\right) + \lambda\left(\widetilde{x}_h, \widetilde{t}_h\right) \in B_h^*\left(p^*, t_{\backslash h}^*\right). \tag{24}$$

`insert picture from equilibrium with bound with equilibrium without bound.pdf`

From semistrict quasi-concavity of $u_{\mathcal{B}_h}$ (see Definition 30), (22) and (24), we have

$$\forall \lambda \in (0,1), \quad .u_{\mathcal{B}_h}\left(\widehat{x}_h(\lambda), w\left(p^*, \widehat{t}_h(\lambda), t_{\backslash h}^*\right)\right) > .u_{\mathcal{B}_h}\left(x_h^*, w\left(p^*, t_h^*, t_{\backslash h}^*\right)\right). \tag{25}$$

Now, $\left(\widehat{x}_h, \widehat{t}_h\right) - \left(x_h^*, t_h^*\right) \overset{\text{Def. } \left(\widehat{x}_h, \widehat{t}_h\right)}{=} \lambda \cdot \left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\| < \delta^*$ if and only if $\lambda < \frac{\delta^*}{\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\|} \in \mathbb{R}_{++}$, where $\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\| > 0$ from (23). Then, for any $\lambda \in \left(0, \frac{\delta}{\left\|\left(x_h^*, t_h^*\right) - \left(\widetilde{x}_h, \widetilde{t}_h\right)\right\|}\right)$, and using (24), we have that

$$\left(\widehat{x}_h, \widehat{t}_h\right)(\lambda) \in B_h^*\left(p^*, t_{\backslash h}^*\right) \cap \mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta^*\right) \cap \left(\mathbb{R}_+^C \times [0, k_h]\right), \tag{26}$$

where the last intersection follow from the fact that the constraints $x_h \geq 0$ and $t_h \in [0, k_h]$ are part of the definition of $B_h^*\left(p^*, t_{\backslash h}^*\right)$. Observe that

$$B_h\left(p^*, t_{\backslash h}^*\right) = B_h^*\left(p^*, t_{\backslash h}^*\right) \cap \left([0, k_x] \times [0, k_h]\right). \tag{27}$$

Then,

$$B_h^*\left(p^*, t_{\backslash h}^*\right) \cap \mathcal{B}\left(\left(x_h^*, t_h^*\right), \delta^*\right) \cap \left(\mathbb{R}_+^C \times [0, k_h]\right) \overset{(21)}{\subseteq} B_h^*\left(p^*, t_{\backslash h}^*\right) \cap \left([0, k_x] \times [0, k_h]\right) \overset{(27)}{=} B_h\left(p^*, t_{\backslash h}^*\right). \tag{28}$$

From (26) and (28), we have

$$\left(\widehat{x}_h, \widehat{t}_h\right) \in B_h\left(p^*, t_{\backslash h}^*\right). \tag{29}$$

(29) and (25) contradict assumption (18.b). ■

Summarizing, we did prove the following results.

Generalized Nash equilibria exist	(Proposition 22);		
Generalized Nash equilibria exist	$\overset{\text{Prop. 29}}{\Rightarrow}$	equilibria with upper bound on consumption exists	$\overset{\text{Prop. 33}}{\Rightarrow}$
equilibria exist			

We can then get the desired result of the section.

Theorem 34 *For any economy $\mathcal{E} := \left(e_h, .u_{\mathcal{B}_h}, k_h\right)_{h \in \mathcal{H}}$, such that for any $h \in \mathcal{H}$,*
$u_{\mathcal{B}_h}$ is Lipschitz continuous and concave,
$u_{\mathcal{B}_h}$ is strictly increasing in x_h and increasing in $\theta_{\mathcal{B}_h}$,,
$e_h \in \mathbb{R}_{++}^C$,
an equilibrium $(x^*, p^*, t^*) \in \mathbb{R}_+^{CH} \times S \times T$ exists and $p^* >> 0$.[6]

Remark 35 *From Corollary 119, the above result holds true if $h \in \mathcal{H}$, $u_{\mathcal{B}_h}$ is continuous and concave, strictly increasing in x_h, increasing in $\theta_{\mathcal{B}_h}$, and $\exists L \in \mathbb{R}_{++}$ such that defined $y = \left(x_h, \theta_{\mathcal{B}_h}\right) \in \mathbb{R}_{++}^{C+CB_h}$ for any $i \in \{1, ...C + CB_h\}$, $y_{\backslash i} \in \mathbb{R}_+^{C+CB_h-1}$, $u'_{\mathcal{B}_h|\{y_{\backslash i}\}}{}'(0^+) \leq L$.*

Remark 36 *If the upper bound k_h is equal to e_h, consistently with Remark 6, it is enough to assume that $u_{\mathcal{B}_h}$ is continuous, quasi-concave,*

Theorem 37 *strictly increasing in x_h and increasing in $\theta_{\mathcal{B}_h}$.*

[6] The existence theorem presented by Kranich (1988) is as follows. For any $h \in \mathcal{H}$, assume that: the consumption set is $\mathbb{R}_+^C$; the set of admissible transfer contains the origin and it is convex and compact; the utility function U_h is continuous, quasi-concave, strictly increasing in x_h; and $e_h \in \mathbb{R}_{++}^C$. Then an equilibrium exists for any economy.

2.3 Existence under some other assumptions on the utility functions

In this section, we show existence of equilibria under different assumptions on the utility functions. The strategy of proof is the same as the one presented in the above section. Below, we provide the proofs of the steps which are peculiar to the chosen specification of the utility function. We proceed as follows. We first state the theorem we want to prove. Then, 1. we state the extension theorem we use; 2. we check the assumptions of that theorem are satisfied; 3. we check the desired properties of the specific choice of the newly defined budget correspondence. Observe that, by construction, the equilibrium value of the arguments of the extended utility function are in the domain of the function before the extension - as done in detail in Proposition 26 for the model presented in the previous section.

We study the case of the utility function of the form

$$U_h : \mathbb{R}^C_{++} \times \mathbb{R}^{B_h}_{++} \to \mathbb{R},$$
$$(x_h, \theta_{\mathcal{B}_h}) \mapsto u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}(\theta_{h'}) . \tag{30}$$

Definition 38 *An economy is* $\mathcal{E}'' := (u_h, v_{hh'}, \beta_h, k_h, e_h)_{h\in\mathcal{H}}, \in \left(\mathcal{U}_h \times \mathbb{R}^{B_h}_{++} \times \mathbb{R}^{CB_h}_{++} \times \mathbb{R}^C_{++}\right)_{h\in\mathcal{H}} := \mathbb{E}''$, *where properties describing* u_h *and* $v_{hh'}$, *are presented below.*

Definition 39 *The vector* $(x^*, t^*, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{BC}_+ \times S$ *is an* ***equilibrium*** *for the economy* $\mathcal{E}'' \in \mathbb{E}''$ *if*

1. for any $h \in \mathcal{H}$, *household* h *maximizes, i.e., for given* $\mathcal{E} \in \mathbb{E}$, $p^* \in S$, $t^*_{\setminus h} \in T_{\setminus h}$, $(x^*_h, t^*_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ *solves*

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \quad u_h(x_h) + \sum_{h'\in\mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}\left(p^*\left(e_{h'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}} t_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}\right)\right)$$
s.t.
$$(x_h, t_h) \in B^*_h\left(p^*, t^*_{\setminus h}\right),$$

2.
Markets clear.

Theorem 40 *For for any economy* $\mathcal{E}''$, *if for any* $h \in \mathcal{H}$,
$u_h : \mathbb{R}^C_{++} \longrightarrow \mathbb{R}$ *is continuous, strictly increasing, concave and*
for any $\alpha \in \mathbb{R}$, $\mathrm{Cl}_{\mathbb{R}^C}\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\} \subseteq \mathbb{R}^C_{++}$, *and it is unbounded below;*
$v_{hh'} : (0,1) \longrightarrow \mathbb{R}$ *is (continuous,) increasing, concave and satisfies the condition*

$$\exists \varepsilon > 0 \text{ and } k > 0 \text{ such that } \forall t \in (0, \varepsilon),\ v'(t^-) < k, \tag{31}$$

$e_h \in \mathbb{R}^C_{++}$,
then an equilibrium $(x^*, p^*, t^*) \in \mathbb{R}^{CH}_+ \times S \times T$ *exists and* $p^* >> 0$.

1.

Proposition 41 *If* $v : (0,1) \to \mathbb{R}$, $t \longmapsto v(t)$ *is (continuous,) increasing, concave and satisfies Condition 31, then there exists a continuous, concave, increasing function* $V : \mathbb{R} \to \mathbb{R}$, $t \longmapsto V(t)$ *which is an extension of* v.

The proof of this result is presented in Proposition 142.

2.

Since $\beta >> 0$, u_h is concave and $v_{hh'}$is concave, then $u_h + \sum_{h'\in\mathcal{B}_h} v_{hh'}$ is concave. Indeed, what is needed in the proof is that a. u_h is quasi-concave, b. $v_{hh'}$ are concave, and c. $u_h + \sum_{h'\in\mathcal{B}_h} \beta_{hh'} \cdot v_{hh'}$ is quasi-concave. Then we can apply Debreu's Theorem.

3.

We present here the main idea in the construction of the budget correspondence, following for example Section 8.9 in Villanacci and others (2022). To get compactness, we have to add a $u(x) \geq u(e)$ type constraint, which has the following properties:

1. it is satisfied at a maximizing choice. The idea is that since e_h is affordable, then the solution to the maximization problem does not change if you add a constraint which the solution has to satisfy.

2. In defining the set-valued function which is analogous to $\widetilde{B}_h$, we have to be sure that it is not empty; then the added constraint should be of the type $u(x) \geq u\left(\frac{e}{4}\right)$. Then, clearly $\frac{e}{3}$ satisfies all the constraints defining $\widetilde{B}_h$.

3. The added constraint should allow to use the assumption $\mathrm{Cl}_{\mathbb{R}^C}\left\{x_h \in \mathbb{R}^C_{++} : u_h(x_h) \geq \alpha\right\} \subseteq \mathbb{R}^C_{++}$ which is used to show compactness.

Let's apply the above general procedure to our problem.

Below, $V_{hh'}$ denotes the extension of $v_{hh'}$ and it is therefore continuous, increasing and concave on all $\mathbb{R}$. Observe that if (x_h, t_h) is a solution to household h maximization problem, then

$$\begin{array}{c} u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + t_{hh'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) \geq \\ u_h\left(\frac{e_h}{4}\right) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) \end{array} \tag{32}$$

where the inequality follows from the fact that $(\underline{x}_h, \underline{t}_{hh'}) = \left(\frac{e_h}{4}, 0\right)$ belongs to the constraint set $B^*_h\left(p^*, t^*_{\setminus h}\right)$.

Remark 42 *If inequality* (32) *holds, then we have*

$$\begin{array}{ll} u_h(x_h) & \overset{(32)}{\geq} \\ u_h\left(\frac{e_h}{4}\right) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) & \overset{(1)}{\geq} \\ -\sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + k_{hh'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) & \\ u_h\left(\frac{e_h}{4}\right) + & \\ \min_{\left(p^*, t^*_{\setminus}\right) \in S \times T_{\setminus h}} \quad \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) + & := \\ -\sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + k_{hh'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) & \\ \underline{u}_h(\beta, e), & \end{array} \tag{33}$$

where (1) *follows from the Extreme Value Theorem, the fact that the involved functions are continuous and* $S \times T_h$ *is a compact set.*

Then the new constraint to be added to the constraint set is the one presented in (32).

Then,

1. from (32), the solutions to the (old problem) and the (problem with the added constraint in the constraint set) are the same.

2. from (32) and the fact that the function u_h is strictly increasing and the functions $V_{hh'}$ are increasing, we do have that the~version of the new constraint set is non-empty because $\left(\frac{e_h}{3}, \frac{e_h}{3\ B_H}\right)$ satisfies the old constraints and the new constraint with strict inequalities.

3. from (33) and the assumption on u_h, we are able to put x_h in a compact set.

Summarizing, the constraint for household h's maximization problem is

$$B^*_h : S \times T_{\setminus h} \longrightarrow\!\!\!\longrightarrow \mathbb{R}^C_{++} \times \mathbb{R}^{CB_h},$$

$$\begin{array}{ll} \left(p^*, t^*_{\setminus h}\right) \mapsto\!\!\!\mapsto & \{\ (x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\ & p^* x_h \leq p^*\left(e_h + \sum_{h' \in \mathcal{B}_{\longrightarrow h}} t^*_{h'h} - \sum_{h' \in \mathcal{B}_h} t_{hh'}\right) \\ & x_h \leq k_x, \quad t_h \geq 0, \quad t_h \leq k_h, \\ & \quad u_h(x_h) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + t_{hh'} + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) \geq \\ & \quad u_h\left(\frac{e_h}{4}\right) + \sum_{h' \in \mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\longrightarrow h'} \setminus \{h\}} t^*_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} t^*_{h'h''}\right)\right) \quad \} \end{array}$$

and

$$\widetilde{B}^*_h : S \times T_{\setminus h} \longrightarrow\!\!\!\longrightarrow \mathbb{R}^C_{++} \times \mathbb{R}^{CB_h}$$

is defined as B_h^* with weak inequalities substituted by strict inequalities.

We now want to check that B_h^* and $\widetilde{B}_h^*$ satisfy the conditions stated in Proposition 107, a fact which is verified below, where we denote by f the function which is naturally defined using the left hand sides of the constraints used in the definition of $\widetilde{B}_h^*$.

Proposition 43 *1. $\widetilde{B}$ is non-empty valued,*

2. f is continuous,

3. for any $j = 1, ...m$ and for any $\pi \in \Pi$, $f_{j|\{\pi\}}$is a. Locally NonSatiated and b. quasi-concave,

4. B is compact valued ($\operatorname{Im} B$ is contained in a compact set).

Proof. 1. Take $\left(x_h^{++}, t_{hh'}^{++}\right) = \left(\frac{e_h}{3}, k_h^* := \frac{1}{2}\left(\min\left\{k_{hh'}^c \cdot \frac{e_h^c}{3B_h}\right\}\right)_{c\in\mathcal{C}, h'\in\mathcal{B}_h}\right)$. Then

$$p^*\left(x_h^{++} + \sum_{h'\in\mathcal{B}_h} t_{hh'}\right) < p^*\left(\frac{e_h}{3} + \sum_{h'\in\mathcal{B}_h} \frac{e_h}{3B_h}\right) = p^* \frac{2e_h}{3} < p^* e_h \leq p^* e_h + \sum_{h'\in\mathcal{B}_{\longrightarrow h}} t_{h'h}^*$$

$$x_h^{++} << e_h << r + 1 << k_x$$
$$x_h^{++} >> 0$$
$$t_h^{++} := \frac{1}{2}\left(\min\left\{k_{hh'}^c \cdot \frac{e_h^c}{3B_h}\right\}\right)_{c\in\mathcal{C}, h'\in\mathcal{B}_h} >> 0$$

$$t_h << k_h^* << k_h$$

$$u_h\left(\frac{e_h}{3}\right) + \sum_{h'\in\mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + k_h^* + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\setminus\{h\}} t_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}^*\right)\right)$$
$$>$$
$$u_h\left(\frac{e_h}{4}\right) + \sum_{h'\in\mathcal{B}_h} \beta_{hh'} \cdot V_{hh'}\left(p^*\left(e_{h'} + 0 + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\setminus\{h\}} t_{h''h'}^* - \sum_{h''\in\mathcal{B}_{h'}} t_{h'h''}^*\right)\right).$$

.

2.

Obvious.

3. All the constraints apart from the last are affine and not constant and u is (quasi-)concave by assumption and u quasi-concave and k constant imply $u + k$ quasi-concave

4.

First of all, observe that

$$B_h^1\left(p^*, t_{\setminus h}^*\right) := B \text{ is bounded below by } 0_{CH} \text{ and above by } k := (k_x, k_h) \in \mathbb{R}_{++}^{C+CB_h}. \tag{34}$$

We want to show that B is sequentially compact, i.e., any sequence in B admits a convergent subsequence converging in B. From (34), up to a subsequence, $(x_h^n, t_h^n) \xrightarrow{n} \left(\overline{x}_h, \overline{t}_h\right) \in [0, k] \subseteq \mathbb{R}_+^{C+CB_h}$. We are then left with showing that $\overline{x}_h \in \mathbb{R}_{++}^C$. For any $n \in \mathbb{N}$, $(x_n^n, t_h^n) \in B$ satisfies the added constraint (32). Then, from Remark 42, $x_h^n \in \left\{x_h \in \mathbb{R}_{++}^{CH} : u_h(x_h) \geq \underline{u}_h(\beta, e)\right\}$. Since $x_h^n \xrightarrow{n} \overline{x}_h$, then $\overline{x}_h \in \mathrm{Cl}_{\mathbb{R}^C}\left(\left\{x_h \in \mathbb{R}_{++}^{CH} : u_h(x_h) \geq \underline{u}_h\right\}\right)$ which is contained in $\mathbb{R}_{++}^{CH}$ by the Assumption on the utility functions contained in the statement of existence Theorem 40. ■

2.4 The relative wealth model

We are going to call the equilibrium studied in Section 1.1 as "equilibrium in the Kranich (1988)'s model".

Definition 44 $p^{\setminus} = (p^c)_{c\in\mathcal{C}\setminus\{1\}} \in \mathbb{R}^{C-1}$.

Definition 45 *The vector $\left(x^*, t^*, p^{*\setminus}\right) \in \mathbb{R}_+^{CH} \times \mathbb{R}^B \times \mathbb{R}_{++}^{C-1}$ is a **Relative Wealth equilibrium** for the economy $\mathcal{E} := (\mathcal{B}_h, u_{\mathcal{B}_h}, e_h, k_h)_{h\in\mathcal{H}} \in \left(\times_{h\in\mathcal{H}}\left(\mathcal{P}(\mathcal{H}\setminus\{h\}) \times \mathcal{U}_{\mathcal{B}_h} \times \mathbb{R}_{++}^C \times \mathbb{R}_{++}^{B_h}\right)\right) := \mathbb{E}$. if*

1. for any $h \in \mathcal{H}$, household h maximizes, i.e., for given $\mathcal{E} \in \mathbb{E}$, $p^* \in \mathbb{R}_{++}^{C-1}$, $t_{\setminus h}^* \in \Gamma_{\setminus h}$,

$(x_h^*, t_h^*) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ *solves*

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{B_h}} \quad u_{\mathcal{B}_h}\left(x_h, \left(\frac{p^* e_{h'} + \left(t_{hh'} + \sum_{h''\in\mathcal{B}_{\longrightarrow h'}\setminus\{h\}} t^*_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} t^*_{h'h''}\right)}{pr}\right)_{h'\in\mathcal{B}_h}\right)$$

s.t.

$$(x_h, t_h) \in B_h^*\left(p^*, t^*_{\setminus h}\right),$$

where

$$B_h^* : \mathbb{R}^{C-1}_{++} \times \Gamma_{\setminus h} \longrightarrow\longrightarrow \mathbb{R}^C \times \mathbb{R}^{B_h},$$

$$\left(p^*, t^*_{\setminus h}\right) \mapsto\mapsto \{ \quad (x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \quad p^* x_h \le p^* e_h + \left(\sum_{h'\in\mathcal{B}_{\longrightarrow h}} t^*_{h'h} - \sum_{h'\in\mathcal{B}_h} t_{hh'}\right)$$

$$x_h \ge 0$$

$$t_h \ge 0$$

$$\sum_{h'\in\mathcal{B}_h} t_{hh'} \le k_h \quad \} \tag{35}$$

2.
Markets clear, i.e.,

$$\sum_{h\in\mathcal{H}} (x_h^* - e_h) = 0.$$

Proposition 46 *If (x,t,p) is an equilibrium in Kranich (1988)'s model, then $\left(x, t, \frac{p}{p^1} := \widetilde{p}\right)$ it is an equilibrium in the Relative Wealth model.*

Proof. First of all observe that $\widetilde{p} = 1$, as required by the definition of RW model. Drop h and denote by $\beta_K(p,t)$, $u_K(x,t,p)$ and $\beta_{RW}(p,t)$, $u_{RW}(x,t,p)$ the (budget set and the objective function) in Kranich (1988) and Relative Wealth models, respectively, i.e., for simplicity dropping constraint not containing prices

$$\beta_K(p,t) = \{(x_h, t_h) : p(x_h + t_{h\longrightarrow}) \le p(e_h + t_{\longrightarrow h})\} \qquad \text{with} \quad pr = 1$$

$$\beta_{RW}(\widehat{p}, t) = \{(x_h, t_h) : \widetilde{p}(x_h + t_{h\longrightarrow}) \le \widetilde{p}(e_h + t_{\longrightarrow h}),\} \quad \text{with} \quad \widetilde{p}^1 = 1$$

$$u_K(x,t,p) = u_h\left(x_h, \left(p\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\ne h}\right)$$

$$u_{RW}(x,t,\widetilde{p}) = u_h\left(x_h, \left(\frac{\widetilde{p}}{\widetilde{p}r}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h'\ne h}\right)$$

Step 1. $\beta_K(p,t) = \beta_{RW}\left(\frac{p}{p^1}, t\right)$.
Indeed, the budget equations are the same. In Kranich (1988)'s model, we have

$$p(x_h + t_{h\longrightarrow}) \le p(e_h + t_{\longrightarrow h})$$

In the Relative Wealth model, we have

$$\frac{p}{p^1}(x_h + t_{h\longrightarrow}) \le \frac{p}{p^1}(e_h + t_{\longrightarrow h})$$

Similar argument applies to the other inequalities.

Step 2. For any (x,t,p) such that $p \in S_{++}$, we have $u_{RW}\left(x,t,\frac{p}{p^1}\right) = u_K(x,t,p)$.

$$u_{RW}\left(x,t,\frac{p}{p^1}\right) = u_h\left(x_h, \left(\frac{\frac{p}{p^1}}{\frac{p}{p^1}r}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h' \neq h}\right) =$$

$$= u_h\left(x_h, \left(\frac{p}{pr}\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h' \neq h}\right) \overset{pr=1}{=}$$

$$= u_h\left(x_h, \left(p\left(e_{h'} + t_{hh'} + \tau_{(\neg h),h'}\right)\right)_{h' \neq h}\right) = u_K(x,t,p).$$

Step 3. Desired result.

Let (x,t,p) be an equilibrium in Kranich (1988)'s model. For any $(x_1', t_h') \in \beta_K\left(p, t_{\setminus h}\right) \overset{\text{Step 1}}{=} \beta_{RW}\left(\frac{p}{p^1}, t_{\setminus h}\right)$, we have

$$u_{RW}\left(x,t,\frac{p}{p^1}\right) \overset{\text{Step 2}}{=} u_k(x,t,p) \overset{(x,t,p)\text{ is Kranich equilibrium}}{\geq} u_k(x',t',p) \overset{\text{Step 2}}{=} u_{RW}\left(x',t',\frac{p}{p^1}\right),$$

i.e., households maximize. Clearly markets clear. ■

Corollary 47 *For for any economy $\mathcal{E} := (e_h, u_h, k_h)_{h \in \mathcal{H}}$, if for any $h \in \mathcal{H}$,*
either
$u_{\mathcal{B}_h}$ is Lipschitz continuous and concave,
$u_{\mathcal{B}_h}$ is strictly increasing in x_h and increasing in $\theta_{\mathcal{B}_h}$,,
or
$u_{\mathcal{B}_h}$ is continuous;
$u_{\mathcal{B}_h}$ is strictly increasing in x_h; increasing in $\theta_{h'}$;
$u_{\mathcal{B}_h}$ is quasi concave in $\left(x_h, (\theta_{h'})_{h' \in \mathcal{B}_h}\right)$;

for any $e_h \in \mathbb{R}^C_{++}$, $\underline{\theta}_{\mathcal{B}_h} \in \mathbb{R}^{B_h}_{++}$

$$\mathrm{Cl}_{\mathbb{R}^{C+B_h}}\left\{(x_h, \theta_{\mathcal{B}_h}) \in \mathbb{R}^{C+B_h}_{++} : u_{\mathcal{B}_h}(x_h, \theta_{\mathcal{B}_h}) \geq u_{\mathcal{B}_h}\left(e_h, \underline{\theta}_{\mathcal{B}_h}\right)\right\} \subseteq \mathbb{R}^{C+B_h}_{++},$$

and
$e_h \in \mathbb{R}^C_{++}$,
then an equilibrium $(x^*, p^*, t^*) \in \mathbb{R}^{CH}_{+} \times S \times T$ in the Relative Wealth model exists and $p^* >> 0$.

Proof. It follows from Proposition 46,Theorems 34 and the analysis presented in the previous section. ■

3 Discussion of the Equilibrium set properties

In this section, we want to address a quite reasonable question about equilibria. In the version of the model presented above, could we get existence without imposing an upper bound on transfers? The answer is negative, as the analysis of the following Cobb-Douglas economy shows. Indeed, a first simple intuitive explanation is as follows.

Consider the following informally described game. There are two players: each player chooses one real number, i.e., her strategy set is $\mathbb{R}$. The player who chooses a bigger number than the one chosen by the other player wins 1 euro; player who chooses smaller number gets 0 euros. If both players choose the same number, they both get 0 euro. Since a best response against $x \in \mathbb{R}$ is $x+1 \in \mathbb{R}$, then the game has no Nash equilibria - not even in mixed strategies. Indeed, if the level of altruism of households is sufficiently high, then each of them overbids the other one transfers. We will come back to this statement after the description of the main results in the Cobb-Douglas economy we analyze below.

We present an analysis in the model without and with an upper bound on transfers. The main results are what follows.

1. In the model without upper bounds on transfers, there is a set $\mathcal{N}$ of economies such that $\mathcal{N}$ has nonempty interior and for which equilibria do not exist - see Proposition 51; indeed, $\mathcal{N} = \{(\beta_1, \beta_2) \in \mathbb{R}^2_{++} : \beta_1 \cdot \beta_2 > 1\}$.

2. in the model with upper bounds on transfers,

a. in the set $\mathcal{N}$, equilibria exists (transfer are equal to the upper bound for at least one household) - see regions 2, 8 and 4 in picture equilibrium transfers in beta1-2 plane.pdf.

b. an infinite number of equilibria arise only for a closed and measure zero set $\mathcal{D}$ of economies - see regions 5 and 6; indeed, equilibria associated with a given economy in $\mathcal{D}$ are different one from another just for transfers, while consumption allocations are constant; indeed, $\mathcal{D} = \{(\beta_1, \beta_2) \in \mathbb{R}^2_{++} : \beta_1 \cdot \beta_2 = 1\}$.

c. there is a set $\mathcal{N}'$ of economies such that $\mathcal{N}'$ has nonempty interior and for which only one (or none) of the two households chooses a strictly positive transfer - see - see regions 1 and 3; indeed, $\mathcal{N}' = \{(\beta_1, \beta_2) \in \mathbb{R}^2_{++} : \beta_1 \cdot \beta_2 < 1\}$.

3.1 A Cobb-Douglas economy with no bound on transfers

In this section we describe the problem of nonexistence of equilibria in a 2 household- one good - Cobb-Douglas economy. For given, $(e_1, e_2, \beta_{12}) \in \mathbb{R}^3_{++}$ and $t_{21} \in \mathbb{R}_+$, the utility function of household 1 is

$$v_1 : \mathbb{R}^2_{++} \times \mathbb{R}_{++} \longrightarrow \mathbb{R}, \quad (x_1, t_{12}) \mapsto \log x_1 + \beta_{12} \log\left(\frac{e_2 - t_{21} + t_{12}}{r}\right).$$

Symmetric definition applies to household 2.

Definition 48 *A vector $(x^*, t^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2$ is an equilibrium associated with the economy $(\beta, e) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ if*

1. (x_1^*, t_{12}^*) solve the following problem:

for given $(e_1, e_2, \beta_{12}) \in \mathbb{R}^3_{++}$ and $t_{21} \in \mathbb{R}_+$,

$$\max_{(x_1, t_{12}) \in \mathbb{R}_{++} \times (-e_2 + t_{21}, +\infty)} \quad \log x_1 + \beta_{12} \log(e_2 - t_{21} + t_{12}) \quad s.t. \quad \begin{array}{ll} -x_1 - t_{12} + e_1 + t_{21} & \geq 0 \\ t_{12} & \geq 0, \end{array} \tag{36}$$

and similar condition holds for (x_2^*, t_{21}^*), and

2. markets clear, i.e.,

$$x_1^* + x_2^* = e_1 + e_2.$$

Remark 49 *The proposition below gives conditions under which equilibria do not exist. It is interesting to compare that result with what said by Mercier Ythier (2000) - see the discussion below Assumption 2, page 10, and beginning of Section 4.*

Proposition 50 *A vector $(x^*, t^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2$ is an equilibrium associated with the economy $(\beta, e) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++}$ iff there exists $(\lambda^*, \gamma^*) \in \mathbb{R}^4$ such that $(x^*, t^*, \lambda^*.\gamma^*)$ is a solution to the following system in the exogenous variables (β, e).*

$$\begin{array}{ll} \frac{1}{x_1} - \lambda_1 & = 0 \\ \beta_{12} \frac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} & = 0 \\ -x_1 - t_{12} + e_1 + t_{21} & = 0 \\ \min\{\gamma_{12}, t_{12}\} & = 0 \\ e_2 - t_{21} + t_{12} & > 0 \\ & \\ \frac{1}{x_2} - \lambda_2 & = 0 \\ \beta_{21} \frac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} & = 0 \\ -x_2 - t_{21} + e_2 + t_{12} & = 0 \\ \min\{\gamma_{21}, t_{21}\} & = 0 \\ e_1 + t_{21} - t_{12} & > 0 \\ & \\ \sum_{h \in \mathcal{H}} (x_h - e_h) & = 0 \end{array} \tag{37}$$

Proof. It is easy to verify that the set of maximizers to maximization problem (36) is characterized by the associated Kuhn-Tucker conditions. Indeed, the choice set $\mathbb{R}_{++} \times (-e_2 + t_{21}, +\infty)$ is open and convex, the objective function is strictly concave, constraint functions are linear and $\left(x_1^{++}, t_{12}^{++}\right) = \left(\frac{e_1}{4}, t_{21} + \frac{e_1}{4}\right)$ satisfies the constraints with strict inequalities and belongs to the choice set:

$$-x_1 - t_{12} + e_1 + t_{21} = -\tfrac{e_1}{4} - t_{21} - \tfrac{e_1}{4} + e_1 + t_{21} = \tfrac{e_1}{2} > 0$$

$$t_{12} - (-e_2 + t_{21}) = t_{21} + \tfrac{e_1}{4} + e_2 - t_{21} = \tfrac{e_1}{4} + e_2 > 0$$

Existence of a solution is not insured due to the possibility of, say, $e_2 - t_{21} < 0$. Uniqueness follows from strict concavity of the objective function. ■

Proposition 51 *If $\beta_{12} \cdot \beta_{21} > 1$, then there is no equilibrium.*

Proof. Observe that $x_2 = -t_{21} + e_2 + t_{12} = e_2 + t_{12} - t_{21}$. Then, $\beta_{12} \frac{1}{x_2} = \lambda_1 - \gamma_{12}$ and $\lambda_1 - \beta_{12}\lambda_2 = \gamma_{12}$. Then, using again the symmetry of the problems and observing that $\lambda_2 = \frac{1}{x2} > 0$, we have

$$\begin{cases} \lambda_1 - \beta_{12}\lambda_2 = \gamma_{12} \\ -\beta_{21}\lambda_1 + \lambda_2 = \gamma_{21} \end{cases}$$

$$\begin{cases} \beta_{21}\lambda_1 - \beta_{12}\beta_{21}\lambda_2 = \beta_{21}\gamma_{12} \\ -\beta_{21}\lambda_1 + \lambda_2 = \gamma_{21} \end{cases}$$

$$0 > \overset{(<0)}{(1 - \beta_{12}\beta_{21})} \overset{(>0)}{\lambda_2} = \overset{(>0)}{\beta_{21}} \overset{(\geq 0)}{\gamma_{12}} + \overset{(\geq 0)}{\gamma_{21}} \geq 0,$$

which shows there is no solution to system (37). ■

Remark 52 *Below, we provide some intuition on the nonexistence result. The above analysis says that there is no equilibrium if*

$$\beta_1\beta_2 > 1, \tag{38}$$

Since the objective functions of households 1 is $u_1 + \beta_1 v_1$, then we can interpret

β_1 *as how much* 1 *cares about* 2

and

$\beta_1 > 1$ *means*	1 *cares about* 2 *more than* 1 *cares about* 1,
and	
$0 < \beta_1 < 1$ *means*	1 *cares about* 2 *less than* 1 *cares about* 1.

Then, equilibria do not exist if both household care too much about the other household.

3.2 A Cobb Douglas economy with an upper bound on transfers

3.2.1 Equilibria

Proposition 53 *Let the following equations be given.*

$$\begin{aligned} -x_1 - t_{12} + e_1 + t_{21} &= 0 \\ -x_2 - t_{21} + e_2 + t_{12} &= 0 \\ x_1 + x_2 - e_1 - e_2 &= 0 \end{aligned}$$

If two equations among the above ones hold true, then the third one holds true as well.

Proof. Just add up the two equations you are assuming hold true. ■

In this section, we analyze the case in which we impose an institutional constraint on transfers. The bound is going to be larger than total resource, i.e., equal to $e + k$, with $k \geq 0$.

Definition 54 *A vector $(x^*, t^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2$ is an equilibrium associated with the economy $(\beta, e, k) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++} \times \mathbb{R}_+$ if*

1. (x_1^*, t_{12}^*) solve the following problem:

for given $(e_1, e_2, \beta_{12}, k) \in \mathbb{R}^3_{++}$ and $t_{21} \in \mathbb{R}_+$,

$$\max_{(x_1,t_{12})\in\mathbb{R}_{++}\times(-e_2+t_{21},+\infty)} \quad \log x_1 + \beta_{12} \log (e_2 - t_{21} + t_{12}) \quad s.t. \quad \begin{array}{ll} -x_1 - t_{12} + e_1 + t_{21} & \geq 0 \\ t_{12} & \geq 0, \\ e_1 + k - t_{12} & \geq 0 \end{array} \tag{39}$$

and similar condition holds for (x_2^*, t_{21}^*), and

2. markets clear, i.e.,

$$x_1^* + x_2^* = e_1 + e_2.$$

Proposition 55 *A vector $(x^*, t^*) \in \mathbb{R}^2_{++} \times \mathbb{R}^2$ is an equilibrium associated with the economy $(\beta, e, k) \in \mathbb{R}^2_{++} \times \mathbb{R}^2_{++} \times \mathbb{R}_+$ iff there exists $(\lambda^*, \gamma^*, \delta^*) \in \mathbb{R}^6$ such that $(x^*, t^*, \lambda^*, \gamma^*, \delta^*)$ is a solution to the following system in the exogenous variables (β, e).*

$$\begin{array}{ll} \frac{1}{x_1} - \lambda_1 & = 0 \\ \beta_{12} \frac{1}{e_2 - t_{21} + t_{12}} - \lambda_1 + \gamma_{12} & = 0 \\ -x_1 - t_{12} + e_1 + t_{21} & = 0 \\ \min\{\gamma_{12}, t_{12}\} & = 0 \\ \min\{\delta_{12}, e_1 + k - t_{12}\} & = 0 \\ e_2 - t_{21} + t_{12} & > 0 \\ & \\ \frac{1}{x_2} - \lambda_2 & = 0 \\ \beta_{21} \frac{1}{e_1 + t_{21} - t_{12}} - \lambda_2 + \gamma_{21} & = 0 \\ -x_2 - t_{21} + e_2 + t_{12} & = 0 \\ \min\{\gamma_{21}, t_{21}\} & = 0 \\ \min\{\delta_{21}, e_2 + k - t_{21}\} & = 0 \\ e_1 + t_{21} - t_{12} & > 0 \\ & \\ \sum_{h\in\mathcal{H}} (x_h - e_h) & = 0 \end{array} \tag{40}$$

Existence of a solution is not insured. It is easy to verify that the set of maximizers of the above maximization problem is characterized by the associated Kuhn-Tucker conditions and the solution is unique. Indeed, the choice set $\mathbb{R}_{++} \times (-e_2 + t_{21}, +\infty)$ is open and convex, the objective function is strictly concave, constraint functions are linear and $\left(x_1^{++}, t_{12}^{++}\right) = \frac{1}{4}(e_1, e_1)$ belongs to the choice set and satisfies the constraints with strict inequalities:

$$t_1 = \frac{1}{4} e_1 > 0$$

$$-x_1 - t_{12} + e_1 + t_{21} = -\frac{1}{2} e_1 + e_1 + t_{21} \geq \frac{1}{2} e_1 > 0,$$

$$e_1 + k - t_{12} > 0$$

Proposition 56 *The equilibrium values are described in the Table below.*

1	*2*	*3*	*4*
$\beta_1\beta_2 < 1$, $\beta_1 < 1$, $\beta_2 < 1$	$\beta_1\beta_2 > 1$, $\beta_1 > 1$, $\beta_2 > 1$	$\beta_1\beta_2 < 1$, $\beta_1 > 1$, $\beta_2 < 1$	$\beta_1\beta_2 > 1$, $\beta_1 > 1$, $\beta_2 < 1$
x $\quad 1 \quad 1$ t $\quad 0 \quad 0$ γ $\quad 1-\beta_1 \quad 1-\beta_2$ δ $\quad 0 \quad 0$	$1 \quad 1$ $1+k \quad 1+k$ $0 \quad 0$ $\beta_1 - 1 \quad \beta_2 - 1$	$\frac{2}{1+\beta_1} \quad \frac{2\beta_1}{1+\beta_1}$ $\frac{\beta_1-1}{1+\beta_1} \quad 0$ $0 \quad \frac{(\beta_1+1)(1-\beta_1\beta_2)}{2\beta_1}$ $0 \quad 0$	$\frac{2\beta_2}{1+\beta_2} \quad \frac{2}{1+\beta_2}$ $1+k \quad 1+k-\frac{1-\beta_2}{1+\beta_2}$ $0 \quad 0$ $\frac{(1+\beta_2)(\beta_1\beta_2-1)}{2\beta_2} \quad 0$

(41)

5	*6*	*7*	*8*
$\beta_1\beta_2 = 1$, $\beta_1 > 1, \beta_2 < 1$	$\beta_1 = 1 = \beta_2$	$\beta_1\beta_2 < 1$, $\beta_1 = 1, \beta_2 < 1$	$\beta_1\beta_2 > 1$, $\beta_1 > 1, \beta_2 = 1$
x $\quad \frac{2}{1+\beta_1} \quad \frac{2}{1+\beta_2}$ t $\quad t_2+\frac{\beta_1-1}{1+\beta_1} \quad \in \left[0, 1-\frac{\beta_1-1}{1+\beta_1}\right]$ γ $\quad 0 \quad 0$ δ $\quad 0 \quad 0$	$1 \quad 1$ $t_2 \quad \in [0, 1+k]$ $0 \quad 0$ $0 \quad 0$	$1 \quad 1$ $0 \quad 0$ $0 \quad 1-\beta_2$ $0 \quad 0$	$1 \quad 1$ $1+k \quad 1+k$ $0 \quad 0$ $\beta_1 - 1 \quad 0$

Only in Cases 5 and 6 in, which $\beta_1\beta_2 = 1$,there is an infinite number of equilibria; in all other cases, there exists a unique equilibrium. Even in the case of infinite equilibria, both households' allocations (and then utility levels) are constant.

We consider the case $\beta_1 \geq \beta_2$, the case $\beta_2 \geq \beta_1$ being perfectly symmetric.

Proof. The main idea to prove the above results is to proceed as follows.

1. Find the best response function of household 1 in the cases $\beta_1 < 1$, $\beta_1 = 1$ and $\beta_1 > 1$; in each case, construction a conjecture, starting from the observation that $\beta_1 \geq 1 \Rightarrow t_1 = 1+k$ and $\beta_1 < 1 \Rightarrow t_1 = 0$; symmetrically, construct the reaction function for household 2;

2. Assuming, without loss of generality, $\beta_1 \geq \beta_2$, combine the different best response functions; that procedure allows to find the equilibrium valued of (t_1, t_2);

3. Then equilibrium is as follows.

$$\begin{array}{ll} t_1 = ... & t_2 = ... \\ x_1 = 1 - t_1 + t_2 & x_2 = 1 - t_2 + t_1 \\ \gamma_1 = \delta_1 - \frac{\beta_1}{x_2} + \frac{1}{x_1} & \gamma_2 = \delta_2 - \frac{\beta_2}{x_1} + \frac{1}{x_2} \\ \delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} & \delta_2 = \gamma_2 + \frac{\beta_2}{x_1} - \frac{1}{x_2} \end{array} \tag{42}$$

Recall that either $\gamma_1 = 0$ or $\delta_1 = 0$ (or both). We use the equilibrium system below.

$$\begin{array}{lll} \text{foc1} & \frac{1}{x_1} - \lambda_1 & = 0 \\ \text{foc2} & \beta_{12}\frac{1}{1-t_{21}+t_{12}} - \lambda_1 + \gamma_{12} - \delta_{12} & = 0 \\ \text{bc} & -x_1 - t_{12} + 1 + t_{21} & = 0 \\ \text{min1} & \min\{t_{12}, \gamma_{12}\} & = 0 \\ \text{min2} & \min\{1 - t_{12}, \delta_{12}\} & = 0 \\ \text{ineq} & x_2 = e_2 - t_{21} + t_{12} & > 0 \\ & & \\ \text{foc1} & \frac{1}{x_2} - \lambda_2 & = 0 \\ \text{foc2} & \beta_{21}\frac{1}{1+t_{21}-t_{12}} - \lambda_2 + \gamma_{21} - \delta_{21} & = 0 \\ \text{bc} & -x_2 - t_{21} + 1 + t_{12} & = 0 \\ \text{min1} & \min\{t_{21}, \gamma_{21}\} & = 0 \\ \text{min2} & \min\{1 - t_{21}, \delta_{21}\} & \\ \text{ineq} & x_1 = e_1 - t_{12} + t_{21} & > 0 \\ & & \\ & x_1 + x_2 - 2 & = 0 \end{array}$$

1.

