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The maximum-weight arborescence network solution

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Abstract

A network solution is a procedure that assigns to each network, i.e., a weighted digraph, a subset of its vertices. We introduce a new solution based on the concept of arborescence: given a network, we consider all maximum-weight arborescences in the network and select the set of vertices that are their roots. This solution is proved to satisfy several desirable properties, including neutrality, cancellation, invariance under translation and positive rescaling of capacities, monotonicity, and polynomial-time computability. Moreover, it is proved to be an extension of the admissible set, a well-known solution for abstract decision problems, to the set of networks. When applied to weighted majority digraphs induced by preference profiles over a set of alternatives, this yields a social choice correspondence that is proved to satisfy the Smith and Condorcet principles, the Condorcet loser principle, as well as anonymity, neutrality, homogeneity, monotonicity, Pareto efficiency, and independence of clones. For three alternatives, we prove that it coincides with both the Simpson and the Schulze social choice correspondences.

Keywords: arborescence; network solution; tournament solution; social choice; weighted majority digraphs; Condorcet method.

JEL classification: D71.

1 Introduction

Consider a society that must select an alternative from a given finite set X on the basis of information about pairwise comparisons among alternatives. A basic way to represent such information is through a dominance relation R on X. The relation R captures the idea that, if an alternative y is being considered by the society and $(x, y) \in R$, then there exists a valid reason to reject y in favor of x. The resulting digraph (X, R) provides a representation of the decision problem and is called an abstract decision problem. A central issue is therefore to determine a method that selects a set of alternatives for any given abstract decision problem. Several solutions have been proposed in the literature, each based on a different rationale. Among them are the uncovered set (Miller 1980, Fishburn 1977, Duggan 2013), the Smith set (Smith 1973, Schwartz 1986), the admissible set (Kalai et al. 1976, Kalai and Schmeidler 1977), the maximum flow value set (Gori 2024).[1]

The previous framework only records whether an alternative dominates another one, without taking into account the intensity of such dominance. In many situations, however, the available information naturally provides a measure of the strength of pairwise comparisons. In this case, a society must select

[1] Other methods select a set of subsets of alternatives, such as the stable set (von Neumann and Morgenstern, 1944), the generalized stable set (van Deemen, 1991), the socially stable set (Delver and Monsuur, 2001), the m-stable set (Peris and Subiza, 2013), and the w-stable set (Han and van Deemen, 2016).

an alternative in X on the basis of a capacity function c that associates with each ordered pair (x, y) of distinct alternatives a number $c(x, y)$ representing the strength of the dominance of x over y. That strength may arise, for instance, from considering several evaluation criteria for alternatives, assigning a suitable numerical weight to each criterion, and summing the weights of the criteria for which x is at least as good as y. The resulting network (X, c) encodes all the relevant information about the decision problem. The problem of finding a sensible method to select a set of alternatives for any given network then naturally arises. Note that, since any digraph can be identified with a 0-1-network, namely a network whose capacity has values in the set $\{0, 1\}$, any such method, which takes the name of network solution, induces a solution for abstract decision problems when restricted to the set of 0-1-networks. Several network solutions have also been introduced and studied (Laslier 1997, Langville and Meyer 2012, González-Díaz et al. 2014, Fischer et al. 2016, Vaziri et al. 2018), most of them assigning a score to each vertex of a network and then selecting the vertices with maximum score.[2]

One way to construct a network solution is obtained by using the notion of the weight of a digraph in a network. Given a network and a digraph with the same vertex set, the weight of the digraph in the network is defined as the sum of the capacities of its arcs. We consider a class of digraphs in which each digraph is endowed with a distinguished vertex, called its root. Given a network, we then consider the family of digraphs in this class having the same set of vertices as the network and we determine in that family the digraphs with maximum weight. The alternatives that are roots of such digraphs are finally selected. The Kemeny network solution is a prominent example fitting this scheme: in this case, the class of digraphs consists of those corresponding to transitive tournaments, where the root is the unique maximal element. Another important example based on the same rationale is the outdegree network solution. In this case, the class of digraphs is given by outstars, namely digraphs whose arcs are exactly the arcs leaving a given vertex, which serves as the root.

In this paper, we propose a new network solution built according to the described scheme, focusing on the class of arborescences. An arborescence is a digraph (X, R) satisfying the following properties: there exists a vertex $r \in X$ such that no arc of (X, R) enters r; for every vertex $x \in X$ different from r, there exists a unique arc of (X, R) entering x; for every vertex $x \in X$ different from r, there exists a path in (X, R) from r to x. The vertex r is called the root of (X, R). Since every outstar is an arborescence, and every arborescence is a subdigraph of a transitive tournament, arborescences occupy a natural intermediate position between outstars and transitive tournaments. Following the general scheme described above, given a network, we consider all the arborescences having the same set of vertices as the network, we determine the ones with maximum weight in the network, and we finally select the alternatives that are roots of such arborescences. We call the resulting network solution the maximum-weight arborescence network solution, denoted by $\mathscr{A}$.

We prove that $\mathscr{A}$ fulfills several desirable properties. In particular, it satisfies neutrality, namely invariance under relabeling of the vertices; cancellation, meaning that every vertex is selected in symmetric networks, that is, networks such that the capacity of an arc (x, y) equals the capacity of the arc (y, x) for every pair of distinct vertices x and y; invariance under translation and positive rescaling of capacities; and a monotonicity property according to which a selected vertex remains selected when the capacities of its outgoing arcs do not decrease, the capacities of its incoming arcs do not increase, and all other capacities remain unchanged. Notably, $\mathscr{A}$ can be computed efficiently using the well-known Chu-Liu/Edmonds algorithm, which runs in polynomial time in the number of vertices (Chu and Liu 1965, Edmonds 1967). We also prove that $\mathscr{A}$, when restricted to the set of digraphs, induces a special solution for abstract decision problems. Specifically, it coincides with the admissible set previously mentioned, namely the solution that, given a digraph (X, R), selects an alternative x if and only if, for every alternative y, the existence of a path from y to x implies the existence of a path from x to y. This fact provides a new characterization of the admissible set in terms of arborescences.

Any network solution induces a method for aggregating individual preferences over a set of alternatives. Indeed, given a preference profile, one can associate with it the weighted majority digraph (Laslier 1997) and then apply the network solution to obtain a subset of alternatives. Under the assumption that

[2] In fact, most of the existing analysis and axiomatic characterizations concern ranking methods, that is, methods that associate with each network a ranking of its vertices, mainly based on scores assigned to the vertices. Any ranking method naturally induces a network solution by selecting the vertices that are ranked first.

individual preferences are expressed as linear orders, applying this procedure to the maximum-weight arborescence network solution yields a new social choice correspondence, called the maximum-weight arborescence social choice correspondence and denoted by A. Interestingly, A is proved to satisfy several remarkable properties. In particular, we prove that A satisfies the Condorcet and Smith principles, the Condorcet loser principle, anonymity, neutrality, homogeneity, monotonicity, and Pareto efficiency. Moreover, we prove that A satisfies independence of clones (Tideman 1987). Finally, in the case of three alternatives, we prove that A coincides with both the Simpson and the Schulze social choice correspondences (Simpson 1969, Schulze 2011).

The paper is organized as follows. Section 2 introduces the necessary preliminaries on relations, digraphs, and networks. In Section 3, we define the maximum-weight arborescence network solution and study its main properties. Section 4 establishes the connection between the proposed network solution and the admissible set. Section 5 introduces a new preference aggregation method based on the maximum-weight arborescence network solution and investigates its properties. The proofs of all stated results are deferred to the appendix.

2 Relations, digraphs and networks

We denote by $\mathbb{N}$ the set of positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The set of nonempty and finite subsets of $\mathbb{N}$ is denoted by $P_f(\mathbb{N})$.

Given a finite set X, we denote by $|X|$ the size of X, by X^2_d the set $\{(x, y) \in X^2 : x = y\}$, by X^2_* the set $X^2 \setminus X^2_d$, by $P(X)$ the set of subsets of X, by $P_*(X)$ the set of nonempty subsets of X. We denote by $\mathrm{Sym}(X)$ the set of permutations of X. Given $\psi_1, \psi_2 \in \mathrm{Sym}(X)$, we denote by $\psi_2\psi_1$ the composition of ψ_2 and ψ_1, namely the permutation of X defined, for every $x \in X$, by $(\psi_2\psi_1)(x) = \psi_2(\psi_1(x))$. The identity function in $\mathrm{Sym}(X)$ is denoted by id_X.