Best response function for household 1 .

A. $\beta_1 < 1$.

We preliminary conjecture that $t_1 = 0$, which implies $\delta_1 = 0$ and $\gamma_1 \geq 0$. Then,

$$0 \leq \gamma_1 = -\frac{\beta_1}{1-t_2} + \frac{1}{1+t_2} = \frac{-\beta_1 - \beta_1 t_2 + 1 - t_2}{(1-t_2)(1+t_2)} = \frac{(1-\beta_1) - (1+\beta_1) t_2}{(1-t_2)(1+t_2)}. \tag{43}$$

Observe if the denominator is positive, then the above inequality (43) is satisfied iff

$$t_2 \leq \frac{1-\beta_1}{1+\beta_1}.$$

Observe that $\frac{1-\beta_1}{1+\beta_1} > 0$ iff $\beta_1 < 1$, which is true in the present case;

$$\frac{1-\beta_1}{1+\beta_1} < 1 \tag{44}$$

iff $1-\beta_1 < 1+\beta_1$ iff $\beta_1 > 0$, which is true by assumption.

Then, we come up with the following conjecture:

$$\text{if } t_2 \in \left[0, \frac{1-\beta_1}{1+\beta_1}\right], \text{ then } t_1 = 0.$$

Observe that

$x_1 = 1 - t_1 + t_2 = 1 + t_2 > 0$, and

$x_2 = 1 + t_1 - t_2 = 1 - t_2 > 1 - \frac{1-\beta_1}{1+\beta_1} \overset{(44)}{>} 0$.

Moreover, $\delta_1 = 0$ and, using (43), $0 \leq \gamma_1 = \frac{(1-\beta_1)-(1+\beta_1)t_2}{(1-t_2)(1+t_2)}$, from the values of x_1, x_2 and the fact that $t_2 \in \left[0, \frac{1-\beta_1}{1+\beta_1}\right]$.

If $t_2 \in \left(\frac{1-\beta_1}{1+\beta_1}, 1+k\right]$, we preliminary conjecture that $t_1 \in (0,1)$ and then $\delta_1 = \gamma_1 = 0$. Then,

$$0 = \delta_1 = \frac{\beta_1}{1-t_2+t_1} - \frac{1}{1+t_2-t_1} = \frac{\beta_1+\beta_1 t_2 - \beta_1 t_1 - 1 + t_2 - t_1}{(1-t_2+t_1)(1+t_2-t_1)}$$

$$= \frac{(\beta_1-1)-(\beta_1+1)t_1+(\beta_1+1)t_2}{(1-t_2+t_1)(1+t_2-t_1)}.$$

Then, if the denominator is positive we have $t_1 = t_2 + \frac{\beta_1-1}{\beta_1+1}$. Then, we come up with the following conjecture:

$$\text{if } t_2 \in \left(\frac{1-\beta_1}{1+\beta_1}, 1+k\right], \text{ then } t_1 = t_2 + \frac{\beta_1-1}{\beta_1+1}.$$

Observe that

$t_1 > 0$ iff $t_2 + \frac{\beta_1-1}{\beta_1+1} > 0$ which we are indeed assuming. Then, $\gamma_1 = 0$

$t_1 \leq 1+k$ iff $t_2 + \frac{\beta_1-1}{\beta_1+1} < 1+k$ or $t_2 < 1+k+\frac{1-\beta_1}{\beta_1+1}$ which is true because we are assuming $t_2 < 1+k$ and, since $\beta_1 < 1$, $\frac{1-\beta_1}{\beta_1+1} > 0$.

Then,

$x_1 = 1 + t_2 - t_1 = 1 + t_2 - t_2 - \frac{\beta_1-1}{\beta_1+1} = 1 - \frac{\beta_1-1}{\beta_1+1} = \frac{2}{1+\beta_1} > 0$;

$x_2 = 1 - t_2 + t_1 = 1 - t_2 + t_2 + \frac{\beta_1-1}{\beta_1+1} = 1 + \frac{\beta_1-1}{\beta_1+1} = \frac{2\beta_1}{1+\beta_1} > 0$.

Moreover,

$$0 = \delta_1 = \frac{-\beta_1}{x_2} + \frac{1}{x_1} = \frac{-\beta_1}{\frac{2\beta_1}{1+\beta_1}} + \frac{1}{\frac{2}{1+\beta_1}} = 0.$$

Summarizing,

$$t_1 = \begin{cases} 0 & \text{if} \quad t_2 \in \left[0, \frac{1-\beta_1}{1+\beta_1}\right] \\ \\ t_2 + \frac{\beta_1-1}{\beta_1+1} & \text{if} \quad t_2 \in \left(\frac{1-\beta_1}{1+\beta_1}, 1+k\right] \end{cases}$$

All the reaction functions picture are presented at the end of the proof.

B. $\beta_1 = 1$.

If $t_2 = 0$, then we conjecture $t_1 = 0$ and $\gamma_1 \geq 0$ (indeed, we are going to show $\gamma_1 = 0$) and $\delta_1 = 0$. Then, $x_1 = 1 + t_2 - t_1 = 1$ and $x_2 = 1 - t_2 + t_1 = 1$ and $\gamma_1 = \delta_1 - \frac{\beta_1}{x_2} + \frac{1}{x_1} = -\frac{1}{1} + \frac{1}{1} = 0$.

If $t_2 = 1 + k$, then we conjecture $t_1 = 1 + k$ and $\delta_1 \geq 0$ (indeed, we are going to show $\delta_1 = 0$) and $\gamma_1 = 0$.

Then, $x_1 = 1 + t_2 - t_1 = 1$ and $x_2 = 1 - t_2 + t_1 = 1$ and $\delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} = -\frac{1}{1} + \frac{1}{1} = 0$.

If $t_2 \in (0, 1 + k)$, then we conjecture $t_1 = t_2$ and $\delta_1 = \gamma_1 = 0$. Then, $x_1 = x_2 = 1$ and $\delta_1 = \gamma_1 = 0$.

C. $\beta_1 > 1$.

First of all, observe that since $\beta_1 > 1$, we have $1 + k + \frac{1-\beta_1}{1+\beta_1} < 1 + k$. We conjecture that $t_1 = 1 + k$. Then,

$$0 \leq \delta_1 = \frac{\beta_1}{1-t_2+1+k} - \frac{1}{1+t_2-1-k} = \frac{\beta_1}{2-t_2+k} - \frac{1}{t_2-k} =$$

$$= \frac{\beta_1 t_2 - \beta_1 k - 2 + t_2 - k}{(2-t_2+k)(t_2-k)} = \frac{(\beta_1+1)t_2 - (\beta_1+1)k - 2}{(2-t_2)t_2}.$$

Then, if the denominator is positive we have $t_2 \geq k + \frac{2}{1+\beta_1} = 1 + k + \left(\frac{2}{1+\beta_1} - 1\right) = 1 + k + \frac{1-\beta_1}{1+\beta_1}$.

Observe that since we are assuming $\beta_1 > 1$, we have $\frac{1-\beta_1}{1+\beta_1} \in (-1, 0)$ and then

$$0 < 1 + k + \frac{1-\beta_1}{1+\beta_1} < 1 + k.$$

Then, we come up with the following conjecture:

$$\text{if } t_2 \in \left[1 + k + \frac{1-\beta_1}{1+\beta_1}, 1 + k\right], \text{ then } t_1 = 1 + k.$$

Then $\gamma_1 = 0$. Observe that

$x_1 = 1 + t_2 - t_1 = 1 + t_2 - 1 - k = t_2 - k > 0$ if $t_2 > k$, as we are assuming. $x_2 = 1 - t_2 + t_1 = 2 - t_2 + k > 0$ if $t < k + 2$, which we are assuming.

Then,

$$0 \leq \delta_1 = \frac{\beta_1}{x_2} - \frac{1}{x_1} = \frac{\beta_1}{2-t_2+k} - \frac{1}{t_2-k} =$$

$$= \frac{\beta_1 t_2 - \beta_1 k - 2 + t_2 - k}{(2-t_2+k)(t_2-k)} = \frac{(\beta_1+1)t_2 - (\beta_1+1)k - 2}{(2-t_2)t_2}$$

which is true if $t_2 \geq k + \frac{2}{1+\beta_1} = 1 + k + \frac{1-\beta_1}{1+\beta_1}$, as we are assuming.

It $0 \leq t_2 < \frac{2}{1+\beta_1} + k$, then the conjecture is $t_1 \in (0, 1 + k)$, which implies $\delta_1 = 0$ and $\gamma_1 = 0$. Then,

$$0 = \frac{\beta_1}{1-t_2+t_1} - \frac{1}{1+t_2-t_1} = \frac{\beta_1 + \beta_1 t_2 - \beta_1 t_1 - 1 + t_2 - t_1}{(1-t_2+t_1)(1+t_2-t_1)} =$$

$$= \frac{(\beta_1-1) - (\beta_1+1)t_1 + (\beta_1+1)t_2}{(1-t_2+t_1)(1+t_2-t_1)}.$$

and $t_1 = t_2 + \frac{\beta_1-1}{\beta_1+1}$.Then, we come up with the following conjecture:

$$\text{if } t_2 \in \left[0, 1 + k + \frac{1-\beta_1}{1+\beta_1}\right), \text{ then } t_1 = t_2 + \frac{\beta_1 - 1}{\beta_1 + 1}.$$

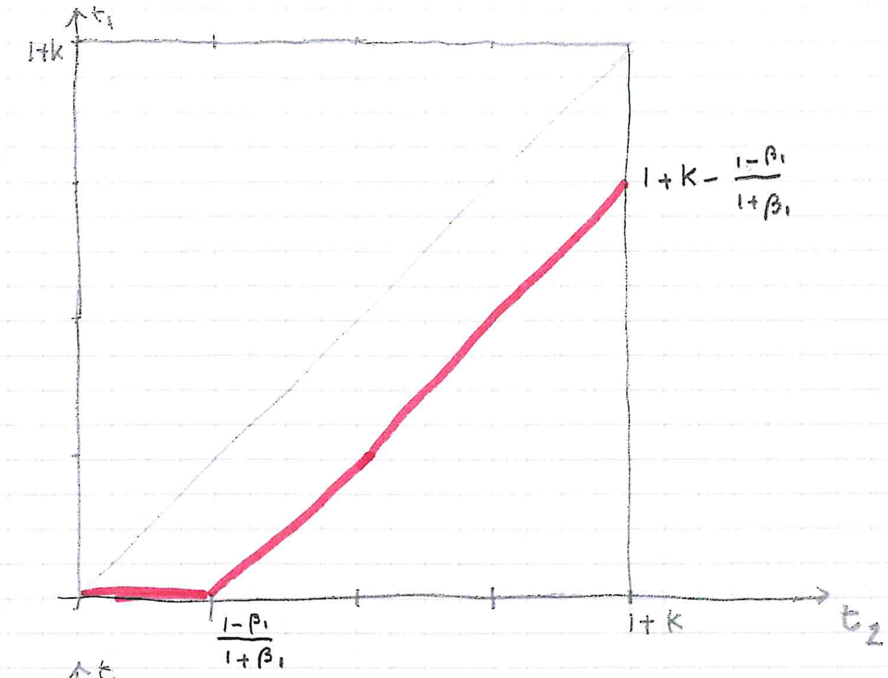
Observe that $t_1 = t_2 + \frac{\beta_1-1}{\beta_1+1} > 0$ and $t_1 = t_2 + \frac{\beta_1-1}{\beta_1+1} < 1 + k + \frac{1-\beta_1}{1+\beta_1} + \frac{\beta_1-1}{\beta_1+1} = 1 + k$. Then, $x_1 = 1 + (t_2 - t_1) = 1 - \frac{\beta_1-1}{\beta_1+1} = \frac{2}{\beta_1+1} > 0$ and $x_2 = 1 - (t_2 - t_1) = 1 + \frac{\beta_1-1}{\beta_1+1} = \frac{2\beta_1}{\beta_1+1} > 0$. Moreover,

$$0 = \gamma_1 = \delta_1 = \frac{\beta_1}{x_2} - \frac{1}{x_1} = \frac{\beta_1}{\frac{2\beta_1}{\beta_1+1}} - \frac{1}{\frac{2}{\beta_1+1}} = 0.$$

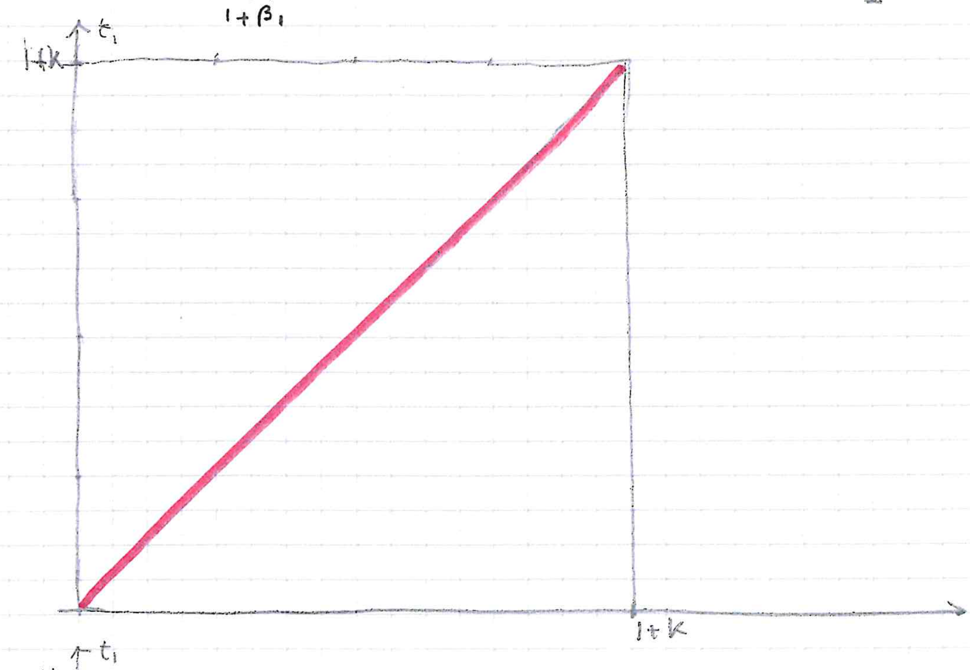
Then, household 1 and, by symmetry, household 2 reaction functions are presented below.

`insert picture reaction function h1 h2.pdf`

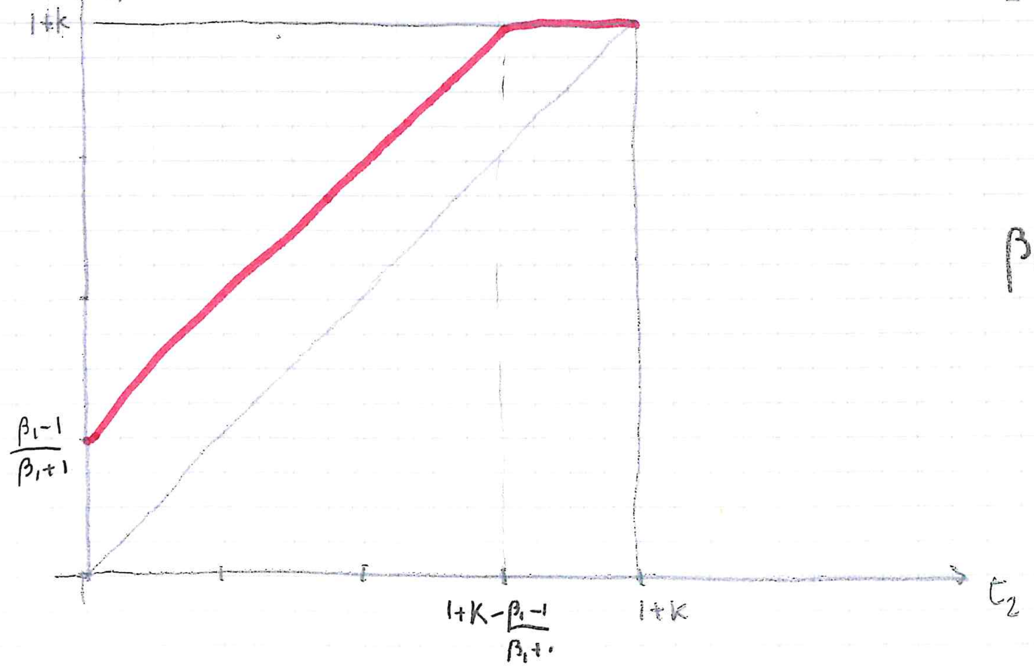
Hoosehold 1



$\beta_1 < 1$

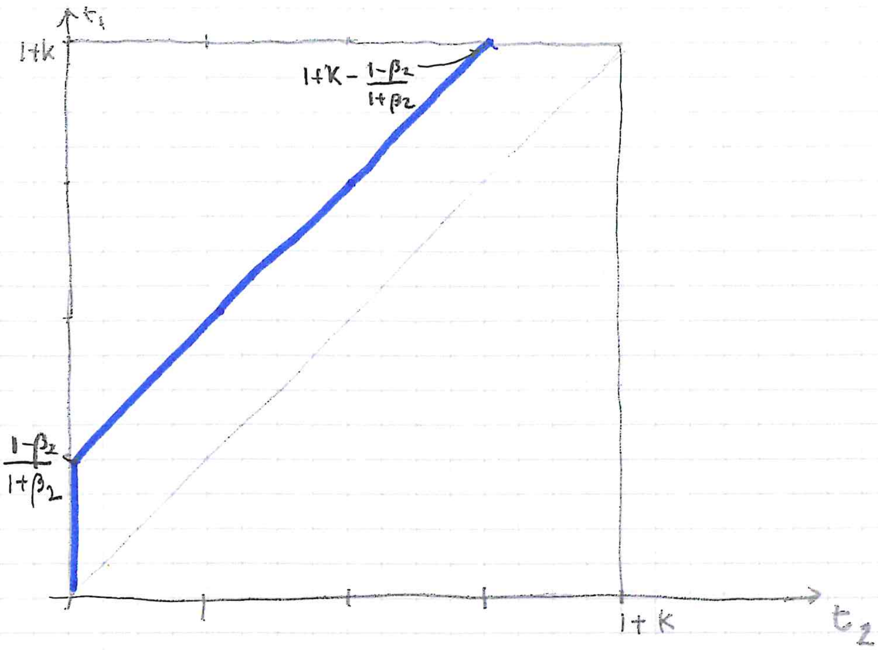


$\beta_1 = 1$

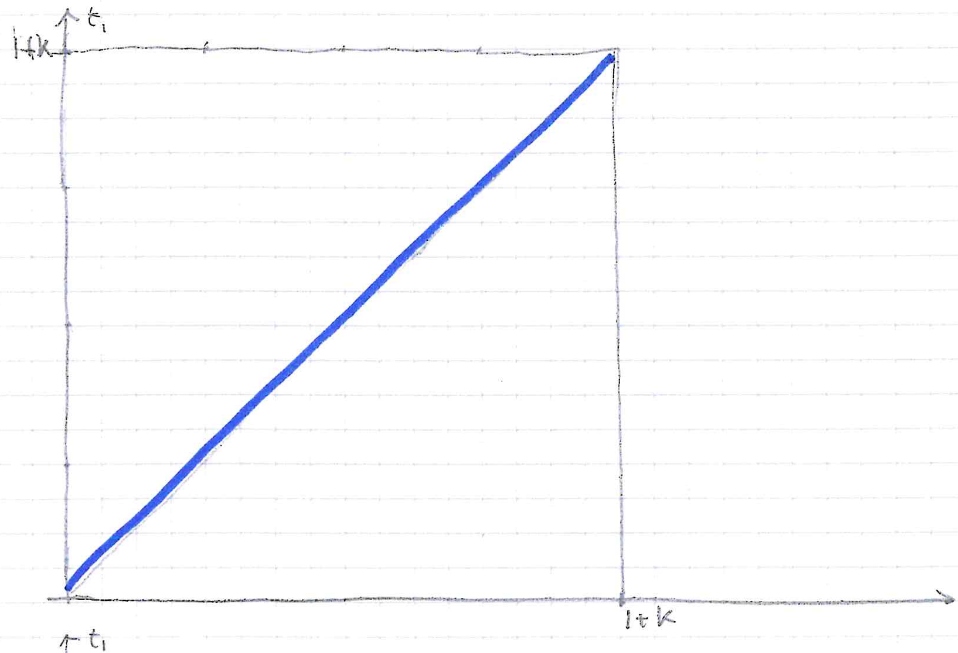


$\beta_1 > 1$

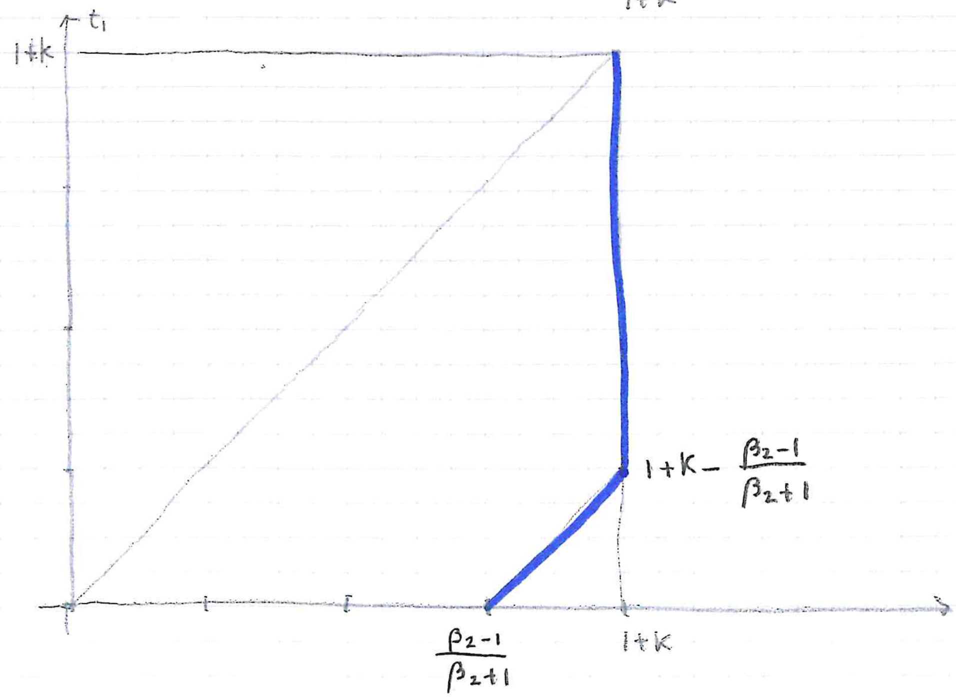
Household 2



$\beta_2 < 1$



$\beta_2 = 1$



$\beta_2 > 1$

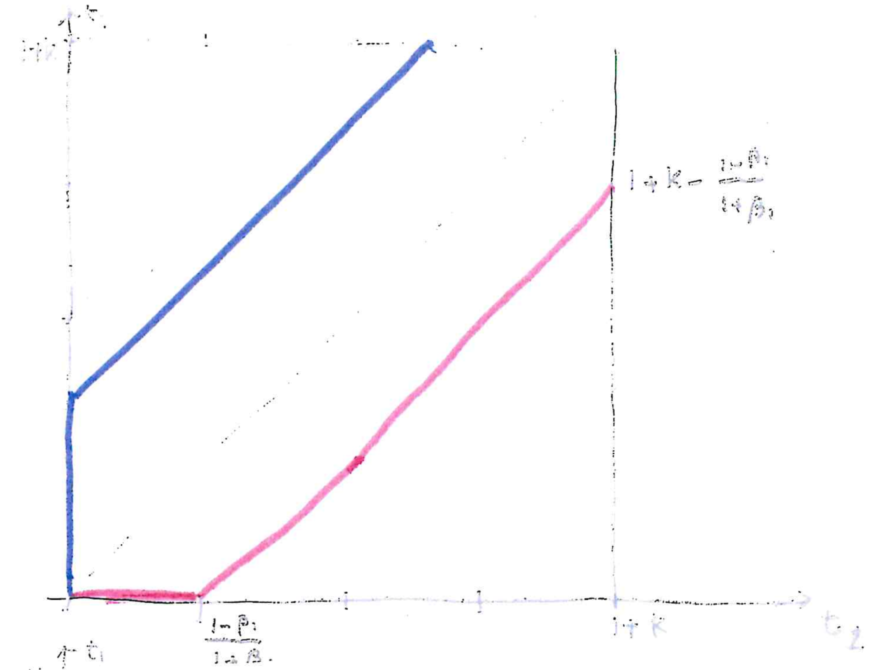
2.

To compute equilibria it is enough to find the intersection between the graphs of the two reaction functions and it is done below. It is crucial to observe that

$$\frac{\beta_1 - 1}{\beta_1 + 1} \le \frac{1 - \beta_2}{1 + \beta_2} \quad \Longleftrightarrow \quad \beta_1 \beta_1 \le 1.$$

Then equilibrium values of t_1 and t_2 and all other variables can be compute easily using the pictures below and Table (42) .

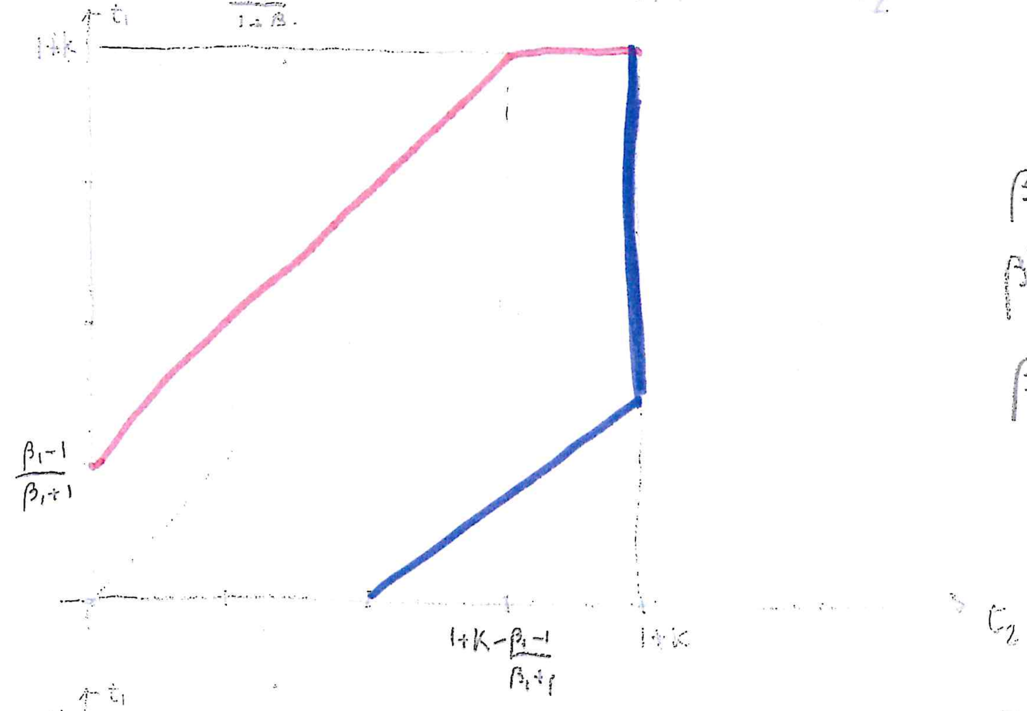
`insert picture` reaction functions and equilibria.pdf



$\beta_1 \beta_2 < 1$

$\beta_1 < 1$

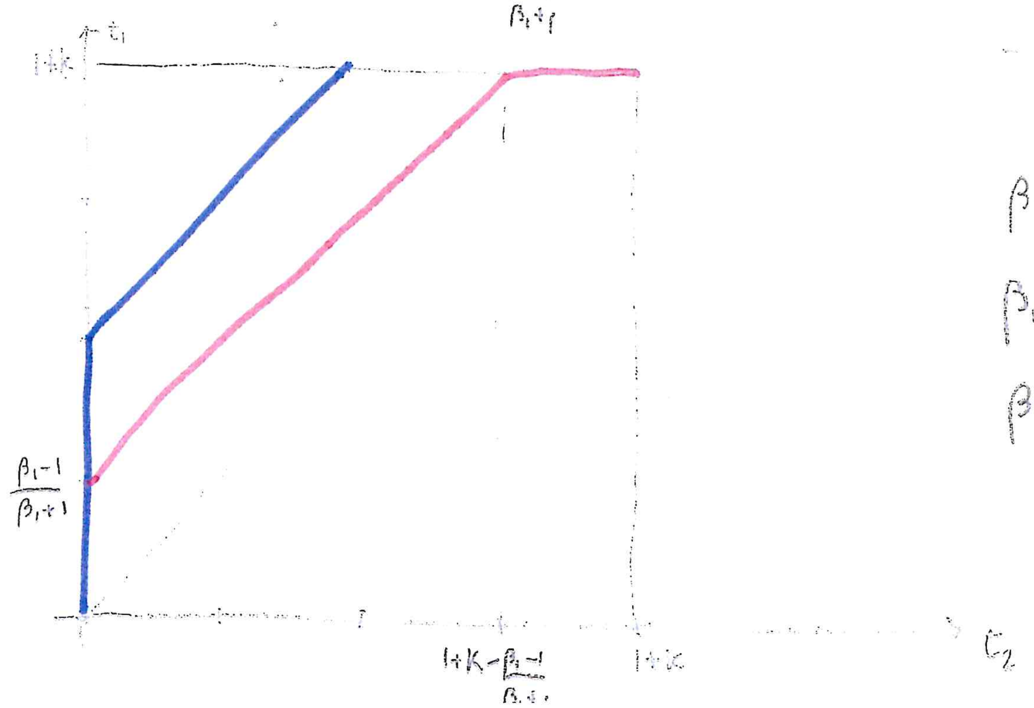
$\beta_2 < 1$



$\beta_1 \beta_2 > 1$

$\beta_1 > 1$

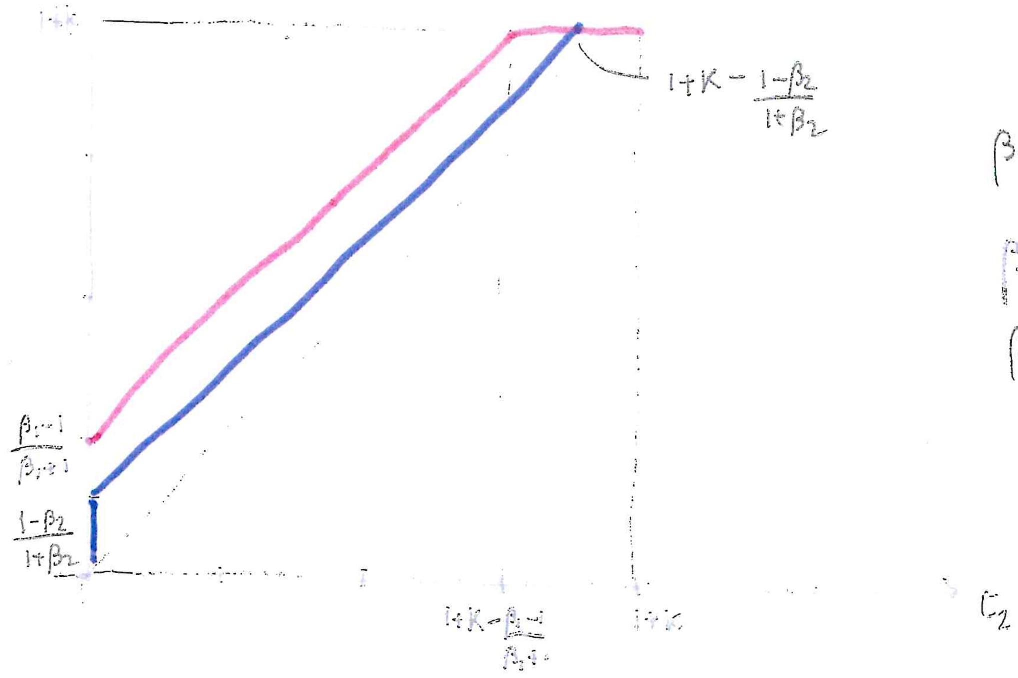
$\beta_2 > 1$



$\beta_1 \beta_2 < 1$

$\beta_1 > 1$

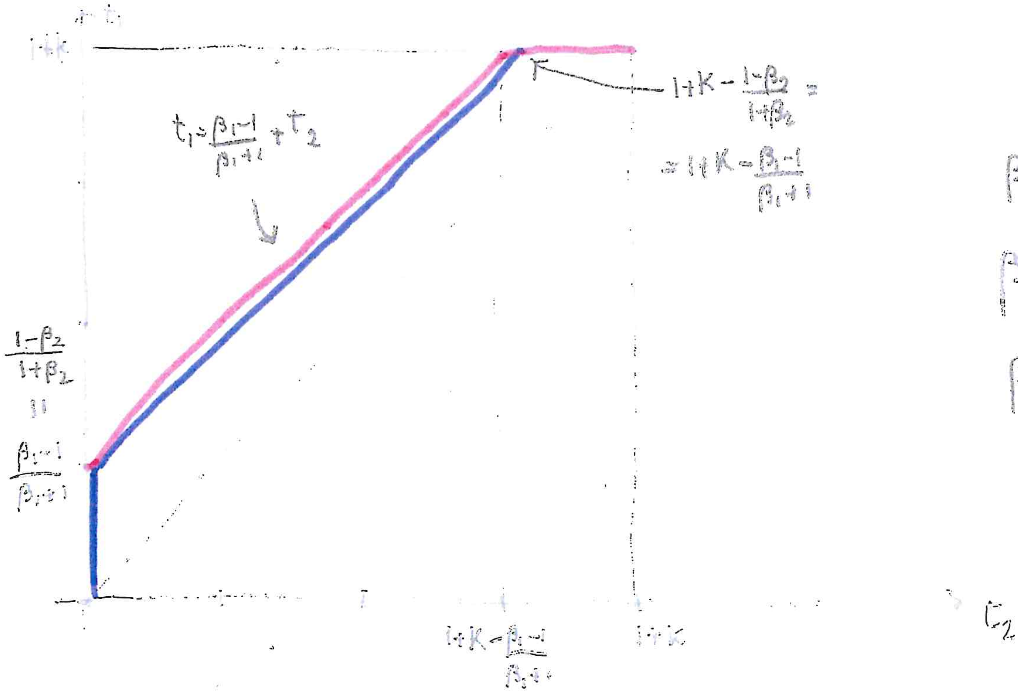
$\beta_2 < 1$



$\beta_1\beta_2 > 1$

$\beta_1 > 1$

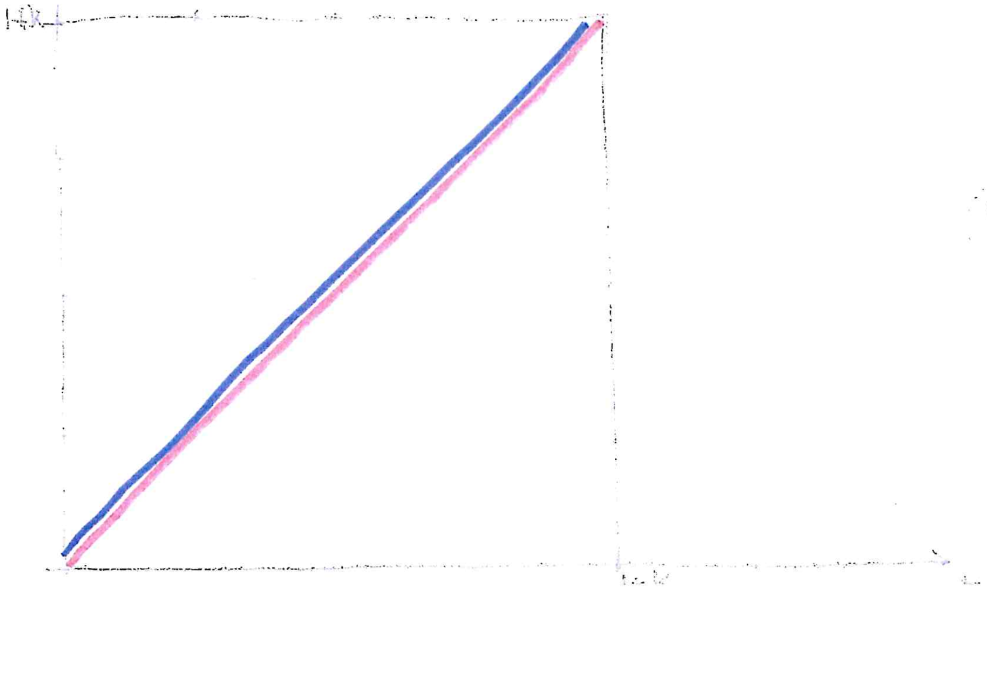
$\beta_2 < 1$



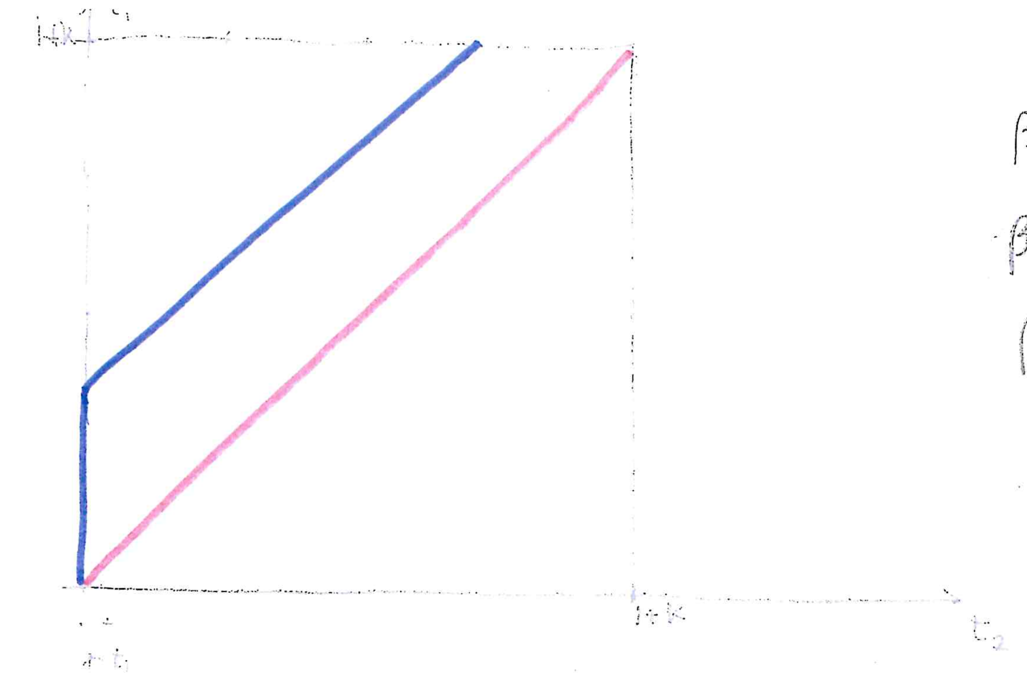
$\beta_1\beta_2 = 1$

$\beta_1 > 1$

$\beta_2 < 1$



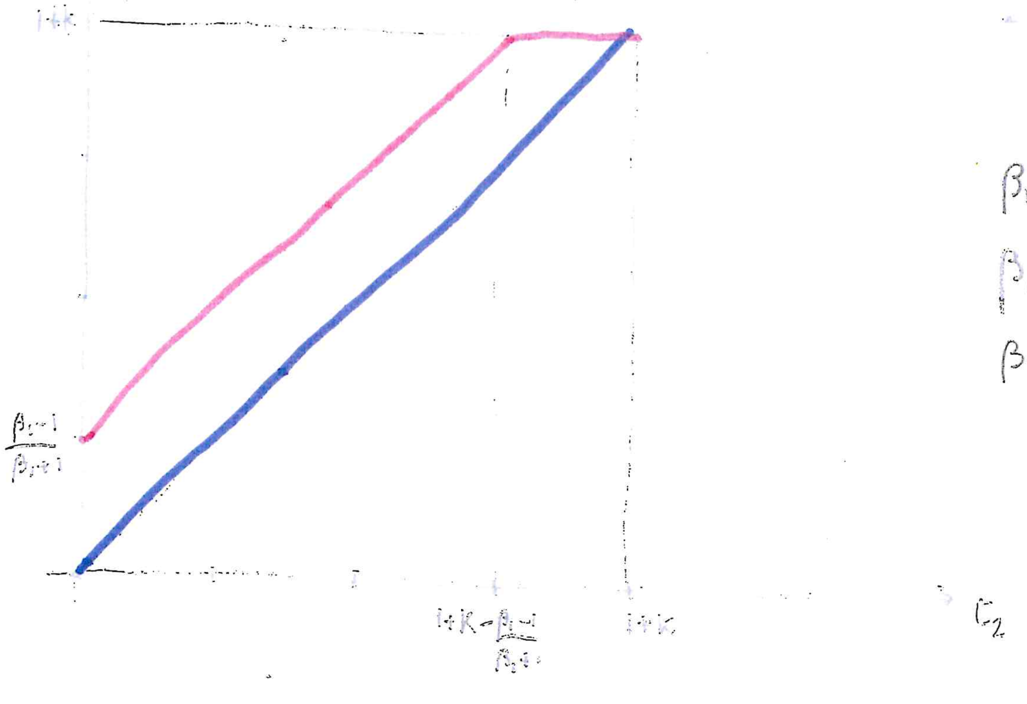
$\beta_1 = 1 = \beta_2$



$\beta_1 \beta_2 < 1$

$\beta_1 = 1$

$\beta_2 < 1$



$\beta_1 \beta_2 > 1$

$\beta_1 > 1$

$\beta_2 = 1$

3.

Observe that $\gamma_h = \delta_h$ implies $\gamma_h = \delta_h = 0$.
Case 1. $\beta_1\beta_2 < 1$, $\beta_1 < 1$ and $\beta_2 < 1$.

$$t_1 = 0 \qquad t_2 = 0$$

$$x_1 = 1 \qquad x_2 = 1$$

$$\gamma_1 = \delta_1 - \frac{\beta_1}{x_2} + \frac{1}{x_1} = 1 - \beta_1 \qquad \gamma_2 = \delta_2 - \frac{\beta_2}{x_1} + \frac{1}{x_2} = 1 - \beta_2$$

$$\delta_1 = 0 \qquad \delta_2 = 0$$

Case 2. $\beta_1\beta_2 > 1$, $\beta_1 > 1$ and $\beta_2 > 1$.

$$t_1 = 1 + k \qquad t_2 = 1 + k$$

$$x_1 = 1 \qquad x_2 = 1$$

$$\gamma_1 = 0 \qquad \gamma_2 = 0$$

$$\delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} = \beta_1 - 1 \qquad \delta_2 = \gamma_2 + \frac{\beta_2}{x_1} - \frac{1}{x_2} = \beta_2 - 1$$

Case 3. $\beta_1\beta_2 < 1$, $\beta_1 > 1$ and $\beta_2 < 1$.

$$t_1 = \frac{\beta_1 - 1}{\beta_1 + 1} \qquad t_2 = 0$$

$$x_1 = 1 - \frac{\beta_1 - 1}{\beta_1 + 1} = \frac{2}{\beta_1 + 1} \qquad x_2 = 1 + \frac{\beta_1 - 1}{\beta_1 + 1} = \frac{2\beta_1}{\beta_1 + 1}$$

$$\gamma_1 = 0 \qquad \gamma_2 = -\frac{\beta_2}{\frac{2}{\beta_1+1}} + \frac{1}{\frac{2\beta_1}{\beta_1+1}} = \frac{(1-\beta_1\beta_2)(\beta_1+1)}{2\beta_1}$$

$$\delta_1 = \frac{\beta_1}{x_2} - \frac{1}{x_1} = \frac{\beta_1}{\frac{2\beta_1}{\beta_1+1}} - \frac{1}{\frac{2}{\beta_1+1}} = 0 \qquad \delta_2 = 0$$

Case 4. $\beta_1\beta_2 > 1$, $\beta_1 > 1$ and $\beta_2 < 1$.

$$t_1 = 1 + k \qquad t_2 = 1 + k - \frac{1-\beta_2}{1+\beta_2}$$

$$x_1 = 1 - \frac{1-\beta_2}{1+\beta_2} = \frac{2\beta_2}{1+\beta_2} \qquad x_2 = 1 + \frac{1-\beta_2}{1+\beta_2} = \frac{2}{1+\beta_2}$$

$$\gamma_1 = 0 \qquad \gamma_2 = 0$$

$$\delta_1 = \frac{\beta_1}{\frac{2}{1+\beta_2}} - \frac{1}{\frac{2\beta_2}{1+\beta_2}} = \frac{(1-\beta_1\beta_2)(\beta_2+1)}{2\beta_2} \qquad \delta_2 = \frac{\beta_2}{\frac{2\beta_2}{1+\beta_2}} - \frac{1}{\frac{2}{1+\beta_2}} = 0$$

Case 5. $\beta_1\beta_2 = 1$, $\beta_1 > 1$ and $\beta_2 < 1$.
Observe that $\frac{\beta_1 - 1}{\beta_1 + 1} = \frac{1-\beta_2}{1+\beta_2}$

$$t_1 = t_2 + \frac{\beta_1 - 1}{\beta_1 + 1} \qquad t_2 \in \left[0, 1 + k - \frac{\beta_1 - 1}{\beta_1 + 1}\right]$$

$$x_1 = 1 - \frac{\beta_1 - 1}{\beta_1 + 1} = \frac{2}{\beta_1 + 1} = \frac{2\beta_2}{\beta_2 + 1} \qquad x_2 = 1 + \frac{\beta_1 - 1}{\beta_1 + 1} = \frac{2\beta_1}{\beta_1 + 1} = \frac{2}{\beta_2 + 1}$$

$$\gamma_1 = 0 \qquad \gamma_2 = \delta_2 - \frac{\beta_2}{x_1} + \frac{1}{x_2} = -\frac{\beta_2}{\frac{2\beta_2}{\beta_2+1}} + \frac{1}{\frac{2}{\beta_2+1}} = 0$$

$$\delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} = \frac{\beta_1}{\frac{2\beta_1}{\beta_1+1}} - \frac{1}{\frac{2}{\beta_1+1}} = 0 \qquad \delta_2 = 0$$

Case 6. $\beta_1 = 1$ and $\beta_2 = 1$.

$t_1 = t_2$ $\qquad$ $t_2 \in [0, 1+k]$

$x_1 = 1$ $\qquad$ $x_2 = 1$

$\gamma_1 = \delta_1 - \frac{\beta_1}{x_2} + \frac{1}{x_1} = \delta_1 = 0$ $\qquad$ $\gamma_2 = \delta_2 = 0$

$\delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} = \gamma_1 = 0$ $\qquad$ $\delta_2 = \gamma_2 + \frac{\beta_2}{x_1} - \frac{1}{x_2} = \gamma_2 = 0$

Case 7. $\beta_1\beta_2 < 1$, $\beta_1 = 1$ and $\beta_2 < 1$.

$t_1 = 0$ $\qquad$ $t_2 = 0$

$x_1 = 1$ $\qquad$ $x_2 = 1$

$\gamma_1 = \delta_1 - \frac{\beta_1}{x_2} + \frac{1}{x_1} = 0$ $\qquad$ $\gamma_2 = \delta_2 - \frac{\beta_2}{x_1} + \frac{1}{x_2} = -\frac{\beta_2}{1} + \frac{1}{1} = 1 - \beta_2$

$\delta_1 = 0$ $\qquad$ $\delta_2 = 0$

Case 8. $\beta_1\beta_2 > 1$, $\beta_1 > 1$ and $\beta_2 = 1$.

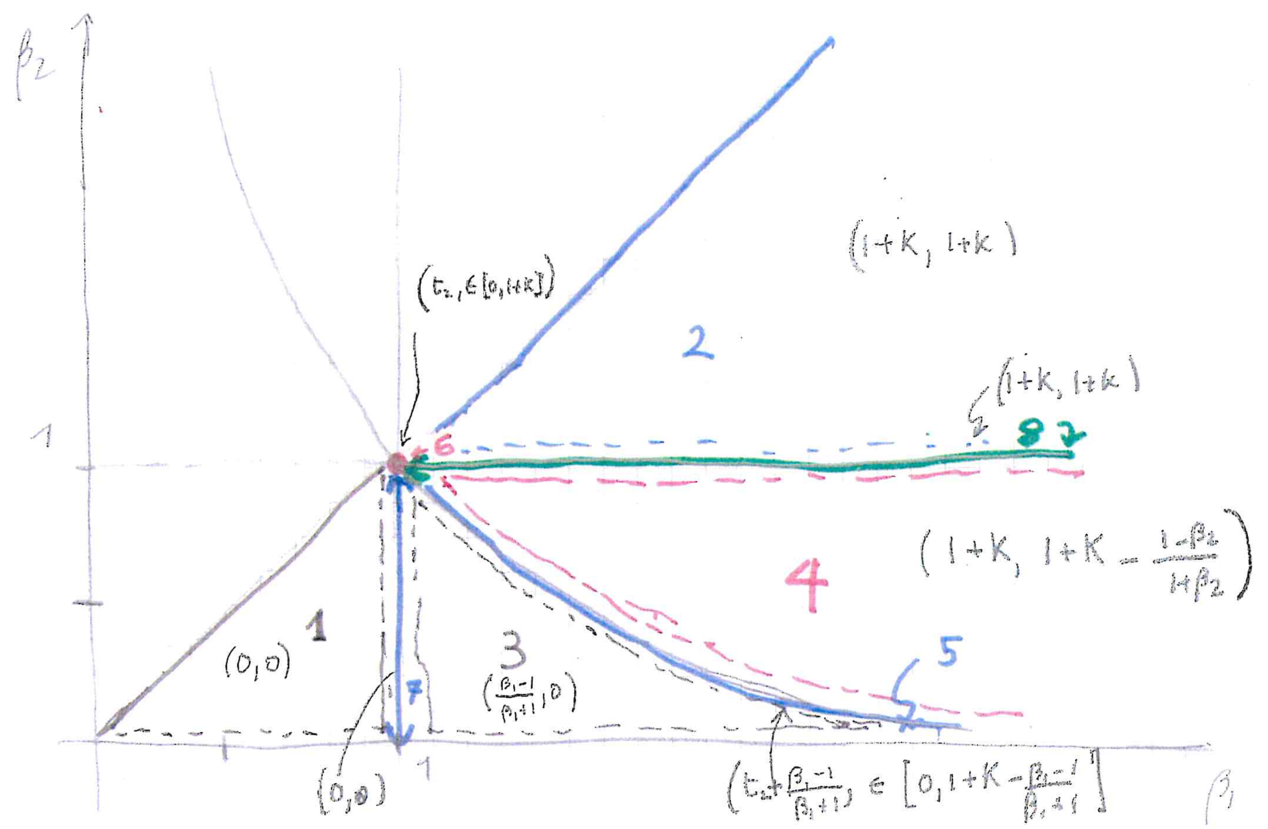
$t_1 = 1 + k$ $\qquad$ $t_2 = 1 + k$

$x_1 = 1$ $\qquad$ $x_2 = 1$

$\gamma_1 = 0$ $\qquad$ $\gamma_2 = 0$

$\delta_1 = \gamma_1 + \frac{\beta_1}{x_2} - \frac{1}{x_1} = \beta_1 - 1$ $\qquad$ $\delta_2 = \gamma_2 + \frac{\beta_2}{x_1} - \frac{1}{x_2} = 0$

The equilibrium values of transfer are summarized in the following picture.
`insert picture` equilibrium transfers in beta1-2 plane.pdf ■



β_2
β_1
1
1
$(1+K, 1+K)$
$(t_2, \in [0, 1+K])$
2
$(1+K, 1+K)$
8 7
6
$\left(1+K, 1+K - \frac{1-\beta_2}{1+\beta_2}\right)$
4
1
$(0,0)$
3
$\left(\frac{\beta_1 - 1}{\beta_1+1}, 0\right)$
7
5
$(0,0)$
$\left(t_2 + \frac{\beta_1 - 1}{\beta_1+1}, \in \left[0, 1+K - \frac{\beta_1 - 1}{\beta_1 + 1}\right]\right)$

3.2.2 Pareto optimality of equilibria

In general, the main difficulty of the analysis of the present model is that prices are an argument of the utility function, which is not the case if there is only one good as in the version of the model we are analyzing.

Below, we present the, obvious, definition of Pareto Optimality, see Definition 59. Then, we present two different approaches to study Pareto optimality of equilibria. The first one seems to be applicable to more general cases and it allows to show Pareto Optimality only in some of the presented cases of our Cobb-Douglas example. The second approach deals with the specific example we are analyzing and it allows to show Pareto optimality of equilibria for any economy - parametrized by $(\beta_1, \beta_2) \in \mathbb{R}^2_{++}$.

Definition 57 *Utility function of household* 1 *is*

$$U_1 : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}, \quad (x_1, x_2) \mapsto \log x_1 + \beta_1 \log x_2.$$

Symmetric definition applies for household 2*'s utility function. Define* $U = (U_h)_{h=1,2}$. *An economy is* (e, U)*, where* $e \in \mathbb{R}^2_{++}$.

Definition 58 *The set of feasible allocations associated with an endowment* $e \in \mathbb{R}^2_{++}$ *is denoted and defined as follows*

$$\mathcal{F}_e = \left\{ x \in \mathbb{R}^2_{++} : x_1 + x_2 \le e_1 + e_2 \right\}.$$

Definition 59 *Given an economy* (e, U)*, an allocation* $x^* \in \mathbb{R}^2_{++}$ *is* (e, U)*-Pareto optimal if*

1. $x^* \in \mathcal{F}_e$*, and*
2. *for any* $x' \in \mathcal{F}_e$ *either* $U(x^*) = U(x')$ *or there exists* $h \in \mathcal{H}$ *such that* $U_h(x^*) > U_h(x')$.

Remark 60 *Observe that* $x^* \in \mathbb{R}^2_{++}$ *is not* (e, U)*-Pareto optimal if either*

1. $x^* \notin \mathcal{F}_e$*, or*
2. *there exists* $x' \in \mathcal{F}_e$ *such that (* $U(x^*) \neq U(x')$ *and for any* $h \in \mathcal{H}$, $U_h(x^*) \le U_h(x')$ *), i.e.,* $U(x') > U(x^*)$.

A more general approach A simple, well-known result relates Pareto Optimal allocations to solutions of well chosen maximization problems.

Proposition 61 *An allocation* $(x_1^*, x_2^*) \in \mathbb{R}^2_{++}$ *is* (e, U)*-Pareto optimal*

$\Leftrightarrow$

$(x_1^*, x_2^*) \in \mathbb{R}^2_{++}$ *solves both the following problems.*

$$\left(\begin{array}{lll} (M1) & \textit{for given } (e,U), & \max_{(x_1,x_2)\in\mathbb{R}^2_{++}} U_1(x_1,x_2) \\ & & s.t. \\ & & U_2(x_1,x_2) - U_2(x_1^*,x_2^*) \ge 0 \\ & & -x_1 - x_2 + e_1 + e_2 \ge 0 \end{array} \right) \tag{45}$$

$$\left(\begin{array}{lll} (M2) & \textit{for given } (e,U), & \max_{(x_1,x_2)\in\mathbb{R}^2_{++}} U_2(x_1,x_2) \\ & & s.t. \\ & & U_1(x_1,x_2) - U_1(x_1^*,x_2^*) \ge 0 \\ & & -x_1 - x_2 + e_1 + e_2 \ge 0 \end{array} \right)$$

Proof. [$\Rightarrow$]

Suppose otherwise. Then, say, (x_1^*, x_2^*) does not solve $(M1)$. Then, there exists a feasible allocation $x' \in \mathbb{R}^2_{++}$ such that $U_1(x_1', x_2') > U_1(x_1^*, x_2^*)$ and $U_2(x_1', x_2') \ge U_2(x_1^*, x_2^*)$, contradicting the Pareto Optimality (x_1^*, x_2^*).

[$\Leftarrow$]

Suppose otherwise. Then there exists $x' \in \mathbb{R}^2_{++}$ which is feasible and such that

$$(U_1(x_1', x_2'), U_2(x_1', x_2')) > (U_1(x_1^*, x_2^*), U_2(x_1^*, x_2^*)).$$

Then, say, $U_1(x_1', x_2') > U_1(x_1^*, x_2^*)$, and $U_2(x_1', x_2') \ge U_2(x_1^*, x_2^*)$, which contradicts the fact that (x_1^*, x_2^*) solves $(M1)$. ■

Remark 62 *Since utility functions are strictly increasing, the solution set to each problem in (45) coincides with the solution set to the corresponding problem with equalities in the place of inequalities.*

Under our specification of the utility functions, Problem in (45) are as follows.

$$\left(\begin{array}{llll} (M1) & \text{for given } (e,\beta)\in\mathbb{R}^4_{++}, & \max_{(x_1,x_2)\in\mathbb{R}^2_{++}} & \log x_1+\beta_1\log x_2 \\ & & \text{s.t.} & \\ & & \log x_2+\beta_2\log x_1-\log x_2^*+\beta_2\log x_1^*\geq 0 & \chi_2 \\ & & -x_1-x_2+e_1+e_2\geq 0 & \gamma_1 \end{array}\right) \tag{46}$$

$$\left(\begin{array}{llll} (M2) & \text{for given } (e,\beta)\in\mathbb{R}^4_{++}„ & \max_{(x_1,x_2)\in\mathbb{R}^2_{++}} & \log x_2+\beta_2\log x_1 \\ & & \text{s.t.} & \\ & & \log x_1+\beta_1\log x_2-\log x_1^*+\beta_1\log x_2^*\geq 0 & \chi_1 \\ & & -x_1-x_2+e_1+e_2\geq 0 & \gamma_2 \end{array}\right)$$

Kuhn-Tucker conditions associated with problems in (46) are presented below.

$$\left(\begin{array}{lll} (K1) & \frac{1}{x_1}+\chi_2\beta_2\frac{1}{x_1}-\gamma_1 & =0 \\ & \beta_1\frac{1}{x_2}+\chi_2\frac{1}{x_2}-\gamma_1 & =0 \\ & \min\{...,\chi_2\} & =0 \\ & \min\{...,\gamma_1\} & =0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{lll} (K2) & \frac{1}{x_2}+\chi_1\beta_1\frac{1}{x_2}-\gamma_2 & =0 \\ & \beta_2\frac{1}{x_1}+\chi_1\frac{1}{x_1}-\gamma_2 & =0 \\ & \min\{...,\chi_1\} & =0 \\ & \min\{...,\gamma_2\} & =0 \end{array}\right)$$

Lemma 63 *The following statements on problem $(M1)$ and Kuhn-Tucker conditions $(K1)$ hold true. Symmetric results hold true for $(M2)$ and $(K2)$.*

1. *For any given (e,U), if $(x_1^*,x_2^*)\in\mathbb{R}^2_{++}$ is feasible, then there exists a unique solution to problem $(M1)$;*
2. *If there exists $(\chi_2^*,\gamma_1^*)\in\mathbb{R}^2$ such that $(x_1^*,x_2^*,\chi_2^*,\gamma_1^*)$ solves $(K1)$, then (x_1^*,x_2^*) solves $(M1)$;*
3. *If (x_1^*,x_2^*) solves $(M1)$ and $\boxed{x_1^*\neq\beta_2x_2^*}$, then there exists (χ_2^*,γ_1^*) such that $(x_1^*,x_2^*,\chi_2^*,\gamma_1^*)$ solves $(K1)$.*

Proof. 1.

Existence. Let C_1 be the constraint set of problem $(M1)$. Since $(x_1^*,x_2^*)\in\mathbb{R}^2_{++}$ is feasible, then (x_1^*,x_2^*) belongs to C_1, which is not empty. From the extreme value theorem, we are left with showing that C_1 is compact. From the first constraint in problem $(M1)$ and from the definition of the log functions, we conclude that C_1 is contained in a closed subset of $\mathbb{R}^C$. Moreover, from the fact that the consumption set is $\mathbb{R}^C_{++}$, C_1 is bounded from below by zero. From constraint (2) in problem $(M2)$, we have that C_1 is bounded from above. Hence C_1 is compact.

Uniqueness. Since the constraint functions are quasi-concave, the constraint set is convex. The objective function is strictly concave.

2.

Sufficiency of Kuhn-Tucker conditions follows from the fact that the objective function is strictly concave and the constraint functions are quasi-concave.

3.

From Remark 62, necessity of Kuhn-Tucker conditions follows from the fact that the Jacobian matrix of the constraint function has full row rank. The computation of that matrix is described below.

	x_1	x_2
$\log x_2+\beta_2\log x_1$	$\frac{\beta_2}{x_1}$	$\frac{1}{x_2}$
$-x_1-x_2+r$	-1	-1

Then, the full rank condition is satisfied iff it is not the case that

$$\det\begin{bmatrix}\frac{\beta_2}{x_1} & \frac{1}{x_2} \\ -1 & -1\end{bmatrix}=-\frac{\beta_2}{x_1}+\frac{1}{x_2}=0, \text{ or } x_1=\beta_2x_2.$$

Observe that γ_1 is strictly positive. ■

We can summarize what said above in the following result.

Corollary 64 *If* (x_1^*, x_2^*) *is such that* $\boxed{x_1^* \neq \beta_2 x_2^* \text{ and } x_2^* \neq \beta_1 x_1^*}$, *then the following statements are equivalent.*

1. (x_1^*, x_2^*) *is Pareto Optimal;*

2. (x_1^*, x_2^*) *solves* $(M1)$ *and* $(M2)$*;*

3. There exist (χ_2^*, γ_1^*) *and* (χ_1^*, γ_2^*) *such that* $(x_1^*, x_2^*, \chi_2^*, \gamma_1^*)$ *solves* $(K1)$ *and* $(x_1^*, x_2^*, \chi_1^*, \gamma_2^*)$*solves* $(K2)$.

Using Corollary 64, we can analyze the Pareto optimality of equilibrium allocations if $x_1^* \neq \beta_2 x_2^*$ and $x_2^* \neq \beta_1 x_1^*$.

Proposition 65 . *In cases 1 and 2 of Proposition* **??**, *the equilibrium allocation* $(1,1)$ *is Pareto Optimal.*

Proof. As an application of Corollary 64, we have check if the equilibrium allocations under analysis do or do not satisfy the appropriate Kuhn-Tucker conditions. Observe that since we are assuming $\beta_1 \neq 1$ and $\beta_2 \neq 1$, it suffices to check that $(x_1^*, x_2^*) = (1,1)$ does satisfy both sets of Kuhn-Tucker conditions below.

$$\left(\begin{array}{lll} (K1) & \frac{1}{x_1} + \chi_2 \beta_2 \frac{1}{x_1} - \gamma_1 & = 0 \\ & \beta_1 \frac{1}{x_2} + \chi_2 \frac{1}{x_2} - \gamma_1 & = 0 \\ & \min\{..., \chi_2\} & = 0 \\ & \min\{..., \gamma_1\} & = 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{lll} (K2) & \frac{1}{x_2} + \chi_1 \beta_1 \frac{1}{x_2} - \gamma_2 & = 0 \\ & \beta_2 \frac{1}{x_1} + \chi_1 \frac{1}{x_1} - \gamma_2 & = 0 \\ & \min\{..., \chi_1\} & = 0 \\ & \min\{..., \gamma_2\} & = 0 \end{array}\right)$$

First of all, observe that $(1,1)$ satisfies both inequality constraints in $(K1)$ and $(K2)$. Moreover, just by substitution, we have

$$\left(\begin{array}{ll} 1 + \chi_2 \beta_2 - \gamma_1 & = 0 \\ \beta_1 + \chi_2 - \gamma_1 & = 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ll} 1 + \chi_1 \beta_1 - \gamma_2 & = 0 \\ \beta_2 + \chi_1 - \gamma_2 & = 0 \end{array}\right)$$

or

$$\left(\begin{array}{ll} \gamma_1 - \chi_2 \beta_2 & = 1 \\ \gamma_1 - \chi_2 & = \beta_1 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ll} \gamma_2 - \chi_1 \beta_1 & = 1 \\ \gamma_2 - \chi_1 & = \beta_2 \end{array}\right)$$

About $(K1)$, observe that

$$\chi_2 = \frac{\beta_1 - 1}{\beta_2 - 1} > 0$$

because in the Cases under analysis we have either both β_1 and β_2 greater than one or both of them smaller than one. Moreover,

$$\gamma_1 = 1 + \chi_2 \beta_2 > 0.$$

About $(K2)$, observe that

$$\chi_1 = \frac{\beta_2 - 1}{\beta_1 - 1} > 0$$

because in the Cases under analysis we have either both β_1 and β_2 greater than one or both of them smaller than one. Moreover,

$$\gamma_2 = 1 + \chi_1 \beta_1 > 0.$$

■

A specific approach

Proposition 66 *Equilibrium allocations are Pareto Optimal.*

Proof. We proceed as follows.

1. We write Maximization Problems $(M1)$ and $(M2)$ without the constraint on the utility and incorporate the constraint about feasibility into the objective function, i.e.,

$$\left(\ (M1-1) \quad \text{for given } \beta_1 \in \mathbb{R}_{++}, \quad \max_{x_2 \in (0,2)} \ \log(2 - x_2) + \beta_1 \log x_2 \ \right)$$
$$\left(\ (M2-2) \quad \text{for given } \beta_2 \in \mathbb{R}_{++},, \quad \max_{x_2 \in (0,2)} \ \log x_2 + \beta_2 \log(2 - x_2) \ \right)$$

■

A simple analysis allows to find the global maximum in each case.

2. Using the above analysis, we use show Pareto Optimality of each equilibria.

Proof. 1.

Since $\beta_1 > 0$, the possible utility levels for household 1 which are compatible with feasibility are described by the following function of the value of household 2's consumption.

$$U_1 : (0,2) \longrightarrow \mathbb{R}, \qquad U_1(x_2) = \log(2 - x_2) + \beta_1 \log x_2.$$

Then,

$$U_1'(x) = -\tfrac{1}{2-x} + \tfrac{\beta_1}{x}$$

$$U_1''(x) = -\tfrac{1}{(2-x)^2} - \tfrac{\beta_1}{x^2} < 0$$

Then, solution x^* to the equation $U_1'(x) = -\frac{1}{2-x} + \frac{\beta_1}{x} = 0$ is the solution to the maximization problem if $x^* \in (0,2)$. Indeed, the solution is

$$x_2^{*1} = 2\frac{\beta_1}{\beta_1 + 1} \in (0,2).$$

Observe that x_2^{*1} is the value of the consumption of household 2 which maximizes the utility of household 1.

Given $\beta_2 > 0$, the possible utility levels for household 2 which are compatible with feasibility are described by the following function of the value of household 2's consumption.

$$U_2 : (0,2) \longrightarrow \mathbb{R}, \qquad U_2(x_2) = \log x_2 + \beta_2 \log(2 - x_2).$$

Then

$$U_2'(x) = \tfrac{1}{x} - \tfrac{\beta_2}{r-x}$$

$$U_2''(x) = -\tfrac{1}{x^2} - \tfrac{\beta_2}{(r-x)^2} < 0$$

Then, solution x^* to the equation $f'(x) = \frac{1}{x} - \frac{\beta_2}{r-x} = 0$ is the solution to the maximization problem if $x^* \in (0,2)$. Indeed, the solution is

$$x_2^{*2} = \frac{2}{\beta_2 + 1} \in (0,2).$$

x_2^{*2} is the value of the consumption of household 2 which maximizes the utility of household 2 herself.

2.

Observe that

$$\begin{array}{l} 2\frac{\beta_1}{\beta_1+1} \geq \frac{2}{\beta_2+1} \\ \Leftrightarrow \\ 0 \leq 2\beta_1(\beta_2+1) - 2(\beta_1+1) = 2(\beta_1 - 1)(\beta_2 + 1) \\ \Leftrightarrow \\ \beta_1 \geq 1 \end{array}$$

which is true in all cases we are analyzing but Case 1. Then in all but that case we have

$$---0------\left(\begin{array}{l} x_2^{*2} = \frac{2}{1+\beta_2} = \\ (\text{max for } h = 2) \end{array}\right)------\left(\begin{array}{l} x_2^{*1} = \frac{2\beta_1}{1+\beta_1} = \\ (\text{max for } h = 1) \end{array}\right)------2---$$

Observe that by definition of global maximum point,

$$\begin{array}{llll} U_1 \text{ is increasing} & \text{and} & U_2 \text{ is increasing} & \text{on } \left(0, x_2^{*2}\right]; \\ U_1 \text{ is increasing} & \text{and} & U_2 \text{ is decreasing} & \text{on } \left[x_2^{*2}, x_2^{*1}\right]; \\ U_1 \text{ is decreasing} & \text{and} & U_2 \text{ is decreasing} & \text{on } \left[x_2^{*1}, 2\right). \end{array}$$

Then, by definition of Pareto Optimality,

any $x_2 \in \left(0, x_2^{*2}\right)$ is not Pareto Optimal: any $x_2' \in \left(x_2, x_2^{*2}\right]$ gives a higher utility to both households;

any $x_2 \in \left(x_2^{*1}, 2\right)$ is not Pareto Optimal: any $x_2' \in \left(x_2^{*1}, x_2,\right)$ gives a higher utility to both households;

any $x_2 \in \left[x_2^{*2}, x_2^{*1}\right]$ is Pareto Optimal: for any $x_2' < x_2$, we have $U_1\left(x_2'\right) < U_1\left(x_2\right)$ and for any $x_2' > x_2$, we have $U_2\left(x_2'\right) > U_2\left(x_2\right)$.

Symmetric situation arises in Case 1.

Then, to verify Pareto Optimality in the five Cases we presented, we have to check

$$\begin{array}{lll} x_2^* \in \left[x_2^{*2} = \frac{2}{1+\beta_2}, x_2^{*1} = \frac{2\beta_1}{1+\beta_1}\right] & \text{if} & \beta_1 \geq 1 \\ x_2^* \in \left[x_2^{*1} = \frac{2\beta_1}{1+\beta_1}, x_2^{*2} = \frac{2}{1+\beta_2},\right] & \text{if} & \beta_1 \leq 1 \end{array} \tag{47}$$

From Table (41), the values of x_2^* are the ones presented below.

Case	1	2	3	4	5	6	7	8
x_2^*	1	1	$\frac{2\beta_1}{1+\beta_1}$	$\frac{2\beta_2}{1+\beta_2}$	$\frac{2}{1+\beta_2}$	1	1	$\frac{2\beta_1}{1+\beta_1}$

Checking conditions presented in (47) is immediate apart for Cases 1 and 2 which are analyzed below.

Case 1. $\beta_1 \leq 1$, $\beta_2 \leq 1$ and $x_2^* = 1$.

Indeed, $x_2^* = 1 \geq \frac{2\beta_1}{1+\beta_1} \Leftrightarrow 1+\beta_1 \geq 2\beta_1 \Leftrightarrow \beta_1 \leq 1$ and $x_2^* = 1 \leq \frac{2}{1+\beta_1} \Leftrightarrow 1+\beta_1 \leq 2 \Leftrightarrow \beta_2 \leq 1$.

Case 2. $\beta_1 \geq 1$, $\beta_2 \geq 1$ and $x_2^* = 1$. ■

Remark 67 *The results presented in Table 41 allow to make some observations, which could be used to get conjectures about a more general framework. If an artificial upper bound on transfers is imposed, then pippo*

1. All equilibrium allocations are Pareto Optimal; this conjecture is false as shown by Kranich (1988);

2. An infinite number of equilibria occurs only in a closed measure zero subset of the set of economies, but equilibrium allocations are constant across equilibria.

3. If $\beta_1\beta_2 > 1$, then equilibria exist (in contrast to the case presented in Proposition 51) and at least one household chooses her transfer to be equal to the upper bound.

4 Appendices

4.1 The set-up of the model by Mercier Ythier

Another paper which studies a model somehow similar to the one introduced by Kranich (1988) is that one presented in Mercier Ythier (2000). Below, we present the definition of equilibrium. In a companion paper (in progress), we discuss that model. Here we want only to underline that a basic observation about Kranich (1988)'s model applies to Mercier Ythier 's model as well: in the maximization problem of each household there is an inconsistency about the assumption that consumption vectors have to be nonnegative and the possibility that consumption vectors of other household may be negative consistently with other households admissible choices. In what follows, we try to make clear the above statement.

The model by Mercier Ythier is presented in terms of excess demand vectors instead of consumption vectors; to make his model more understandable, we rewrite the standard exchange economy model in terms of excess demand vectors.

Definition 68 $(x^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **allocation-**price equilibrium for an economy* (e, u) *if*

(i) households maximize, i.e., $\forall h \in \mathcal{H}$, for given p^*, e_h, u_h, we have that x_h^* solves the problem

$$\max_{x_h \in \mathbb{R}^C} \quad u_h\left(x_h\right) \qquad s.t. \qquad \begin{array}{l} p^* e_h - p^* x_h \geq 0 \\ x_h \geq 0 \end{array} \tag{48}$$

(ii) x^* satisfies the following (market clearing) conditions (at e)

$$\sum_{h=1}^{H} x_h = \sum_{h=1}^{H} e_h$$

To better understand the model in terms of excess demand vectors define and denote it as follows

$$z_h = x_h - e_h,$$
and then
$$x_h = z_h + e_h.$$

We can then give the following definition.

Definition 69 $(z^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **excess–demand**-price equilibrium for an economy* (e, u) *if*

(i) households maximize, i.e., $\forall h \in \mathcal{H}$, for given p^*, e_h, u_h, we have that z_h^* solves the problem

$$\max_{z_h \in \mathbb{R}^C} \quad u_h (z_h + e_h) \qquad s.t. \qquad \begin{array}{l} -p^* z_h \geq 0 \\ z_h + e_h \geq 0 \end{array} \tag{49}$$

(ii) z^* satisfies the following (market clearing) conditions (at e)

$$\sum_{h=1}^{H} z_h^* = 0$$

Proposition 70 *1.* $(x^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **allocation-**price equilibrium for an economy* (e, u)
$\Rightarrow$
$(x^* - e, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **excess–demand**-price equilibrium for an economy* (e, u).
2.
$(z^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **excess–demand**-price equilibrium for an economy* (e, u)
$\Rightarrow$
$(z^* + e, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^C_{++}$ *is an **allocation-**price equilibrium for an economy* (e, u).

Proof. 1.
Clearly market clearing is satisfied.
We want to show that
a. $p^* z_h^* \leq 0$ and $z_h^* + e_h \geq 0$;
b. if $p^* z_h \leq 0$ and $z_h + e_h \geq 0$, then $u_h (z_h^* + e_h) > u_h (z_h + e_h)$.
Indeed,
a. $p^* z_h^* := p^* (x_h^* - e_h) \overset{\text{Assu.}}{\leq} 0$; $z_h^* + e_h := x_h^* - e_h + e_h = x_h^* \overset{\text{Assu.}}{\geq} 0$.
b. Suppose otherwise, i.e., there exists $\widehat{z}_h \in \mathbb{R}^C$ such that $p^* \widehat{z}_h \leq 0$ and $\widehat{z}_h + e_h \geq 0$ and $u_h (\widehat{z}_h + e_h) > u_h (z_h^* + e_h)$.
Define $\widehat{x}_h = \widehat{z}_h + e_h$. Then $p^* \widehat{z}_h \leq 0 \Rightarrow p^* (\widehat{x}_h - e_h) \leq 0$; $\widehat{z}_h + e_h \geq 0 \Rightarrow \widehat{x}_h \geq 0$ and
$u_h (\widehat{z}_h + e_h) > u_h (z_h^* + e_h) \Rightarrow u_h (\widehat{x}_h) > u_h (x_h^*)$, contradicting the definition of x_h^* as a maximizing choice.
2.
Exercise. ■
We can now write the definition of equilibrium in terms of excess demand as done in M-Y (2000).
Let the functions below be given:

$$u_h : \mathbb{R}^C_+ \longrightarrow \mathbb{R}, \qquad x_h \mapsto u_h (x_h)$$

$$w_h : \mathbb{R}^H \longrightarrow \mathbb{R} \qquad y \mapsto w_h(y_h, y_{\setminus h})$$

Definition 71 $(z^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{C(H-1)H} \times \mathbb{R}^C_+$ *is an excess demand-transfer-price equilibrium for the economy* (e, u, w) *if* *(i) households maximize, i.e., $\forall h \in \mathcal{H}$, for given $z^*_{\setminus h} \in \mathbb{R}^{C(H-1)}, p^* \in \mathbb{R}^C_+, , t^*_{\setminus h} \in \mathbb{R}^{H(C-1)C}, e \in \mathbb{R}^{CH}_{++}, u, w_h$ we have that (z^*_h, t^*_h) solves the problem*

$$\max_{(z_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{C(H-1)}}$$
$$w_h\left(u_h\left(z_h+e_h+t^*_{\longrightarrow h}-t_{h\longrightarrow}\right),\ \left(u_{h'}\left(z^*_{h'}+e_{h'}+t_{hh'}+t^*_{noth\longrightarrow h'}-t^*_{h'\longrightarrow}\right)\right)_{h'\in\setminus\{h\}}\right)$$

s.t.

$$-p^*z_h\geq 0$$
$$t_h\geq 0$$
$$z_h+e_h+t^*_{\longrightarrow h}-t_{h\longrightarrow}\geq 0$$

(ii) z^* satisfies the following (market clearing) conditions (at e)

$$\sum_{h=1}^{H} z^*_h = 0$$

Observe that consumption is $x_h = z_h + e_h + t_{\longrightarrow h} - t_{h\longrightarrow}$.

In the above model as in Kranich (1988) model, the problem is not well written simply because we have the same problem presented in Remark **??**.

Let's rewrite the budget constraint, keeping into account that consumption has to be nonnegative for any household different from h as well:

$$\begin{array}{ll} -p^*z_h\geq 0 & \\ t_h\geq 0 & \\ z_h+e_h+t^*_{\longrightarrow h}-t_{h\longrightarrow}\geq 0 & \\ z^*_{h'}+e_{h'}+t_{hh'}+t^*_{noth\longrightarrow h'}-t^*_{h'\longrightarrow}\geq 0 & h'\neq h \end{array}$$

or since for any $h\in\mathcal{H}$, $x_h = z_h + e_h + t_{\longrightarrow h} - t_{h\longrightarrow}$, we have

$$\begin{array}{ll} -p^*\left(x_h-e_h-t_{\longrightarrow h}+t_{h\longrightarrow}\right)\geq 0 & \\ t_h\geq 0 & \\ x_h\geq 0 & \\ z^*_{h'}+e_{h'}+t_{hh'}+t^*_{noth\longrightarrow h'}-t^*_{h'\longrightarrow}\geq 0 & h'\neq h \end{array}$$

0bserve that the very definition of the maximization problem allows $z^*_{\setminus h}\in\mathbb{R}^{C(H-1)}$. Then use exactly the same numerical example provided in Remark adding only $z^*_2 = 0$ and again you get the constraint set to be empty. The main equations above for household 1 are

$x_1 - 1 + t_{12} + t_{13} \leq 1$ and then $t_{12}\leq 1$,

$z_2 + e_2 + t_{12} + t_{32} - t_{21} - t_{23} \geq 0$ and then $0 + 1 + t_{12} + 0 - 0 - 3 \geq 0$ or $1 + t_{12} - 3 \geq 0$ or $t_{12}\geq 2$

and again the budget set is empty.

Remark 72 *Observe that imposing the "legal constraint" $pt_{h\rightarrow}\leq pe_h$ does not seems to solve the problem, differently from what happens in Kranich (1988)'s model. Indeed, by Walras law $pz_h = 0$; then*

$$px_h = p\left(z_h+e_h+t_{\longrightarrow h}-t_{h\longrightarrow}\right) = \overset{=0}{pz_h} + p\left(\overset{\geq 0}{e_h - t_{h\longrightarrow}}\right) + \overset{\geq 0}{pt_{\longrightarrow h}} \geq 0,$$

which is consistent with $x_h\geq 0$, but does not imply it.

Remark 73 *Of course, the problem described above does not arise if the utility function is assumed to be defined for negative value of the consumption vector as well. Indeed, on page 47, line 3, Mercier Ythier says that the consumption vector is an element of $\mathbb{R}^C$ (using our notation). We believe the assumption that the utility function is defined for negative values of consumption is economically inconsistent.*

4.2 Kranich (1988)'s definition and our definition of equilibrium

Our definition.

Definition 74 *The vector* $(x^*, t^*, p^*) \in X \times T \times \mathbb{R}^C_+$ *is an* ***equilibrium*** *vector for the economy* $(X_h, T_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathcal{E}$ *if*

1.

For any $h \in \mathcal{H}$, *household h maximizes, i.e., for given* $(X_h, T_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathcal{E}$, $p^* \in \mathbb{R}^C_+, t^*_{\backslash h} \in T_{\backslash h}, (x^*_h, t^*_h) \in X_h \times T_h$ *solves*

$$\begin{array}{ll} \max_{(x_h,t_h)\in X_h\times T_h} & u_h\left(x_h, w\left(p^*, t_h, t^*_{\backslash h}\right)\right) \\ s.t. & \\ p^* x_h \leq p^*\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t^*_{h'h} - t_{hh'}\right)\right) & \end{array}$$

2.