2.1 Relations

Let X be a finite set. A relation on X is a subset of X^2. The set of relations on X is denoted by $\mathbf{R}(X)$. Let $R \in \mathbf{R}(X)$. Given $x, y \in X$, we sometimes write $x \succeq_R y$ instead of $(x, y) \in R$; $x \succ_R y$ instead of $(x, y) \in R$ and $(y, x) \notin R$. We say that R is reflexive if, for every $x \in X$, $(x, x) \in R$; irreflexive if, for every $x \in X$, $(x, x) \notin R$; asymmetric if, for every $x, y \in X$, $(x, y) \in R$ implies $(y, x) \notin R$; transitive if, for every $x, y, z \in X$, $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$; antisymmetric if, for every $x, y \in X$, $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$; quasi-complete if, for every $x, y \in X$ with $x \neq y$, $(x, y) \in R$ or $(y, x) \in R$; complete if, for every $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$; an order if R is complete and transitive; a linear order if R is complete, transitive and antisymmetric. The set of orders on X is denoted by $\mathbf{O}(X)$; the set of linear orders on X is denoted by $\mathbf{L}(X)$. The set of maximal elements of R is

$$\mathrm{Max}(R) = \{x \in X : \forall y \in X,\ y \not\succ_R x\}.$$

Given $\psi \in \mathrm{Sym}(X)$, we define $R^\psi = \{(x, y) \in X^2 : (\psi^{-1}(x), \psi^{-1}(y)) \in R\}$. Observe that, for every $\psi_1, \psi_2 \in \mathrm{Sym}(X)$, $(R^{\psi_1})^{\psi_2} = R^{\psi_2\psi_1}$. We also define $R^r = \{(x, y) \in X^2 : (y, x) \in R\}$.

There are several ways to represent an order on a finite set X. For instance, if $X = \{1, 2, 3, 4\}$, the tables

$$R_1 = \begin{bmatrix} 1 \\ 2,3 \\ 4 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1,2,4 \\ 3 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} \tag{1}$$

represent orders on X according to the following rule: for every $x, y \in X$, $(x, y) \in R_i$ if and only if x is above or at the same level as y in the corresponding table. Thus, for example, $(1, 1) \in R_1$, $(2, 4) \in R_1$, $(4, 3) \in R_2$, $(1, 2) \in R_3$. Note that R_3 represents a linear order.

We can also encode these tables on a single line by writing each level as a set, omitting curly brackets for singletons; levels are separated by commas and are listed from left to right in the same order as from top to bottom in the table. Thus, we can write the orders in (1) as

$$R_1 = [1, \{2, 3\}, 4], \quad R_2 = [\{1, 2, 4\}, 3], \quad R_3 = [1, 4, 3, 2].$$

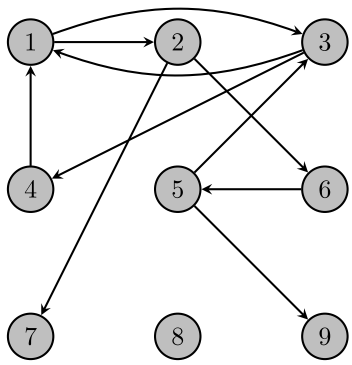

Figure 1: The digraph D_1 in (2).

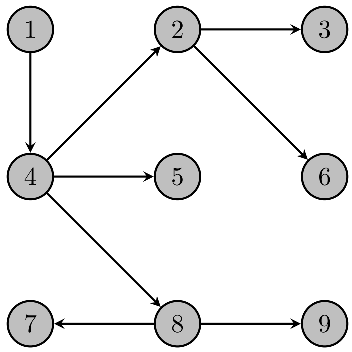

Figure 2: The digraph D_2 in (2).

2.2 Digraphs and arborescences

Let $X \in P_f(\mathbb{N})$. A digraph on X is a pair (X, R), where R is an irreflexive relation on X. We say that X is the set of vertices of (X, R), and R is the set of arcs of (X, R). The set of digraphs on X is denoted by $\mathcal{D}(X)$. A digraph is an element of the set

$$\mathcal{D} = \bigcup_{X \in P_f(\mathbb{N})} \mathcal{D}(X).$$

Given $D \in \mathcal{D}$, we denote by $\mathrm{Ve}(D)$ the set of vertices of D and by $\mathrm{Ar}(D)$ the set of arcs of D. Thus, $D = (\mathrm{Ve}(D), \mathrm{Ar}(D)) \in \mathcal{D}(\mathrm{Ve}(D))$. Of course, if $D \in \mathcal{D}(X)$, then $\mathrm{Ve}(D) = X$.

A digraph D can be graphically represented in the plane by points and arrows: the elements of $\mathrm{Ve}(D)$ correspond to distinct points in the plane, and each arc $(x, y) \in \mathrm{Ar}(D)$ corresponds to an arrow from x to y. The pictures in Figures 1 and 2 respectively represent the digraphs

$$\begin{aligned} D_1 &= (\{1,2,3,4,5,6,7,8,9\}, \{(1,2),(1,3),(2,6),(2,7),(3,1),(3,4),(4,1),(5,3),(5,9),(6,5)\}), \\ D_2 &= (\{1,2,3,4,5,6,7,8,9\}, \{(1,4),(2,3),(2,6),(4,2),(4,5),(4,8),(8,7),(8,9)\}). \end{aligned} \tag{2}$$

Let $D \in \mathcal{D}$. If $(x, y) \in \mathrm{Ar}(D)$, then (x, y) is said to leave x and enter y. A path in D is a sequence $(x_j)_{j=1}^m$, where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of $\mathrm{Ve}(D)$ and $(x_j, x_{j+1}) \in \mathrm{Ar}(D)$ for all $j \in \{1, \ldots, m-1\}$. If $x, y \in \mathrm{Ve}(D)$ are distinct, a path from x to y in D is a path $(x_j)_{j=1}^m$ in D such that $x_1 = x$ and $x_m = y$. If there exists a path in D from x to y, then we say that y is reachable from x in D. A cycle in D is a sequence $(x_j)_{j=1}^m$, where $m \geq 3$, $x_1, \ldots, x_{m-1}$ are distinct elements of $\mathrm{Ve}(D)$, $x_m = x_1$, and $(x_j, x_{j+1}) \in \mathrm{Ar}(D)$ for all $j \in \{1, \dots, m-1\}$. If D does not admit any cycles, it is called acyclic. For instance, referring to the digraphs D_1 and D_2 in (2), we have that the sequence $(1, 3, 4, 1)$ is a cycle in D_1, and the sequence $(1, 4, 8, 9)$ is a path from 1 to 9 in D_2. Given now $x \in \mathrm{Ve}(D)$, we set

$$\begin{aligned}
\pi^D(x) &= \{y \in \mathrm{Ve}(D) : (y, x) \in \mathrm{Ar}(D)\},\\
\gamma^D(x) &= \{y \in \mathrm{Ve}(D) : (x, y) \in \mathrm{Ar}(D)\},\\
\alpha^D(x) &= \{y \in \mathrm{Ve}(D) : \text{ there exists a path from } y \text{ to } x \text{ in } D\},\\
\delta^D(x) &= \{y \in \mathrm{Ve}(D) : \text{ there exists a path from } x \text{ to } y \text{ in } D\}.
\end{aligned}$$

The sets $\pi^D(x)$, $\gamma^D(x)$, $\alpha^D(x)$, and $\delta^D(x)$ are called, respectively, the sets of parents, children, ancestors, and descendants of x in D. Of course, $y \in \pi^D(x)$ if and only if $x \in \gamma^D(y)$, and $y \in \alpha^D(x)$ if and only if $x \in \delta^D(y)$. Moreover, $\pi^D(x) \subseteq \alpha^D(x)$, $\gamma^D(x) \subseteq \delta^D(x)$, $\pi^D(x) = \varnothing$ implies $\alpha^D(x) = \varnothing$, and $\gamma^D(x) = \varnothing$ implies $\delta^D(x) = \varnothing$. Referring to the digraphs D_1 and D_2 in (2), we have that $\pi^{D_1}(1) = \{3, 4\}$, $\alpha^{D_1}(3) = \{1, 2, 4, 5, 6\}$, $\gamma^{D_1}(5) = \{3, 9\}$, $\delta^{D_1}(6) = \{1, 2, 3, 4, 5, 7, 9\}$, $\pi^{D_2}(2) = \{4\}$, $\alpha^{D_2}(2) = \{1, 4\}$, $\gamma^{D_2}(2) = \delta^{D_2}(2) = \{3, 6\}$.