Markets clear, i.e.,

$$\sum_{h=1}^{H} x^*_h = \sum_{h=1}^{H} e_h.$$

Kranich's definition.

Let (e, u) be given.

Definition 75

$$w : X_h \times T_h \times T_{\backslash h} \longrightarrow \mathbb{R}^H,$$

$$\left(p, t_h, t_{\backslash h},\right) \mapsto \left(w_h\left(p, t_h, t_{\backslash h}\right)_{h'\in\mathcal{H}}\right) := \left(p\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t_{h'h} - t_{hh'}\right)\right)\right)_{h\in\mathcal{H}}.$$

Definition 76 *The set of feasible outcome at* $r \in \mathbb{R}^C_{++}$ *is denoted and defined as follows*

$$H(r) = \left\{(x, \theta) \in X \times \mathbb{R}^H : \sum_{h=1}^{H} x^*_h \leq r \text{ and for any } h \in \mathcal{H}, \text{ there exists } p \in S \text{ such that } \theta_h = p x_h\right\}$$

Definition 77 *The vector* $(x^*, t^*, p^*) \in X \times T \times \mathbb{R}^C_+$ *is an* ***equilibrium*** *vector for the economy* $(X_h, T_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathcal{E}$ *if*

1.

For any $h \in \mathcal{H}$, *household h maximizes, i.e., for given* $(X_h, T_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathcal{E}$, $p^* \in \mathbb{R}^C_+, t^*_{\backslash h} \in T_{\backslash h}$, $(x^*_h, t^*_h) \in X_h \times T_h$ *solves*

$$\begin{array}{ll} \max_{(x_h,t_h)\in X_h\times T_h} & u_h\left(x_h, w\left(p^*, t_h, t^*_{\backslash h}\right)\right) \\ s.t. & \\ p^* x_h \leq p^*\left(e_h + \sum_{h'\in\mathcal{H}\backslash\{h\}}\left(t^*_{h'h} - t_{hh'}\right)\right) & \end{array}$$

2".

$$\left(x^*_h, w\left(p^*, t_h, t^*_{\backslash h}\right)\right)_{h\in\mathcal{H}} \in H\left(\sum_{h\in\mathcal{H}} e_h\right)$$

Remark 78 *The two concepts are equivalent.*

Proposition 79 *Given an economy* $(X_h, T_h, e_h, u_h)_{h\in\mathcal{H}} \in \mathcal{E}$,

a. $(x^*, t^*, p^*) \in X \times T \times \mathbb{R}^C_+$ *is an equilibrium* $\Rightarrow (x^*, t^*, p^*) \in X \times T \times \mathbb{R}^C_+$ *is a Kranich equilibrium, and*

b. if u_h *is Locally Nonsatiated and* $p \in \mathbb{R}^C_{++}$, *then the opposite implication hods true.*

Proof. a.

We have to show that $\sum_{h=1}^{H} x_h^* \leq r$ and for any $h \in \mathcal{H}$, there exists $p \in S$ such that $w\left(p^*, t_h, t_{\setminus h}^*\right) = px_h^*$. The first inequality follows from 2. and the second equality from the definition of $w\left(p^*, t_h, t_{\setminus h}^*\right)$.

b.

The two assumptions are needed to get weak inequality in the budget constraints and in the market clearing conditions to hold as *equalities.* ■

4.3 The generalized game proposed by Rosen (1964)

Another attempt to show existence of equilibria using a generalize game is to follow the approach proposed by Rosen (1965), an approach which fails to work as described below.

Below,

1. we present the objects describing the "Rosen generalized game";
2. we list the assumption introduced by Rosen and we state the existence theorem;
3. we show that a (crucial) assumption is not satisfied in the version of the Rosen generalized game applied to our model.

1.

There are $n \in \mathbb{N}$ players denoted by $i \in \{1, ..., n\} := N$. For any $i \in N$, the strategy vector of player i is a subset of $\mathbb{R}^{m_i}$. Define $m = \sum_{i=1}^{n} m_i$. Indeed, the set of "allowed" (see page 522) strategies *for all players* is

$$R \subseteq \mathbb{R}^m.$$

Moreover, define the projection function $pr_i : R \longrightarrow \mathbb{R}^{m_i}$, $(x_k)_{k \in N} \mapsto x_i$, $P_i = pr_i(R)$ and

$$P := \times_{k \in N} P_i$$

See simple picture on page 522 in Rosen (1965).

Then the strategy set of player $i \in \{1, ..., n\}$ is

$$A_i = \left\{ x_i^0 \in \mathbb{R}^{m_i} : \text{there exists } \left(x_j^0\right)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \mathbb{R}^{m_j} \text{ such that } \left(x_k^0\right)_{k \in N} \in R \right\}.$$

Define $x_{\setminus i} = (x_j)_{j \in N \setminus \{i\}}$ and with innocuous abuse of notation we do not distinguish between $(x_i)_{i \in N}$ and $\left(x_i, x_{\setminus i}\right)$. The definition of utility function presented by Rosen has some ambiguity. My understanding is what follows (and maybe the proof of Theorem 1 should be read carefully to understand what in that proof is needed).

For any $i \in N$, the utility function is

$$u_i : P \longrightarrow \mathbb{R}, \; x \mapsto u_i(x)$$

A Rosen generalized game is a pair $G^* = \{R, (u_i)_{i=1}^n\}$

Definition 80 *A Nash equilibrium for the generalized game $G^* = \{R, (u_i)_{i=1}^n\}$ is $x^0 \in A$ such that for any $i \in \{1, ..., n\}$, x_i^0 solves the following problem.*

$$\max_{x_i \in P_i} \; u_i\left(x_i, x_{\setminus i}^0\right) \qquad s.t. \qquad \left(x_i, x_{\setminus i}^0\right) \in R.$$

2.

Theorem 81 *Let $G^* = \{R, (u_i)_{i=1}^n\}$ be given. If for any $i \in N$,*

1. R is a nonempty, compact, convex subset of $\mathbb{R}^m$;

2. . is continuous and

for any $\left(x_j^0\right)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \mathbb{R}^{m_j}$, $u_{i \; | \; x_{\setminus i}} : P_i \longrightarrow \mathbb{R}$, $x_i \mapsto u_i\left(x_i, x_{\setminus i}\right)$ is ***concave****,*

then G has a Nash equilibrium.

Remark 82 *The assumption of concavity cannot be weakened in the approach followed by Rosen: his proof is based on the fact $\sum_{i=1}^{n} u_i(x)$ is concave which follows from the fact that each term in the sum is concave and the sum of* ***concave*** *functions is concave (a fact which is not true for quasi-concave functions).*

3.

We are now going to define a "Rosen generalized" game for our model and to show the set R which seems to be the only possible choice consistent with the model we are analyzing is not convex.

Indeed, we define R as follows. For any economy $e \in \mathbb{R}^{CH}_{++}$, R is

$$\Big\{(p,x,t) \in \mathbb{R}^C \times \mathbb{R}^{CH} \times \mathbb{R}^{CB} : \quad \text{for any } h \in \mathcal{H},$$

$$p\left(x_h + \textstyle\sum_{h' \in \mathcal{B}_h} t_{hh'}\right) - p\left(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\longrightarrow h}} t_{h'h}\right) \leq 0$$

$$x_h \geq 0$$

$$x \leq k_x$$

$$t_h \geq 0$$

$$t_h \leq k_h$$

$$p \textstyle\sum_{h' \in \mathcal{B}_h} t_{hh'} \leq p e_h$$

$$p \geq 0$$

$$p \textstyle\sum_{h \in \mathcal{H}} e_h = 1 \qquad \Big\}$$

For $H = 2$, we have what follows.

$$R := \Big\{(p, x_1, t_1, x_2, t_2) \in \mathbb{R}^C \times \mathbb{R}^{2C} \times \mathbb{R}^{2C} \quad \begin{array}{l} p(x_1 + t_{12}) - p(e_1 + t_{21}) \leq 0 \\ 0 \leq x_1 \leq k_x \\ 0 \leq t_1 \leq k_1 \\ pt_1 \leq pe_1 \\ \\ p(x_2 + t_{21}) - p(e_2 + t_{12}) \leq 0 \\ 0 \leq x_2 \leq k_x \\ 0 \leq t_2 \leq k_2 \\ pt_2 \leq pe_2 \\ \\ p \geq 0 \\ p \sum_{h \in \mathcal{H}} e_h = 1 \qquad \Big\} \end{array}$$

Observe that

$$A_i = \left\{ x_i^0 \in \mathbb{R}^{m_i} : \text{there exists } \left(x_j^0\right)_{j \in N \setminus \{i\}} \in \times_{j \in N \setminus \{i\}} \mathbb{R}^{m_j} \text{ such that } \left(x_k^0\right)_{k \in N} \in R \right\},$$

see also Definition (2) page 462 in Ausell and Dutta (2008) - is indeed for $h = 0$,

$$A_0 = \left\{ p \in \mathbb{R}^C : p \geq 0 \text{ and } p \sum_{h \in \mathcal{H}} e_h = 1 \right\} := S$$

and for any $h \in \mathcal{H}$, in the simpler case $H = 2$, we have

$$\Big\{(x_1,t_1)\in\mathbb{R}^{2C} \quad \begin{array}{l} \text{there exists } p\in\mathbb{R}^C \text{ such that } p\geq 0 \text{ and } p\sum_{h\in\mathcal{H}} e_h = 1, \\ \text{there exists } x_2\in\mathbb{R}^C \text{ and } t_2\in\mathbb{R}^C \text{ such that } \begin{array}{l} p(x_2+t_{21})-p(e_2+t_{12})\leq 0 \\ 0\leq x_2\leq k_x \\ 0\leq t_2\leq k_2 \\ pt_2\leq pe_2 \end{array} \\ \text{such that} \\ p(x_1+t_{12})-p(e_1+t_{21})\leq 0 \\ 0\leq x_1\leq k_x \\ 0\leq t_1\leq k_1 \\ pt_1\leq pe_1 \end{array} \Big\}$$

We claim that R is not convex.

Consider the inequalities $pt_1 \leq pe_1$ and $pt_2 \leq pe_2$. Then, we have $p(t_1+t_2) \leq p(e_1+e_2) \overset{p\in S}{=} 1$, or $pt \leq 1$. Then if $C = 2$ and $r = (1,1)$ and then $p_1 + p_2 = 1$,take

$$p' = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and } t' = (1,1)$$

$$p' = \left(\frac{1}{10}, \frac{9}{10}\right) \text{ and } t' = \left(5, \frac{5}{9}\right)$$

Observe that
$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$ and $\begin{pmatrix} \frac{1}{10} & \frac{9}{10} \end{pmatrix}\begin{pmatrix} 5 \\ \frac{5}{9} \end{pmatrix} = 1.$

Then, we want to show that for any $\lambda \in (0,1)$, we have

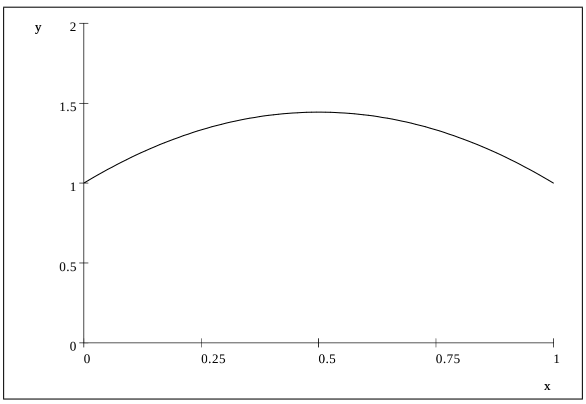
$$\left((1-\lambda)\left(\frac{1}{2},\frac{1}{2}\right)+\lambda\left(\frac{1}{10},\frac{9}{10}\right)\right)\left((1-\lambda)(1,1)+\lambda\left(5,\frac{5}{9}\right)\ \right) > 1,$$

as verified below;
$\left((1-\lambda)\left(\frac{1}{2},\frac{1}{2}\right)+\lambda\left(\frac{1}{10},\frac{9}{10}\right)\right) = \begin{bmatrix} -\frac{2}{5}\lambda+\frac{1}{2} & \frac{2}{5}\lambda+\frac{1}{2} \end{bmatrix}$
$\left((1-\lambda)(1,1)+\lambda\left(5,\frac{5}{9}\right)\ \right) = \begin{bmatrix} 4\lambda+1 & -\frac{4}{9}\lambda+1 \end{bmatrix}$
and

$$\begin{pmatrix} -\frac{2}{5}\lambda+\frac{1}{2} & \frac{2}{5}\lambda+\frac{1}{2} \end{pmatrix}\begin{pmatrix} 4\lambda+1 \\ -\frac{4}{9}\lambda+1 \end{pmatrix} = (4\lambda+1)\left(-\frac{2}{5}\lambda+\frac{1}{2}\right)+\left(-\frac{4}{9}\lambda+1\right)\left(\frac{2}{5}\lambda+\frac{1}{2}\right) =$$

$$\left(-\frac{1}{9}\right)\left(16\lambda^2-16\lambda-9\right) = \frac{16}{9}\lambda-\frac{16}{9}\lambda^2+1 > 1$$

for any $\lambda \in (0,1)$.



4.4 Assuming away the loss of quasi-concavity

The goal of this section is to show Proposition 5. Let's recall the following definitions.

$$
\begin{array}{lll}
 & \mathcal{B}_h \subseteq \mathcal{H}\backslash\{h\}, & \text{set of households } h \text{ likes (exogenous)} \\
 & \mathcal{B}_h^{\backslash} = (\mathcal{H}\backslash\{h\}) \setminus \mathcal{B}_h, & \text{set of households } h \text{ dislikes} \\
(a) & \mathcal{B}_{\rightarrow h} = \{h' \in \mathcal{H}\backslash\{h\} : h \in \mathcal{B}_{h'}\}, & \text{set of households who like } h \\
(b) & \mathcal{B}_{\rightarrow h}^{\backslash} = (\mathcal{H}\backslash\{h\}) \setminus \mathcal{B}_{\rightarrow h} = \left\{h' \in \mathcal{H}\backslash\{h\} : h \in \mathcal{B}_{h'}^{\backslash}\right\} & \text{set of households who dislike } h
\end{array}
\tag{50}
$$

Observe that from the definition of $\mathcal{B}_{\rightarrow h}^{\backslash}$, we have that

$$
\text{for any } h, h' \in \mathcal{H} \text{ such that } h \neq h', \quad h' \in \mathcal{B}_{\rightarrow h}^{\backslash} \quad \Leftrightarrow \quad h \in \mathcal{B}_{h'}^{\backslash}. \tag{51}
$$

We are now going to prove the following intuitive result: if households give nothing to households they dislike, then households get nothing from people they dislike them. We present a formal statement and a proof of that result and also an example verifying the statement itself.

Lemma 83 *Assume that*

$$
\text{for any } h \in \mathcal{H},\ h' \in \mathcal{H}\backslash\{h\} \text{ and } h'' \in \mathcal{B}_{h'}^{\backslash}, \text{ we have } t_{h'h''} = 0. \tag{52}
$$

Then for any $h \in \mathcal{H}$, $h' \in \mathcal{H}$ and $h'' \in \mathcal{B}_{\longrightarrow h'}^{\backslash}\backslash\{h\}$, $t_{h''h'} = 0$.

Proof. Assumption (52) says

$$
\begin{array}{l}
h \in \mathcal{H} \\
h' \in \mathcal{H}\backslash\{h\} \\
h'' \in \mathcal{B}_{h'}^{\backslash} \\
\Downarrow \\
t_{h'h''} = 0.
\end{array}
\tag{53}
$$

Case 1. $h' \neq h$.

Take $h' \in \mathcal{H}$ and $h'' \in \mathcal{B}_{\longrightarrow h'}^{\backslash}\backslash\{h\}$; then $h'' \in \mathcal{B}_{\longrightarrow h'}^{\backslash}$ and from (51), we have that $h' \in \mathcal{B}_{h''}^{\backslash}$. Therefore,

$$
\begin{array}{l}
h \in \mathcal{H} \\
h' \in \mathcal{H}\backslash\{h\} \\
h' \in \mathcal{B}_{h''}^{\backslash} \\
h'' \in \mathcal{B}_{\longrightarrow h'}^{\backslash}\backslash\{h\} \Rightarrow\ h'' \in \mathcal{H}\backslash\{h\}
\end{array}
\tag{54}
$$

Then, h'' and h' described in (54) satisfy the preliminary conditions in (53) (identifying h', h'' there with h'', h' here) and then, we get

$$
\text{for any } h \in \mathcal{H},\ h' \in \mathcal{H}\backslash\{h\} \text{ and } h'' \in \mathcal{B}_{\longrightarrow h'}^{\backslash}\backslash\{h\},\ t_{h''h'} = 0,
$$

as desired.

Case 2. $h' = h$.

We want to show that

$$
h'' \in \mathcal{B}_{\longrightarrow h}^{\backslash}\backslash\{h\} = \mathcal{B}_{\longrightarrow h}^{\backslash} \quad \Rightarrow \quad t_{h''h} = 0.
$$

From (51), we have that for any $h, h'' \in \mathcal{H}$ such that $h \neq h''$, we have $h'' \in \mathcal{B}^{\setminus}_{\longrightarrow h} \Leftrightarrow h \in \mathcal{B}^{\setminus}_{h''}$. Then, it is enough to show that

$$h \in \mathcal{B}^{\setminus}_{h''} \quad \Rightarrow \quad t_{h''h} = 0,$$

which is true by Assumption (52). ■

Example 84 *Let present an example in which we verify the statement in the Lemma above.*

Assume that there are 4 households and that for any $h \in \mathcal{H}$, $\mathcal{B}_h$ is exogenously given as described in the first column of the table below. Then we have what follows.

$\mathcal{B}_1 = \{2,3,4\}$	$\mathcal{B}^{\setminus}_1 = \varnothing$	$\mathcal{B}_{\longrightarrow 1} = \{2\}$	$\mathcal{B}^{\setminus}_{\longrightarrow 1} = \{3,4\}$
$\mathcal{B}_2 = \{1,4\}$	$\mathcal{B}^{\setminus}_2 = \{3\}$	$\mathcal{B}_{\longrightarrow 2} = \{1,3,4\}$	$\mathcal{B}^{\setminus}_{\longrightarrow 2} = \varnothing$
$\mathcal{B}_3 = \{2,4\}$	$\mathcal{B}^{\setminus}_3 = \{1\}$	$\mathcal{B}_{\longrightarrow 3} = \{1\}$	$\mathcal{B}^{\setminus}_{\longrightarrow 3} = \{2,4\}$
$\mathcal{B}_4 = \{2\}$	$\mathcal{B}^{\setminus}_4 = \{1,3\}$	$\mathcal{B}_{\longrightarrow 4} = \{1,2,3\}$	$\mathcal{B}^{\setminus}_{\longrightarrow 4} = \varnothing$

The assumption of the Lemma is: for any $h \in \mathcal{H}$, $h' \in \mathcal{H}\setminus\{h\}$ and $h'' \in \mathcal{B}^{\setminus}_{h'}$, we have $t_{h'h''} = 0$.

Choose $h = 1$.

Then, $h' \in \{2,3,4\}$ and then since $\mathcal{B}^{\setminus}_2 = \{3\}$, $\mathcal{B}^{\setminus}_3 = \{1\}$ and $\mathcal{B}^{\setminus}_4 = \{1,3\}$, we have $t_{23} = t_{31} = t_{41} = t_{43} = 0$.

The conclusion of the Lemma is: for any $h \in \mathcal{H}$, $h' \in \mathcal{H}$ and $h'' \in \mathcal{B}^{\setminus}_{\longrightarrow h'}\setminus\{h\}$, $t_{h''h'} = 0$.

Again, of course, we choose $h = 1$.

Then, since $\mathcal{B}^{\setminus}_{\longrightarrow 1} = \{3,4\}$, $\mathcal{B}^{\setminus}_{\longrightarrow 2} = \varnothing, \mathcal{B}^{\setminus}_{\longrightarrow 3} = \{2,4\}$ and $\mathcal{B}^{\setminus}_{\longrightarrow 4} = \varnothing$, we must have $t_{31} = t_{41} = t_{23} = t_{43} = 0$, as assumed.

Example 85 *Let's also verify that the statement of Proposition 10 is verified in the above case. Indeed,*

$$\mathcal{S} = \{(1,2),(1,3),(1,4) \quad (2,1),(2,4), \quad (3,2),(3,4) \quad (4,2)\};$$
$$\mathcal{T} = \{(2,1), \quad (1,2),(3,2),(4,2), \quad (1,3), \quad (1,4),(2,4),(3,4)\}.$$

Definition 86 $\left(\widetilde{x}, \widetilde{t}, \widetilde{p}\right) \in \mathbb{R}^{CH}_{+} \times \mathbb{R}^{H(H-1)C}_{+} \times S$ *is an* ***equilibrium*** *for the economy $\mathcal{E}' \in \mathbb{E}'$ if*

a. for any $h \in \mathcal{H}$, , for given $\mathcal{E} \in \mathbb{E}, \widetilde{p} \in S$,

$\widetilde{t}_{\setminus h} \in \widehat{T}_{\setminus h}(\widetilde{p}, e_h) := \left\{ t_{\setminus h} \in \mathbb{R}^{C(H-1)(H-1)} : \text{for any } h' \in \mathcal{H}\setminus\{h\}, \quad \widetilde{p}\sum_{h'' \in \mathcal{H}\setminus\{h'\}} t_{h'h''} \leq \widetilde{p}e_{h'} \right\}$,

$\left(\widetilde{x}_h,\ \widetilde{t}_h\right) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)}$ *solves*

$$\max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{C(H-1)}}$$
$$u_{\mathcal{B}_h}\left(x_h, \left(\widetilde{p}\left(e_{h'} + t_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \widetilde{t}_{h''h'} + \sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}} \widetilde{t}_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h''\in\mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''}\right)\right)_{h'\in\mathcal{B}_h}\right) +$$
$$+v_h\left(\left(\widetilde{p}\left(e_{h'} + t_{hh'} + \sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}} \widetilde{t}_{h''h'} + \sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}} \widetilde{t}_{h''h'} - \sum_{h''\in\mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h''\in\mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''}\right)\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)$$

s.t.

$$\widetilde{p}\left(\sum_{h'\in\mathcal{B}_h} t_{hh'} + \sum_{h'\in\mathcal{B}^{\setminus}_h} t_{hh'}\right) \leq \widetilde{p}e_h$$

$$\widetilde{p}x_h \leq \widetilde{p}\left(e_h + \sum_{h'\in\mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h'\in\mathcal{B}^{\setminus}_{\to h}} \widetilde{t}_{h'h} - \sum_{h'\in\mathcal{B}_h} t_{hh'} - \sum_{h'\in\mathcal{B}^{\setminus}_h} t_{hh'}\right)$$

$$x_h \geq 0$$

$$0 \leq t_h \leq k'_h \tag{55}$$

b. markets clear.

Definition 87 $(\widetilde{x}, \widetilde{\tau}, \widetilde{p}) \in \mathbb{R}_{+}^{CH} \times \mathbb{R}_{+}^{C\sum_h B_h} \times S$ *is a* $\mathcal{B}$-***equilibrium*** *for the economy* $\mathcal{E} \in \mathbb{E}$ *if a. for any* $h \in \mathcal{H}$, *for given* $\mathcal{E} \in \mathbb{E}$, $\widetilde{p} \in S$ *and*

$\widetilde{\tau}_{\setminus h} \in \widehat{T}_{\setminus h}(\widetilde{p}, e, \mathcal{B}) := \left\{ (t_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C\sum_{h' \neq h} B_{h'}} : \text{for any } h' \in \mathcal{H} \setminus \{h\}, \quad \widetilde{p} \sum_{h'' \in \mathcal{B}_{h'}} t_{h'h''} \leq \widetilde{p} e_{h'} \right\}$,

$(\widetilde{x}_h, \widetilde{\tau}) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ *solves*

$$\begin{aligned}
&\max_{(x_h, \tau_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}} \quad u_{\mathcal{B}_h}\left(x_h, \widetilde{p}\left(e_{h'} + \tau_{hh'} + \textstyle\sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''} \right)_{h' \in \mathcal{B}_h} \right) \\
&s.t. \\
&\widetilde{p}\left(\textstyle\sum_{h' \in \mathcal{B}_h} \tau_{hh'} \right) \leq \widetilde{p} e_h \\
&\widetilde{p} x_h \leq \widetilde{p}\left(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\rightarrow h}} \widetilde{\tau}_{h'h} - \sum_{h' \in \mathcal{B}_h} \tau_{hh'} \right) \\
&x_h \geq 0 \\
&0 \leq \tau_h \leq (k_{hh'})_{h' \in \mathcal{B}_h} := k_h,
\end{aligned} \tag{56}$$

and

b. markets clear

Proposition 88 *Let* $\mathcal{E}' \in \mathbb{E}'$ *and* $\widetilde{p} \in S$ *given.*

1. *If* $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{H-1}$ *solves problem* (55) *at* $\widetilde{t}_{\setminus h} \in \widehat{T}_h(p, e)$, *then* $\left(\widetilde{t}_{h,h'} \right)_{h' \in \mathcal{B}_h^{\setminus}} = 0$.

2. *If* $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{H-1}$ *solves problem* (55) *at*

$$\widetilde{t}_{\setminus h} \in \widehat{T}_h(p, e) \text{ such that for any } h' \in \mathcal{H} \setminus \{h\} \text{ and } h'' \in \mathcal{B}_{h'}^{\setminus} \setminus \{h\} \text{ we have } \widetilde{t}_{h'h''} = 0, \tag{57}$$

then $(\widetilde{x}_h, (\widetilde{\tau}_{hh'})_{h' \in \mathcal{B}_h}) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ *with* $(\widetilde{\tau}_{hh'})_{h' \in \mathcal{B}_h} = \left(\widetilde{t}_{hh'} \right)_{h' \in \mathcal{B}_h}$ *solves problem* (56) *at* $\widetilde{\tau}_{\setminus h} := \left(\widetilde{t}_{h'h''} \right)_{h' \in \mathcal{H} \setminus \{h\},\ h'' \in \mathcal{B}_{h'}} \in \widehat{T}_h(p, e, \mathcal{B})$.

3. *If* $\left(\widetilde{x}_h, \left(\widetilde{\widetilde{\tau}}_h \right)_{h \in \mathcal{B}_h} \right) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ *solves problem* (56) *at* $\widetilde{\tau}_{\setminus h} \in \widehat{T}_h(p, e, \mathcal{B})$, *then* $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{H-1}$, *with for any* $h \in \mathcal{H} \setminus \{h'\}$

$$\widetilde{t}_{hh'} = \begin{cases} \widetilde{\widetilde{\tau}}_{h,h'} & \text{if} \quad h' \in \mathcal{B}_h, \\ \\ 0 & \text{if} \quad h' \in \mathcal{B}_h^{\setminus}, \end{cases} \tag{58}$$

solves problem (55) *at* $\widetilde{t}_{\setminus h} \in T_h(p, e)$ *such that for any* $h' \in \mathcal{H} \setminus \{h\}$ *and for any* $h'' \neq h'$,

$$\widetilde{t}_{h'h''} = \begin{cases} \widetilde{\tau}_{h'h''} & \text{if} \quad h'' \in \mathcal{B}_{h'}, \\ \\ 0 & \text{if} \quad h'' \in \mathcal{B}_{h'}^{\setminus}, \end{cases} \tag{59}$$

Proof. 1.

We want to show that for any $h' \in \mathcal{B}_h^{\setminus}$, we have $\widetilde{t}_{h,h'} = 0$. If $\mathcal{B}_h^{\setminus} = \varnothing$, then we are done. Suppose now that $\mathcal{B}_h^{\setminus} \neq \varnothing$ and that our claim is false, i.e.,[7]

$$\left(\widetilde{t}_{h,h'} \right)_{h' \in \mathcal{B}_h^{\setminus}} > 0. \tag{60}$$

Then, the simple idea of the proof is to use those transfers as consumption of household h to get the desired contradiction.

For lighter notation, define U_h as the objective function of the maximization problem under analysis. We are going to show that $(\widetilde{x}_h, \widetilde{t}_h)$ does not solve problem (55) at $\widetilde{t}_{\setminus h} \in \widehat{T}_h(p, e)$, verifying that

$$\exists\, (x_h^*, t_h^*) \in \mathbb{R}^C \times \mathbb{R}^{C(H-1)} \text{ such that } \boxed{\text{a.}}\ U_h(x_h^*, t_h^*) > U_h(x_h^*, t_h^*) \text{ and } \boxed{\text{b.}}\ (x_h^*, t_h^*) \in B_h\left(p, t_{\setminus h}^* \right).$$

[7] We use the "standard" definitions for $\geq, >, >>$ between vectors in $\mathbb{R}^n$.

Take

$$x_h^* = \widetilde{x}_h + \sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{h,h'} \overset{(60)}{>} \widetilde{x}_h$$

and $t_h^* = \left(t_{h,h'}^*\right)_{h' \neq h}$ such that

$$t_{h,h'}^* = \begin{cases} \widetilde{t}_{h,h'} & \text{if} \quad h' \in \mathcal{B}_h \\ 0 & \text{if} \quad h' \in \mathcal{B}_h^{\setminus}. \end{cases} \tag{61}$$

a.

$U_h\left((x_h^*, t_h^*)\right) :=$

$$u_{\mathcal{B}_h}\left(\left(\widetilde{x}_h + \textstyle\sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{h,h'}\right), \left(\widetilde{p}e_{h'} + t_{hh'}^* + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}_{\to h'}^{\setminus} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{t}_{h'h''}\right)_{h' \in \mathcal{B}_h}\right) +$$

$$+ v_h\left(\left(\widetilde{p}e_{h'} + 0 + \textstyle\sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}_{\to h'}^{\setminus} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{t}_{h'h''}\right)_{h' \in \mathcal{B}_h^{\setminus}}\right)$$

$\overset{(1)}{>}$

$$u_{\mathcal{B}_h}\left(\widetilde{x}_h, \left(\widetilde{p}e_{h'} + t_{hh'}^* + \textstyle\sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}_{\to h'}^{\setminus} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{t}_{h'h''}\right)_{h' \in \mathcal{B}_h}\right) +$$

$$+ v_h\left(\left(\widetilde{p}e_{h'} + \widetilde{t}_{hh'} + \textstyle\sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}_{\to h'}^{\setminus} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{t}_{h'h''}\right)_{h' \in \mathcal{B}_h^{\setminus}}\right).$$

or defined $m_{h'} = \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}_{\to h'}^{\setminus} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{t}_{h'h''}$,

$$U_h\left((x_h^*, t_h^*)\right) := u_{\mathcal{B}_h}\left(\left(\widetilde{x}_h + \textstyle\sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{h,h'}\right), \left(\widetilde{p}e_{h'} + t_{hh'}^* + m_{h'}\right)_{h' \in \mathcal{B}_h}\right) + v_h\left(\left(\widetilde{p}e_{h'} + 0 + m_{h'}\right)_{h' \in \mathcal{B}_h^{\setminus}}\right)$$

$\overset{(1)}{>}$

$$u_{\mathcal{B}_h}\left(\widetilde{x}_h, \left(\widetilde{p}e_{h'} + t_{hh'}^* + m_{h'}\right)_{h' \in \mathcal{B}_h}\right) + v_h\left(\left(\widetilde{p}e_{h'} + \widetilde{t}_{hh'} + m_{h'}\right)_{h' \in \mathcal{B}_h^{\setminus}}\right)$$

where (1) follows from the facts that $u_{\mathcal{B}_h}$ is strictly increasing in x_h; v_h is decreasing in $\theta_{\mathcal{B}_h^{\setminus}}$ and $0 < \left(\widetilde{t}_{h,h'}\right)_{h' \in \mathcal{B}_h^{\setminus}}$.

b.

The constraint $x_h^* \geq 0$ and $0 \leq t_h^* \leq k_h'$ are clearly satisfied. Moreover,

$$\widetilde{p}\left(\textstyle\sum_{h' \in \mathcal{B}_h} t_{hh'}^* + \sum_{h' \in \mathcal{B}_h^{\setminus}} t_{hh'}^*\right) - \widetilde{p}e_h \overset{\text{def. } (x_h^*, t_h^*)}{=} \widetilde{p}\left(\sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'} + \sum_{h' \in \mathcal{B}_h^{\setminus}} 0\right) - \widetilde{p}e_h \quad \leq$$

$$\widetilde{p}\left(\textstyle\sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'} + \sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{hh'}\right) - \widetilde{p}e_h \overset{\text{def. } (\widetilde{x}_h, \widetilde{t}_h)}{\leq} 0;$$

$$\widetilde{p}x_h^* - \widetilde{p}\left(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h' \in \mathcal{B}_{\to h}^{\setminus}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} t_{hh'}^* - \sum_{h' \in \mathcal{B}_h^{\setminus}} t_{hh'}^*\right) \overset{\text{def. } (x_h^*, t_h^*)}{=}$$

$$\widetilde{p}\left(\widetilde{x}_h + \textstyle\sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{h,h'}\right) - \widetilde{p}\left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h' \in \mathcal{B}_{\to h}^{\setminus}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'} - \sum_{h' \in \mathcal{B}_h^{\setminus}} 0\right) =$$

$$\widetilde{p}\widetilde{x}_h - \widetilde{p}\left(e_h + \textstyle\sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h' \in \mathcal{B}_{\to h}^{\setminus}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'} - \sum_{h' \in \mathcal{B}_h^{\setminus}} \widetilde{t}_{h,h'}\right) \overset{\text{def. } (\widetilde{x}_h, \widetilde{t}_h)}{\leq} 0$$

2.

Let $B1$ and $B2$ be the constraint sets presented in Definitions 86 of equilibrium and 87 of $\mathcal{B}$-equilibrium, respectively.

Suppose our claim is false, i.e., there exists $\left(x_h^*, \left(\tau_{h,h'}^*\right)_{h' \in \mathcal{B}}\right) \in \mathbb{R}^C \times \mathbb{R}^{B_h}$ such that

$$\left(x_h^*, \left(\tau_{h,h'}^*\right)_{h' \in \mathcal{B}_h}\right) \in B2 \tag{62}$$

and

$$\begin{array}{l} u_{\mathcal{B}_h}\left(x_h^*, \left(\widetilde{p}e_{h'} + \tau_{hh'}^* + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{\tau}_{hh''}\right)_{h' \in \mathcal{B}_h}\right) > \\ u_{\mathcal{B}_h}\left(\widetilde{x}_h, \left(\widetilde{p}e_{h'} + \widetilde{\tau}_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}^{\setminus}} \widetilde{\tau}_{hh''}\right)_{h' \in \mathcal{B}_h}\right). \end{array} . \tag{63}$$

Now choose

$$\widehat{x}_h = x_h^* \quad \text{and} \quad \widehat{t}_{h,h'} = \begin{cases} \tau_{h,h'}^* & \text{if} \quad h' \in \mathcal{B}_h, \\ 0 & \quad h' \in \mathcal{B}_h^{\setminus}. \end{cases} \tag{64}$$

We want to show that $\left(\widehat{x}_h, \widehat{t}_h\right)$ belongs to $B1$ and gives a higher utility than $\left(\widetilde{x}_h, \widetilde{t}_h\right)$ which contradicts the fact that $\left(\widetilde{x}_h, \widetilde{t}_h\right)$ is assumed to be a maximizer.

Observe that

$$
\begin{aligned}
&u_{\mathcal{B}_h}\left(\widehat{x}_h,\left(\widetilde{p}e_{h'}+\widehat{t}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+\widehat{t}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{(64)}{=}\\
&u_{\mathcal{B}_h}\left(x^*_h,\left(\widetilde{p}e_{h'}+\tau^*_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+0+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{(1)}{=}\\
&u_{\mathcal{B}_h}\left(x^*_h,\left(\widetilde{p}e_{h'}+\tau^*_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+0+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{\text{def. }\widetilde{\tau}}{=}\\
&u_{\mathcal{B}_h}\left(x^*_h,\left(\widetilde{p}e_{h'}+\tau^*_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{\tau}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+0+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{\tau}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{(63)}{>}\\
&u_{\mathcal{B}_h}\left(\widetilde{x}_h,\left(\widetilde{p}e_{h'}+\widetilde{\tau}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{\tau}_{hh''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+0+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{\tau}_{h''h'}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{\tau}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{(1)}{=}\\
&u_{\mathcal{B}_h}\left(\widetilde{x}_h,\left(\widetilde{p}e_{h'}+\widetilde{\tau}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+0+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}\setminus\{h\}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)\overset{\text{1. above}}{=}\\
&u_{\mathcal{B}_h}\left(\widetilde{x}_h,\left(\widetilde{p}e_{h'}+\widetilde{\tau}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}_h}\right)+\\
&+v_h\left(\left(\widetilde{p}e_{h'}+\widetilde{t}_{hh'}+\textstyle\sum_{h''\in\mathcal{B}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}+\sum_{h''\in\mathcal{B}^{\setminus}_{\to h'}\setminus\{h\}}\widetilde{t}_{h''h'}-\sum_{h''\in\mathcal{B}_{h'}}\widetilde{t}_{h'h''}-\sum_{h''\in\mathcal{B}^{\setminus}_{h'}}\widetilde{t}_{h'h''}\right)_{h'\in\mathcal{B}^{\setminus}_h}\right)
\end{aligned}
\tag{65}
$$

where (1) follows from Assumption (57) and Lemma 83.

Clearly, every constraint in $B1$ is satisfied because the consumption is the same and the chosen transfers are smaller, as formalized below.

$$
\widetilde{p}\left(\sum_{h'\in\mathcal{B}_h}\widehat{t}_{hh'}+\sum_{h'\in\mathcal{B}^{\setminus}_h}\widehat{t}_{hh'}-e_h\right)\overset{\text{def }\widehat{t}_h}{=}\widetilde{p}\left(\sum_{h'\in\mathcal{B}_h}\tau^*_{hh'}-e_h\right)\overset{(62)}{\le}0;
$$

$$
\begin{aligned}
&\widetilde{p}\left(\widehat{x}_h-e_h+\textstyle\sum_{h'\in\mathcal{B}_{\to h}}\widetilde{t}_{h'h}+\sum_{h'\in\mathcal{B}^{\setminus}_{\to h}}\widetilde{t}_{h'h}-\sum_{h'\in\mathcal{B}_h}\widehat{t}_{hh'}-\sum_{h'\in\mathcal{B}^{\setminus}_h}\widehat{t}_{hh'}\right)\overset{(1)}{=}\\
&=\widetilde{p}\left(\widehat{x}_h-e_h+\textstyle\sum_{h'\in\mathcal{B}_{\to h}}\widetilde{t}_{h'h}-\sum_{h'\in\mathcal{B}_h}\widehat{t}_{hh'}\right)\overset{(62)}{\le}0
\end{aligned}
\tag{66}
$$

where (1) follows from the following facts:
$\sum_{h'\in\mathcal{B}^{\setminus}_h}\widehat{t}_{hh'}=0$ from the definition of $\left(\widehat{x}_h,\widehat{t}_h\right)$ and $\sum_{h'\in\mathcal{B}^{\setminus}_{\to h}}\widetilde{t}_{h'h}=0$, from what said below.

From Assumption (57), we have that the assumptions of Lemma 83 are satisfied. Then, in that Lemma, identifying h'' with h', h' with h and h with h, we have that for any $h' \in \mathcal{B}^{\setminus}_{\longrightarrow h} \setminus \{h\} \stackrel{\text{def.}}{=} \mathcal{B}^{\setminus}_{\longrightarrow h}$, we get $\widehat{t}_{h'h} = 0$, as desired.

$$\widehat{x}_h = x^*_h \geq 0;$$

$$0 \leq \widehat{t}_h := \left(\left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h} \right) \stackrel{\text{def. } \widehat{t}_h}{=} \left(\tau^*_{hh'h' \in \mathcal{B}_h}, \left(\widehat{t}_{hh'} = 0\right)_{h' \in \mathcal{B}_h} \right)$$

Then,

$$\left(\widehat{x}_h, \left(\widehat{t}_{h,h'}\right)_{h' \neq h} \right) \in B1 \tag{67}$$

(65) and (67) contradicts the fact that $(\widetilde{x}_h, \widetilde{t}_h)$ solves problem (55) .

3.

Suppose otherwise, i.e., there exists $\left(\widehat{x}_h, \widehat{t}_h\right) \in B1$ and such that

$$U_h \left(\widehat{x}_h, \widehat{t}_h\right) > U_h \left(\widetilde{x}_h, \widetilde{t}_h\right). \tag{68}$$

Then, from Statement 1 above, we have

$$\left(\widehat{x}_h, \widehat{t}_h\right) := \left(\widehat{x}_h, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}, \left(\widehat{t}_{hh'} = 0\right)_{h' \in \mathcal{B}^{\setminus}_h} \right). \tag{69}$$

Then, we go through two steps (a. and b.).

a.

Roughly speaking, we are going to show that

$U_h \left(\widehat{x}_h, \widehat{t}_h\right) = U_{\mathcal{B}_h} \left(\widehat{x}_h, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h} \right)$ +(a constant) $> U_h \left(\widetilde{x}_h, \widetilde{t}_h\right) = U_{\mathcal{B}_h} \left(\widetilde{x}_h, \left(\widetilde{\tau}_{hh'}\right)_{h' \in \mathcal{B}_h}\right)$ +(same constant)."

$$U_h \left(\widehat{x}_h, \widehat{t}_h\right) \stackrel{\text{def.} \widehat{x}_h, \widehat{t}_h \text{and } \widetilde{\tau}_{\setminus h}}{=}$$

$$u_{\mathcal{B}_h} \left(x_h, \widetilde{p} \left(e_{h'} + \widehat{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{\tau}_{h'h''} \right)_{h' \in \mathcal{B}_h} \right) +$$
$$+ v_h \left(\widetilde{p} \left(e_{h'} + \widehat{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''} \right)_{h' \in \mathcal{B}^{\setminus}_h} \right)$$

$$\stackrel{\text{Lemma 83 and (59)}}{=}$$

$$u_{\mathcal{B}_h} \left(x_h, \widetilde{p} \left(e_{h'} + \widehat{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''} \right)_{h' \in \mathcal{B}_h} \right) +$$
$$+ v_h \left(\widetilde{p} \left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} \right)_{h' \in \mathcal{B}^{\setminus}_h} \right) \stackrel{(68)}{>}$$

$$U_h \left(\widetilde{x}_h, \widetilde{t}_h\right) =$$

$$u_{\mathcal{B}_h} \left(x_h, \widetilde{p} \left(e_{h'} + \widetilde{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{\tau}_{h'h''} \right)_{h' \in \mathcal{B}_h} \right) +$$
$$+ v_h \left(\widetilde{p} \left(e_{h'} + \widetilde{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} + \sum_{h'' \in \mathcal{B}^{\setminus}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} - \sum_{h'' \in \mathcal{B}^{\setminus}_{h'}} \widetilde{t}_{h'h''} \right)_{h' \in \mathcal{B}^{\setminus}_h} \right) \stackrel{\text{Lemma 83 and (59)}}{=}$$

$$u_{\mathcal{B}_h} \left(x_h, \widetilde{p} \left(e_{h'} + \widetilde{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''} \right)_{h' \in \mathcal{B}_h} \right) +$$
$$+ v_h \left(\widetilde{p} \left(e_{h'} + 0 + \sum_{h'' \in \mathcal{B}_{\rightarrow h'} \setminus \{h\}} \widetilde{t}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{t}_{h'h''} \right)_{h' \in \mathcal{B}^{\setminus}_h} \right).$$

Then,

$$u_{\mathcal{B}_h}\left(x_h, \widetilde{p}\left(e_{h'} + \widehat{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}\right)_{h' \in \mathcal{B}_h}\right) > \\ u_{\mathcal{B}_h}\left(x_h, \widetilde{p}\left(e_{h'} + \widetilde{t}_{hh'} + \sum_{h'' \in \mathcal{B}_{\to h'} \setminus \{h\}} \widetilde{\tau}_{h''h'} - \sum_{h'' \in \mathcal{B}_{h'}} \widetilde{\tau}_{h'h''}\right)_{h' \in \mathcal{B}_h}\right) \tag{70}$$

b. $\left(\widehat{x}_h, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}\right) \in B2$.

We want to show that if $\left(\widehat{x}_h, \widehat{t}_h\right) := \left(\widehat{x}_h, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}, \left(\widehat{t}_{hh'} = 0\right)_{h' \in \mathcal{B}_h^{\setminus}}\right) \in B1$, then $\left(\widehat{x}_h, \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}\right) \in B2$.

Indeed,

$$0 \geq \widetilde{p}\left(\sum_{h' \in \mathcal{B}_h} \widehat{t}_{hh'} + \sum_{h' \in \mathcal{B}_h^{\setminus}} \widehat{t}_{hh'}\right) - \widetilde{p}e_h = \widetilde{p}\left(\sum_{h' \in \mathcal{B}_h} \widehat{t}_{hh'}\right) - \widetilde{p}e_h$$

$$0 \geq \widetilde{p}\widehat{x}_h - \widetilde{p}\left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} + \sum_{h' \in \mathcal{B}_{\to h}^{\setminus}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} \widehat{t}_{hh'} - \sum_{h' \in \mathcal{B}_h^{\setminus}} \widehat{t}_{hh'}\right) \overset{(1)}{=} \\ = \widetilde{p}\widehat{x}_h - \widetilde{p}\left(e_h + \sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} \widehat{t}_{hh'}\right) \tag{71}$$

$$x_h \geq 0$$

$$0 \leq \left(\widehat{t}_{hh'}\right)_{h' \in \mathcal{B}_h}, \left(\widehat{t}_{hh'} = 0\right)_{h' \in \mathcal{B}_h^{\setminus}} \leq k_h,$$

where (1) follows from said below. From (69), for any $h' \in \mathcal{B}_h^{\setminus}$, $\widehat{t}_{hh'} = 0$. Moreover, from Definition (59) and Lemma 83, we get $\sum_{h' \in \mathcal{B}_{\to h}^{\setminus}} \widetilde{t}_{h'h}$ (following the same strategy using to show condition (66), using Assumption (57) and Lemma 83.

(70) and (71) contradict the definition of $\left(\widetilde{x}_h, \left(\widetilde{\tau}_{hh'}\right)_{h' \in \mathcal{B}_h}\right)$ as a maximum. ■

Proposition 89 *1. If $\left(\widetilde{x}, \widetilde{t}, \widetilde{p}\right)$ is an equilibrium, then for any $h \in \mathcal{H}$ and any $h' \in \mathcal{B}_h^{\setminus}$, we have $\widetilde{t}_{hh'} = 0$.*

2. $\left(\widetilde{x}, \widetilde{t}, \widetilde{p}\right)$ is an equilibrium $\Leftrightarrow$ $\left(\widetilde{x}, \left(\widetilde{t}_{hh'}\right)_{h' \in \mathcal{B}_h}, \widetilde{p}\right)$ is a $\mathcal{B}$-equilibrium.

Proof. 1.

It follows from Proposition 88.1.

2.

[⇒] It follows from Proposition 88.1 and 2.

[⇐] It follows from Proposition 88.3. ■

4.5 Some basic facts on set valued functions, convex analysis, topology and measure theory

Definition 90 *Given two topological spaces $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$, then*

$$\mathcal{B} = \{U \times V : U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Z\}$$

is a basis for a topology on $X \times Y$, called the box or product topology $\mathcal{T}_{X \times Z}$.

Remark 91 *Therefore if $(x, z) \in S \in \mathcal{T}_{X \times Z}$, then there exist $U_x \in \mathcal{T}_X$ and $V_z \in \mathcal{T}_Z$ such that $(x, z) \in U_x \times V_z \subseteq S$.*

Proposition 92 *Given a topological space $(X, \mathcal{T}_X)$ and sets B, Y, if*

1. $B \subseteq Y \subseteq X$,

2. B is Y-closed, and

3. $\mathrm{Cl}_{(X, \mathcal{T}_X)}(B) \subseteq Y$,

then B is X-closed.

Proof. See page 7 in my handwritten notes "basic product-relative topologies.pdf". ■

Remark 93 *In the case analyzed in the paper, we have*

$$B = \left\{x \in \mathbb{R}^C_{++} : u(x) \geq \alpha\right\},$$
$$Y = \mathbb{R}^C_{++}, \quad X = \mathbb{R}^C,$$
$$\mathrm{Cl}(B) \subseteq \mathbb{R}^C_{++}.$$

Definition 94 *Given a topological space $(X, \mathcal{T}_X)$ and $x \in X$, the family of neighborhoods of x is denoted and defined as follows.*

$$\mathcal{N}_{(X,\mathcal{T}_X)}(x) = \{U \subseteq X : x \in U \text{ and } U \in \mathcal{T}_X\}.$$

Definition 95 *Given a set $A \subseteq (X, \mathcal{T}_X)$,the set of adherent points to A is denoted and defined as follows.*

$$\mathrm{Ad}_{(X,\mathcal{T}_X)}(A) = \left\{x \in X : \text{ for any } U \in \mathcal{N}_{(X,\mathcal{T}_X)}(x),\ U \cap A \neq \varnothing\right\}.$$

Proposition 96

$$\mathrm{Ad}_{(X,\mathcal{T}_X)}(A) = \mathrm{Cl}_{(X,\mathcal{T}_X)}(A).$$

Proof. See Math 2 notes. The proof is presented there for metric space, but it does generalize. ■

Proposition 97 *The intersection of a finite number of open and dense sets is open and dense.*

Proposition 98 *The intersection of a finite (in fact countable) number of full measure sets has full measure.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a family of sets with full measure, i.e., such that for any $n \in \mathbb{N}$, $\mu\left(A_n^C\right) = 0$. We want to show that $\cap_{n\in\mathbb{N}} A_n$ has full measure, i.e., $\mu\left((\cap_{n\in\mathbb{N}} A_n)^C\right) = 0$. Indeed, $\mu\left((\cap_{n\in\mathbb{N}} A_n)^C\right) = \mu\left(\cup_{n\in\mathbb{N}} A_n^C\right) \leq \sum_{n\in\mathbb{N}} \mu\left(A_n^C\right) = 0$, where the weak inequality follows from countable subadditivity of measure. ■

Proposition 99 *(Corollary 2.3.9, page 64, in Webster (1984)[33])If K is a convex subset of $\mathbb{R}^n$ such that* $\mathrm{Int}(K) \neq \varnothing$, *then* $\mathrm{Cl}(\mathrm{Int}\ K) = \mathrm{Cl}\ K$.

Proposition 100 *(Proposition 2.38, page 50 in Hu and Papageorgiou (1997) [13]) A set-valued function $\varphi : X \rightrightarrows Y$ is lower hemi-continuous if and only if* $\mathrm{Cl}(\varphi)$ *is lower hemi-continuous.*

Proposition 101 *([12], bottom page 23) If $\varphi : X \rightrightarrows Y$ is a set valued function which is closed graph and if $\varphi(X)$ is contained in a compact set, then φ is upper hemi-continuous.*

Proposition 102 *(Lemma 1, page 33, in Hildebrand (1974) [12]) If a set-valued function φ of a metric space in $\mathbb{R}^n$ is non-empty valued, compact valued, convex valued, closed and lower hemi-continuous Then φ is upper hemi-continuous.*

4.6 Sets-valued functions defined using function inequalities

Preliminary definitions and results taken from Villanacci (2022).

Definition 103 *Given (an utility) function $f : X \to \mathbb{R}$, we say that f is*
Locally NonSatiated, or LNS, if $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x' \in B(x, \varepsilon) \cap X$ such that $u(x') > u(x)$;
NonSatiated, or NS, if $\forall x \in X\ \exists x' \in X$ such that $u(x') > u(x)$.

Proposition 104 *If X is a convex metric space and $u : X \to \mathbb{R}$ is continuous, then*

u is semistrictly quasi-concave and Non-Satiated $\Leftrightarrow$ u is quasi-concave and Locally NonSatiated.

Proof. See, for example, Villanacci (2022), Corollary 42, page 15. ■

Definition 105 *Let the following objects be given:*
for any $j \in \{1, ..., m\}$, functions $f_j : \mathbb{R}^C \longrightarrow \mathbb{R}$, $x \mapsto f_j(x)$, and
the function $f : \mathbb{R}^C \to \mathbb{R}^m$, $x \longmapsto (f_j(x))_{j=1}^m$, and
the sets $B = \left\{x \in \mathbb{R}^C : f(x) \geq 0\right\}$ and $\widetilde{B} = \left\{x \in \mathbb{R}^C : f(x) >> 0\right\}$.

Proposition 106 *If $\widetilde{B} \neq \varnothing$ and for any $j \in \{1, ...m\}$, either*
f_j is continuous, NonSatiated and semistrictly quasi-concave, or
f_j is continuous, Locally NonSatiated and quasi-concave,[8]

[8] Keep in mind that from Proposition 104, assuming any of the two lists of conditions implies the other list.

then
1. $\widetilde{B} = \text{Int}(B)$*, and*
2. $B = \text{Cl}\left(\widetilde{B}\right)$.

Proof. For any $j \in \{1, ..., m\}$, define $B_j := \left\{x \in \mathbb{R}^C : f_j(x) \geq 0\right\}$ and $\widetilde{B}_j := \left\{x \in \mathbb{R}^C : f_j(x) > 0\right\}$; observe that $B = \cap_{j=1}^m B_j$ and $\widetilde{B} = \cap_{j=1}^m \widetilde{B}_j$.
1.
First of all observe that if f_j is is semistrictly quasi-concave, continuous and nonsatiated, then

$$\varnothing \neq \widetilde{B}_j = \text{jnt}(B_j). \tag{72}$$

The above result is relatively well-known and obvious - see for example Lemma 11 page 7 in Border (2017) or Proposition 38 in Villanacci 2022. Therefore,

$$\widetilde{B} = \cap_{j=1}^m \widetilde{B}_j \overset{(1)}{=} \cap_{j=1}^m \text{Int}(B_j) \overset{(2)}{=} \text{Int}\left(\cap_{j=1}^m (B_j)\right) = \text{Int}(B),$$

where (1) follows from (72), and
and (2) follows from Exercise 78, page 85, Lipschutz (1965) or see the file interior of intersection.pdf.
2.
Since it is false that $\cap_{j=1}^m \text{Cl}(B_j) = \text{Cl}\left(\cap_{j=1}^m (B_j)\right)$, we cannot use an approach similar to the above one.
By assumption, B is closed and $\widetilde{B} \subseteq B$; then $\text{Cl}\left(\widetilde{B}\right) \subseteq \text{Cl}(B) = B$.

We are left with showing that $B \subseteq \text{Cl}\left(\widetilde{B}\right)$. We want to show that: $x \in B \Rightarrow \forall \varepsilon > 0, \mathcal{B}(x,\varepsilon) \cap B \neq \varnothing$. Suppose otherwise, i.e.,

$$\begin{array}{l} x \in B, \text{ or } f(x) \geq 0 \\ \text{and} \\ \exists \varepsilon > 0 \text{ such that } \mathcal{B}(x,\varepsilon) \cap B = \varnothing, \end{array}$$

or

$$\begin{array}{l} f(x) \geq 0 \\ \text{and} \\ \exists \varepsilon > 0 \text{ such that } \mathcal{B}(x,\varepsilon) \subseteq B^C, \end{array}$$

or

$$\begin{array}{l} \text{for any } j = 1, ..., m, \;\; f_j(x) \geq 0 \\ \text{and} \\ \exists \varepsilon > 0 \text{ such that } \forall y \in \mathcal{B}(x,\varepsilon) \text{ we have } \neg \langle \; \forall ij = 1, ..., m, \;\; f_j(y) \geq 0 \rangle, \end{array}$$

or

$$\begin{array}{l} \text{for any } j = 1, ..., m, \;\; f_j(x) \geq 0 \\ \text{and} \\ \exists \varepsilon > 0 \text{ such that } \forall y \in \mathcal{B}(x,\varepsilon), \exists j^* \in \{1, ..., m\} \text{ such that } f_{j^*}(y) < 0 \end{array}$$

Then,

$$\exists x \in B \subseteq \mathbb{R}^C, \;\; \exists j^* \in \{1, ..., m\}, \; \exists \varepsilon \in \mathbb{R}_{++} \quad \text{such that}$$

$$\forall y \in \mathcal{B}(x,\varepsilon) \;\; \text{we have } f_{j^*}(x) \geq 0 > f_{j^*}(y).$$

Then x is a local maximum point for f_{j^*} which contradicts the fact that f_{j^*} is Locally Nonsatiated.
■

Proposition 107 *Let a subset* Π *of* $\mathbb{R}^p$ *and a function* $f : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}^m$, $x \mapsto (f_j(\pi, x))_{j=1}^m$ *be given (with* $f_j : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}$*). Let also the following set valued function be given.*

$$B : \Pi \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C,$$

$$\pi \mapsto\!\!\!\!\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) \geq 0\right\};$$

$$\widetilde{B} : \Pi \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C,$$

$$\pi \mapsto\!\!\!\!\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) >> 0\right\}.$$

$\boxed{I.}$

If $\widetilde{B}$ is non-empty valued, f is continuous and for any $j = 1, ...m$ and for any $\pi \in \Pi$, either $f_{j|\{\pi\}}$ is NonSatiated and semistrictly quasi-concave, or
$f_{j|\{\pi\}}$ is Locally NonSatiated and quasi-concave,
then B is non-empty valued convex valued, closed graph and lower hemicontinuous.

$\boxed{II.}$

If in addition either a. B is compact valued or b. $\operatorname{Im} B$ is contained in a compact set, then B is upper hemicontinuous.

Proof. $\boxed{\text{I.}}$

1. B is non-empty valued.

Since $\widetilde{B} \subseteq B$ by definition, and $\widetilde{B} \neq \varnothing$, by assumption, then the desired result follows.

2. B is convex valued.

Since for any $j = 1, ...m$ and for any $\pi \in \Pi$, $f_{j|\{\pi\}}$is quasi-concave, then $\left\{x \in \mathbb{R}^n : f_{j|\{\pi\}}(x) \geq 0\right\}$ is convex and then $B(\pi) = \cap_{j=1}^{m} \left\{x \in \mathbb{R}^n : f_{j|\{\pi\}}(x) \geq 0\right\}$ is convex as well.

3. B is closed graph.

We want to show that for and $(\pi^n, x^n)_{n \in \mathbb{N}} \in \left(\Pi \times \mathbb{R}^C\right)^\infty$ such that for any $n \in \mathbb{N}$, $x_n \in B(\pi^n)$ and such that $(\pi^n, x^n) \longrightarrow (\pi, x)$, then $x \in B(\pi)$.

Since for any $n \in \mathbb{N}$, $x_n \in B(\pi^n)$, we do have that for any $n \in \mathbb{N}$, $f(\pi^n, x^n) \geq 0$. Taking limits of both sides of that inequality and using the continuity of f, we do have $f(x, \pi) \geq 0$, i.e., $x \in B(\pi)$, as desired.

4. B is lower hemicontinuous.

First of all observe that from Proposition 100, it is enough to show that a. $\widetilde{B}$ is lower hemicontinuous, and b. $B = \operatorname{Cl}\left(\widetilde{B}\right)$.

a.

First proof.

Recall the definition of lower hemi-continuity for set valued functions: $\varphi : X \rightarrow\rightarrow Y$ is lower hemi-continuous at $x \in X$ if $\varphi(x) \neq \varnothing$ and for any open set V in Y such that $\varphi(x) \cap V \neq \varnothing$, there exists an open neighborhood U of x such that for every $x' \in U$, $\varphi(x') \cap V \neq \varnothing$.

$\widetilde{B}$ is not empty valued by assumption. Suppose our Claim is false. Then, taking the negation of the main statement in the definition of lower hemi-continuity, we have

$$\begin{array}{l}\text{there exists } \pi \in \Pi \text{ and an open set } V \text{ in } \mathbb{R}^C \text{ such that } \widetilde{B}(\pi) \cap V \neq \varnothing \text{ and} \\ \text{for any } m \in \mathbb{N} \text{ there exists } \pi^m \in \mathcal{B}\left(\pi, \frac{1}{m}\right) \text{ such that } \widetilde{B}(\pi^m) \cap V = \varnothing.\end{array} \tag{73}$$

Since $\widetilde{B}(\pi) \cap V \neq \varnothing$, we can take $x \in \widetilde{B}(\pi) \cap V$. Observe that V is open by assumption and $\widetilde{B}(\pi)$ is open because it is defined in terms of continuous functions and strict inequalities (here we are using the very definition of $\widetilde{B}$ in terms of strict inequalities).Then $\widetilde{B}(\pi) \cap V$ is open as well. Therefore,

$$\exists\ \lambda^* \in (0,1) \text{ such that for any } \lambda \in (\lambda^*, 1)\text{, we have } \lambda x \in \widetilde{B}(\pi) \cap V. \tag{74}$$

From (73), we do have that for any $m \in \mathbb{N}$, $\pi^m \in \mathcal{B}\left(\pi, \frac{1}{m}\right)$ and therefore $\pi^m \longrightarrow \pi$ and

$$f(\pi^m, \lambda x) \stackrel{m}{\longrightarrow} f(\pi, \lambda x) \stackrel{(??)}{>>} 0$$

Then, for m large enough, (here we are using the very definition of $\widetilde{B}$ in terms of strict inequalities,again),

$$\exists\ \lambda^* \in (0,1) \text{ such that for any } \lambda \in (\lambda^*, 1),\ \lambda x \in \widetilde{B}(\pi^m). \tag{75}$$

Then (74) and (75) contradict the fact that $\widetilde{B}(\pi^m) \cap V = \varnothing$, as stated in (73).

Second proof.

We now want to use a well known characterization of lower hemicontinuity[9]. Indeed, we want to show that

for any sequence $(\pi^n)_{n \in \mathbb{N}} \in \Pi^\infty$ such that $\pi^n \to \pi$ and any $x \in \widetilde{B}(\pi)$,

there exists a sequence $(x^n)_{n \in \mathbb{N}} \in \left(\mathbb{R}^C\right)^\infty$ such that $\forall n \in \mathbb{N}, (x^n) \in \widetilde{B}(\pi^n)$ and $x^n \to x$.

[9]See Proposition 4, page 229 in Ok (2007) and Theorem AIII.2, page 197 in Hildebrand and Kirman (1976).

Observe that

$$f(\pi^n, x) \longrightarrow f(\pi, x) >> 0$$

where strict inequalities follows from the fact that $x \in \widetilde{B}(\pi)$. Then, there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$f(\pi^n, x) >> 0.$$

Then, for any $n > N$, $x \in B(\pi^n)$ and taking $x^n = x$ for any $n > N$ (and arbitrary values of x^n for $n \leq N$) completes the proof.

b.

First proof.

We go through three substeps:

i. for any $\pi \in \Pi$

$$\text{Int}\,(B(\pi)) = \widetilde{B}(\pi) \overset{\text{Assumption}}{\neq} \varnothing; \tag{76}$$

ii. $\text{Cl}(\text{Int}(B(\pi))) = \text{Cl}(B(\pi))$;

iii. desired result.

i.

It is the content of Proposition 106.

ii.

Since $\text{Int}\,(B(\pi))$ is nonempty from (76) and since $B(\pi)$ is convex, we can apply Proposition 99, to get the desired result.

iii.

Observe that $B(\pi)$ is a closed set and $\widetilde{B}(\pi)$ is an open set. Then

$$\varnothing \overset{(76)}{\neq} \text{Int}\,(B(\pi)) = \widetilde{B}(\pi) = \text{Int}\left(\widetilde{B}(\pi)\right), \tag{77}$$

and

$$\text{Cl}\left(\widetilde{B}(\pi)\right) \overset{(77)}{=} \text{Cl}\,(\text{Int}B(\pi)) \overset{\text{ii. above}}{=} \text{Cl}\,(B(\pi)) = B(\pi).$$

Second proof.

It is the content of Proposition 106, identifying f there with $f_{|\{\pi\}}$ here.

II.

Conclusion a. and b. follow from the four results contained in I. above and Proposition 102 and 101, respectively. ■

4.7 Price normalization in the case of separable utility functions

The goal of this section is to support the statement that "price normalizations do matter" in a relatively simple example.

Consider the case of an economy with only one good. Then, household 1's maximization problem is as follows. For given, $\beta_1, \beta_2, e_1, e_2 \in \mathbb{R}_{++}$ and $t_2 \in [0, e_2]$, and the price is expressed in units of account.

$$\max_{(x_1,t_1)\in\mathbb{R}^2} \quad u_1(x_1) + \beta_1 v_1(p(e_2 - t_2 + t_1)) \quad \text{s.t.} \quad \begin{array}{ll} p(-x_1 - t_1 + e_1 + t_2) & \geq 0 \\ t_1 & \geq 0 \\ t_1 & \leq e_1 \\ x_1 & \geq 0 \end{array}$$

It is natural to conjecture that solutions to the above problem and equilibria as well are affected by price normalization. To be more precise, it is easy to conjecture that the set of equilibria allocations does contain the image of open interval in $\mathbb{R}$ via a one-to-one function, i.e., the set of equilibria exhibits a degree equal to one of real indeterminacy - see Villanacci and others (2002), page 343, for further details.[10]

[10] A general equilibrium model exhibits real indeterminacy or allocation indeterminacy if the following condition holds. There exists an open and full measure subset O of the set of economies such that if $e \in O$, then the set of equilibrium allocations associated with e contains the image of an open subset of $\mathbb{R}^d$ via a $\mathcal{C}^1$ one-to-one function, with $d > 0$. d is called the degree of real or allocation indeterminacy.

Instead of presenting a heavy proof of that very reasonable result under some sort of general assumptions, we present the desired result in a simple specification of the above model: the case of a two household-one good-CARA economy for a specific choice of the exogenous variables, as explained in detail in what follows. The utility function of household 1 is defined as follows: for any $\beta_1, e_1, e_2, a, p \in \mathbb{R}_{++}$ and $t_2 \in [0, e_2]$

$$u_1 : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}, \qquad (x_1, t_1) \mapsto -e^{-ax_1} - \beta_1 e^{-p(e_2 - t_2 + t_1)}.$$

Then household 1's maximization problem is as follows (with multipliers of the associated constraints)

$$\begin{array}{llllll} \max_{(x_1,t_1)\in\mathbb{R}^2} & -e^{-ax_1} - \beta_1 e^{-p(e_2-t_2+t_1)} & \text{s.t.} & -x_1 - t_1 + e_1 + t_2 & \geq 0 & \lambda_1 \\ & & & t_1 & \geq 0 & \gamma_1 \\ & & & e_1 - t_1 & \geq 0 & \delta_1 \\ & & & x_1 & \geq 0 & \mu_1 \end{array}$$

Observe that the constraint set is compact and therefore a solution exists. The objective function has the following Hessian matrix

$$\begin{bmatrix} -a^2 e^{-ax_1} & 0 \\ 0 & -\beta_1 p^2 e^{-p(e_2-t_2+t_1)} \end{bmatrix}$$

which negative semidefinite and therefore the objective function is strictly concave; the constraint functions are affine; $\left(x_1^{++}, t_1^{++}\right) := \left(\frac{e_1}{4}, \frac{e_1}{4}\right)$ satisfies each constraint with strict inequality. Therefore, the solution to the maximization problem is unique and characterized by the associated Kuhn-Tucker conditions. Therefore, we can present the following Definition of equilibrium.

Definition 108 *$((x_1^*, t_1, \lambda_1^*, \gamma_1^*, \mu_1^*), (x_2^*, t_2, \lambda_2^*, \gamma_2^*, \mu_2^*))$ is an equilibrium for the economy $(\beta_1, \beta_2, e_1, e_2) \in \mathbb{R}^4_{++}$ if it is a solution to the following system.*

$$\begin{array}{ll} \frac{a}{e^{ax_1}} - \lambda_1 + \mu_1 & = 0 \\ \frac{\beta_1 p}{e^{-p(e_2-t_2+t_1)}} - \lambda_1 + \gamma_1 - \delta_1 & = 0 \\ \min\{-x_1 - t_1 + e_1 + t_2, \lambda_1\} & = 0 \\ \min\{t_1, \gamma_1\} & = 0 \\ \min\{e_1 - t_1, \delta_1\} & = 0 \\ \min\{x_1, \mu_1\} & = 0 \\ & \\ \frac{a}{e^{ax_2}} - \lambda_2 + \mu_2 & = 0 \\ \frac{\beta_2 p}{e^{-p(e_1-t_1+t_2)}} - \lambda_2 + \gamma_2 - \delta_2 & = 0 \\ \min\{-x_2 - t_2 + e_2 + t_1, \lambda_2\} & = 0 \\ \min\{t_2, \gamma_2\} & = 0 \\ \min\{e_2 - t_2, \delta_2\} & \\ \min\{x_2, \mu_2\} & = 0 \\ & \\ \sum_{h\in\mathcal{H}} (x_h - e_h) & = 0 \end{array}$$

Conjecture. If $\beta_1 > 1 > \beta_2$, then $x_1 > 0, \lambda_1 > 0, \mu_1 = 0, t_1 > 0, \gamma_1 = 0, \delta_1 = 0$; $x_2 > 0, \lambda_2 = 0, \mu_2 = 0, t_2 = 0, \gamma_2 > 0, \delta_2 = 0$.

Then, the system becomes

$$\begin{array}{ll} \frac{a}{e^{ax_1}} - \lambda_1 & = 0 \\ \frac{\beta_1 p}{e^{p(e_2+t_1)}} - \lambda_1 & = 0 \\ & = 0 \\ & \\ \frac{a}{e^{ax_2}} - \lambda_2 & = 0 \\ \frac{\beta_2 p}{e^{p(e_1-t_1)}} - \lambda_2 + \gamma_2 & = 0 \\ -x_2 + e_2 + t_1 & = 0 \\ & \\ \sum_{h\in\mathcal{H}} (x_h - e_h) & = 0 \end{array}$$

Then,

$$\begin{aligned}
&\frac{a}{e^{ax_1}} - \lambda_1 = 0\\
&e^{ax_1} = \frac{a}{\lambda_1}\\
&ax_1 = \log a - \log \lambda_1\\
&x_1 = \frac{\log a - \log \lambda_1}{a}
\end{aligned}$$

$$\begin{aligned}
&\frac{\beta_1 p}{e^{p(e_2+t_1)}} - \lambda_1 = 0\\
&e^{p(e_2+t_1)} = \frac{\beta_1 p}{\lambda_1}\\
&e_2 + t_1 = \frac{\log(\beta_1 p) - \log \lambda_1}{p}\\
&t_1 = \frac{\log(\beta_1 p) - \log \lambda_1}{p} - e_2
\end{aligned}$$

$$x_1 + t_1 - e_1 = 0$$

$$\begin{aligned}
&\frac{\log a - \log \lambda_1}{a} + \frac{\log(\beta_1 p) - \log \lambda_1}{p} - e_2 - e_1 = 0\\
&p \log a - p \log \lambda_1 + a \log(\beta_1 p) - a \log \lambda_1 - apr = 0\\
&\log \lambda_1 = \frac{p \log a + a \log(\beta_1 p) - apr}{a+p}
\end{aligned}$$

Then,

$$x_1 = \frac{\log a}{a} - \frac{p \log a + a \log(\beta_1 p) - apr}{a(a+p)} =$$

$$= \frac{a \log a + p \log a - p \log a - a \log(\beta_1 p) + apr}{a(a+p)} = \frac{\log a - \log(\beta_1 p) + pr}{a+p}$$

$$t_1 = \frac{\log(\beta_1 p) - \log \lambda_1}{p} - e_2 = \frac{\log(\beta_1 p)}{p} - \frac{p \log a + a \log(\beta_1 p) - apr}{p(a+p)} - e_2 =$$

$$= \frac{a \log(\beta_1 p) + p \log(\beta_1 p) - p \log a - a \log(\beta_1 p) + apr}{p(a+p)} - e_2 = \frac{\log(\beta_1 p) - \log a + ar}{a+p} - e_2 =$$

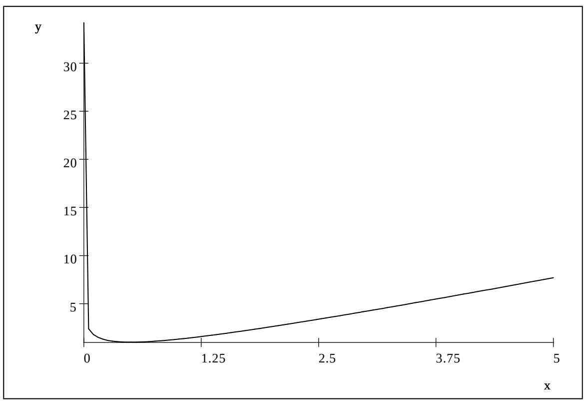
$$= \frac{\log(\beta_1 p) - \log a + ae_1 + ae_2 - ae_2 - pe_2}{a+p} = \frac{\log(\beta_1 p) - \log a + ae_1 - pe_2}{a+p}$$

Sciword check: $\frac{\log(\beta_1 p)}{p} - \frac{p \log a + a \log(\beta_1 p) - ape_1 - ape_2}{p(a+p)} - e_2 = \frac{\log(\beta_1 p) - \log a + ae_1 - pe_2}{a+p}$ is true.

We are going to assume that
$a = 1,\ \beta_1 = 2 > \frac{1}{2} = \beta_2$ and $e_1 = e_2 = 1$.

Let's check that $x_1 > 0$.

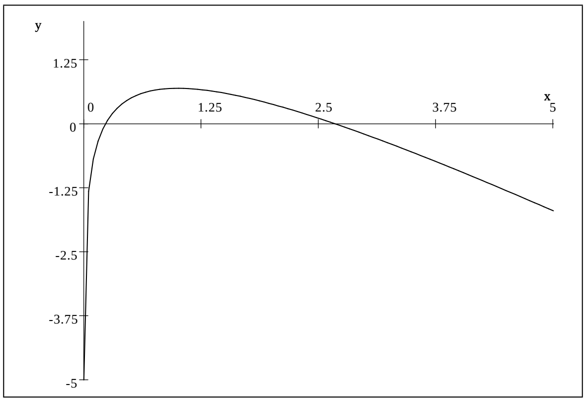
$$\begin{aligned}
&\log a - \log(\beta_1 p) + pr \overset{a=1}{=} -\log(\beta_1 p) + pr > 0\\
&r = 2 = \beta_1\\
&f(p) = -\log(2p) + 2p\\
&f'(p) = 2 - \frac{1}{p}\\
&f\left(\frac{1}{2}\right) = 1
\end{aligned}$$



Then $x_1 > 0$ for any value of $p > 0$.

Let's check that $t_1 > 0$.

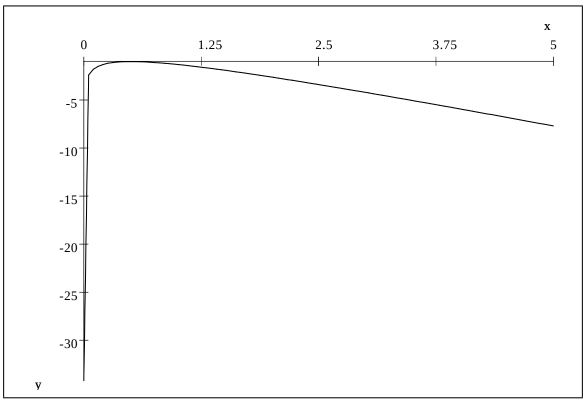
$\log(\beta_1 p) - \log a + ae_1 - pe_2 = \log(2p) + 1 - p$
$\log(2p) + 1 - p = 0$
$f(x) := 1 - x + \ln(2x) = 0$, Solution are $x \approx 0.231960952986534...x \approx 2.67834699001666...$

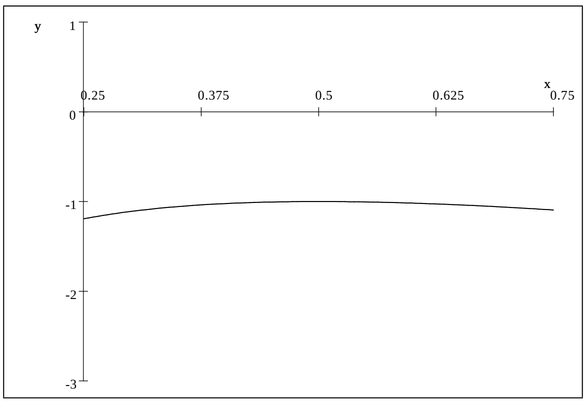


Then $t_1 > 0$ if $p \in (0.23\ ,\ 2.67)$.

Let's check that $t_1 < 1$.

Observe that $t_1 < 1$ iff $= \frac{\ln(2p)+1-p}{1+p} < 1$ iff $\ln(2p) + 1 - p - 1 - p = \ln 2p - 2p < 0$. Defined $f(p) = \ln(2p) - 2p$, we have $f'(p) = \frac{1}{p} - 2$. Then the function has a global maximum in $p = \frac{1}{2}$ and since $f\left(\frac{1}{2}\right) = -1$, we do have $f(p) < 0$ for any $p \in \mathbb{R}_{++}$, as desired.





Let's check that $x_2 > 0$.

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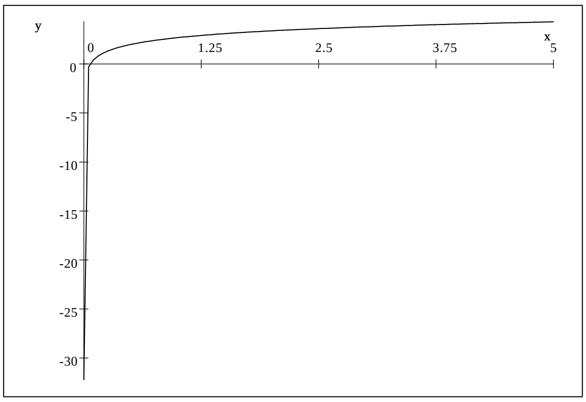
Using Proposition 53, we can compute x_2 using market clearing:

$$x_2 = r - \frac{\log a - \log(\beta_1 p) + pr}{a+p} = \frac{ar + pr - \log a + \log(\beta_1 p) - pr}{a+p} = \frac{ar - \log a + \log(\beta_1 p)}{a+p}$$

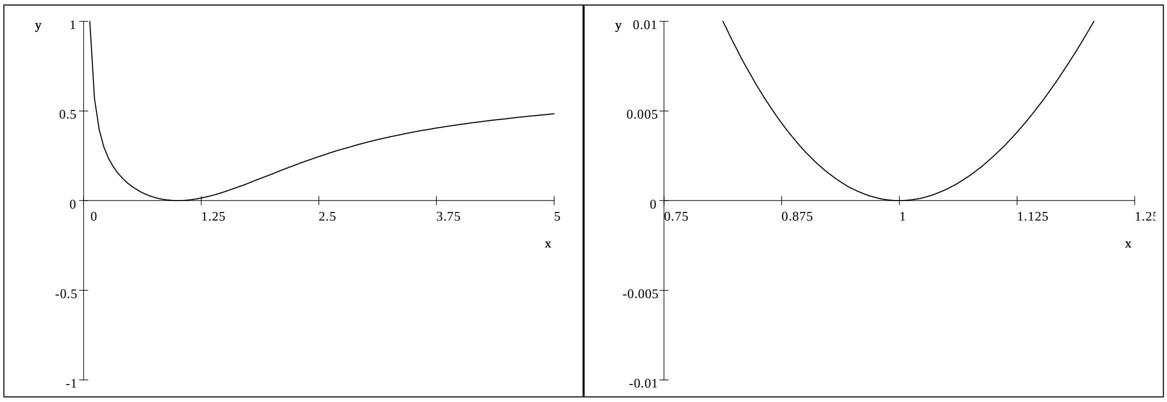
$ar - \log a + \log(\beta_1 p) = 2 + \log(2p)$
$2 + \log(2p) = 0$, Solution is: $\left\{ \left[p = 6.7668 \times 10^{-2} \right] \right\}$



Then $x_1 > 0$ if $p > 0.06$.

Let's check that $\gamma_2 > 0$.

$\gamma_2 = -\frac{\beta_2 p}{e^{p(e_1 - t_1)}} + \lambda_2 = -\frac{\beta_2 p}{e^{px_1}} + \frac{a}{e^{ax_2}} = -\frac{1}{2}\frac{p}{e^{px_1}} + \frac{1}{e^{2-x_1}} = -\frac{1}{2}pe^{-px_1} + e^{x_1 - 2}$
$x_1 = \frac{\log a - \log(\beta_1 p) + pr}{a+p} = \frac{-\log(2p) + 2p}{1+p}$
$x_1 - 2 = \frac{-\log(2p) + 2p - 2 - 2p}{1+p} = \frac{-\log(2p) - 2}{1+p}$
$\gamma_2 = -\frac{1}{2}pe^{p\frac{\log(2p) - 2p}{1+p}} + e^{\frac{-\log(2p) - 2}{1+p}}$

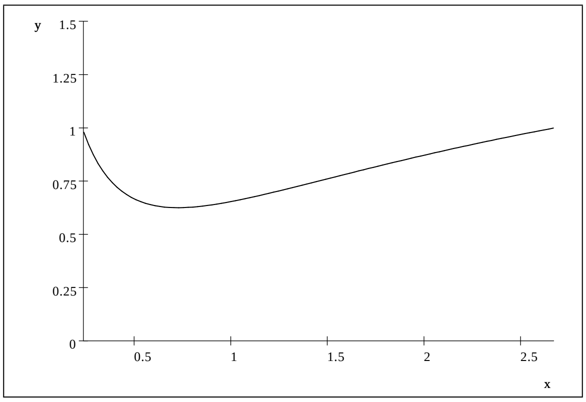


Then $\gamma_2 > 0$ for any value of p.