Given $\psi \in \mathrm{Sym}(\mathrm{Ve}(D))$, we define $D^\psi = (\mathrm{Ve}(D), \mathrm{Ar}(D)^\psi)$. Of course, by the preceding discussion about relations, for every $\psi_1, \psi_2 \in \mathrm{Sym}(\mathrm{Ve}(D))$, $(D^{\psi_1})^{\psi_2} = D^{\psi_2\psi_1}$. We also define $D^r = (\mathrm{Ve}(D), \mathrm{Ar}(D)^r)$.

Let $D \in \mathcal{D}$. We say that D is an arborescence if there exists $x_0 \in \mathrm{Ve}(D)$ such that

- for every $y \in \mathrm{Ve}(D)$, $(y, x_0) \notin \mathrm{Ar}(D)$,
- for every $y \in \mathrm{Ve}(D) \setminus \{x_0\}$, there exists a unique $x \in \mathrm{Ve}(D)$ such that $(x, y) \in \mathrm{Ar}(D)$,
- for every $y \in \mathrm{Ve}(D) \setminus \{x_0\}$, there exists a path from x_0 to y in D.

The element x_0 is a root of D. The set of arborescences is denoted by $\mathcal{A}$.[3]

Let $D \in \mathcal{A}$. It is well known that D is acyclic, and $|\mathrm{Ar}(D)| = |\mathrm{Ve}(D)| - 1$. Moreover, D admits a unique root, which is then called the root of D and denoted by $r(D)$; for every $x \in \mathrm{Ve}(D) \setminus \{r(D)\}$, there exists a unique path from $r(D)$ to x in D; $\pi^D(r(D)) = \alpha^D(r(D)) = \varnothing$; for every $x \in \mathrm{Ve}(D) \setminus \{r(D)\}$, $|\pi^D(x)| = 1$. We also have that the set $\{x \in \mathrm{Ve}(D) : \gamma^D(x) = \varnothing\}$ is nonempty: its elements are called the leaves of D. For example, the digraph D_2 in (2) is an arborescence with $r(D_2) = 1$ and whose leaves are the vertices 3, 5, 6, 7 and 9.

Given $X \in P_f(\mathbb{N})$ and $x_0 \in X$, we denote by $\mathcal{A}(X)$ the set of arborescences whose set of vertices is X, and by $\mathcal{A}_{x_0}(X)$ the set of the arborescences in $\mathcal{A}(X)$ whose root is x_0. It can be checked that if $D \in \mathcal{A}_{x_0}(X)$ and $\psi \in \mathrm{Sym}(X)$, then $D^\psi \in \mathcal{A}_{\psi(x_0)}(X)$.

Let $D \in \mathcal{D}$. We say that D is an outstar if there exists $x_0 \in \mathrm{Ve}(D)$ such that $\mathrm{Ar}(D) = \{(x_0, y) : y \in \mathrm{Ve}(D) \setminus \{x_0\}\}$. The element x_0 is called a root of D. The set of outstars is denoted by $\mathcal{O}$. Given $D \in \mathcal{O}$, it admits a unique root, again denoted by $r(D)$. Given $X \in P_f(\mathbb{N})$ and $x_0 \in X$, we denote by $\mathcal{O}(X)$ the set of outstars whose set of vertices is X, and by $\mathcal{O}_{x_0}(X)$ the set of outstars in $\mathcal{O}(X)$ whose root is x_0. Note that $\mathcal{O}_{x_0}(X)$ has a unique element.

Let $D \in \mathcal{D}$. We say that D is a transitive tournament if $\mathrm{Ar}(D)$ is transitive, quasi-complete and asymmetric. The set of transitive tournaments is denoted by $\mathcal{T}$. Given $D \in \mathcal{T}$, there exists a unique $x_0 \in \mathrm{Ve}(D)$ such that, for every $y \in \mathrm{Ve}(D) \setminus \{x_0\}$, $(x_0, y) \in \mathrm{Ar}(D)$. We call such a vertex the root of D and we again denote it by $r(D)$. Given $X \in P_f(\mathbb{N})$ and $x_0 \in X$, we denote by $\mathcal{T}(X)$ the set of transitive tournaments whose set of vertices is X, and by $\mathcal{T}_{x_0}(X)$ the set of transitive tournaments in $\mathcal{T}(X)$ whose root is x_0.

We finally note that, for every $X \in P_f(\mathbb{N})$ and $x_0 \in X$, $\mathcal{O}_{x_0}(X) \subseteq \mathcal{A}_{x_0}(X)$. Moreover, due to Szpilrajn's extension theorem (Szpilrajn 1930), if $D \in \mathcal{A}_{x_0}(X)$ (resp. $D \in \mathcal{O}_{x_0}(X)$), then there exists $E \in \mathcal{T}_{x_0}(X)$ such that $\mathrm{Ar}(D) \subseteq \mathrm{Ar}(E)$.

[3]The terminology is not uniform in the literature. The term arborescence is used, for example, by Berge (1962), Edmonds (1967), and Deo (1974), whereas Harary (1969) uses the term out-tree. In the context of spanning subdigraphs of a prescribed digraph, Bang-Jensen and Gutin (2007) use the term out-branching.

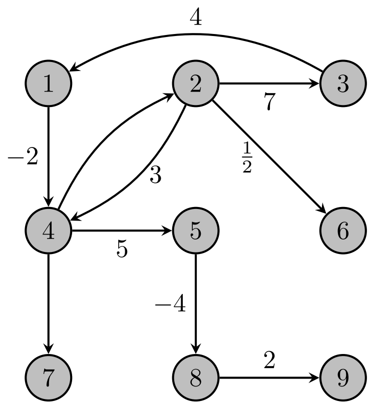

Figure 3: The network N_1.

2.3 Networks

Let $X \in P_f(\mathbb{N})$. A network on X is a pair (X, c), where c is a function from X^2_* to $\mathbb{R}$. We say that X is the set of vertices of (X, c), X^2_* is the set of arcs of (X, c), and c is the capacity of (X, c). The set of networks on X is denoted by $\mathcal{N}(X)$. A network is an element of the set

$$\mathcal{N} = \bigcup_{X \in P_f(\mathbb{N})} \mathcal{N}(X).$$

Given $N \in \mathcal{N}$, we denote by $\mathrm{Ve}(N)$ the set of vertices of N and by c^N the capacity of N. Thus, $N = (\mathrm{Ve}(N), c^N) \in \mathcal{N}(\mathrm{Ve}(N))$. Of course, if $N \in \mathcal{N}(X)$, then $\mathrm{Ve}(N) = X$.

A network N can be graphically represented in the plane by points and arrows with numbers attached: the elements of $\mathrm{Ve}(N)$ correspond to distinct points in the plane, each arc $(x, y) \in (\mathrm{Ve}(N))^2_*$ corresponds to an arrow from x to y, and the value $c^N(x, y)$ is written next to the arrow representing (x, y). We also adopt the convention that if an arc has capacity zero, it is not drawn; if an arc has capacity one, the capacity is not written. The picture in Figure 3 represents a network, which we denote by N_1. From the picture, we see that $\mathrm{Ve}(N_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and that, for example, $c^{N_1}(1, 2) = 0$, $c^{N_1}(2, 3) = 7$, and $c^{N_1}(4, 2) = 1$.