Summarizing

if $a = 1,\ \beta_1 = 2 > \frac{1}{2} = \beta_2$ and $e_1 = e_2 = 1$,
then for any $p \in (0.23\ ,\ 2.67)$ the following vector is an equilibrium:
$x_1 = \frac{-\log(2p)+2p}{1+p} > 0$
$t_1 = \frac{\log(2p)+1-p}{1+p} > 0$
$\gamma_1 = 0$
$\delta_1 = 0$
$\mu_1 = 0$
$x_2 = \frac{2+\log(2p)}{1+p} > 0$
$t_2 = 0$
$\gamma_2 = -\frac{1}{2} p e^{p \frac{\log(2p)-2p}{1+p}} + e^{\frac{-\log(2p)-2}{1+p}} > 0$
$\delta_2 = 0$
$\mu_2 = 0$

Then, since $x_1 = \frac{-\log(2p)+2p}{1+p}$,



then there is indeterminacy of x_1 with respect to prices. [11]

4.8 Extension of continuous quasi-concave and concave functions

We want to analyze the problem of extending a quasi-concave or concave, continuous function. We consider three cases.[12]

4.8.1 The case of Lipschitz and concave functions

Let's start the section with the definition and some properties of Lipschitz function.

Definition 109 *A function $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is said to be L-Lipschitz (continuous) on S or to satisfy a Lipschitz condition on S if*

$$\exists L \in \mathbb{R}_{++} \ \text{ such that } \ \forall x^1, x^2 \in S, \quad \left\| f\left(x^1\right) - f\left(x^2\right) \right\| \leq L \cdot \left\| x^1 - x^2 \right\|. \tag{78}$$

Remark 110 *A continuous function $f : [0,1] \longrightarrow \mathbb{R}$ is not necessarily Lipschitz, the standard counter-example being $f(x) = \sqrt{x}$. Below we present some sufficient conditions for a function to be Lipschitz.*

Proposition 111 *Let an open, convex set S in $\mathbb{R}^m$ and a function $f : S \to \mathbb{R}^n$ be given. If f is differentiable and $\exists \gamma > 0$ such that $\forall x \in A \subseteq S$, $\|Df(x)\| < \gamma$, then f is Lipschitz on A.*

[11] In the version of the paper ge-prosocial-2023-08-14-existence-with-wrong sections.tex
we provide another, less convincing, definition of "normalizations do matter" and we prove that it is the case.
[12] The present section could not be written without Carlo De Bernardi's help.

Proof. Take $x_1, x_2 \in A$ and without loss of generality take $x_1 \neq x_2$. Denote by $[x_1, x_2]$ the segment from x_1 to x_2. From the Multivariate Mean Value Theorem, for any $a \in \mathbb{R}^n$, there exists $c \in [x_1, x_2]$ such that

$$a \cdot [f(x_2) - f(x_2)] = a \cdot Df(c) \cdot (x_2 - x_1).$$

Then,

$$\frac{\|f(x_2) - f(x_1)\|}{\|x_2 - x_1\|} = \|Df(c)\| < \gamma,$$

as desired. ■

Proposition 112 *If A is an open subset of $\mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}$ is a C^1 function, then f is Lipschitz on any non-empty compact subset C of A.*

Proof. We are going to use the following results.

a. Given $f : (X, d_X) \to (Y, d_Y)$, if S is a compact subset of X and f is continuous, then $f(S)$ is a compact subset (of Y).

b. Let an open set S in $\mathbb{R}^m$and a function $f : S \to \mathbb{R}^n$ be given. If f is differentiable and $\exists \gamma > 0$ such that $\forall x \in S$, $\|Df(x)\| < \gamma$, then f is Lipschitz;

Since f is C^1, from result a. applied to the continuous function[13]

$$g : A \longrightarrow \mathbb{R}, \quad x \mapsto \|Df(x)\|$$

on the non-empty compact set C, we have that $g(C)$ is a compact set and therefore bounded. Then the desired result follows from result b. ■

We can now present an important result about extension of convex, Lipschitz functions.

Proposition 113 *Let A be a convex subset of a normed space X. If $g : A \to \mathbb{R}$ is an L-Lipschitz convex function then it admits an L-Lipschitz convex extension G to the whole X; moreover, such an extension G can be defined by the infimal-convolution formula*

$$G(x) = \inf_{y \in A}\big[g(y) + L\|x - y\|\big], \qquad x \in X.$$

For a proof of the above result, see McShane (1934). A more recent reference is Borwein and Vanderwerff (2020), Exercise 8.3.4, page 399.

As usual, it simple to go from results on convex functions to those ones on concave functions:

If $f : A \to \mathbb{R}$ is an L-Lipschitz concave function then $g = -f$ is an L-Lipschitz convex function. Our formula

$$G(x) = \inf_{y \in A}\big[g(y) + L\|x - y\|\big] = \inf_{y \in A}\big[-f(y) + L\|x - y\|\big], \qquad x \in X.$$

gives an L-Lipschitz convex extension of g to the whole X. Now, consider the function $F := -G$. Then,

$$F(x) = -G(x) = -\inf_{y \in A}\big[-f(y) + L\|x - y\|\big] = \sup_{y \in A}\big[f(y) - L\|x - y\|\big], \qquad x \in X,$$

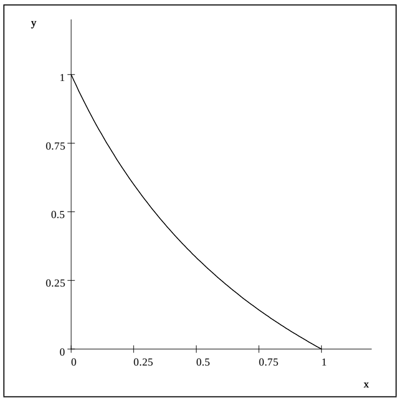
and F is an L-Lipschitz concave extension of f to the whole X. We then have the following result.

Proposition 114 *Let A be a convex subset of a normed space X. If $f : A \to \mathbb{R}$ is an L-Lipschitz concave function, then it admits an L-Lipschitz convex extension G to the whole X; moreover, such an extension F can be defined by the supremal convolution formula*

$$F(x) = \sup_{y \in A}\big[f(y) - L\|x - y\|\big], \qquad x \in X.$$

We can apply the above result to our case identifying X, A, g with $\mathbb{R}^C$, $\mathbb{R}^C_+$ and u respectively. Let's present an example of utility function does satisfy the conditions assumed in the above Proposition. Let the function $v : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$, $v(x_1, x_2) = \log(x_1 + 1) + \log(x_2 + 1)$ be given. A level curve of that function is presented below.

[13] $Df(x) := \left(\frac{\partial f(x)}{\partial x_i}\right)_{i=1}^n$ is the gradient at $x \in A$.



Observe that $V : (-1, +\infty)^2 \longrightarrow \mathbb{R}$, $V(x_1, x_2) = \log(x_1 + 1) + \log(x_2 + 1)$ is C^∞ and its Hessian is negative definite and therefore V and v are strictly concave. The fact that v is Lipschitz is shown in Proposition 115 which follows from the well known result stated in Proposition 111.

Proposition 115 $v : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$, $v(x_1, x_2) = \log(x_1 + 1) + \log(x_2 + 1)$ *is Lipschitz.*

Proof. Let $\varepsilon > 0$ be given; from Proposition 111, it suffices to show that $\|Dv(x)\|$ is bounded above on $\mathbb{R}^2_{-\varepsilon} := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > -\varepsilon \text{ and } x_2 > -\varepsilon\right\}$. Indeed, $Dv(x) = \left(\frac{1}{1+x_1}, \frac{1}{1+x_2}\right)$. For any $z \in \mathbb{R}_{-\varepsilon}$, since $\frac{1}{1+z}$ is strictly decreasing, we have $\frac{1}{1+z} \leq \frac{1}{1-\varepsilon}$. Since for any $(a, b) \in \mathbb{R}^2$, $\|(a, b)\| = \|(a, 0) + (0, b)\| \leq \|(a, 0)\| + \|(0, b)\| = |a| + |b|$, then the desired result follows. ■

4.9 A sufficient condition for Lipschitz continuity

Remark 116 *For any $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, we have $\sum_{i=1}^n |x_i| \leq \sqrt{n} \cdot \|x\|$ and $\|x\| \leq \sum_{i=1}^n |x_i|$. Indeed,*

$$\sum_{i=1}^n |x_i| = (1, ..., , 1)(|x_1|, ..., |x_n|) = |(1, ..., , 1)(|x_1|, ..., |x_n|)| \overset{(1)}{\leq} \|(1, ..., , 1)\| \cdot \|x\| \leq \sqrt{n} \cdot \|x\|,$$

where (1) follows from Cauchy-Schwarz inequality. The other inequality is a well known result.

Proposition 117 *Let a function $f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, $x = (x_i)_{i=1}^n \mapsto f(x)$ be given. Then, f is Lipschitz continuous $\Leftrightarrow$*

$$\exists L \in \mathbb{R}_{++} \text{ such that for any } i \in \{1, ...n\},\ x_{\setminus i} := (x_j)_{j \in \{1,...n\} \setminus \{i\}} \in \mathbb{R}^{n-1}_+,$$

$$f_{\{x_{\setminus i}\}} : \mathbb{R} \longrightarrow \mathbb{R},\ x_i \mapsto f\left(x_i, x_{\setminus i}\right) \quad \text{is } L\text{-Lipschitz continuous.}$$

Proof. We want to show that

$$\begin{array}{l} \left\langle \exists L' \in \mathbb{R}_{++} \text{ such that for any } x, y \in \mathbb{R}^n_+,\ |f(x) - f(y)| \leq L_n \cdot \|x - y\| \right\rangle \\ \Leftrightarrow \\ \left\langle \begin{array}{l} \exists L \in \mathbb{R}_{++} \text{ such that for any } i \in \{1, ..., n\},\ x_{\setminus i} \in \mathbb{R}^{n-1}_+,\ x_i, y_i \in \mathbb{R}_+, \\ \left|f\left(x_i, x_{\setminus i}\right) - f\left(y_i, x_{\setminus i}\right)\right| \leq L \cdot |x_i - y_i| \end{array} \right\rangle. \end{array} \tag{79}$$

[⇒]
Take $L = L'$.Then, by assumption,

$$\text{for any } i \in \{1, ..., n\},\ x_{\setminus i} \in \mathbb{R}^{n-1}_+,\ x_i, y_i \in \mathbb{R}_+,$$

$$\left|f\left(x_i, x_{\setminus i}\right) - f\left(y_i, x_{\setminus i}\right)\right| \leq L' \cdot \left\|\left(x_i, x_{\setminus i}\right) - \left(y_i, x_{\setminus i}\right)\right\| = L' \cdot \|(x_i - y_i, 0_{\mathbb{R}^{n-1}})\| = L' \cdot |x_i - y_i|.$$

[⇐]
We show the desired result by (the complete version) of the principle of mathematical induction: Let P be a proposition defined on $\mathbb{N}$ such that (i) $P(1)$ is true, and (ii) $P(k)$ is true for any $k \in \{1, ..., n-1\} \Rightarrow P(n)$ is true. Then, P is true for any $n \in \mathbb{N}$.

To get a better understanding of the main idea of the proof, let's prove the desired result for $n = 2$. Take $L' = L \cdot \sqrt{2}$. Then for any $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$,

$$|f(x_1, x_2) - f(y_1, y_2)| \leq |f(x_1, x_2) - f(y_1, x_2)| + |f(y_1, x_2) - f(y_1, y_2)| \overset{\text{Assu.}}{\leq} L \cdot |x_1 - y_1| + L \cdot |x_2 - y_2| =$$

$$= L \cdot (|x_1 - y_1| + |x_2 - y_2|) \overset{(1)}{\leq} L \cdot \sqrt{2} \cdot \|(x_1 - y_1, x_2 - y_2)\| = L \cdot \sqrt{2} \cdot \|x - y\|,$$

where (1) follows from Remark 116.

Now we are going to assume that (79) holds true for $k \in \{1, ..., n-1\}$ and we show it holds for n. Take $L' = L\sqrt{2n}$. For any $x, y \in \mathbb{R}^n_+$,

$$|f(x_1, ..., x_n) - f(y_1, ..., y_n)| \leq$$

$$\leq |f(x_1, ..., x_{n-1}, x_n) - f(x_1, ..., x_{n-1}, y_n)| + |f(x_1, ..., x_{n-1}, y_n) - f(y_1, ..., y_{n-1}, y_n)|,$$

or

$$|f(x) - f(y)| \leq \left|f\left(x_{\setminus n}, x_n\right) - f\left(x_{\setminus n}, y_n\right)\right| + \left|f\left(x_{\setminus n}, y_n\right) - f\left(y_{\setminus n}, y_n\right)\right|. \tag{80}$$

Define $g_{x_{\setminus n}} : \mathbb{R} \longrightarrow \mathbb{R}$, $x_n \mapsto f\left(x_{\setminus n}, x_n\right)$; by assumption of the induction argument for $k = 1$, we have that for any $x_n, y_n \in \mathbb{R}$, $\left|g_{x_{\setminus n}}(x_n) - g_{x_{\setminus n}}(y_n)\right| \leq L \cdot |x_n - y_n|$ and by definitionn of $g_{x_{\setminus n}}$, we have

$$\left|f\left(x_{\setminus n}, x_n\right) - f\left(x_{\setminus n}, y_n\right)\right| \leq L \cdot |x_n - y_n|. \tag{81}$$

Define $h_{y_n} : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, $x_{\setminus n} \mapsto f\left(x_{\setminus n}, y_n\right)$; by assumption of the induction argument for $k = n-1$, we have that for any $x_{\setminus n}, y_{\setminus n} \in \mathbb{R}^{n-1}$, $\left|h_{y_n}\left(x_{\setminus n}\right) - h_{y_n}\left(y_{\setminus n}\right)\right| \leq L \cdot \left|x_{\setminus n} - y_{\setminus n}\right|$ and by definitionn of h_{y_n}, we have

$$\left|f\left(x_{\setminus n}, y_n\right) - f\left(y_{\setminus n}, y_n\right)\right| \leq L \cdot \left|x_{\setminus n} - y_{\setminus n}\right|. \tag{82}$$

Then, from (80), (81) and (82), we get

$$|f(x) - f(y)| \leq L \cdot |x_n - y_n| + L \cdot \left|x_{\setminus n} - y_{\setminus n}\right| =$$

$$= (L, L) \cdot \left(|x_n - y_n|, \left|x_{\setminus n} - y_{\setminus n}\right|\right) \overset{(1)}{\leq} \|(L, L)\| \cdot \left\|\left(|x_n - y_n|, \left|x_{\setminus n} - y_{\setminus n}\right|\right)\right\| \overset{(2)}{\leq}$$

$$\leq L\sqrt{2}\left(|x_n - y_n| + \left|x_{\setminus n} - y_{\setminus n}\right|\right) \overset{(3)}{\leq} L\sqrt{2}\left(\sum_{i=1}^n |x_i - y_i|\right) \overset{(4)}{\leq} L\sqrt{2n}\left(\|x - y\|\right),$$

where (1) follows from Cauchy-Schwarz inequality, (2), (3) and (4) from Remark 116. ■

Proposition 118 *Let a concave, continuous and increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $x \mapsto f(x)$ be given. If $f'(0^+)$ is finite, then f is $f'(0^+)$ - Lipschitz continuous.*

Proof. Using Proposition 135, it is easy to show that for any $s \in \mathbb{R}_+$, the function

$$\psi_s : \mathbb{R}\setminus\{s\} \longrightarrow \mathbb{R},\ t \mapsto \frac{f(t) - f(s)}{t - s}$$

is decreasing. Then from basic analysis, see for example Lebl (2023), page 149, $\lim_{t \longrightarrow 0^+} \psi_0(t)$ exists (finite or infinite) and indeed

$$\lim_{t \longrightarrow 0^+} \psi_0(t) = \lim_{t \longrightarrow 0^+} \frac{f(t) - f(0)}{t} := f'\left(0^+\right) \in \mathbb{R}.$$

Observe that for any $u > 0$, $f'(0^+) \geq f'(u^-)$; indeed, from the proof of Lemma 134 (which applies also if φ is defined on a closed interval), for $h > 0$ and sufficiently small $0 < h < u - h < u$ and

$$\frac{f(h) - f(0)}{h} \geq \frac{f(u) - f(u - h)}{h}$$

and taking limits (which do exist), we get

$$f'\left(0^{+}\right) := \lim_{h\longrightarrow 0+} \frac{f\left(h\right)-f\left(0\right)}{h} \geq \lim_{h\longrightarrow 0+} \frac{f\left(u\right)-f\left(u-h\right)}{h} = f'\left(u^{-}\right);$$

indeed, $f'\left(u^{-}\right) := \lim_{x\longrightarrow u^{-}-} \frac{f(x)-f(u)}{x-u} \overset{h=u-x>0}{=} \lim_{h\longrightarrow 0+} \frac{f(u-h)-f(u)}{-h} = \lim_{h\longrightarrow 0+} \frac{f(u)-f(u-h)}{h}$.
Then, for any $x, y > 0$ with $x < y$, we have

$$f'\left(0^{+}\right) \geq f'\left(x^{-}\right) \overset{\text{Lemma 137}}{\geq} \frac{f\left(y\right)-f\left(x\right)}{y-x} \overset{f \text{ increasing}}{=} \left|\frac{f\left(y\right)-f\left(x\right)}{y-x}\right|,$$

or

$$\frac{f\left(y\right)-f\left(x\right)}{y-x} \leq f'\left(0^{+}\right). \tag{83}$$

We are left with showing that for any $y > 0$, $f\left(y\right)-f\left(0\right) \geq f'\left(0^{+}\right)\cdot y$; indeed, from (83)

$$\text{for any } n \in \mathbb{N},\ f\left(y\right) - f\left(\frac{1}{n}\right) \geq f'\left(0^{+}\right)\left(y - \frac{1}{n}\right).$$

Taking limits for $n \longrightarrow +\infty$ and using the assumption of continuity of f, we get the desired result.
■

Corollary 119 *Let a function* $f : \mathbb{R}^{n}_{+} \longrightarrow \mathbb{R}$, $x = (x_i)_{i=1}^{n} \mapsto f\left(x\right)$ *be given. If*

$$f \text{ is continuous, concave, increasing and } \exists L \in \mathbb{R}_{++} \text{ such that for any } i \in \{1, ...n\},\ x_{\setminus i} \in \mathbb{R}^{n-1}_{+},\ \ f'_{\{x_{\setminus i}\}}\left(0^{+}\right) \leq L \tag{84}$$

then f *is Lipschitz continuous.*

Proof. From (84) and Proposition 118, we do have that $\exists L \in \mathbb{R}_{++}$ such that for any $i \in \{1, ...n\}$, $x_{\setminus i} \in \mathbb{R}^{n-1}_{+}$, $f_{\{x_{\setminus i}\}}$ is L-Lipschitz continuous. Then from Proposition 117, the desired result follows.

We are of course also using the fact that concavity, monotniciy and continuity are preserved by the functions $f_{\{x_{\setminus i}\}}$. ■

4.9.1 The case of uniformly convex sets and uniformly continuous quasi-concave functions

The following statement is false - see De Bernardi and Vesely (2023), page 7.

Conjecture 120 *Let the following objects be given.*

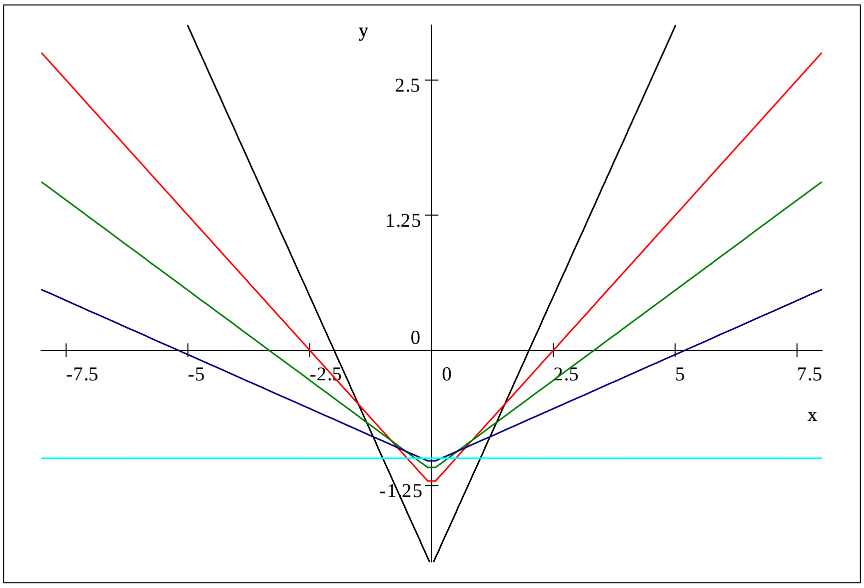
1. A convex set $S \subseteq \mathbb{R}^{n}$*;*

2. a (Lipschitz) continuous, quasi-convex function $f : S \longrightarrow \mathbb{R}$.

Then there exists a function $F : \mathbb{R}^{n} \longrightarrow \mathbb{R}$ *such that 1.* F *is an extension of* f *(i.e.,* $F_{|S} = f$*), 2.* F *is quasi-convex and* 3. F *is continuous.*

Let's present some intuition about the above statement, following Example 2.10, page 7 in De Bernardi and Vesely (2023). Consider the function $f : A \longrightarrow \mathbb{R}$, where $A := \left\{(x, y) \in \mathbb{R}^2 : y > -1\right\}$, and f is defined in terms of the following lower contour sets: for any $n \in \mathbb{N}$ are

$$\{f \leq n\} := \{(x, y) \in A : f\left(x, y\right) \leq n\} := D_n = A \cap \left\{\ (x, y) \in \mathbb{R}^2 : y \geq \frac{1}{n}\left|x\right| - \frac{n^2+1}{n^2}\right\}$$



It is possible to show that f is quasi-convex, Lipschitz and such that for any extension F of f, we have $\{F \leq n\} \subseteq \left\{ (x,y) \in \mathbb{R}^2 : y \geq \frac{1}{n}|x| - \frac{n^2+1}{n^2} \right\} := K_n$.Then take $(x_0, y_0) = (0, -1-\varepsilon)$. Then, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $(x_0, y_0) \notin K_n$ and therefore $(x_0, y_0) \notin \{F \leq n\}$. Summarizing, for any $n \geq n_0$, $F(x_0, y_0) > n$, which contradicts the fact the codomain of F is $\mathbb{R}$.

We are going to use the following result (again, see De Bernardi and Vesely (2023)).

(on what follows, see also the file most complete account of extension 2023-11-22.tex)

Proposition 121 *Let the following objects be given.*

1. *A normed space* $(X, \|\|)$*;*
2. *a bounded, (open or) closed and uniformly convex subset* S *of* X*;*
3. *a uniformly continuous, quasi-convex function* $f : S \longrightarrow \mathbb{R}$*.*

Then f *can be extended to a uniformly continuous, quasi-convex function on* X*.*

Before proceeding to state a useful corollary of the above Proposition, let's define and discuss the concept of uniformly convex set.

Definition 122 $S \subseteq (X, \|\|)$ *is uniformly convex if*

$$\forall \varepsilon \in (0, \operatorname{diam}(S)), \ \exists \delta > 0 \text{ such that } \langle\, x, y \in \mathcal{F}(S), \ \|x-y\| \geq \varepsilon \rangle \Rightarrow \ d_{\|\|}\left(\frac{x+y}{2}, \mathcal{F}(S) \right) > \delta.$$

Remark 123 $\mathbb{R}^2_{++}$ *and* $\mathbb{R}^2_{+}$ *are not uniformly convex: take* $\varepsilon = 1$*; then for any* $\delta > 0$ *and any* $x, y \in \mathcal{F}(S)$*, we do have* $\frac{x+y}{2} \in \mathcal{F}(S)$ *and therefore* $d_{\|\|}\left(\frac{x+y}{2}, \mathcal{F}(S)\right) = 0 < \delta$*. Indeed the following results hold.*

1. *If* $S \subseteq \mathbb{R}^n$ *is a bounded, closed with nonempty interior set then*

$$S \text{ is strictly convex} \quad \Leftrightarrow \quad S \text{ is uniformly convex.}$$

2. *If* $S \subseteq \mathbb{R}^n$ *is not bounded, it is false that "*S *is strictly convex* $\Rightarrow$ S *is uniformly convex."*
3. *If* $S \subseteq \mathbb{R}^n$ *is uniformly convex, then* S *is bounded.*

For the first result, which is the most important one,
see the handwritten file by Carlo: uniformly and strictly convedx-debernardi.pdf

On the basis of the above discussion, we have the following result.

Proposition 124 *Let the following objects be given.*

1. *a bounded,* $\mathbb{R}^n-$*closed with nonempty interior and strictly convex subset* A *of* $\mathbb{R}^n$*;*
2. *a (uniformly) continuous*[14]*, quasi-convex function* $f : A \longrightarrow \mathbb{R}$*.*

Then f *can be extended to a uniformly continuous, quasi-convex function on* $\mathbb{R}^n$*.*

[14] A bounded, $\mathbb{R}^n$-closed set is a compact set.
A continuous function on a compact set is uniformly continuous.

Proof. It follows from Proposition 121 and Remark 123.1. ■

We still need a further result (see Proposition 125 below), whose proof requires Lemma 127 and related Proposition 128.

Proposition 125 *Let the following objects be given.*
1. a bounded, $\mathbb{R}^C-$closed and with nonempty interior subset A of $\mathbb{R}^C_{++}$;
2. a continuous, quasi-concave function $f : \mathbb{R}^C_{++} \longrightarrow \mathbb{R}$.
Then $f_{|A}$ can be extended to a uniformly continuous, quasi-concave function on $\mathbb{R}^C$.

We now introduce the needed results to show the above Proposition

Definition 126 *A body in $\mathbb{R}^C$ is a convex, compact with nonempty interior set.*

Lemma 127 *Let Q be a closed* hyperrectangle *in $\mathbb{R}^C$, i.e., a set of the form*

$$Q = [a_1, b_1] \times \ldots \times [a_C, b_C],$$

where $a_i < b_i$, whenever $1 \leq i \leq C$. Then there exists a sequence $(B_k)_{k\in\mathbb{N}}$ of strictly convex bodies in $\mathbb{R}^C$ such that
(i) for any $k \in \mathbb{N}$, $B_k \subseteq Q$;
(ii) $\bigcup_{k\in\mathbb{N}} B_k = \text{int}(Q)$;
(iii) for any $k \in \mathbb{N}$, $B_k \subseteq \text{int}(B_{k+1})$.

Proof. The proof goes through six main steps : each of them is proved in detail in the Appendix.
1. By considering a translation, if necessary, we can assume without any loss of generality that $Q = -Q$.
2. Hence, Q is the closed unit ball of a norm $\|\cdot\|$ on $\mathbb{R}^C$. Let us denote by $\|\cdot\|_2$ the euclidean norm on $\mathbb{R}^C$ and, for $k \in \mathbb{N}$, define a norm $\|\cdot\|_k$ on $\mathbb{R}^C$ by

$$\|x\|_k = \|x\| + \frac{1}{k}\|x\|_2, , \qquad x \in \mathbb{R}^C.$$

3. By the strict convexity of $\|\cdot\|_2$,
4. it is easy to see that $\|\cdot\|_k$ is a strictly convex norm (see also [9, Fact 7.7]). For $k \in \mathbb{N}$, let us denote by B_k the closed unit ball with respect to $\|\cdot\|_k$
(5. which is clearly a strictly convex body). By definition, it is clear that
6. conditions (i)-(iii) are satisfied. ■

Proposition 128 *For each compact set $S \subset \mathbb{R}^C_{++}$, there exists a strictly convex body S' such that $S \subset S' \subset \mathbb{R}^C_{++}$.*

Proof. For $n \in \mathbb{N}$, define $Q_n = [\frac{1}{n}, n]^C = [\frac{1}{n}, n] \times \ldots \times [\frac{1}{n}, n]$. Then $\{Q_n\}$ is a sequence of *hyperrectangles* in $\mathbb{R}^C_{++}$ such that

1. for any $n \in \mathbb{N}$, $Q_n \subseteq \mathbb{R}^C_{++}$;
2. $\bigcup_{n\in\mathbb{N}} Q_n = \mathbb{R}^C_{++}$;
3. for any $n \in \mathbb{N}$, $Q_n \subseteq \text{Int}(Q_{n+1})$.

Since S is compact, there exists $n_0 \in \mathbb{N}$ such that $S \subseteq Q_{n_0}$. By Lemma 127, there exists a sequence $\{B_k\}$ of strictly convex bodies in $\mathbb{R}^C$ such that

(i) for any $k \in \mathbb{N}$, $B_k \subseteq Q_{n_0+1}$;

(ii) $\bigcup_{k\in\mathbb{N}} B_k = \text{Int}(Q_{n_0+1})$;

(iii) for any $k \in \mathbb{N}$, $B_k \subseteq \text{Int}(B_{k+1})$.

We now claim that there exists $k_0 \in \mathbb{N}$ such that $B_{k_0} \supseteq Q_{n_0} \subseteq B_{k_0}$, as explained below. Observe from 3.,

$$\text{for any } n_0 \in \mathbb{N},\, Q_{n_0} \subseteq \operatorname{Int}\left(Q_{n_0+1}\right), \tag{85}$$

from (ii) above

$$\bigcup_{k\in\mathbb{N}} B_k = \operatorname{Int}(Q_{n_0+1}), \tag{86}$$

from (iii)

$$\text{for any } k \in \mathbb{N},\;\; B_k \subseteq \operatorname{Int}(B_{k+1}) \subseteq B_{k+1}. \tag{87}$$

Then,

$$Q_{n_0} \overset{(85)}{\subseteq} \operatorname{Int}\left(Q_{n_0+1}\right) \overset{(86)}{=} \bigcup_{k\in\mathbb{N}} B_k \overset{(87)}{\subseteq} \bigcup_{k\in\mathbb{N}} \operatorname{Int}(B_{k+1}).$$

Since $Q_{n_0}$is compact, then there exists $\mathcal{N} \subseteq \mathbb{N}$ such that $\#\mathcal{N} = n \in \mathbb{N}$ such that

$$Q_{n_0} \subseteq \bigcup_{i\in\mathcal{N}} \operatorname{Int}(B_{i+1}) \overset{(87)}{\subseteq} \bigcup_{k\in\mathbb{N}} B_{i+2} \overset{(87)}{\subseteq} B_{k_0},$$

where $k_0 := \max\left\{i+2 : i \in \mathcal{N}\right\}$.

Define $S' = B_{k_0}$ and observe that S' is a compact strictly convex body, from the previous proposition, such that

$$S \subseteq Q_{n_0} \subseteq S' = B_{k_0} \subseteq Q_{n_0+1} \subseteq \mathbb{R}^C_{++}.$$

The proof is concluded. ■

We can now present the desired result.

Proof. of Proposition 125. Given A satisfying Assumptions 1, from Proposition 128, there exists a compact strictly convex with nonempty interior set S' such that $A \subset S' \subset \mathbb{R}^C_{++}$. Then, from Proposition 124, the desired result follows. ■

4.9.2 The case of a function of a real variable

As a Corollary of Proposition 114, we have the following result.

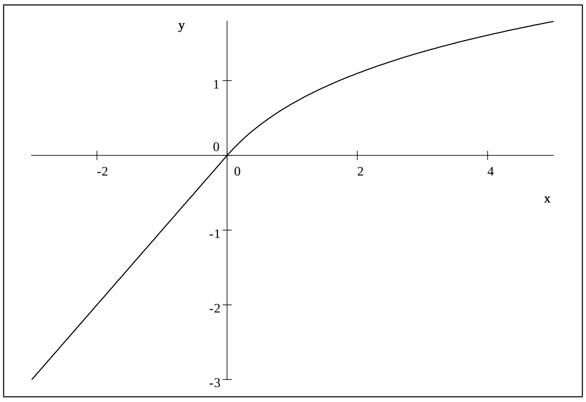
Proposition 129 *Let I be an interval in $\mathbb{R}$. If $g : I \to \mathbb{R}$ is an L-Lipschitz concave function, then it admits an L-Lipschitz concave extension G to the whole $\mathbb{R}$, such an extension $G : \mathbb{R} \longrightarrow \mathbb{R}$ can be defined by the supremal-convolution formula*

$$G(x) = \sup_{y\in I}\bigl[g(y) - L\|x-y\|\bigr]. \tag{88}$$

Observe that the above formula coincides with some simple intuition on how to extend a Lipschitz concave function from $\mathbb{R}$ to $\mathbb{R}$, as explained in details below. Consider the function $v : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $x \mapsto v(x)$ which is continuous, concave and increasing[15] and such that the derivative in zero exists and it finite. Then, v is clearly $v'(0)$-Lipschitz and the "obvious" concave, LIpschitz extension is

$$V : \mathbb{R} \longrightarrow \mathbb{R},\, x \mapsto \begin{cases} v(0) + v'(0)\cdot x & \text{if} \quad x \leq 0 \\ \\ v(x) & \text{if} \quad x \geq 0 \end{cases}$$

[15] A similar argument follows if v is decrasing.



Then we want to check that the above intuitive extension coincides with the supremal-convolution formula (88), i.e.,

$$\sup_{y\in\mathbb{R}_+}\{v(y)-v'(0)\cdot|x-y|\}=\begin{cases}v(0)+v'(0)\cdot x & \text{if} \quad x\leq 0\\ v(x) & \text{if} \quad x\geq 0\end{cases}$$

Observe that since v is $v'(0)$-Lipschitz, by definition,

$$\text{for any } x,y\in\mathbb{R}_+,\ |v(y)-v(x)|\leq v'(0)\cdot|x-y|. \tag{89}$$

Recall that $s=\sup A$ iff a. for any $a\in A$, $s\geq a$, and b. for any $\varepsilon>0$, $\exists a\in A$ such that $s-\varepsilon<a$.

Assume that $x\leq 0$. We want to show that for any $x\leq 0$, $v(0)+v'(0)\cdot x=\sup_{y\in\mathbb{R}_+}\{v(y)-v'(0)\cdot|x-y|\}$.

a. we want to show that for any $y\in\mathbb{R}_+$, $v(0)+v'(0)\cdot x\geq v(y)-v'(0)\cdot|x-y|$.

$$v(0)+v'(0)\cdot x\geq v(y)-v'(0)\cdot|x-y|\overset{x\leq 0,y\geq 0}{=}v(y)-v'(0)\cdot(y-x)=v(y)-v'(0)\cdot y+v'(0)\cdot x \quad \text{iff}$$

$$v(0)\geq v(y)-v'(0)\cdot y \quad \text{iff}$$

$$v(y)-v(0)\leq v'(0)\cdot y.$$

If $y=0$, then the inequality holds in the form $0\leq 0$; if $y>0$, then, since v is increasing by assumption, and v is Lipschitz, using (89) we do have $v(y)-v(0)\leq v'(0)\cdot y$, as desired.

b. we want to show that for any $\varepsilon>0$ there exists $y\in\mathbb{R}_+$ such that $v(0)+v'(0)\cdot x-\varepsilon<v(y)-v'(0)\cdot|x-y|$. Taking $y=0$, we have $v(0)+v'(0)\cdot x-\varepsilon<v(0)+v'(0)\cdot x$ or $-\varepsilon<0$.

Assume that $x>0$. We want to show that for any $x>0$, $v(x)=\sup_{y\in\mathbb{R}_+}\{v(y)-v'(0)\cdot|x-y|\}$.

a. we want to show that for any $y\in\mathbb{R}_+$, $v(x)\geq v(y)-v'(0)\cdot|x-y|$.

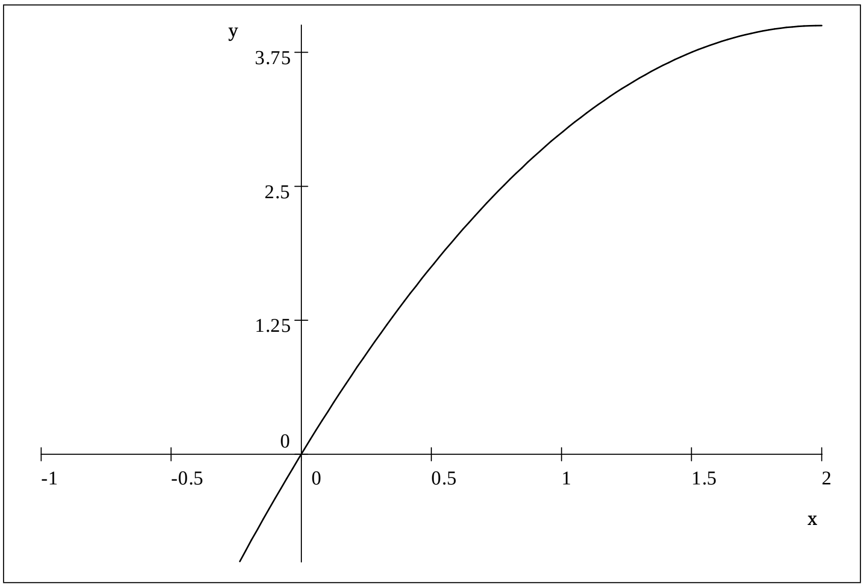
$$v(x)\geq v(y)-v'(0)\cdot|x-y| \quad \text{iff}$$

$$v(x)-v(y)\geq -v'(0)\cdot|x-y|.$$

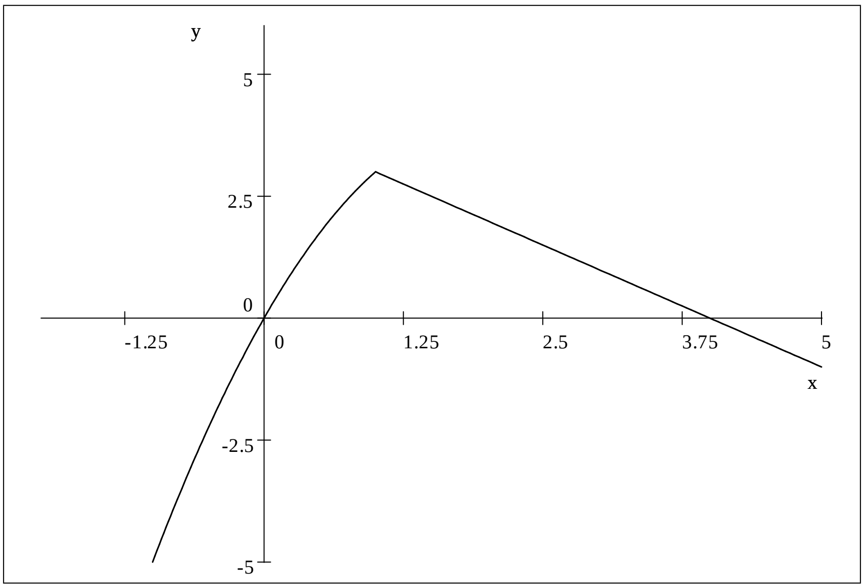
Since v is $v'(0)$-Lipschitz, then $|v(x)-v(y)|\leq v'(0)\cdot|x-y|$ and then, by definition of absolute value, $v(x)-v(y)\geq -v'(0)\cdot|x-y|$, as desired.

b. we want to show that for any $\varepsilon>0$ there exists $y\in\mathbb{R}_+$ such that $v(x)-\varepsilon<v(y)-v'(0)\cdot|x-y|$. Taking $y=x$, we have $v(x)-\varepsilon<v(x)-v'(0)\cdot|x-x|$ or $-\varepsilon<0$.

Conjecture 130 *Given the function* $v:(0,1)\to\mathbb{R}$*,* $v(x)=-x(x+1)$,



then the supremal convolution function of v is



4.9.3 Extending the real function of real variable v

In this section we present some result on extending the function $v : (0, 1) \longrightarrow \mathbb{R}$, first in the case in which we assume v is differentiable and then in the case in which v is only continuous. Of course, the second result is stronger than the first one, but 1. the first result is what we really need in future analysis and 2. the first result is easier and then the probability of mistakes is lower.

The differentiable case

Proposition 131 *If $v : (0, 1) \to \mathbb{R}$, $t \longmapsto v(t)$ is a differentiable, concave, increasing function such that*

$$\exists \varepsilon > 0 \text{ and } k > 0 \text{ such that } \forall t \in (0, \varepsilon),\ v'(t) < k,$$

then there exists a differentiable, concave, increasing function $V : \mathbb{R} \to \mathbb{R}$, $t \longmapsto V(t)$ which is an extension of v.

Remark 132 *Using Theorem 502 and Corollary 507 in my notes on "Measure, abstract measure and probability", we should be able to eliminate the word differentiable (indeed, concave on an open set implies continuous) and substitute the additional assumption with something like the following one*

$$\exists \varepsilon > 0 \text{ and } k \in \mathbb{R}_+ \text{ such that } \forall t \in (0, \varepsilon),\ v'(t^-) < k.$$

Then the way to change the proof below is as follows:

1. substitute $v'(t)$ with $v'(t^-)$ and use the result in Theorem 502 which says that, since v is concave, then $v'(t^+) \leq v(t^-)$.