Let $N \in \mathcal{N}$. We say that N is a 0-1-network if $\mathrm{Im}(c^N) \subseteq \{0, 1\}$. We say that N is balanced if there exists $k \in \mathbb{R}$ such that, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $c^N(x, y) + c^N(y, x) = k$. Given $k \in \mathbb{R}$, we denote by $N + k$ the element of $\mathcal{N}$ such that $\mathrm{Ve}(N + k) = \mathrm{Ve}(N)$ and, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $c^{N+k}(x, y) = c^N(x, y) + k$; we denote by kN the element of $\mathcal{N}$ such that $\mathrm{Ve}(kN) = \mathrm{Ve}(N)$ and, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $c^{kN}(x, y) = kc^N(x, y)$. Given $\psi \in \mathrm{Sym}(\mathrm{Ve}(N))$, we define N^ψ as the element of $\mathcal{N}$ such that $\mathrm{Ve}(N^\psi) = \mathrm{Ve}(N)$ and, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $c^{N^\psi}(x, y) = c^N(\psi^{-1}(x), \psi^{-1}(y))$. Observe that, for every $\psi_1, \psi_2 \in \mathrm{Sym}(\mathrm{Ve}(N))$, $(N^{\psi_1})^{\psi_2} = N^{\psi_2\psi_1}$. We define N^r as the element of $\mathcal{N}$ such that $\mathrm{Ve}(N^r) = \mathrm{Ve}(N)$ and, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $c^{N^r}(x, y) = c^N(y, x)$. A path in N is a sequence $(x_j)_{j=1}^m$, where $m \geq 2$, $x_1, \ldots, x_m$ are distinct elements of $\mathrm{Ve}(N)$. If $x, y \in \mathrm{Ve}(N)$ are distinct, a path from x to y in N is a path $(x_j)_{j=1}^m$ in N such that $x_1 = x$ and $x_m = y$.

0-1-networks can naturally be identified with digraphs. Indeed, let $\mathcal{N}_{01}$ be the set of 0-1-networks and consider $\Phi : \mathcal{N}_{01} \to \mathcal{D}$ defined, for every $N \in \mathcal{N}_{01}$, by $\Phi(N) = (\mathrm{Ve}(N), R^N) \in \mathcal{D}$, where

$$R^N = \left\{(x, y) \in (\mathrm{Ve}(N))^2_* : c^N(x, y) = 1\right\}.$$

It is a simple exercise to verify that Φ is a bijection and that, for every $D \in \mathcal{D}$, $\Phi^{-1}(D) = (\mathrm{Ve}(D), c^D) \in \mathcal{N}_{01}$, where c^D is defined, for every $(x, y) \in (\mathrm{Ve}(D))^2_*$, by $c^D(x, y) = 1$ if $(x, y) \in \mathrm{Ar}(D)$ and by $c^D(x, y) = 0$ if $(x, y) \notin \mathrm{Ar}(D)$. Throughout the paper, we identify the sets $\mathcal{D}$ and $\mathcal{N}_{01}$ so that $\mathcal{D} \subseteq \mathcal{N}$. Observe that, for every $N \in \mathcal{N}_{01}$ and $\psi \in \mathrm{Sym}(\mathrm{Ve}(N))$, $\Phi(N^\psi) = \Phi(N)^\psi$ and $\Phi(N^r) = \Phi(N)^r$.

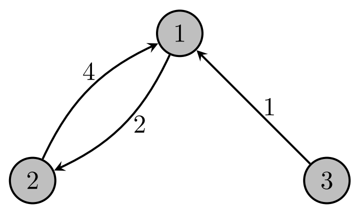

Figure 4: The network N_2.

Consider now $N \in \mathcal{N}$. For every $D \in \mathcal{D}$ with $\mathrm{Ve}(N) = \mathrm{Ve}(D)$, we set

$$N \circ D = \sum_{(x,y)\in \mathrm{Ar}(D)} c^N(x, y).$$

The number $N \circ D$ is called the weight of the digraph D in the network N. We also set

$$\mathcal{A}^N = \underset{D\in\mathcal{A}(\mathrm{Ve}(N))}{\mathrm{argmax}}\ N \circ D, \qquad \mathcal{O}^N = \underset{D\in\mathcal{O}(\mathrm{Ve}(N))}{\mathrm{argmax}}\ N \circ D, \qquad \mathcal{T}^N = \underset{D\in\mathcal{T}(\mathrm{Ve}(N))}{\mathrm{argmax}}\ N \circ D.$$

Since the sets $\mathcal{A}(\mathrm{Ve}(N))$, $\mathcal{O}(\mathrm{Ve}(N))$, and $\mathcal{T}(\mathrm{Ve}(N))$ are nonempty and finite, the sets $\mathcal{A}^N$, $\mathcal{O}^N$, and $\mathcal{T}^N$ are nonempty and finite.

As an example, let us consider $X = \{1, 2, 3\}$ and $N_2 \in \mathcal{N}(X)$ represented in Figure 4 and compute the sets $\mathcal{A}^{N_2}$, $\mathcal{O}^{N_2}$, and $\mathcal{T}^{N_2}$.

The elements of $\mathcal{A}(X)$ are the digraphs

$$E_1 = (\{1, 2, 3\}, \{(1, 2), (2, 3)\}), \quad E_2 = (\{1, 2, 3\}, \{(1, 3), (3, 2)\}), \quad E_3 = (\{1, 2, 3\}, \{(1, 2), (1, 3)\}),$$

$$E_4 = (\{1, 2, 3\}, \{(2, 1), (1, 3)\}), \quad E_5 = (\{1, 2, 3\}, \{(2, 3), (3, 1)\}), \quad E_6 = (\{1, 2, 3\}, \{(2, 1), (2, 3)\}),$$

$$E_7 = (\{1, 2, 3\}, \{(3, 1), (1, 2)\}), \quad E_8 = (\{1, 2, 3\}, \{(3, 2), (2, 1)\}), \quad E_9 = (\{1, 2, 3\}, \{(3, 1), (3, 2)\}).$$

Moreover, we have

$$N_2 \circ E_1 = 2, \quad N_2 \circ E_2 = 0, \quad N_2 \circ E_3 = 2,$$

$$N_2 \circ E_4 = 4, \quad N_2 \circ E_5 = 1, \quad N_2 \circ E_6 = 4,$$

$$N_2 \circ E_7 = 3, \quad N_2 \circ E_8 = 4, \quad N_2 \circ E_9 = 1.$$

Thus, we conclude that $\mathcal{A}^{N_2} = \{E_4, E_6, E_8\}$.

The elements of $\mathcal{O}(X)$ are the digraphs E_3, E_6, and E_9. Using the computations above, we conclude that $\mathcal{O}^{N_2} = \{E_6\}$.

The elements of $\mathcal{T}(X)$ are the digraphs

$$H_1 = (\{1, 2, 3\}, \{(1, 2), (2, 3), (1, 3)\}), \quad H_2 = (\{1, 2, 3\}, \{(1, 3), (3, 2), (1, 2)\}),$$

$$H_3 = (\{1, 2, 3\}, \{(2, 1), (1, 3), (2, 3)\}), \quad H_4 = (\{1, 2, 3\}, \{(2, 3), (3, 1), (2, 1)\}),$$

$$H_5 = (\{1, 2, 3\}, \{(3, 1), (1, 2), (3, 2)\}), \quad H_6 = (\{1, 2, 3\}, \{(3, 2), (2, 1), (3, 1)\}).$$

Moreover, we have that

$$N_2 \circ H_1 = 2, \quad N_2 \circ H_2 = 2, \quad N_2 \circ H_3 = 4,$$

$$N_2 \circ H_4 = 5, \quad N_2 \circ H_5 = 3, \quad N_2 \circ H_6 = 5.$$

Thus, we conclude that $\mathcal{T}^{N_2} = \{H_4, H_6\}$.

3 A new network solution

Let $\mathcal{K} \subseteq \mathcal{N}$. A network solution (NS) on $\mathcal{K}$ is a function $\mathscr{F}$ from $\mathcal{K}$ to $P_*(\mathbb{N})$ such that, for every $N \in \mathcal{K}$, $\mathscr{F}(N) \subseteq \mathrm{Ve}(N)$. Thus, a NS on $\mathcal{K}$ is a procedure that associates with any network belonging to $\mathcal{K}$ a nonempty subset of its vertices. A NS on $\mathcal{N}$ is simply called a NS. Several notable NSs have been proposed in the literature. The following are particularly interesting and rich in desirable properties.

- The outdegree NS (van den Brink and Gilles 2009), denoted by $\mathscr{O}$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{O}(N) = \underset{x \in \mathrm{Ve}(N)}{\operatorname{argmax}} \sum_{y \in \mathrm{Ve}(N) \setminus \{x\}} c^N(x, y).$$
- The net-outdegree NS (Bouyssou, 1992; Bubboloni and Gori, 2025), denoted by $\mathscr{N}$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{N}(N) = \underset{x \in \mathrm{Ve}(N)}{\operatorname{argmax}} \sum_{y \in \mathrm{Ve}(N) \setminus \{x\}} \left(c^N(x, y) - c^N(y, x) \right).$$
- The maximin NS, denoted by $\mathscr{M}^*$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{M}^*(N) = \begin{cases} \underset{x \in \mathrm{Ve}(N)}{\operatorname{argmax}} \ \underset{y \in \mathrm{Ve}(N) \setminus \{x\}}{\min} c^N(x, y) & \text{if } |\mathrm{Ve}(N)| \geq 2 \\ \mathrm{Ve}(N) & \text{if } |\mathrm{Ve}(N)| = 1 \end{cases}$$
- The minimax NS, denoted by $\mathscr{M}_*$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{M}_*(N) = \begin{cases} \underset{x \in \mathrm{Ve}(N)}{\operatorname{argmin}} \ \underset{y \in \mathrm{Ve}(N) \setminus \{x\}}{\max} c^N(y, x) & \text{if } |\mathrm{Ve}(N)| \geq 2 \\ \mathrm{Ve}(N) & \text{if } |\mathrm{Ve}(N)| = 1 \end{cases}$$
- The Schulze NS (Schulze, 2011, 2018), denoted by $\mathscr{S}$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{S}(N) = \left\{ x \in \mathrm{Ve}(N) : s^N_{x,y} \geq s^N_{y,x} \text{ for all } y \in \mathrm{Ve}(N) \setminus \{x\} \right\},$$
where, for every $x, y \in \mathrm{Ve}(N)$ with $x \neq y$, $s^N_{x,y} = \max\{\lambda^N(\gamma) : \gamma \in \Gamma(N, x, y)\}$. Here $\Gamma(N, x, y)$ denotes the set of the paths from x to y in N and, for every $\gamma = (x_j)_{j=1}^m \in \Gamma(N, x, y)$, $\lambda^N(\gamma) = \min\{c^N(x_j, x_{j+1}) : j \in \{1, \ldots, m-1\}\}$.
- The Kemeny NS (Kemeny, 1959), denoted by $\mathscr{K}$, is the NS that associates with each $N \in \mathcal{N}$, the set
$$\mathscr{K}(N) = \left\{ r(D) \in \mathrm{Ve}(N) : D \in \mathcal{T}^N \right\}.$$

We observe that, given $N \in \mathcal{N}$ balanced, we have $\mathscr{M}^*(N) = \mathscr{M}_*(N)$ and $\mathscr{O}(N) = \mathscr{N}(N)$.

Note that the rationale behind $\mathscr{O}$ and $\mathscr{K}$ is similar. Indeed, it is simple to verify that, for every $N \in \mathcal{N}$, we have
$$\mathscr{O}(N) = \left\{ r(D) \in \mathrm{Ve}(N) : D \in \mathcal{O}^N \right\}.$$
This shows that both $\mathscr{O}$ and $\mathscr{K}$ focus on a special family of digraphs, each element of which is characterized by the presence of a distinguished vertex, namely the root. Among the digraphs in the family, they consider those with maximum weight and then select their roots. In this paper, we introduce a new NS based on the same principle, but focusing on a different family of digraphs. Specifically, instead of considering outstars or transitive tournaments, we focus on arborescences.

We are now ready to define the main object of the paper. The maximum-weight arborescence NS, denoted by $\mathscr{A}$, is the NS that associates with each $N \in \mathcal{N}$ the set
$$\mathscr{A}(N) = \left\{ r(D) \in \mathrm{Ve}(N) : D \in \mathcal{A}^N \right\}.$$

Since $\mathcal{A}^N$ is nonempty, it follows that $\mathscr{A}(N) \neq \varnothing$; hence, $\mathscr{A}$ is well defined.

Given $N \in \mathcal{N}$, we define, for every $x \in \mathrm{Ve}(N)$,

$$\mu^N(x) = \max\{N \circ D : D \in \mathcal{A}_x(\mathrm{Ve}(N))\}.$$

Using the function μ^N, we can equivalently write

$$\mathscr{A}(N) = \underset{x \in \mathrm{Ve}(N)}{\operatorname{argmax}} \, \mu^N(x). \tag{3}$$

The value of $\mu^N(x)$ can be computed efficiently using the well-known Chu–Liu/Edmonds algorithm, which runs in polynomial time in the number of vertices. Consequently, $\mathscr{A}(N)$ can also be computed in polynomial time in the number of vertices. The same holds for $\mathscr{O}(N)$, whereas the computation of $\mathscr{K}(N)$ is known to be computationally hard (Fischer et al. 2016).

Considering the network N_2 in Figure 4 and the computations made in Section 2.3, we obtain $\mathscr{A}(N_2) = \{r(E_4), r(E_6), r(E_8)\} = \{2, 3\}$. Note also that

$$\mu^{N_2}(1) = \max\{N_2 \circ E_1, N_2 \circ E_2, N_2 \circ E_3\} = 2,$$
$$\mu^{N_2}(2) = \max\{N_2 \circ E_4, N_2 \circ E_5, N_2 \circ E_6\} = 4,$$
$$\mu^{N_2}(3) = \max\{N_2 \circ E_7, N_2 \circ E_8, N_2 \circ E_9\} = 4.$$

We also have $\mathscr{O}(N_2) = \{2\}$ and $\mathscr{K}(N_2) = \{2, 3\}$.

3.1 Properties of $\mathscr{A}$

This section is devoted to analyzing the properties of the maximum-weight arborescence NS.

Propositions 1, 2, and 3 describe regularity and invariance properties of the maximum-weight arborescence NS. In particular, Proposition 1 establishes the neutrality property: any permutation of the vertices affects the outcome only through a relabeling of the alternatives induced by the same permutation. Proposition 2 states that adding the same constant to the capacity of every arc does not change the outcome. Finally, Proposition 3 shows that $\mathscr{A}$ is homogeneous, in the sense that multiplying all arc capacities by the same positive constant leaves the outcome unchanged.

Proposition 1. *Let $N \in \mathcal{N}$ and $\psi \in \mathrm{Sym}(\mathrm{Ve}(N))$. Then, $\mathscr{A}(N^\psi) = \psi(\mathscr{A}(N))$.*

Proposition 2. *Let $N \in \mathcal{N}$ and $k \in \mathbb{R}$. Then, $\mathscr{A}(N + k) = \mathscr{A}(N)$.*

Proposition 3. *Let $N \in \mathcal{N}$ and $k \in \mathbb{R}$ with $k > 0$. Then, $\mathscr{A}(kN) = \mathscr{A}(N)$.*

Propositions 4, 5, 6, 7, and 8 provide conditions under which an alternative is either necessarily selected or necessarily excluded. Note that Propositions 6 and 8 specifically refer to balanced networks.

Proposition 4. *Let $N \in \mathcal{N}$ and $x_0, y_0 \in \mathrm{Ve}(N)$ with $x_0 \neq y_0$.*

(i) *Assume that $c^N(x_0, y_0) \geq c^N(z, x_0)$ for all $z \in \mathrm{Ve}(N) \setminus \{x_0\}$. Then $y_0 \in \mathscr{A}(N)$ implies $x_0 \in \mathscr{A}(N)$.*

(ii) *Assume that $c^N(x_0, y_0) > c^N(z, x_0)$ for all $z \in \mathrm{Ve}(N) \setminus \{x_0\}$. Then $y_0 \notin \mathscr{A}(N)$.*

Proposition 5. *Let $N \in \mathcal{N}$, and $x_0, y_0 \in \mathrm{Ve}(N)$ with $x_0 \neq y_0$.*

(i) *Assume that $c^N(x_0, y_0) \geq c^N(y_0, x_0)$, and that, for every $z \in \mathrm{Ve}(N) \setminus \{x_0, y_0\}$, $c^N(x_0, z) \geq c^N(y_0, z)$ and $c^N(z, y_0) \geq c^N(z, x_0)$. Then $y_0 \in \mathscr{A}(N)$ implies $x_0 \in \mathscr{A}(N)$.*

(ii) *Assume that $c^N(x_0, y_0) > c^N(y_0, x_0)$, and that, for every $z \in \mathrm{Ve}(N) \setminus \{x_0, y_0\}$, $c^N(x_0, z) > c^N(y_0, z)$ and $c^N(z, y_0) \geq c^N(z, x_0)$. Then $y_0 \notin \mathscr{A}(N)$.*