2. Instead of using the characterization of concave functions in terms of derivative, use the "supporting line result", i.e., Corollary 506 that says you can substitute $v'(t)$ with $p_t \in [v'(t^+), v(t^-)]$.

Proof. of Proposition 131.

We need to go through several steps.

Step 1. v' is bounded.

By assumption,

$$\forall t \in \left(0, \frac{\varepsilon}{2}\right), v'(t) < k. \tag{90}$$

From Calculus 1 - see for example Marcellini and Sbordone, Calcolo 1, page ... or Salsa and Squellati page 264 - we have that, since v is concave and differentiable, then $\forall t \in \left(\frac{\varepsilon}{4}, 1\right)$, v' is decreasing and

$$\forall t \in \left(\frac{\varepsilon}{4}, 1\right),\ v'(t) \leq v'\left(\frac{\varepsilon}{4}\right) < k\ . \tag{91}$$

Then, from (90) and (91), we do have

$$\forall t \in \left(0, \frac{\varepsilon}{2}\right) \cup \left(\frac{\varepsilon}{4}, 1\right) = (0,1),\ v'(t) < k.$$

By assumption, v is increasing and therefore v' is bounded below by 0.

Step 2. v is bounded below.

Suppose otherwise, i.e.,

$$\text{not} \quad \langle \exists n \in \mathbb{N} \text{ such that } \forall x \in (0,1), \quad f(x) \geq -n \rangle\ ,$$

i.e. $\forall n \in \mathbb{N}\ \exists x_n \in (0,1)$ such that $f(x_n) < -n$. Since v is increasing, then for any $x \in (0, x_n)$, $f(x) \leq f(x_n) < -n$. We can then construct a sequence in $(0,1)$ as follows.

$$y_1 = x_1$$

$$y_2 = \min\left\{x_2, \tfrac{1}{2}y_1, \tfrac{1}{2}\right\} \quad \Rightarrow \quad f(y_2) < -2, \quad y_2 < y_1, \quad y_2 \leq \tfrac{1}{2} \quad y_2 > 0$$

...

$$y_n = \min\left\{x_n, \tfrac{1}{2}y_{n-1}, \tfrac{1}{n}\right\} \quad \Rightarrow \quad f(y_n) < -n, \quad y_n < y_{n-1}, \quad y_n \leq \tfrac{1}{n} \quad y_n > 0$$

Now applying the mean value theorem for differentiable function to v on $[y_n, y_1]$ for any $n \in \mathbb{N} \setminus \{1\}$, we that

$$\forall n \in \mathbb{N},\ \exists y_{1n} \in (y_n, y_1) \text{ such that } \frac{f(y_1) - f(y_n)}{y_1 - y_n} = f'(y_{1n})\,.$$

Then, since, by construction of the above sequence, $\lim_{n \to +\infty} f(y_n) = -\infty$ and $\lim_{n \to +\infty} y_n = 0$, we have

$$\lim_{n \to +\infty} \frac{f(y_1) - f(y_n)}{y_1 - y_n} = +\infty. \tag{92}$$

Since for any $n \in \mathbb{N}$, $v'(y_{1n}) < k$, then[16] $\limsup_{n \longrightarrow +} f'(y_{1n}) \leq k$, contradicting (92).

Step 3. v is bounded above.

[16] See file limsup of a bdd fcn.doc for details.

Since v is concave, increasing and differentiable on $(0,1)$, then for any $t \in (0,1)$

$$v(t) \leq v\left(\frac{1}{2}\right) + \overset{(>0)}{v'\left(\frac{1}{2}\right)}\left(t - \frac{1}{2}\right) \overset{\text{Step 1, } t<1}{\leq} v\left(\frac{1}{2}\right) + k \cdot \frac{1}{2},$$

as desired.

Step 4. There exists $v_0 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 0^+} v(t) = v_0$.

Since v is increasing and bounded, then $\left(v\left(\frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is a decreasing bounded below sequence[17]. Then,

$$\lim_{n \longrightarrow +\infty} v\left(\frac{1}{n+1}\right) = \inf\left\{v\left(\frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v_0. \tag{93}$$

Then, from (93) and the definition of inf, we have

$$\forall n \in \mathbb{N},\ v\left(\frac{1}{n+1}\right) \geq v_0; \tag{94}$$

from (93) and the definition of limit, we have

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\ v\left(\frac{1}{n+1}\right) - v_0 \overset{(94)}{=} \left| v\left(\frac{1}{n+1}\right) - v_0 \right| < \varepsilon. \tag{95}$$

We want to show that $\lim_{t \longrightarrow 0^+} v(t) = v_0$, i.e.,

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \text{ such that if } x \in (0, \delta_\varepsilon)\text{, then } |v(x) - v_0| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v is increasing, then

$$\forall x \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v(x) \leq v\left(\frac{1}{N_\varepsilon + 1}\right). \tag{96}$$

Moreover, for any $x \in (0,1)$, there exists $n_x \in \mathbb{N}$ such that $\frac{1}{n_x+1} < x$. Since v is increasing, then

$$v(x) \geq v\left(\frac{1}{n_x + 1}\right) \overset{(94)}{\geq} v_0. \tag{97}$$

Then,

$$|v(x) - v_0| \overset{(97)}{=} v(x) - v_0 \overset{(96)}{\leq} v\left(\frac{1}{N_\varepsilon + 1}\right) - v_0 \overset{(94)}{=} \left|v\left(\frac{1}{N_\varepsilon + 1}\right) - v_0\right| \overset{(95) \text{ with } n = N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 5. There exists $v_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v(t) = v_1$.

The proof is quite similar to the proof of Step 4.

Since v is increasing and bounded above, then $\left(v\left(1 - \frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is an increasing bounded above sequence[18]. Then,

$$\lim_{n \longrightarrow +\infty} v\left(1 - \frac{1}{n+1}\right) = \sup\left\{v\left(1 - \frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v_1 \in \mathbb{R}. \tag{98}$$

Then, from (127) and the definition of sup, we have

$$\forall n \in \mathbb{N},\ v\left(1 - \frac{1}{n+1}\right) \leq v_1; \tag{99}$$

[17] We have to take $v\left(\frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

[18] We have to take $v\left(1 - \frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

from (98) and the definition of limit, we have

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\, v_1 - v\left(\frac{1}{n+1}\right) \overset{(??)}{=} \left| v_1 - v\left(\frac{1}{n+1}\right)\right| < \varepsilon. \tag{100}$$

We want to show that $\lim_{t \longrightarrow 1^-} v(t) = v_1$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \text{ such that if } x \in (1 - \delta_\varepsilon, 1)\,, \text{ then } \; |v(x) - v_1| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v is increasing, then

$$\forall x \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v(x) \geq v\left(1 - \frac{1}{N_\varepsilon + 1}\right) \text{ and } - v(x) \leq -v\left(1 - \frac{1}{N_\varepsilon + 1}\right). \tag{101}$$

Moreover, for any $x \in (0,1)$, there exists $n_x \in \mathbb{N}$ such that $1 - \frac{1}{n_x + 1} > x$. Since v is increasing, then

$$v(x) \leq v\left(1 - \frac{1}{n_x + 1}\right) \overset{(99)}{\leq} v_1. \tag{102}$$

Then,

$$|v(x) - v_1| \overset{(102)}{=} v_1 - v(x) \overset{(101)}{\leq} v_1 - v\left(1 - \frac{1}{N_\varepsilon + 1}\right) \overset{(99)}{=} \left| v\left(\frac{1}{N_\varepsilon + 1}\right) - v_1 \right| \overset{(100) \text{ with } n = N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 6. The function $\widehat{v}$ defined below is continuous.

Define $\widehat{v} : [0,1] \longrightarrow \mathbb{R}$,

$$\widehat{v}(t) = \begin{cases} v_0 & \text{if} \quad t = 0 \\ v(t) & \text{if} \quad t \in (0,1) \\ v_1 & \text{if} \quad t = 1 \end{cases}$$

It follows from Steps 4 and 5.

Step 7. There exists $v_0' \in \mathbb{R}$ such that $\lim_{t \longrightarrow 0^+} v'(t) = v_0'$.

Observe that v':$(0,1) \longrightarrow \mathbb{R}$ is bounded (from Step 1) and decreasing (by assumption). We can then mimic the argument presented in Steps 4 or 5 - observe that there we did NOT use the concavity of v.

Since v' is decreasing and bounded, then $\left(v'\left(\frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is an increasing bounded above sequence[19]. Then,

$$\lim_{n \longrightarrow +\infty} v'\left(\frac{1}{n+1}\right) = \sup\left\{ v'\left(\frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v_0'. \tag{103}$$

Then, from (103) and the definition of sup, we have

$$\forall n \in \mathbb{N},\, v'\left(\frac{1}{n+1}\right) \leq v_0'; \tag{104}$$

from (103) and the definition of limit, we have

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\, v_0' - v'\left(\frac{1}{n+1}\right) \overset{(104)}{==} \left| v_0' - v'\left(\frac{1}{n+1}\right)\right| < \varepsilon. \tag{105}$$

We want to show that $\lim_{t \longrightarrow 0^+} v'(t) = v_0'$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \text{ such that if } x \in (0, \delta_\varepsilon)\,, \text{ then } \; |v'(x) - v_0'| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v' is decreasing, then

$$\forall x \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v'(x) \leq v'\left(\frac{1}{N_\varepsilon + 1}\right). \tag{106}$$

[19] We have to take $v\left(\frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

Moreover, for any $x \in (0,1)$, there exists $n_x \in \mathbb{N}$ such that $\frac{1}{n_x+1} < x$. Since v' is decreasing, then

$$v'(x) \le v'\left(\frac{1}{n_x+1}\right) \overset{(104)}{\le} v'_0. \tag{107}$$

Then,

$$|v'(x) - v'_0| \overset{(107)}{=} v'_0 - v'(x) \overset{(106)}{\le} v'_0 - v'\left(\frac{1}{N_\varepsilon+1}\right) \overset{(104)}{=} \left|v'_0 - v'\left(\frac{1}{N_\varepsilon+1}\right)\right| \overset{(105) \text{ with } n=N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 8. There exists $v'_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v'(t) = v'_1$.

It is quite similar to the proof of Step 4.

Since v' is decreasing and bounded below, then $\left(v'\left(1 - \frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is a decreasing bounded below sequence[20]. Then,

$$\lim_{n \longrightarrow +\infty} v'\left(1 - \frac{1}{n+1}\right) = \inf\left\{v'\left(1 - \frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v'_1 \in \mathbb{R}. \tag{108}$$

Then, from (108) and the definition of inf, we have

$$\forall n \in \mathbb{N},\ v'\left(1 - \frac{1}{n+1}\right) \ge v'_1; \tag{109}$$

from (108) and the definition of limit, we have

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \ge N_\varepsilon,\ v'\left(1 - \frac{1}{n+1}\right) - v'_1 \overset{(104)}{=} \left|\ v'\left(1 - \frac{1}{n+1}\right) - v'_1\right| < \varepsilon. \tag{110}$$

We want to show that $\lim_{t \longrightarrow 1^-} v'(t) = v'_1$, i.e.,

$$\forall \varepsilon > 0\ \ \exists \delta_\varepsilon > 0 \text{ such that if } x \in (1 - \delta_\varepsilon, 1)\text{, then } |v'(x) - v'_1| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon+1}$; since v' is decreasing, then

$$\forall x \in \left(0, 1 - \frac{1}{N_\varepsilon+1}\right), \quad v'(x) \le v'\left(1 - \frac{1}{N_\varepsilon+1}\right). \tag{111}$$

Moreover, for any $x \in (0,1)$, there exists $n_x \in \mathbb{N}$ such that $1 - \frac{1}{n_x+1} > x$. Since v' is decreasing, then

$$v'(x) \ge v'\left(1 - \frac{1}{n_x+1}\right) \overset{(109)}{\ge} v'_1. \tag{112}$$

Then,

$$|v'(x) - v'_1| \overset{(112)}{=} v'(x) - v'_1 \overset{(111)}{\le} v'\left(1 - \frac{1}{N_\varepsilon+1}\right) - v'_1 \overset{(109)}{=} \left|v'\left(1 - \frac{1}{N_\varepsilon+1}\right) - v'_1\right| \overset{(110) \text{ with } n=N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 9. $\widehat{v}$ is differentiable in 0 and $\widehat{v}'(0) = v'_0$.

From Step 6, we have that $\widehat{v}_{|\left[0,\frac{1}{2}\right]}$ is continuous; by assumption, $\widehat{v}_{|\left[0,\frac{1}{2}\right]}$ is differentiable on $\left(0, \frac{1}{2}\right)$. From Step 7, $\lim_{t \longrightarrow 0^+} v'(t) = v'_0$. Then, from, for example Theorem 103.5, page 408 in Marcellini and Sbordone (), we get the desired result. Keep also in mind the definition of left and right derivative presented on page 351 in Marcellini and Sbordone ().

Step 10. $\widehat{v}$ is differentiable in 1 and $\widehat{v}'(1) = v'_1$.

Same proof as in Step 9.

Step 11. The function V defined below is the desired extension.

[20] We have to take $v\left(1 - \frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

$V : \mathbb{R} \longrightarrow \mathbb{R}$,

$$V(t) = \begin{cases} \widehat{v}(0) + \widehat{v}'(0) \cdot t & \text{if} \quad t \leq 0 \\ \widehat{v}(t) & \text{if} \quad t \in [0,1] \\ \widehat{v}(1) + \widehat{v}'(1) \cdot t & \text{if} \quad t \geq 1. \end{cases}$$

a. V is differentiable.

V is continuous from the Pasting Lemma. Below, we state a version of that theorem (see Munkres (1975)) and we apply it to our case.

(The pasting lemma) Let X and Y be topological spaces. Let $X = A \cup B$, where A and B are closed in X and $f : A \to Y$, $g : B \to Y$ be continuous functions. If $\forall x \in A \cap B$, $f(x) = g(x)$, then "f and g combine to give a continuous function", i.e., $h : X \to Y$,

$$h(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ \\ g(x) & \text{if} \quad x \in B \end{cases}$$

is a continuous function.

We can apply the above lemma identifying

X	with	$\mathbb{R}$
A	with	$(-\infty, 0] \cup [1, +\infty)$
B	with	$[0,1]$
f	with	$\begin{cases} v(0) + \widehat{v}'(0) \cdot t & \text{if} \quad t \leq 0 \\ v(1) + \widehat{v}'(1) \cdot t & \text{if} \quad t \geq 1. \end{cases}$
g	with	$\widehat{v}$
$A \cap B$	with	$\{0,1\}$

Then, the desired result follows again from Theorem 103.5, page 408 in Marcellini and Sbordone () and

the facts that $\lim_{t \longrightarrow 0^-} (v(0) + \widehat{v}'(0) \cdot t)' = \widehat{v}'(0)$ and Step 9.

Similar proof applies to show differentiability in 1.

b. V is increasing.

Observe that

$$V'(t) = \begin{cases} \widehat{v}'(0) \geq 0 & \text{if} \quad t \leq 0 \\ \widehat{v}'(t) \geq 0 & \text{if} \quad t \in [0,1] \\ \widehat{v}'(1) \geq 0 & \text{if} \quad t \geq 1. \end{cases}$$

c. V is concave.

We are going to use the following result - see, for example, Theorem 3.3.a, page 290 in Pagani and Salsa, or Marcellini and Sbordone, page 387: given a differentiable function defined on an open interval, we have that f' decreasing $\Leftrightarrow$ f concave.

Indeed, $V'$is decreasing from Lemma 133 below. Let's check the assumptions. Identify φ and u there with V' and $\widehat{v}'$. Indeed, $\widehat{v}$ is decreasing on $[0,1]$ as an application of the Lagrange theorem; $V'$is continuous from **a.** above. ■

Lemma 133 *If the function $u : [0,1] \longrightarrow \mathbb{R}$ is decreasing and the function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$*

$$\varphi(t) = \begin{cases} u(0) & \text{if} \quad t \leq 0 \\ u(t) & \text{if} \quad t \in [0,1] \\ u(1) & \text{if} \quad t \geq 1 \end{cases}$$

is continuous (and then bounded), then φ is decreasing.

Proof. We have to show that for any $t_0, t_1 \in \mathbb{R}$ such that $t_0 < t_1$, we have $\varphi(t_0) \geq \varphi(t_1)$. The case to be analyzed more carefully is the case in which $t_0 \leq 0$ and $t_1 \in (0,1)$. Suppose our claim is false and $\varphi(t_0) < \varphi(t_1)$, i.e., $u(0) < u(t_1)$. Choose $n_1 \in \mathbb{N}$ such that $\frac{1}{n_1} \in (0, t_1)$. Then, since φ is bounded, then $\left(u\left(\frac{1}{n}\right)\right)_{n>n_1, n\in\mathbb{N}}$ is an increasing sequence converging to its sup. Then,

$$\forall n > n_1, \quad u\left(\frac{1}{n}\right) \geq u(t_1) > u(0), \tag{113}$$

$$\lim_{n\longrightarrow+\infty} u\left(\frac{1}{n}\right) = \sup\left\{u\left(\frac{1}{n}\right) : n \geq n_1, n \in \mathbb{N}\right\} \geq u(t_1), \tag{114}$$

and, since φ is continuous

$$\lim_{n\longrightarrow+\infty} u\left(\frac{1}{n}\right) = u(0). \tag{115}$$

Then,

$$u(t_1) \overset{(114)}{\leq} \lim_{n\longrightarrow+\infty} u\left(\frac{1}{n}\right) \overset{(115)}{=} u(0) \overset{(113)}{<} u(t_1),$$

which is the desired contradiction. ■

The continuous case We first of all present some important results about concave functions we are going to use in our analysis. The proof all results stated below are presented in my notes on measure theory.

Proposition 134 *Let φ be a concave function on (a,b) and*

$$a < s < t < u < b. \tag{116}$$

Then

$$\frac{\varphi(t)-\varphi(s)}{t-s} \geq \frac{\varphi(u)-\varphi(s)}{u-s} \geq \frac{\varphi(u)-\varphi(t)}{u-t}.$$

Proposition 135 *The function*

$$\psi : [a,b]^2 \setminus \{(s',t') \in \mathbb{R}^2 : s' = t'\} \to \mathbb{R}, \qquad \psi : (s,t) \mapsto \frac{\varphi(t)-\varphi(s)}{t-s}$$

is componentwise increasing, i.e., $\forall s,t$ such that $s \neq t$ and $s,t \in [a,b]$, both $\psi_{|\{s\}}$ and $\psi_{|\{t\}}$ are increasing.

Theorem 136 *Let $\varphi : (a,b) \to \mathbb{R}$ be a concave function. Then*

1. *$\forall t \in (a,b)$, the following limits exist and are finite:*

$$\varphi'(t^+) := \lim_{h\to 0^+} \frac{\varphi(t+h)-\varphi(t)}{h} \qquad \varphi'(t^-) := \lim_{h\to 0^-} \frac{\varphi(t+h)-\varphi(t)}{h}.$$

2. *φ is continuous on (a,b).*
3. *$\forall t \in (a,b)$, $\varphi'(t^-) \geq \varphi'(t^+)$.*
4. *φ' is defined everywhere in (a,b) except at most a countable set of points. Moreover φ' is decreasing.*

Lemma 137 *If $\varphi : (a,b) \to \mathbb{R}$ is a concave function and $a < u < \nu < b$, then*

$$\varphi'\left(u^-\right) \geq \varphi'\left(u^+\right) \geq \frac{\varphi(\nu)-\varphi(u)}{\nu - u} \geq \varphi'\left(\nu^-\right) \geq \varphi'\left(\nu^+\right).$$

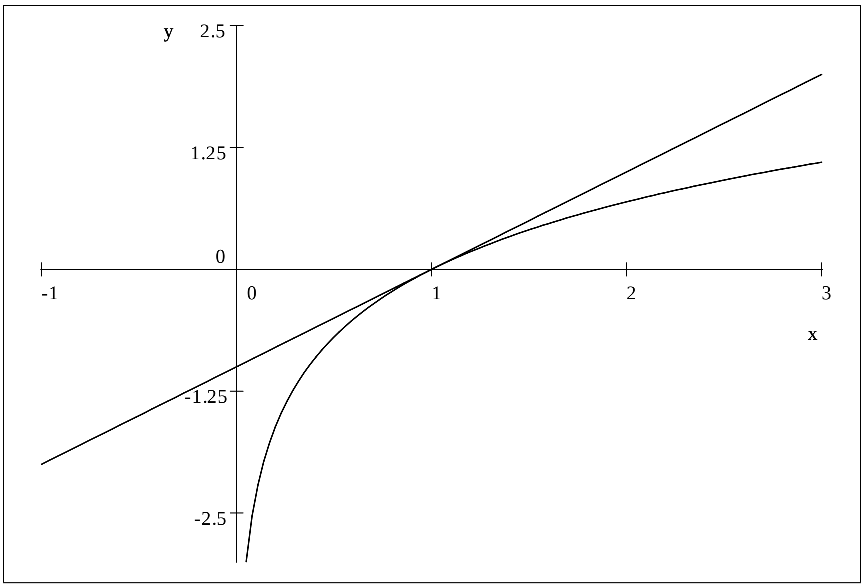
Corollary 138 *Let the concave function $\varphi : [a,b] \to \mathbb{R}$ be given, then*
1. φ is Lipschitz, and 2. absolutely continuous on any interval $[c,d] \subseteq (a,b)$.

Definition 139 *Given the convex function $\varphi : (a,b) \to \mathbb{R}$ and $t \in (a,b)$, then the line with equation*

$$y = \varphi(t) + m(t-s)$$

is called a supporting line at t for the graph of φ if

$$\forall s \in (a,b), \quad \varphi(s) \le \varphi(t) + m(t-s).$$



Remark 140 *The following result says that if φ is convex, then its graph admits a supporting line at any point of its domain.*

Corollary 141 *Given a concave function $\varphi : (a,b) \to \mathbb{R}$ and $s, t \in (a,b)$ such that $s \neq t$, then for any $p_t \in [\varphi'(t^+, \varphi'(t^-))]$ we have*

$$\varphi(s) \le \varphi(t) + p_t(s-t).$$

Proposition 142 *If $v : (0,1) \to \mathbb{R}$, $t \longmapsto v(t)$ is (continuous,) increasing, concave and satisfies the condition*

$$\exists \varepsilon > 0 \text{ and } k > 0 \text{ such that } \forall t \in (0,\varepsilon),\ v'(t^-) < k, \tag{117}$$

then there exists a continuous, concave, increasing function $V : \mathbb{R} \to \mathbb{R}$, $t \longmapsto V(t)$ which is an extension of v.

Proof. We need to go through several steps.

Step 0. v is continuous.

We present three supporting statements.

1. Exercise 23, page 101, in Rudin (1976) : Any convex function $f : (a,b) \longrightarrow \mathbb{R}$ is continuous.
2. Theorem 5 in de Barra (1981), page 112.
3. a complete proof is obtained as follows. a. $v_{|\left[\frac{1}{3},\frac{2}{3}\right]}$ is concave and then, from Theorem 136, $v_{|\left(\frac{1}{3},\frac{2}{3}\right)}$ is continuous.

b. (see Lemma 2.13, page 16, in Jahn (2007) and my handwritten notes on it). If $S \subseteq (X, \|\|)$ is open and convex, $f : S \longrightarrow \mathbb{R}$ is concave and there exists $\overline{x} \in S$ such that f is continuous at $\overline{x}$, then f is continuous (on S).

Then the desired result follows from a. and b.

Step 1. (Left and right derivatives are bounded) For any $t \in (a,b)$**,** $v'(t^+), v'(t^-) \in [0,k]$.

Boundedness above.

$$\forall t \in \left(0, \frac{\varepsilon}{2}\right), \qquad v'\left(t^+\right) \overset{\text{Thm. } 136}{\le} v'\left(t^-\right) \overset{\text{Assu.}}{<} k. \tag{118}$$

$$\forall t \in \left(\frac{\varepsilon}{4}, 1\right) \quad , v'\left(t^{+}\right) \overset{\text{Thm. 136}}{\leq} v'\left(t^{-}\right) \overset{\text{Lemma 137}}{\leq} v'\left(\frac{\varepsilon}{4}^{-}\right) < k \; . \tag{119}$$

Then, from (118) and (119), we do have

$$\forall t \in \left(0, \frac{\varepsilon}{2}\right) \cup \left(\frac{\varepsilon}{4}, 1\right) = (0, 1), \; v'(t) < k.$$

Boundedness below by 0.

Suppose otherwise, i.e., there exists $t \in (0,1)$ such that either $v'(t^{+}) < 0$ or $v'(t^{-}) < 0$. Then, in both cases we get a contradiction, as verified below.

$$0 > v'\left(t^{+}\right) := \lim_{h \longrightarrow 0^{+}} \frac{v(t+h) - v(t)}{h} \geq 0,$$

where the weak inequality follows from the fact that $h > 0$ and, since v is increasing, we have $v(t+h) - v(t) \geq 0$; similarly,

$$0 > v'\left(t^{-}\right) := \lim_{h \longrightarrow 0^{-}} \frac{v(t+h) - v(t)}{h} \geq 0,$$

where the weak inequality follows from the fact that $h < 0$ and, since v is increasing, we have $v(t+h) - v(t) \leq 0$.

Step 2. v is bounded below.

Suppose otherwise, i.e.,

$$\text{not} \quad \langle \exists n \in \mathbb{N} \text{ such that } \forall t \in (0,1), \quad v(t) \geq -n \rangle \; ,$$

i.e. $\forall n \in \mathbb{N}\ \exists t_n \in (0,1)$ such that $v(t_n) < -n$. Since v is increasing, then for any $t \in (0, t_n)$, $v(t) \leq v(t_n) < -n$. We can then construct a sequence in $(0,1)$ as follows.

$y_1 = t_1$

$y_2 = \min\left\{t_2, \frac{1}{2}y_1, \frac{1}{2}\right\}$ and then $v(y_2) < -2, \quad y_2 < y_1, \quad y_2 \leq \frac{1}{2} \quad y_2 > 0$

...

$y_n = \min\left\{t_n, \frac{1}{2}y_{n-1}, \frac{1}{n}\right\}$ and then $v(y_n) < -n, \quad y_n < y_{n-1}, \quad y_n \leq \frac{1}{n} \quad y_n > 0$

Now, since $y_n < y_1$, from Lemma 137, we have

$$v'\left(y_n^{-}\right) \geq \frac{v(y_1) - v(y_n)}{y_1 - y_n},$$

and from Step 1,

$$\forall n \in \mathbb{N}, \qquad k \geq v'\left(y_n^{-}\right) \geq \frac{v(y_1) - v(y_n)}{y_1 - y_n} \tag{120}$$

Since, by construction of the above sequence, $\lim_{n \to +\infty} v(y_n) = -\infty$ and $\lim_{n \to +\infty} y_n = 0$, we have

$$\lim_{n \to +\infty} \frac{v(y_1) - v(y_n)}{y_1 - y_n} = +\infty. \tag{121}$$

(120) and (121) are the desired contradiction.

Step 3. v is bounded above.

Since v is concave and increasing, using Corollary 507, we have that for any $t \in (0,1)$ and any $p_{\frac{1}{2}} \in \left[v'\left(\frac{1}{2}^{+}\right), v'\left(\frac{1}{2}^{-}\right)\right] \subseteq [0, k]$,

$$v(t) \leq v\left(\frac{1}{2}\right) + \overset{(\geq 0)}{p_{\frac{1}{2}}}\left(t - \frac{1}{2}\right) \overset{\text{Step 1, } t<1}{\leq} v\left(\frac{1}{2}\right) + k \cdot \frac{1}{2},$$

as desired.

Step 4. There exists $v_0 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 0^+} v(t) = v_0$.[21]

Since v is increasing and bounded, then $\left(v\left(\frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is a decreasing bounded below sequence[22]. Then,

$$\lim_{n \longrightarrow +\infty} v\left(\frac{1}{n+1}\right) = \inf\left\{v\left(\frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v_0. \tag{122}$$

Then, from (122) and the definition of inf, we have

$$\forall n \in \mathbb{N},\, v\left(\frac{1}{n+1}\right) \geq v_0; \tag{123}$$

from (122) and the definition of limit, we have

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\, v\left(\frac{1}{n+1}\right) - v_0 \stackrel{(123)}{=} \left| v\left(\frac{1}{n+1}\right) - v_0 \right| < \varepsilon. \tag{124}$$

We want to show that $\lim_{t \longrightarrow 0^+} v(t) = v_0$, i.e.,

$$\forall \varepsilon > 0 \ \ \exists \delta_\varepsilon > 0 \text{ such that if } t \in (0, \delta_\varepsilon)\,, \text{ then } \ |v(t) - v_0| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v is increasing, then

$$\forall t \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v(t) \leq v\left(\frac{1}{N_\varepsilon + 1}\right). \tag{125}$$

Moreover, for any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $\frac{1}{n_t+1} < t$. Since v is increasing, then

$$v(t) \geq v\left(\frac{1}{n_t + 1}\right) \stackrel{(123)}{\geq} v_0. \tag{126}$$

Then,

$$|v(t) - v_0| \stackrel{(126)}{=} v(t) - v_0 \stackrel{(125)}{\leq} v\left(\frac{1}{N_\varepsilon + 1}\right) - v_0 \stackrel{(123)}{=} \left|v\left(\frac{1}{N_\varepsilon + 1}\right) - v_0\right| \stackrel{(124) \text{ with } n = N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 5. There exists $v_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v(t) = v_1$.

The proof is quite similar to the proof of Step 4.

Since v is increasing and bounded above, then $\left(v\left(1 - \frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is an increasing bounded above sequence[23]. Then,

$$\lim_{n \longrightarrow +\infty} v\left(1 - \frac{1}{n+1}\right) = \sup\left\{v\left(1 - \frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v_1 \in \mathbb{R}. \tag{127}$$

Then, from (127) and the definition of sup, we have

$$\forall n \in \mathbb{N},\, v\left(1 - \frac{1}{n+1}\right) \leq v_1; \tag{128}$$

from (127) and the definition of limit, we have

$$\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\, v_1 - v\left(\frac{1}{n+1}\right) \stackrel{(??)}{=} \left| v_1 - v\left(\frac{1}{n+1}\right) \right| < \varepsilon. \tag{129}$$

[21] See also Theorem 4.5.1, page 200, in Zakon (2020).

[22] We have to take $v\left(\frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

[23] We have to take $v\left(1 - \frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

We want to show that $\lim_{t\longrightarrow 1^-} v(t) = v_1$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \text{ such that if } t \in (1-\delta_\varepsilon, 1), \text{ then } |v(t) - v_1| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v is increasing, then

$$\forall t \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v(t) \geq v\left(1 - \frac{1}{N_\varepsilon + 1}\right) \text{ and } -v(t) \leq -v\left(1 - \frac{1}{N_\varepsilon + 1}\right). \tag{130}$$

Moreover, for any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $1 - \frac{1}{n_t + 1} > t$. Since v is increasing, then

$$v(t) \leq v\left(1 - \frac{1}{n_t + 1}\right) \overset{(128)}{\leq} v_1. \tag{131}$$

Then,

$$|v(t) - v_1| \overset{(131)}{=} v_1 - v(t) \overset{(130)}{\leq} v_1 - v\left(1 - \frac{1}{N_\varepsilon + 1}\right) \overset{(128)}{=} \left|v\left(\frac{1}{N_\varepsilon + 1}\right) - v_1\right| \overset{(129) \text{ with } n = N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 6. The function $\widehat{v} : [0,1] \longrightarrow \mathbb{R}$,

$$\widehat{v}(t) = \begin{cases} v_0 & \text{if } t = 0 \\ v(t) & \text{if } t \in (0,1) \\ v_1 & \text{if } t = 1 \end{cases}$$

is continuous.

It follows from Steps 0, 4 and 5.

Step 7. Defined $v'_- : (0,1) \longrightarrow \mathbb{R}$ $t \mapsto v'(t^-) := \lim_{h \longrightarrow 0^-} \frac{v(t+h) - v(t)}{h}$, i.e., the left derivative in t, then there exists $v'_0 \in \mathbb{R}_+$ such that $\lim_{t \longrightarrow 0^+} v'_-(t) = v'_0$.

Observe that $v'_- : (0,1) \longrightarrow \mathbb{R}$ $t \mapsto v'(t^-)$ is bounded (from Step 1) and decreasing (by Lemma 137). We can then mimic the argument presented in Steps 4 or 5 - observe that there we did NOT use the concavity of v.

Since, from Lemma 137, v'_- is decreasing and bounded, then $\left(v'_-\left(\frac{1}{n+1}\right)\right)_{n \in \mathbb{N}}$ is an increasing bounded above sequence[24]. Then,

$$\lim_{n \longrightarrow +\infty} v'_-\left(\frac{1}{n+1}\right) = \sup\left\{v'_-\left(\frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v'_0 \in \mathbb{R}. \tag{132}$$

Then, from (132) and the definition of sup, we have

$$\forall n \in \mathbb{N},\; v'_-\left(\frac{1}{n+1}\right) \leq v'_0; \tag{133}$$

from (132) and the definition of limit, we have

$$\forall \varepsilon > 0 \; \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\; v'_0 - v'_-\left(\frac{1}{n+1}\right) \overset{(133)}{=} \left| v'_0 - v'_-\left(\frac{1}{n+1}\right)\right| < \varepsilon. \tag{134}$$

We want to show that $\lim_{t \longrightarrow 0^+} v'_-(t) = v'_0$, i.e.,

$$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0 \text{ such that if } t \in (0, \delta_\varepsilon), \text{ then } \left|v'_-(t) - v'_0\right| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon + 1}$; since v'_- is decreasing, then

$$\forall t \in \left(0, \frac{1}{N_\varepsilon + 1}\right), \quad v'_-(t) \leq v'_-\left(\frac{1}{N_\varepsilon + 1}\right). \tag{135}$$

[24] We have to take $v\left(\frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

Moreover, for any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $\frac{1}{n_t+1} < t$. Since v'_- is decreasing, then

$$v'_-(t) \leq v'_-\left(\frac{1}{n_t+1}\right) \overset{(133)}{\leq} v'_0. \tag{136}$$

Then,

$$\left|v'_-(t) - v'_0\right| \overset{(136)}{=} v'_0 - v'_-(t) \overset{(135)}{\leq} v'_0 - v'_-\left(\frac{1}{N_\varepsilon+1}\right) \overset{(133)}{=} \left|v'_0 - v'_-\left(\frac{1}{N_\varepsilon+1}\right)\right| \overset{(134)\text{ with } n=N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 8. Defined $v'_+ :(0,1) \longrightarrow \mathbb{R}$ $t \mapsto v'(t^+)$, there exists $v'_1 \in \mathbb{R}$ such that $\lim_{t \longrightarrow 1^-} v'(t^+) = v'_1$.

Observe that $v'_+ :(0,1) \longrightarrow \mathbb{R}$ $t \mapsto v'(t^+)$ is bounded (from Step 1) and decreasing (by Lemma 503). We can then mimic the argument presented in Steps 4 or 5 or 7.

Since v'_+ is decreasing and bounded below, then $\left(v'_+\left(1-\frac{1}{n+1}\right)\right)_{n\in\mathbb{N}}$ is a decreasing bounded below sequence[25]. Then,

$$\lim_{n\longrightarrow+\infty} v'_+\left(1-\frac{1}{n+1}\right) = \inf\left\{v'_+\left(1-\frac{1}{n+1}\right) : n \in \mathbb{N}\right\} := v'_1 \in \mathbb{R}. \tag{137}$$

Then, from (137) and the definition of inf, we have

$$\forall n \in \mathbb{N},\ v'_+\left(1-\frac{1}{n+1}\right) \geq v'_1; \tag{138}$$

from (137) and the definition of limit, we have

$$\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N} \text{ such that if } n \geq N_\varepsilon,\ v'_+\left(1-\frac{1}{n+1}\right) - v'_1 \overset{(133)}{=} \left| v'_+\left(1-\frac{1}{n+1}\right) - v'_1\right| < \varepsilon. \tag{139}$$

We want to show that $\lim_{t\longrightarrow 1^-} v'_+(t) = v'_1$, i.e.,

$$\forall\varepsilon > 0\ \ \exists\delta_\varepsilon > 0 \text{ such that if } t \in (1-\delta_\varepsilon, 1)\text{, then } \left|v'_+(t) - v'_1\right| < \varepsilon.$$

Take $\delta_\varepsilon = \frac{1}{N_\varepsilon+1}$; since v'_+ is decreasing, then

$$\forall t \in \left(0, 1-\frac{1}{N_\varepsilon+1}\right), \quad v'_+(t) \leq v'_+\left(1-\frac{1}{N_\varepsilon+1}\right). \tag{140}$$

Moreover, for any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $1-\frac{1}{n_t+1} > t$. Since v'_+ is decreasing, then

$$v'_+(t) \geq v'_+\left(1-\frac{1}{n_t+1}\right) \overset{(138)}{\geq} v'_1. \tag{141}$$

Then,

$$\left|v'_+(t) - v'_1\right| \overset{(141)}{=} v'_+(t) - v'_1 \overset{(140)}{\leq} v'_+\left(1-\frac{1}{N_\varepsilon+1}\right) - v'_1 \overset{(138)}{=} \left|v'_+\left(1-\frac{1}{N_\varepsilon+1}\right) - v'_1\right| \overset{(139)\text{ with } n=N_\varepsilon}{<} \varepsilon,$$

as desired.

Step 9. The function V defined below is the desired extension.

$$V:\mathbb{R}\longrightarrow\mathbb{R}, \qquad V(t) = \begin{cases} v_0 + v'_0 \cdot t & \text{if} \quad t \leq 0 \\ \widehat{v}(t) & \text{if} \quad t \in [0,1] \\ v_1 + v'_1 \cdot t & \text{if} \quad t \geq 1. \end{cases}$$

a. V is continuous.

[25] We have to take $v\left(1-\frac{1}{n+1}\right)$ because v is defined on $(0,1)$.

V is continuous from the Pasting Lemma. Below, we state a version of that theorem (see Theorem 18.3, page 108, in Munkres (1975)) and we apply it to our case.

(The pasting lemma) Let X and Y be topological spaces. Let $X = A \cup B$, where A and B are closed in X and $f : A \to Y$, $g : B \to Y$ be continuous functions. If $\forall x \in A \cap B$, $f(x) = g(x)$, then "f and g combine to give a continuous function", i.e., $h : X \to Y$,

$$h(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ \\ g(x) & \text{if} \quad x \in B \end{cases}$$

is a continuous function.

We can apply the above lemma identifying

$$\begin{array}{lll} X & \text{with} & \mathbb{R} \\ \\ A & \text{with} & (-\infty, 0] \cup [1, +\infty) \\ \\ B & \text{with} & [0,1] \\ \\ f & \text{with} & \begin{cases} v_0 + v_0' \cdot t & \text{if} \quad t \leq 0 \\ v_1 + v_1' \cdot t & \text{if} \quad t \geq 1. \end{cases} \\ \\ g & \text{with} & \widehat{v} \\ \\ A \cap B & \text{with} & \{0,1\} \end{array}$$

b. V is increasing.

We want to show that for any $t_1, t_2 \in \mathbb{R}$, if $t_1 < t_2$, then $V(t_1) \leq V(t_2)$. We distinguish the following cases. The other cases follow from the facts that $v_0', v_1' \in [0, k]$ and that v is increasing by assumption.

Case 1. $t_1 \leq 0$ and $t_2 \in (0,1)$; Case 2. $t_1 \in (0,1)$ and $t_2 \geq 1$; Case 3. $t_1 \leq 0$ and $t_2 \geq 1$.

Case 1.

We want to show that $V(t_1) \leq V(t_2)$ or $v_0 + v_0' \cdot t_1 \leq v(t_2)$. Indeed,

$$V(t_1) = v_0 + v_0' \cdot t_1 \overset{v_{0+}' \geq 0,\ t_1 \leq 0}{\leq} v_0.$$

Then, it is enough to show that for any $t_2 \in (0,1)$, $v_0 \leq v(t_2)$, or, from Step 4, by definition of v_0, $v(t_2) \geq \inf_{n \in \mathbb{N}} v\left(\frac{1}{n+1}\right)$. Indeed, for any $t_2 \in (0,1)$, there exists $n_2 \in \mathbb{N}$ such that $t_2 > \frac{1}{1+n_2}$ Then, since v is increasing on $(0,1)$, we have $v(t_2) \geq v\left(\frac{1}{1+n_2}\right) \geq \inf_{n \in \mathbb{N}} v\left(\frac{1}{n+1}\right)$, as desired.

Case 2.

We want to show that $V(t_1) \leq V(t_2)$ or $v(t_1) \leq v_1 + v_1'(t_2 - 1)$. Indeed,

$$V(t_2) = v_1 + v_1'(t_2 - 1) \overset{v_{1-}' \geq 0,\ t_2 \geq 1}{\geq} v_1.$$

Then, it is enough to show that for any $t_1 \in (0,1)$, $v(t_1) \leq v_1$, or, from Step 5, by definition of v_1, $v(t_1) \leq \sup_{n \in \mathbb{N}} v\left(1 - \frac{1}{n+1}\right)$. Indeed, for any $t_1 \in (0,1)$, there exists $n_1 \in \mathbb{N}$ such that $t_1 < 1 - \frac{1}{1+n_1}$ Then, since v is increasing on $(0,1)$, we have $v(t_1) \leq v\left(1 - \frac{1}{1+n_1}\right) \geq \sup_{n \in \mathbb{N}} v\left(1 - \frac{1}{n+1}\right)$, as desired.