(iii) *Assume that $c^N(x_0, y_0) > c^N(y_0, x_0)$, and that, for every $z \in \mathrm{Ve}(N) \setminus \{x_0, y_0\}$, $c^N(x_0, z) \geq c^N(y_0, z)$ and $c^N(z, y_0) > c^N(z, x_0)$. Then $y_0 \notin \mathscr{A}(N)$.*

Proposition 6. *Let $N \in \mathcal{N}$ be balanced, and $x_0, y_0 \in \mathrm{Ve}(N)$ with $x_0 \neq y_0$.*

(i) *Assume that $c^N(x_0, y_0) \geq c^N(y_0, x_0)$, and that, for every $z \in \mathrm{Ve}(N) \setminus \{x_0, y_0\}$, $c^N(x_0, z) \geq c^N(y_0, z)$. Then $y_0 \in \mathscr{A}(N)$ implies $x_0 \in \mathscr{A}(N)$.*

(ii) *Assume that $c^N(x_0, y_0) > c^N(y_0, x_0)$, and that, for every $z \in \mathrm{Ve}(N) \setminus \{x_0, y_0\}$, $c^N(x_0, z) > c^N(y_0, z)$. Then $y_0 \notin \mathscr{A}(N)$.*

Proposition 7. *Let $N \in \mathcal{N}$ with $|\mathrm{Ve}(N)| \geq 2$ and $x_0 \in \mathrm{Ve}(N)$.*

(i) *If $\min\{c^N(x_0, z) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\} \geq \max\{c^N(z, x_0) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\}$, then $x_0 \in \mathscr{A}(N)$.*

(ii) *If $\min\{c^N(x_0, z) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\} > \max\{c^N(z, x_0) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\}$, then $\mathscr{A}(N) = \{x_0\}$.*

(iii) *If $\min\{c^N(z, x_0) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\} > \max\{c^N(x_0, z) : z \in \mathrm{Ve}(N) \setminus \{x_0\}\}$, then $x_0 \notin \mathscr{A}(N)$.*

Proposition 8. *Let $N \in \mathcal{N}$ be balanced with $|\mathrm{Ve}(N)| \geq 2$ and $x_0 \in \mathrm{Ve}(N)$.*

(i) *If $c^N(x_0, y) \geq c^N(y, x_0)$ for all $y \in \mathrm{Ve}(N) \setminus \{x_0\}$, then $x_0 \in \mathscr{A}(N)$.*

(ii) *If $c^N(x_0, y) > c^N(y, x_0)$ for all $y \in \mathrm{Ve}(N) \setminus \{x_0\}$, then $\mathscr{A}(N) = \{x_0\}$.*

(iii) *If $c^N(x_0, y) < c^N(y, x_0)$ for all $y \in \mathrm{Ve}(N) \setminus \{x_0\}$, then $x_0 \notin \mathscr{A}(N)$.*

The following proposition, also concerning balanced networks, states that the alternatives selected by $\mathscr{A}$ are contained in every set of vertices satisfying a specific property.

Proposition 9. *Let $N \in \mathcal{N}$ be balanced and $S \subseteq \mathrm{Ve}(N)$ with $S \neq \varnothing$. Assume that, for every $x \in S$ and $y \in \mathrm{Ve}(N) \setminus S$, we have $c^N(x, y) > c^N(y, x)$. Then $\mathscr{A}(N) \subseteq S$.*

The next result is a monotonicity property with respect to all arcs incident to a selected vertex. It shows that the vertex remains selected when the capacities of its outgoing arcs do not decrease, the capacities of its incoming arcs do not increase, and all other capacities remain unchanged.

Theorem 10. *Let $N, M \in \mathcal{N}$ with $\mathrm{Ve}(N) = \mathrm{Ve}(M)$ and $x_0 \in \mathscr{A}(N)$. Assume that*

- *for every $y \in \mathrm{Ve}(N) \setminus \{x_0\}$, $c^M(x_0, y) \geq c^N(x_0, y)$ and $c^N(y, x_0) \geq c^M(y, x_0)$,*
- *for every $(x, y) \in (\mathrm{Ve}(N) \setminus \{x_0\})^2_*$, $c^M(x, y) = c^N(x, y)$.*

Then $x_0 \in \mathscr{A}(M)$.

The next proposition shows that the maximum-weight arborescence NS satisfies a property known as cancellation.

Proposition 11. *Let $N \in \mathcal{N}$ be such that $N = N^r$. Then $\mathscr{A}(N) = \mathrm{Ve}(N)$.*

Let $N \in \mathcal{N}$, let $u_0 \in \mathrm{Ve}(N)$, and let $C \in P_f(\mathbb{N})$ be such that $C \cap \mathrm{Ve}(N) = \varnothing$. A network $M \in \mathcal{N}$ is said to be obtained from N by adding the set C of clones of u_0 if $\mathrm{Ve}(M) = \mathrm{Ve}(N) \cup C$ and

- for every $(x, y) \in \mathrm{Ve}(N)^2_*$, $c^M(x, y) = c^N(x, y)$,
- for every $x \in C$ and $y \in \mathrm{Ve}(N) \setminus \{u_0\}$, $c^M(x, y) = c^N(u_0, y)$,
- for every $x \in C$ and $y \in \mathrm{Ve}(N) \setminus \{u_0\}$, $c^M(y, x) = c^N(y, u_0)$,

while no restriction is imposed on $c^M(x, y)$ for distinct $x, y \in C \cup \{u_0\}$.

The next result describes how the maximum-weight arborescence NS behaves when a set of clones of a vertex is added to a network.

Theorem 12. *Let $N, M \in \mathcal{N}$, $u_0 \in \mathrm{Ve}(N)$, and $C \in P_f(\mathbb{N})$ be such that $C \cap \mathrm{Ve}(N) = \varnothing$. Suppose that M is obtained from N by adding the set C of clones of u_0. Then:*

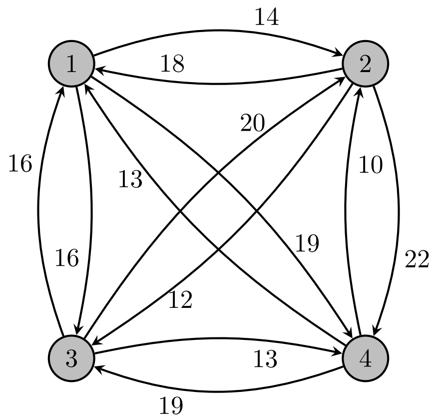

Figure 5: The network N_3.

(i) *for every* $x_0 \in \mathrm{Ve}(N) \setminus \{u_0\}$, $x_0 \in \mathscr{A}(N)$ *if and only if* $x_0 \in \mathscr{A}(M)$*;*

(ii) $u_0 \in \mathscr{A}(N)$ *if and only if* $(C \cup \{u_0\}) \cap \mathscr{A}(M) \neq \varnothing$*.*

While for networks with at most two vertices all previously introduced network solutions select the same vertices, differences begin to emerge when there are three or more vertices. The following result shows that, when there are exactly three vertices, the maximum-weight arborescence NS coincides on balanced networks with the maximin NS, the minimax NS, and the Schulze NS.

Proposition 13. *Let* $N \in \mathcal{N}$ *be balanced with* $|\mathrm{Ve}(N)| = 3$*. Then* $\mathscr{A}(N) = \mathscr{M}_*(N) = \mathscr{M}^*(N) = \mathscr{S}(N)$*.*

The balanced network N_3 in Figure 5 shows that, when the number of vertices exceeds three, the equality in Proposition 13 generally fails for balanced networks. Indeed, $\mathscr{A}(N_3) = \{3\}$, whereas $\mathscr{M}_*(N_3) = \mathscr{M}^*(N_3) = \mathscr{S}(N_3) = \{1\}$. Network N_2 in Figure 4 further shows that, if networks are not required to be balanced, the equality in Proposition 13 may fail even when the vertices are three. Indeed, $\mathscr{A}(N_2) = \{2, 3\}$, $\mathscr{M}^*(N_2) = \{1, 2, 3\}$, $\mathscr{M}_*(N_2) = \mathscr{S}(N_2) = \{3\}$.

4 Application to abstract decision problems

A digraph (X, R) is called an abstract decision problem when X is interpreted as a set of alternatives from which a society has to choose, and R is interpreted as a dominance relation among the alternatives. The relation R is meant to capture the idea that, if an alternative y is currently being considered by the society during the decision process, the existence of another alternative x such that $(x, y) \in R$ provides a valid reason to reject y in favor of x. Conversely, if $(x, y) \notin R$ for all alternatives x, then the society continues to consider y, since there is no reason to reject it.

A NS on $\mathcal{D}$ is also called a solution for abstract decision problems: it represents a method for selecting, for any abstract decision problem, a subset of alternatives, interpreted as the potential outcomes of the decision process. Several solutions for abstract decision problems have been proposed in the literature. The admissible set is a well-known solution introduced by Kalai et al. (1976) and Kalai and Schmeidler (1977).[4] Such a solution, here denoted by $\mathscr{A}_*$, is defined, for every $D \in \mathcal{D}$, by

$$\mathscr{A}_*(D) = \left\{ x \in \mathrm{Ve}(D) : \forall y \in \mathrm{Ve}(D) \setminus \{x\},\ y \in \alpha^D(x) \text{ implies } x \in \alpha^D(y) \right\}.$$

In other words, given $D \in \mathcal{D}$ and $x \in \mathrm{Ve}(D)$, we have that $x \in \mathscr{A}_*(D)$ if and only if, for every $y \in \mathrm{Ve}(D) \setminus \{x\}$, the existence of a path from y to x in D implies the existence of a path from x to y in D.

The next result shows that the admissible set coincides with the maximum-weight arborescence NS restricted to $\mathcal{D}$.

[4] Schwartz (1972, 1986) and Shenoy (1979, 1980) gave equivalent definitions of this concept.

Theorem 14. *Let $D \in \mathcal{D}$. Then, $\mathscr{A}_*(D) = \mathscr{A}(D)$.*

As proved by Gori (2023, Proposition 6), $\mathscr{A}_*(D) = \mathscr{S}(D)$ for all $D \in \mathcal{D}$, where $\mathscr{S}$ is the Schulze NS. Theorem 14 provides a further characterization of the admissible set. In particular, we understand that both $\mathscr{A}$ and $\mathscr{S}$ can be seen as extensions of the admissible set from the set of digraphs to the whole set of networks.

5 Application to preference aggregation

5.1 Preference profiles

Consider a countably infinite set whose elements are to be interpreted as potential voters and a countably infinite set whose elements are to be interpreted as potential alternatives. For simplicity, we identify both sets with $\mathbb{N}$, as no confusion may arise. We then define the set

$$\mathbf{L}^* = \bigcup_{X \in P_f(\mathbb{N})} \bigcup_{I \in P_f(\mathbb{N})} \mathbf{L}(X)^I.$$

An element of $\mathbf{L}^*$ is called a preference profile. Given $p \in \mathbf{L}^*$, there exists a unique pair $(X, I) \in P_f(\mathbb{N})^2$ such that $p \in \mathbf{L}(X)^I$: the set X, the set of alternatives, is denoted by $\mathrm{Al}(p)$ and the set I, the set of individuals, is denoted by $\mathrm{In}(p)$. For every $i \in \mathrm{In}(p)$, $p(i) \in \mathbf{L}(\mathrm{Al}(p))$ is interpreted as the preference relation of individual i on the set of alternatives $\mathrm{Al}(p)$. Typically an element p of $\mathbf{L}^*$ is represented by means of a table where in the first row there are the elements of $\mathrm{In}(p)$ and below the name of each individual i there is the preference of i on $\mathrm{Al}(p)$, that is $p(i)$, represented as a ranking of the elements of $\mathrm{Al}(p)$. For instance, the table

3	4	8	11	17	21
7	2	7	2	3	7
3	10	3	3	2	3
2	7	10	7	10	2
10	3	2	10	7	10

(4)

represents the element p of $\mathbf{L}^*$ such that $\mathrm{Al}(p) = \{2, 3, 7, 10\}$, $\mathrm{In}(p) = \{3, 4, 8, 11, 17, 21\}$, and such that $p(3)$ is the unique linear order such that $7 \succ_{p(3)} 3 \succ_{p(3)} 2 \succ_{p(3)} 10$, $p(4)$ is the unique linear order such that $2 \succ_{p(4)} 10 \succ_{p(4)} 7 \succ_{p(4)} 3$, and so on. The order of the columns is irrelevant.

Let $p \in \mathbf{L}^*$. Given $\psi \in \mathrm{Sym}(\mathrm{Al}(p))$, we denote by p^ψ the element of $\mathbf{L}^*$ such that $\mathrm{Al}(p^\psi) = \mathrm{Al}(p)$, $\mathrm{In}(p^\psi) = \mathrm{In}(p)$ and, for every $i \in \mathrm{In}(p)$, $p^\psi(i) = p(i)^\psi$. The preference profile p^ψ is then obtained from p by renaming each alternative $x \in \mathrm{Al}(p)$ as $\psi(x)$. Given $\varphi \in \mathrm{Sym}(\mathrm{In}(p))$, we denote by p_φ the element of $\mathbf{L}^*$ such that $\mathrm{Al}(p_\varphi) = \mathrm{Al}(p)$, $\mathrm{In}(p_\varphi) = \mathrm{In}(p)$ and, for every $i \in \mathrm{In}(p)$, $p_\varphi(i) = p(\varphi^{-1}(i))$. The preference profile p_φ is then obtained from p by renaming each individual $i \in \mathrm{In}(p)$ as $\varphi(i)$. Given $q \in \mathbf{L}^*$ and $n \in \mathbb{N}$, we say that q is an n-fold replication of p if $\mathrm{Al}(q) = \mathrm{Al}(p)$ and there exists a partition $\{I_1, \dots, I_n\}$ of $\mathrm{In}(q)$ such that, for every $k \in \{1, \dots, n\}$, there exists a bijection $\varphi_k : \mathrm{In}(p) \to I_k$ such that, for every $i \in \mathrm{In}(p)$, we have $q(\varphi_k(i)) = p(i)$. Given $q \in \mathbf{L}^*$, we say that p and q are disjoint if $\mathrm{Al}(p) = \mathrm{Al}(q)$ and $\mathrm{In}(p) \cap \mathrm{In}(q) = \varnothing$. If $p, q \in \mathbf{L}^*$ are disjoint, we denote by $p+q$ the element of $\mathbf{L}^*$ such that $\mathrm{Al}(p+q) = \mathrm{Al}(p)$, $\mathrm{In}(p+q) = \mathrm{In}(p) \cup \mathrm{In}(q)$, $(p+q)(i) = p(i)$ for all $i \in \mathrm{In}(p)$, and $(p+q)(i) = q(i)$ for all $i \in \mathrm{In}(q)$. Note that if $p, q \in \mathbf{L}^*$ are disjoint, then $p + q = q + p$. Given $q \in \mathbf{L}^*$ with $\mathrm{In}(q) = \mathrm{In}(p)$ and $\mathrm{Al}(q) = \mathrm{Al}(p)$, and $x \in \mathrm{Al}(p)$, we say that x improves its position from p to q if

- for every $i \in \mathrm{In}(p)$ and $y \in \mathrm{Al}(p)$, $x \succeq_{p(i)} y$ implies $x \succeq_{q(i)} y$;
- for every $i \in \mathrm{In}(p)$ and $y_1, y_2 \in \mathrm{Al}(p) \setminus \{x\}$, $y_1 \succeq_{p(i)} y_2$ if and only if $y_1 \succeq_{q(i)} y_2$.

Given $q \in \mathbf{L}^*$, $u_0 \in \mathrm{Al}(p)$, and $C \in P_f(\mathbb{N})$ such that $C \cap \mathrm{Al}(p) = \varnothing$, we say that q is obtained from p by adding the set of clones C of u_0 if $\mathrm{In}(q) = \mathrm{In}(p)$, $\mathrm{Al}(q) = \mathrm{Al}(p) \cup C$ and

- for every $(x, y) \in \mathrm{Al}(p)^2_*$ and $i \in \mathrm{In}(p)$, $x \succeq_{q(i)} y$ if and only if $x \succeq_{p(i)} y$,

- for every $x \in C$, $y \in \mathrm{Al}(p) \setminus \{u_0\}$ and $i \in \mathrm{In}(p)$, $x \succeq_{q(i)} y$ if and only if $u_0 \succeq_{p(i)} y$,
- for every $x \in C$, $y \in \mathrm{Al}(p) \setminus \{u_0\}$ and $i \in \mathrm{In}(p)$, $y \succeq_{q(i)} x$ if and only if $y \succeq_{p(i)} u_0$,

while no restriction is imposed on $q(i)$ for distinct $x, y \in C \cup \{u_0\}$ and $i \in \mathrm{In}(p)$.