Case 3.

It is enough to observe that

$$V(t_1) \overset{\text{Case 1}}{\leq} V\left(\frac{1}{2}\right) \overset{\text{Case 2}}{\leq} V(t_2).$$

c. V is concave.

It is showed in Lemma 149 below, which requires some other Lemmas. ■

Lemma 143 *If* $\varphi_1, \varphi_2 : [a,b] \longrightarrow \mathbb{R}$ *are concave functions, then* $\varphi := \min\{\varphi_1, \varphi_2\}$ *is concave.*

Proof. For any t_1, $t_2 \in [a,b]$, $\lambda \in (0,1)$,

$$\varphi((1-\lambda)t_1 + \lambda t_2) := \min\{\varphi_1((1-\lambda)t_1 + \lambda t_2),\ \varphi_2((1-\lambda)t_1 + \lambda t_2)\} \overset{\varphi_i \text{ concave}}{\geq}$$

$$\min\{(1-\lambda)\varphi_1(t_1) + \lambda\varphi_1(t_2),\ (1-\lambda)\varphi_2(t_1) + \lambda\varphi_2(t_2)\} \overset{(1)}{\geq}$$

$$(1-\lambda)\min\{\varphi_1(t_1), \varphi_2(t_1)\} + \lambda\min\{\varphi_1(t_2), \varphi_2(t_2)\} :=$$

$$(1-\lambda)\varphi(t_1) + \lambda\varphi(t_2),$$

as desired and where (1) follows from the facts that

$$(1-\lambda)\varphi_1(t_1) + \lambda\varphi_1(t_2) \geq (1-\lambda)\min\{\varphi_1(t_1), \varphi_2(t_1)\} + \lambda\min\{\varphi_1(t_2), \varphi_2(t_2)\},$$

$$(1-\lambda)\varphi_2(t_1) + \lambda\varphi_2(t_2) \geq (1-\lambda)\min\{\varphi_1(t_1), \varphi_2(t_1)\} + \lambda\min\{\varphi_1(t_2), \varphi_2(t_2)\}$$

and

$$\langle\ a \geq c \wedge b \geq c\rangle \overset{\text{def.}}{\Leftrightarrow} \min\{a,b\} \geq c.$$

■

Lemma 144 *Given* $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ *and* $\alpha_0, \alpha_1 \in \mathbb{R}_+$ *such that*

$$(x_0, y_0) << (x_1, y_1) \quad \text{and} \quad \alpha_0 > \alpha_1,$$

defined $g_0, g_1 : \mathbb{R} \longrightarrow \mathbb{R}$,

$$g_0(x) = y_0 + \alpha_0(x - x_0),$$

$$g_1(x) = y_1 + \alpha_1(x - x_1),$$

if

$$\alpha_0 \geq \frac{y_1 - y_0}{x_1 - x_0} \geq \alpha_1,$$

then 1. $g := \min\{g_0, g_1\}$ *is concave, and*
2. there exists $x^* \in \mathbb{R}$ *such that a.* $g_0(x^*) = g_1(x^*)$, *b.* $x^* \in [x_0, x_1]$ *and c.*

$$g(x) = \begin{cases} g_0(x) & \text{if} \quad x \leq x^* \\ g_1(x) & \quad\quad x \geq x^*. \end{cases}$$

`see also` *picture for extension of v.pdf*

Proof. 1. Since g_0 and g_1 are linear, then the desired result follows from Lemma 143.
2. a. x^* solves

$$y_0 + \alpha_0(x - x_0) = y_1 + \alpha_1(x - x_1)$$
or
$$(\alpha_0 - \alpha_1)x = y_1 - \alpha_1 x_1 - (y_0 - \alpha_0 x_0)$$
and then
$$x^* = \tfrac{y_1 - \alpha_1 x_1 - (y_0 - \alpha_0 x_0)}{\alpha_0 - \alpha_1}.$$

b.

$$\tfrac{y_1 - \alpha_1 x_1 - (y_0 - \alpha_0 x_0)}{\alpha_0 - \alpha_1} \geq x_0$$

$$y_1 - \alpha_1 x_1 - y_0 + \alpha_0 x_0 - \alpha_0 x_0 + \alpha_1 x_0 \geq 0$$

$$y_1 - \alpha_1 x_1 - y_0 + \alpha_1 x_0 \geq 0$$

$$\alpha_1 \leq \tfrac{y_1 - y_0}{x_1 - x_0},$$

as assumed.

$$\frac{y_1-\alpha_1 x_1-y_0+\alpha_0 x_0}{\alpha_0-\alpha_1} \leq x_1$$

$$y_1-\alpha_1 x_1-y_0+\alpha_0 x_0-\alpha_0 x_1+\alpha_1 x_1 \leq 0$$

$$y_1-y_0+\alpha_0 x_0-\alpha_0 x_1 \leq 0$$

$$\alpha_0 \geq \frac{y_1-y_0}{x_1-x_0},$$

as assumed.
c.
We verify that $x \leq x^*$ iff $g_0(x) \leq g_1(x)$. Indeed,

$$y_0+\alpha_0(x-x_0) \leq y_1+\alpha_1(x-x_1)$$

$$y_1-y_0-\alpha_1 x_1+\alpha_0 x_0 \geq \overset{(>0)}{(\alpha_0-\alpha_1)}x$$

$$x^* := \frac{y_1-\alpha_1 x_1-(y_0-\alpha_0 x_0)}{\alpha_0-\alpha_1} \geq x,$$

as desired. Similar proof applies to the other case. ■

Remark 145 *Under the assumptions of Lemma 144, we have what follows.*

$$g_0(x_1) \geq y_1 \quad \Leftrightarrow \quad y_0+\alpha_0(x_1-x_0) \geq y_1 \quad \Leftrightarrow \quad \alpha_0(x_1-x_0) \geq y_1-y_0 \quad \Leftrightarrow \quad \alpha_0 \geq \frac{y_1-y_0}{x_1-x_0};$$

$$g_1(x_0) \geq y_0 \quad \Leftrightarrow \quad y_1+\alpha_1(x_0-x_1) \geq y_0 \quad \Leftrightarrow \quad y_1-y_0 \geq \alpha_1(x_1-x_0) \quad \Leftrightarrow \quad \frac{y_1-y_0}{x_1-x_0} \geq \alpha_1.$$

The above analysis allows to give the definition of the following derivative set-valued function.

Definition 146 *The derivative set-valued function associated with V is denoted and defined as follows:*

$$p: \mathbb{R} \longrightarrow\!\!\!\!\longrightarrow \mathbb{R},$$

$$t \mapsto\!\!\!\!\mapsto \begin{cases} \{v_0'\} & \text{if} \quad t \leq 0 \\ \left[v_-'(t^+), v(t^-)\right] & \text{if} \quad t \in (0,1) \\ \{v_1'\} & \text{if} \quad t \geq 1. \end{cases}$$

Lemma 147 *For any $t_0, t_1 \in \mathbb{R}$, if $t_0 < t_1$, then for any $p_0 \in p(t_0)$ and any $p_1 \in p(t_1)$, we have $p_0 \geq p_1$.*

Proof. We want to show that for any $t \in (0,1)$,

$$v_0' \overset{(1)}{\geq} v_-'(t) \geq v_+'(t) \overset{(2)}{\geq} v_1'.$$

Proof of (1).
From Step 7, we have

$$v_0' = \sup\left\{v_-'\left(\frac{1}{n+1}\right) : n \in \mathbb{N}\right\}. \tag{142}$$

For any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $\frac{1}{n_t+1} < t$ and since v_-' is decreasing, we do have

$$v_-'\left(\frac{1}{n+1}\right) \geq v_-'(t). \tag{143}$$

(142) and (143) imply the desired result.
Proof of (2).
From Step 8, we have

$$v_1' = \inf\left\{v_+'\left(1-\frac{1}{n+1}\right) : n \in \mathbb{N}\right\}. \tag{144}$$

For any $t \in (0,1)$, there exists $n_t \in \mathbb{N}$ such that $1 - \frac{1}{n_t+1} > t$ and since v'_- is decreasing, we do have

$$v'\left(1 - \frac{1}{n_t + 1}\right) \leq v'\left(t^-\right). \tag{145}$$

(144) and (145) imply the desired result. ■

Lemma 148 *For any $t_0, t_1 \in \mathbb{R}$, for any $p_{t_0} \in p(t_0)$*

$$V(t_1) \leq V(t_0) + p_{t_0}(t_1 - t_0).$$

Proof. The strategy of the proof is as follows. We distinguish several cases; in each of them,

a. compute the equation of the line going through $(t_0, V(t_0))$ and slope $p_{t_0} \in p(t_0)$; call g_0 the associated function.

b. compute the equation of the line going through $(t_1, V(t_1))$ and slope $p_{t_1} \in p(t_1)$; call g_1 the associated function.

c. choose "simple values of" (x_0, y_0) and (x_1, y_1).

d. verify assumptions of Lemma 144 are satisfied if $y_1 > y_0$ and $\alpha_0 > \alpha_1$; verify the desired result also in the case in which some equalities hold true.

e. apply Corollary 141 to $g := \min\{g_0, g_1\}$ with t, t_0 in the place of s, t, which is just the desired result.

We distinguish the following results.

Case 1. $t_0 \in (-\infty, 0]$. a. $t_1 \in (-\infty, 0]$; b. $t_1 \in (0,1)$; c. $t_1 \in [1, +\infty)$.

Case 2. $t_0 \in (0,1)$. a. $t_1 \in (-\infty, 0]$; b. $t_1 \in (0,1)$; c. $t_1 \in [1, +\infty)$.

Case 3. $t_0 \in [1, +\infty)$. a. $t_1 \in (-\infty, 0]$; b. $t_1 \in (0,1)$; c. $\in t_1 [1, +\infty)$.

Case 1.

Case 1.a.

Obvious.

Case 1.b. . $t_0 \in (-\infty, 0]$ and $t_1 \in (0,1)$

a. $g_0(t) = v_0 + v'_0 \cdot t$; b. $g_1(t) = v(t_1) + p_{t_1}(t - t_1)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
0	v_0	t_1	$v(t_1)$	v'_0	p_{t_1}

d.1.

$$g_0(x_1) \geq y_1 : \quad v_0 + v'_0 \cdot t_1 \geq v(t_1).$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v(t_1) \leq v\left(\frac{1}{n}\right) + v'_-\left(\frac{1}{n}\right) \cdot \left(t_1 - \frac{1}{n}\right)$$

and taking limits, we get

$$v(t_1) \leq v_0 + v'_0 \cdot t_1.$$

d.2.

$$g_1(x_0) \geq y_0 : \quad v(t_1) - p_{t_1} \cdot t_1 \geq v_0.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(\frac{1}{n}\right) \leq v(t_1) + p_{t_1} \cdot \left(\frac{1}{n} - t_1\right)$$

and taking limits, we get

$$v_0 \leq v(t_1) - p_{t_1} \cdot t_1.$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[0, t_1]$ - recall that V is increasing from Step 9.b - and since $t_1 \in (0,1)$, $v'_-(t_1) = 0$ and $p(t_1) = \{0\}$ - using Corollary 141. We want to show

$$V(t_1) \leq V(t_0) + p_{t_1}(t_1 - t_0),$$

or

$$y_0 \leq y_0 + 0 \cdot (t_1 - t_0),$$

which is true.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

Since $t_0 \leq 0$, then for any $t \in [t_0, t_1)$, $p_{t_0} = v_0' \geq p_t \geq p_{t_1}$ and then for any $t \in [t_0, t_1)$, $p(t) = v_0'$. Then, the desired result follows from the Claim below.

Claim. If $t_0 < t_1$ and for any $t \in [t_0, t_1)$, we have $V'(t) = v_0' = p_{t_1}$, then $V(t_1) \leq V(t_0) + p_{t_1}(t_1 - t_0)$.

Proof of the Claim. By Assumption, there exists $k \in \mathbb{R}$ such that for any $t \in [t_0, t_1)$, $V(t) = v_0' \cdot t + k$. Then, we want to show

$$v_0' \cdot t_1 + k \leq v_0' \cdot t_0 + k + v_0'(t_1 - t),$$

which is true as an equality.

End of the proof of the Claim.

Case 1.c. $t_0 \in (-\infty, 0]$ and $t \in [1, +\infty)$.

a. $g_0(t) = v_0 + v_0' \cdot t$; b. $g_1(t) = v_1 + v_1'(t-1)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
0	v_0	1	v_1	v_0'	v_1'

d.1.

$$g_0(x_1) \geq y_1: \quad v_0 + v_0' \geq v_1.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(1 - \frac{1}{n}\right) \leq v\left(\frac{1}{n}\right) + v_{\pm}'\left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n} - \frac{1}{n}\right)$$

and taking limits, we get

$$v_1 \leq v_0 + v_0'.$$

d.2.

$$g_1(x_0) \geq y_0: \quad v_1 - v_1' \geq v_0.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(\frac{1}{n}\right) \leq v\left(1 - \frac{1}{n}\right) + v_{\pm}'\left(1 - \frac{1}{n}\right) \cdot \left(\frac{1}{n} - \left(1 - \frac{1}{n}\right)\right)$$

and taking limits, we get

$$v_0 \leq v_1 - v_1'.$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[0,1]$,then V is constant on $\mathbb{R}$, because $v_0' = v_1'$and we are done.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

Since $t_0 \leq 0$ and $t_1 \geq 1$, we have $v_0' = v_1'$ then for any $t \in (0,1)$, $p_{t_0} = v_0' \geq p_t \geq p_{t_1} = v_0'$ and then for any $t \in \mathbb{R}$, $p(t) = v_0'$. Then, V is an affine function.

Case 2.

Case 2.b.

Obvious.

Case 2.a. $t_0 \in (0,1)$ and $t \in (-\infty, 0]$.

a. $g_0(t) = v_0 + v_0' \cdot t$; b. $g_1(t) = v(t_0) + p_{t_0}(t - t_0)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
0	v_0	t_0	$v(t_0)$	v_0'	p_{t_0}

d.1.

$$g_0(x_1) \geq y_1: \quad v_0 + v_0' \cdot t_0 \geq v(t_0).$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v(t_0) \leq v\left(\frac{1}{n}\right) + v'_{\pm}\left(\frac{1}{n}\right) \cdot \left(t_0 - \frac{1}{n}\right)$$

and taking limits, we get

$$v(t_0) \leq v_0 + v'_0 \cdot t_0.$$

d.2.

$$g_1(x_0) \geq y_0 : \quad v(t_0) - p_{t_0} t_0 \geq v_0.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(\frac{1}{n}\right) \leq v(t_0) + p_{t_0} \cdot \left(\frac{1}{n} - t_0\right)$$

and taking limits, we get

$$v_0 \leq v(t_0) - p_{t_0} \cdot t_0.$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[0, t_0]$ - recall that V is increasing from Step ... - and since $t_0 \in (0,1)$, $v'_-(t_1) = 0$ and $p(t_0) = \{0\}$ - using Corollary 141. We want to show

$$V(t_1) \leq V(t_0) + p_{t_0}(t_0 - t_1),$$

or

$$v(t_0) \leq v_0 + 0 \cdot (t_1 - t_0),$$

which is true.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

Since $t_0 \in (0,1)$ and $t_1 \leq 0$, then for any $t \in [t_1, t_0)$, $p_{t_0} = v'_0 \geq p_t \geq p_{t_1}$ and then for any $t \in [t_0, t_1)$, $p(t) = v'_0$. Then, the desired result follows from the Claim above.

Case 2.c. $t_0 \in (0,1)$. and $t \in [1, +\infty)$.

a. $g_0(t) = v_0 + p_{t_0} \cdot (t - t_0)$; b. $g_1(t) = v(t_1) + v'_1(t - 1)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
t_0	v_0	1	v_1	p_{t_0}	v'_1

d.1.

$$g_0(x_1) \geq y_1 : \quad v_0 + p_{t_0} \cdot (1 - t_0) \geq v_1.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(1 - \frac{1}{n}\right) \leq v(t_0) + p_{t_0} \cdot \left(\left(1 - \frac{1}{n}\right) - t_0\right)$$

and taking limits, we get

$$v_1 \leq v(t_0) + p_{t_0} \cdot (1 - t_0)$$

d.2.

$$g_1(x_0) \geq y_0 : \quad v_1 + v'_1(t_0 - 1) \geq v(t_0).$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v(t_0) \leq v\left(1 - \frac{1}{n}\right) + v'_{+-}\left(1 - \frac{1}{n}\right) \cdot \left(t_0 - \left(1 - \frac{1}{n}\right)\right)$$

and taking limits, we get

$$v(t_0) \leq v_1 + v' \cdot (t_0 - 1)$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[t_0, t_1]$ - recall that V is increasing from Step ... - and since $t_0 \in (0,1)$, $v'_+(t_0) = 0$ and $p(t_0) = \{0\}$ - using Corollary 507. We want to show

$$V(t_1) \leq V(t_0) + p_{t_0}(t_1 - t_0),$$

or

$$y_1 \leq y_1 + 0 \cdot (t_1 - t_0),$$

which is true.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

Since $t_0 \in (0,1)$, then for any $t \in [t_0, t_1)$, $p_{t_0} = v'_0 \geq p_t \geq p_{t_1} = v'_1$ and then for any $t \in [t_0, t_1)$, $p(t) = v'_0$. Then, the desired result follows from the Claim above. .

Case 3.

Case 3. c. $t_0 \in [1, +\infty)$.and $t \in [1, +\infty)$.

Obvious.

Case 3. a. $t_0 \in [1, +\infty)$ and $t \in (-\infty, 0]$.

a. $g_0(t) = v_0 + v'_0 \cdot t$; b. $g_1(t) = v_1 + v'_1(t-1)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
0	v_0	1	v_1	v'_0	v'_1

d.1.

$$g_0(x_1) \geq y_1 : \quad v_0 + v'_0 \geq v_1.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(1 - \frac{1}{n}\right) \leq v\left(\frac{1}{n}\right) + v'_{\pm}\left(\frac{1}{n}\right) \cdot \left(1 - \frac{1}{n}\right)$$

and taking limits, we get

$$v_1 \leq v_0 + v'_0$$

d.2.

$$g_1(x_0) \geq y_0 : \quad v_1 - v'_1 \geq v_0.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(\frac{1}{n}\right) \leq v\left(1 - \frac{1}{n}\right) + v'_{+-}\left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{1}{n} - \frac{1}{n}\right)$$

and taking limits, we get

$$v_0 \leq v_1 + v'_1$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[0,1]$,then V is constant on $\mathbb{R}$, and we are done.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

The V is an affine function.

Case 3.c. $t_0 \in [1, +\infty)$ and $t \in (0,1)$.

a. $g_0(t) = v(t_1) + p_{t_1} \cdot (t - t_1)$; b. $g_1(t) = v_1 + v'_1(t-1)$; c.

x_0	y_0	x_1	y_1	α_0	α_1
t_1	$v(t_1)$	1	v_1	p_{t_1}	v'_1

d.1.

$$g_0(x_1) \geq y_1 : \quad v(t_1) + p_{t_1} \cdot (1 - t_1) \geq v_1.$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v\left(1 - \frac{1}{n}\right) \leq v(t_1) + p_{t_1} \cdot \left(\left(1 - \frac{1}{n}\right) - t_1\right)$$

and taking limits, we get

$$v(1) \leq v(t_1) + p_{t_1} \cdot (1 - t_1)$$

d.2.

$$g_1(x_0) \geq y_0 : \quad v_1 + v_1'(t_1 - 1) \geq v(t_1).$$

Indeed, since V is concave on $(0,1)$, we have that for any $n \geq 2$,

$$v(t_1) \leq v\left(1 - \frac{1}{n}\right) + v'_{+-}\left(1 - \frac{1}{n}\right) \cdot \left(t_1 - \left(1 - \frac{1}{n}\right)\right)$$

and taking limits, we get

$$v(t_1) \leq v(1) + v'_{1-} \cdot (t_1 - 1)$$

The equality cases. $y_0 = y_1$.

Then, V is constant on $[t_1, 1]$ and $p_{t_1} = 0 = v_1'$ and then V is constant on $[t_1, +\infty)$. We want to show

$$V(t_1) \leq V(t_0) + p_{t_0}(t_1 - t_0),$$

or

$$v(t_1) \leq V(t_0) + 0 \cdot (t_1 - t_0),$$

which is true because $t_0 > t_1$.

The equality cases. $y_0 < y_1$ and $p_{t_0} = p_{t_1}$.

From Corollary 141, for any $t \in [t_1, t_0)$ and then for any $t \in [t_1, +\infty)$, we have $V'(t) = v_1'$ and then V is affine on $[t_1, +\infty)$ and we are done. ■

Lemma 149 *V is concave.*

Proof. We are going to use Lemma 148 and then mimic the proof on page 263 in my Math 2 Notes.

Lemma 148 says that

for any $t_0, t \in \mathbb{R}$, for any $p_{t_0} \in p(t_0)$, $V(t) \leq V(t_0) + p_{t_0}(t - t_0)$.

For any $t', t'' \in \mathbb{R}$ and $\lambda \in (0,1)$. Define $t^\lambda = (1-\lambda)t' + \lambda x''$. We want to show that

$$V\left(t^\lambda\right) \geq (1-\lambda)V(t') + \lambda V(t'').$$

From Lemma 148,we have

$$V(t'') - V(t^\lambda) \leq p_{t^\lambda} \cdot (t'' - t^\lambda) \text{ and}$$

$$V(t') - V(t^\lambda) \leq p_{t^\lambda} \cdot (t' - t^\lambda)$$

Multiplying the first expression by λ, the second one by $(1-\lambda)$ and summing up, we get

$$\lambda(V(t'') - V(t^\lambda)) + (1-\lambda)(V(t') - V(t^\lambda)) \leq p_{t^\lambda} \cdot (\lambda(t'' - t^\lambda) + (1-\lambda)(t' - t^\lambda))$$

Since

$$\lambda(t'' - t^\lambda) + (1-\lambda)(t' - t^\lambda) = t^\lambda - t^\lambda = 0,$$

we get

$$\lambda V(t'') + (1-\lambda)V(t') \leq V(t^\lambda),$$

i.e., the desired result. ■

4.9.4 On some probably useless results on extending the function $v : \mathbb{R} \longrightarrow \mathbb{R}$

Proposition 150 *If a function* $\varphi : (0,1) \longrightarrow \mathbb{R}$ *is concave and increasing and*

$$\exists \varepsilon > 0 \text{ and } \exists k \in \mathbb{R}_{++} \text{ such that for any } x, y \in \mathbb{R} \text{ such that } 0 < x < y < \varepsilon \text{ we have that } \frac{\varphi(y) - \varphi(x)}{y - x} < k, \tag{146}$$

then φ *is k-Lipschitz.*

Proof. We want to show that $\forall \alpha, \beta \in (0,1), \alpha \neq \beta$, we have $\frac{|\varphi(\beta)-\varphi(\alpha)|}{|\beta-\alpha|} < k$. Without loss of generality, assume $\alpha < \beta$; then, since φ is increasing, we have to show that $\frac{\varphi(\beta)-\varphi(\alpha)}{\beta-\alpha} < k$. The key ingredient of the proof is the following result.

Let the function $\varphi : (a,b) \longrightarrow \mathbb{R}$ be a concave function and assume that

$$a < s < t < u < b. \tag{147}$$

Then

$$\frac{\varphi(t) - \varphi(s)}{t - s} \geq \frac{\varphi(u) - \varphi(s)}{u - s} \geq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

`see picture` debarra prop.pdf

For a proof of the above result see, for example, de Barra (2003), Theorem 2, page 111.

We distinguish the following three cases.

Case 1. $0 < \alpha < \beta < \varepsilon$; Case 2. $0 < \alpha < \varepsilon \leq \beta$; Case 3. $0 < \varepsilon \leq \alpha < \beta$.

Case 1. The desired result is true by assumption.

Case 2. Take $\alpha' \in (\alpha, \varepsilon)$; then,

$$k \overset{\text{Assu.}}{>} \frac{\varphi(\alpha') - \varphi(\alpha)}{\alpha' - \alpha} \overset{(147)}{\geq} \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha},$$

`or`

Take $\alpha' \in (0, \alpha)$; then,

$$k \overset{\text{Assu.}}{>} \frac{\varphi(\alpha) - \varphi(\alpha')}{\alpha - \alpha'} \overset{(147)}{\geq} \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha},$$

Case 3.

$$k \overset{\text{Assu.}}{>} \frac{\varphi(y) - \varphi(x)}{y - x} \overset{(147)}{\geq} \frac{\varphi(\alpha) - \varphi(y)}{\alpha - y} \overset{(147)}{\geq} \frac{\varphi(\beta) - \varphi(\alpha)}{\beta - \alpha}.$$

■

4.10 On quasi-concavity of the utility function

We want to find conditions under which the utility function is quasi-concave in both x_h and $\theta_{h'}$. That for sure it is the case in which $\beta > 0$. In that case, we utility function is strictly concave, since the Hessian matrix is

$$\begin{matrix} u'' & 0 \\ 0 & \beta v'' \end{matrix}$$

and $u'' < 0, v'' < 0$ and $\beta > 0$.

If $\beta < 0$, we have the following proposition.

Proposition 151 *Consider a one-good economies* (β, u, v) *with* $\beta < 0$. *Define*

$$V : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}, \ (x, y) \mapsto \ u(x) + \beta v(y).$$

1. *A sufficient condition for quasi-concavity of* V *is*

$$\overset{(-)}{v''} \cdot \overset{(+)}{(u')^2} + \overset{(-)}{\beta} \cdot \overset{(-)}{u''} \cdot \overset{(+)}{v'} > 0,$$

or, denoted the coefficient of absolute risk aversion associated with function f *by* $R_{A(f)}$,

$$(-\beta) R_A(u) < u' \cdot R_A(v)$$

2. *There exist economies for which* $\beta < 0$ *and* V *is quasi-concave.*

Proof. Preliminary observation. Below, we present two approaches to find sufficient conditions for quasi-concavity.

After the preliminary observation, we present the proof of the desired results.

Approach 1. Implicit Function Theorem.

Let the following equation be given.

$$V(x,y) := u(x) + \beta v(y) - k = 0,$$

with

$$u' > 0, \quad u'' < 0, \quad \beta < 0, \quad v' > 0, \quad v'' < 0.$$

From the Implicit Function Theorem, we have

$$\frac{dx}{dy} := g'(y) = -\frac{\frac{\partial V(x,y}{\partial y}}{\frac{\partial V(x,y}{\partial x}} = -\frac{\beta v'(y)}{u'(x)} > 0.$$

We now want to give conditions under which $g''(y) > 0$.

$$g''(y) = \frac{d\left(-\frac{\beta v'(y)}{u'(g(y))}\right)}{dx} = -\frac{1}{(u'(g(y)))^2}\left(\beta \cdot v''(y) \cdot u'(g(y)) - \beta \cdot v'(y) \cdot u''(g(y)) \cdot g'(y)\right) =$$

$$-\frac{1}{(u'(g(y)))^2}\overset{(-)}{\beta}\left(v'' \cdot u' + v' \cdot u'' \cdot \beta \cdot v' \cdot \frac{1}{u'}\right)$$

Then

$$sign\ g''(y) = sign\ \left(\overset{(-)}{v''} \cdot \overset{(+)}{u'} + \overset{(+)}{v'} \cdot \overset{(-)}{u''} \cdot \overset{(-)}{\beta} \cdot \overset{(+)}{v'} \cdot \overset{(+)}{\frac{1}{u'}}\right) \tag{148}$$

For example for large $|\beta|$, the indifference curve is convex and "therefore" V is quasi-concave.

Approach 2. The bordered Hessian.

We are using the following result. If $n \geq 2$ and $\forall x \in X$, for any $k \in \{3, ..., n+1\}$,

$$sign\,(k - \text{leading principal minor of } Bf(x)) = sign\ (-1)^{k-1},$$

then f is pseudo concave and, therefore, quasi-concave.

$$DV = [u'(x), \beta v'(y)]$$

$$\begin{array}{lll} & x & y \\ u'(x) & u'' & 0 \\ \beta v'(y) & 0 & \beta v'' \end{array}$$

$$\begin{array}{lll} 0 & u' & \beta v' \\ u' & u'' & 0 \\ \beta v' & 0 & b\beta v'' \end{array}$$

row and column 1

$\det \begin{bmatrix} u'' & 0 \\ 0 & \beta v'' \end{bmatrix} = \beta u'' v'' < 0$

r c 2

$\det \begin{bmatrix} 0 & \beta v' \\ \beta v' & \beta v'' \end{bmatrix} = -\beta^2 (v')^2 < 0$

r c 3

$\det \begin{bmatrix} 0 & u' \\ u' & u'' \end{bmatrix} = -(u')^2 < 0.$

$$\det \begin{bmatrix} 0 & u' & \beta v' \\ u' & u'' & 0 \\ \beta v' & 0 & \beta v'' \end{bmatrix} = -\beta (u')^2 v'' - \beta^2 u'' (v')^2 = (-\beta) \cdot \left(\overset{(-)}{v''} \cdot \overset{(+)}{(u')^2} + \overset{(-)}{\beta} \cdot \overset{(-)}{u''} \cdot \overset{(+)}{v'}\right) > 0 \tag{149}$$

if

$$\overset{(-)}{v''} \cdot \overset{(+)}{(u')^2} + \overset{(-)}{\beta} \cdot \overset{(-)}{u''} \cdot \overset{(+)}{v'} < 0, \qquad \frac{\overset{(-)}{v''}}{v'} \cdot \overset{(+)}{u'} + \overset{(-)}{\beta} \cdot \frac{\overset{(-)}{u''}}{u'} < 0, \qquad (-\beta)\left(-\frac{u''}{u'}\right) < u'\left(-\frac{v''}{v'}\right) \tag{150}$$

Observe that (149) has a structure similar to the expression in (148).

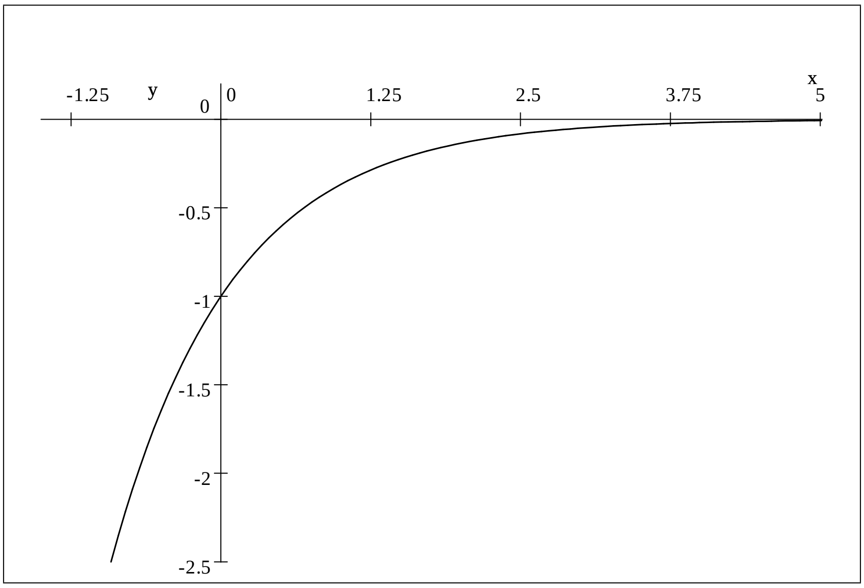
2.

Consider $f(x) = -e^{-ax}$;

Observe that the above assumption does satisfy our existence maintained assumptions, but not our regularity maintained assumption: closure of the upper level set is not closed in $\mathbb{R}$: take $k > 0$.

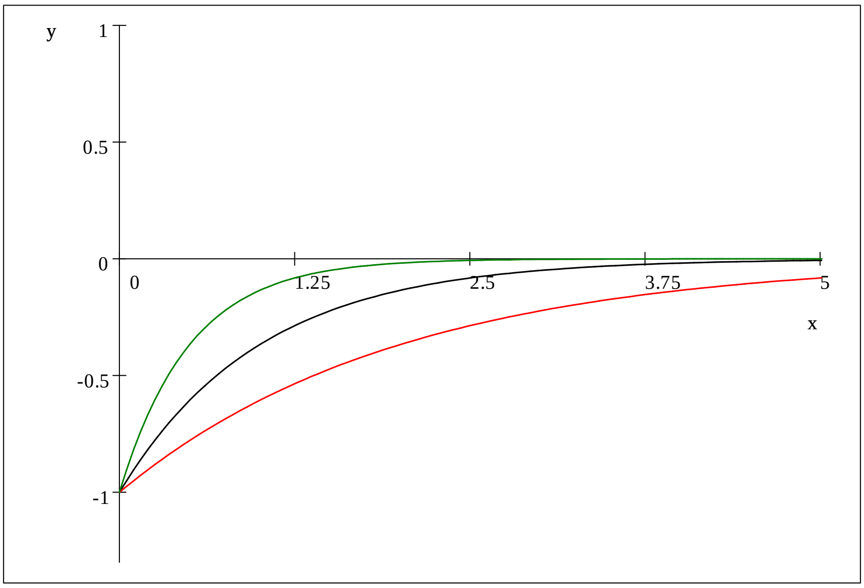
$$\{x \in \mathbb{R}_{++} : -e^{-ax} \geq -k\} = \left(\frac{e^k}{a}, +\infty\right)$$

$-e^{-x}$



Observe also that[26] $R_A(x) := -\frac{f''(x)}{f'(x)} = -\frac{-a^2 e^{-ax}}{ae^{-ax}} = a$.

The higher a, the more concave the function is (graphs below correspond to $a \in \left\{\frac{1}{2}, 1, 2\right\}$).



[26] See Mas Colell, page 191.

Then assuming $u(x) = -e^{-ax}$ and $v(x) = -e^{-bx}$, condition (150) becomes

$$-\beta a < ae^{-ax}b \quad \text{or} \quad -\beta < e^{-ax}b$$

Now, observe that $v_{12}(pe_2 + t_{12} - t_{21})$ and $pe_2 + t_{12} - t_{21} \leq pe_2 + k_1 \leq pr + k_1 = 1 + k_1$. Then we can express condition (150) as

$$-\beta < e^{-a(1+k_1)}b,$$

i.e., in terms of exogenous variables. That show the nonemptyness statement. ■

Remark 152 *As discussed below, it is not easy to say something as in the above proposition is the utility function are* log.

Assume $v = u = \log$. *Then,*

$$\left(\overset{(-)}{v''} \cdot \overset{(+)}{(u')^2} + \overset{(-)}{\beta} \cdot \overset{(-)}{u''} \cdot \overset{(+)}{v'}\right) = \left(-\frac{1}{y^2}\right)\left(\frac{1}{x}\right)^2 + \beta\left(-\frac{1}{x^2}\right)\frac{1}{y} > 0$$

$1 + \beta y < 0$
$\beta y < -1$
$(-\beta) > \frac{1}{y}$
$y > \frac{1}{(-\beta)}$
$\frac{1}{(-\beta)} < y$ *indeed* $y = pe_2 + t_{12} - t_{21} \overset{\beta \leq 0}{=} pe_2 - t_{21}$
$\leq pr + k = 1 + k$. *Then, we must have*

$$\frac{1}{(-\beta)} < 1 + k$$

In the relative wealth model, we have

$$y = \frac{pe_2 + t_{12} - t_{21}}{pr} = \frac{px_2}{pr} \overset{in\ equilibrium}{\leq} \frac{pr}{pr} = 1$$

4.11 Simple facts on maximization problems

4.11.1 Fact 1

Definition 153 *Let* $f : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $x \mapsto f(x)$ *and* $g : T \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \mapsto g(x)$ *be given. Define* $C := \{x \in T : g(x) \geq 0\}$. *To solve*

$$\max f(x) \quad s.t.\ g(x) \geq 0$$

means to find the set

$$\{x^* \in \mathbb{R}^n : x^* \in S \cap T \cap C \text{ and for any } x \in S \cap T \cap C,\ f(x^*) \geq f(x)\} =$$

$$= \{x^* \in S \cap C : \text{for any } x \in S \cap C,\ f(x^*) \geq f(x)\}.$$

4.11.2 Fact 2

Preliminary Observation.

Let the following sets and functions be given.

$$S \subseteq \mathbb{R}^n, \qquad W \subseteq \mathbb{R}^m, \qquad T \subseteq \mathbb{R}^l, \qquad \Pi \subseteq \mathbb{R}^k,$$

and

$$f : S \times W \times \Pi \longrightarrow \mathbb{R}, \qquad (x, \theta, \pi) \mapsto f(x, \theta, \pi),$$

$$v : T \times \Pi \longrightarrow \mathbb{R}^m \qquad (t, \pi) \mapsto v(t, \pi),$$

$$g : S \times T \times \Pi \longrightarrow \mathbb{R}, \qquad (x, t, \pi) \mapsto g(x, t, \pi).$$

For any $\pi \in \Pi$, define $C(\pi) = \{(x,t) \in S \in T : g(x,t,\pi) \geq 0\}$. Let the following problem be given.

$$\text{For any } \pi \in \Pi, \quad \max_{(x,t)} f(x, v(t.\pi), \pi) \qquad \text{s.t.} \qquad g(x,t,\pi) \geq 0 \tag{151}$$

To solve problem (151) at $\pi \in \Pi$ means to find the set

$\{ (x^*, t^*) \in \mathbb{R}^n \times \mathbb{R}^l :$ 1. $(x^*, t^*) \in C(\pi)$,

2. $v(x^*, t^*) \in W$, and

3. for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^l$ such that $(x,t) \in C(\pi)$ and $v(x,t) \in W$,
$f(x^*, v(t^*, \pi), \pi) \geq f(x, v(t,\pi), \pi)$ $\}$

Defined for any $\pi \in \Pi$, $\widehat{C}(\pi) = \{(x,t) \in S \in T : g(x,t,\pi) \geq 0 \text{ and } v(x,t) \in W\}$, then to solve problem (151) at $\pi \in \Pi$ means to find the set

$\{ (x^*, t^*) \in \mathbb{R}^n \times \mathbb{R}^l :$ 1. $(x^*, t^*) \in \widehat{C}(\pi)$,

2. for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}^l$ such that $(x,t) \in \widehat{C}(\pi)$,
$f(x^*, v(t^*, \pi), \pi) \geq f(x, v(t,\pi), \pi)$ $\}$

If $\overline{\pi} \in \Pi$ is such that $\widehat{C}(\overline{\pi}) = \varnothing$, then problem (151) at $\pi \in \Pi$ has no solution.

The need to have a well defined maximization problem.

The Preliminary observation wants to stress that compositions of functions have to be well defined: in the case described in the Preliminary observation, it needs to be checked that $g(t,\pi) \in W$, i.e., in our model, beliefs of household h about other individuals' wealth have to positive, irrespectively of the choices of household h.

In our model, a drastic way of avoiding the problem above is to make the following "legal" assumption:

Assumption 1. For any $h \in \mathcal{H}$, any $e \in \mathbb{R}^C_{++}$, any $p^* \in S$, the choice of $t_h \in T_h$ has to satisfy the constraint

$$p^* \sum_{h' \in \mathcal{H} \setminus \{h\}} t_{hh'} \leq p^* e_h.$$

For simplicity, consider the case $H = 2$. Then, it must be the case

$$p^* t_{21} \leq p^* e_2,$$

and then

$$p^* (e_2 - t_{21} + t_{12}) \geq 0.$$

Since $u_1 : \mathbb{R}^C_+ \times \overset{\downarrow}{\mathbb{R}_+} \longrightarrow \mathbb{R}$, if $x_1 \geq 0$, then

$$\text{for any } t_{12} \in \mathbb{R}_+, \; u_1(x_1, p^*(e_2 - t_{21} + t_{12})) \text{ is well defined.} \tag{152}$$

Remark 154 *Consider the following milder Assumption.*

Assumption 1'. *For any $h \in \mathcal{H}$, any $e \in \mathbb{R}^C_{++}$, $p^* \in S$, $t_{\setminus h} \in T_{\setminus h}$, the choice of $t_h \in T_h$ has to satisfy the constraint*

$$p^* \sum_{h' \in \mathcal{H} \setminus \{h\}} t_{hh'} \leq p^* e_h + \sum_{h' \in \mathcal{H} \setminus \{h\}} t_{h'h}.$$

Could we substitute Assumption 1 with the above milder assumption?

The answer is negative. Again, for simplicity, take $H = 2$. Then, consistently with Assumption 1'., we have

$$p^* t_{21} \leq p^* (e_2 + t_{12}),$$

and then

$$p^* (e_2 - t_{21} + t_{12}) \geq 0,$$

and it can be

$$p^* (e_2 - t_{21}) < 0.$$

Therefore, condition (152) *does not hold true.*

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