In other words, $q \in \mathbf{L}^*$ is obtained from p by adding the set of clones C of u_0 if, for every individual i, the linear order $q(i)$ is obtained from $p(i)$ by replacing u_0 with a consecutive block whose elements are precisely those of $C \cup \{u_0\}$, arranged in an arbitrary order. The relative order of the alternatives in $\mathrm{Al}(p)$ remains unchanged. For instance, the following two tables represent preference profiles obtained from the profile in (4) by adding the set of clones $\{6, 31\}$ of 7:

3	4	8	11	17	21
31	2	**7**	2	3	**7**
6	10	**31**	3	2	**6**
7	**6**	**6**	**7**	10	**31**
3	**31**	3	**31**	**6**	3
2	**7**	10	**6**	**7**	2
10	3	2	10	**31**	10

3	4	8	11	17	21
7	2	**31**	2	3	**6**
31	10	**7**	3	2	**31**
6	**7**	**6**	**6**	10	**7**
3	**31**	3	**31**	**6**	3
2	**6**	10	**7**	**31**	2
10	3	2	10	**7**	10

.

5.2 Social choice correspondences

Given $p \in \mathbf{L}^*$, we denote by $N(p)$ the network such that $\mathrm{Ve}(N(p)) = \mathrm{Al}(p)$ and such that, for every $(x, y) \in (\mathrm{Al}(p))^2_*$,

$$c^{N(p)}(x, y) = |\{i \in \mathrm{In}(p) : x \succeq_{p(i)} y\}| = |\{i \in \mathrm{In}(p) : x \succ_{p(i)} y\}|.$$

The network $N(p)$ is called the weighted majority digraph associated with p. Note that $N(p)$ is balanced and if $\psi \in \mathrm{Sym}(\mathrm{Al}(p))$, then $N(p^\psi) = N(p)^\psi$.

A social choice correspondence (SCC) is a function F from $\mathbf{L}^*$ to $P_*(\mathbb{N})$ such that, for every $p \in \mathbf{L}^*$, $\mathsf{F}(p) \subseteq \mathrm{Al}(p)$. There are many SCCs in the literature. The following well-known SCCs originate from the network solution introduced in Section 3:

- The Borda SCC is the SCC denoted by Bor and defined, for every $p \in \mathbf{L}^*$, as $\mathsf{Bor}(p) = \mathscr{O}(N(p)) = \mathscr{N}(N(p))$.
- The Schulze SCC is the SCC denoted by Sch and defined, for every $p \in \mathbf{L}^*$, as $\mathsf{Sch}(p) = \mathscr{S}(N(p))$.
- The Simpson SCC is the SCC denoted by Sim and defined, for every $p \in \mathbf{L}^*$, as $\mathsf{Sim}(p) = \mathscr{M}^*(N(p)) = \mathscr{M}_*(N(p))$.
- The Kemeny SCC is the SCC denoted by Kem and defined, for every $p \in \mathbf{L}^*$, as $\mathsf{Kem}(p) = \mathscr{K}(N(p))$.

Clearly, these SCCs have the property that, for every $p, q \in \mathbf{L}^*$ such that $N(p) = N(q)$, the outcome is the same. Hence, they are C2-functions in the sense of Fishburn (1977). Building on the same idea, we can define a new C2 SCC based on the NS $\mathscr{A}$.

The maximum-weight arborescence SCC is the SCC denoted by A and defined, for every $p \in \mathbf{L}^*$, as $\mathsf{A}(p) = \mathscr{A}(N(p))$. To the best of our knowledge, A is a new SCC. In the next section, we investigate the properties of A.

5.3 Properties of A

Let F be a SCC. We say that F satisfies

- anonymity if, for every $p \in \mathbf{L}^*$ and $\varphi \in \mathrm{Sym}(\mathrm{In}(p))$, $\mathsf{F}(p_\varphi) = \mathsf{F}(p)$;
- neutrality if, for every $p \in \mathbf{L}^*$ and $\psi \in \mathrm{Sym}(\mathrm{Al}(p))$, $\mathsf{F}(p^\psi) = \psi(\mathsf{F}(p))$;

- Pareto efficiency if, for every $p \in \mathbf{L}^*$ and $x, y \in \mathrm{Al}(p)$ distinct, we have that $\{i \in \mathrm{In}(p) : y \succ_{p(i)} x\} = \mathrm{In}(p)$ implies $x \notin \mathsf{F}(p)$;
- monotonicity if, for every $p, q \in \mathbf{L}^*$ with $\mathrm{In}(p) = \mathrm{In}(q)$ and $\mathrm{Al}(p) = \mathrm{Al}(q)$ and $x \in \mathsf{F}(p)$, the fact that x improves its position from p to q implies that $x \in \mathsf{F}(q)$;
- homogeneity if, for every $p, q \in \mathbf{L}^*$ and $n \in \mathbb{N}$, the fact that q is an n-fold replication of p implies $\mathsf{F}(q) = \mathsf{F}(p)$;
- independence of clones[5] if, for every $p \in \mathbf{L}^*$, $u_0 \in \mathrm{Al}(p)$, $C \in P_f(\mathbb{N})$ with $C \cap \mathrm{Al}(p) = \varnothing$, and $q \in \mathbf{L}^*$ obtained from p by adding the set of clones C of u_0, we have that
  - for every $x_0 \in \mathrm{Al}(p) \setminus \{u_0\}$, $x_0 \in \mathsf{F}(p)$ if and only if $x_0 \in \mathsf{F}(q)$.
  - $u_0 \in \mathsf{F}(p)$ if and only if $(C \cup \{u_0\}) \cap \mathsf{F}(q) \neq \varnothing$,
- the Condorcet principle if, for every $p \in \mathbf{L}^*$,

$$\mathsf{F}(p) = \left\{x \in \mathrm{Al}(p) : c^{N(p)}(x, y) > \tfrac{|\mathrm{In}(p)|}{2} \text{ for all } y \in \mathrm{Al}(p) \setminus \{x\}\right\},$$

  whenever the set on the right hand side is nonempty;
- the inclusive Condorcet principle[6] if, for every $p \in \mathbf{L}^*$,

$$\left\{x \in \mathrm{Al}(p) : c^{N(p)}(x, y) \geq \tfrac{|\mathrm{In}(p)|}{2} \text{ for all } y \in \mathrm{Al}(p) \setminus \{x\}\right\} \subseteq \mathsf{F}(p);$$

- the Smith principle[7] if, for every $p \in \mathbf{L}^*$ and $S \subseteq \mathrm{Al}(p)$ with $S \neq \varnothing$ and such that $c^{N(p)}(x, y) > \frac{|\mathrm{In}(p)|}{2}$ for all $x \in S$ and $y \in \mathrm{Al}(p) \setminus S$, we have $\mathsf{F}(p) \subseteq S$;
- the Condorcet loser principle if, for every $p \in \mathbf{L}^*$ with $|\mathrm{Al}(p)| \geq 2$,

$$\mathsf{F}(p) \cap \left\{x \in \mathrm{Al}(p) : c^{N(p)}(x, y) < \tfrac{|\mathrm{In}(p)|}{2} \text{ for all } y \in \mathrm{Al}(p) \setminus \{x\}\right\} = \varnothing;$$

- cancellation[8] if, for every $p \in \mathbf{L}^*$ such that $N(p) = N(p)^r$, we have $\mathsf{F}(p) = \mathrm{Al}(p)$;

The aforementioned properties are classical. Note that the Smith principle implies the Condorcet principle.

The next proposition shows that the maximum-weight arborescence SCC satisfies all the aforementioned properties and hence, in particular, is a Condorcet SCC.

Proposition 15. A *satisfies anonymity, neutrality, Pareto efficiency, monotonicity, homogeneity, independence of clones, the Condorcet principle, the inclusive Condorcet principle, the Smith principle, the Condorcet loser principle, and cancellation.*

Condorcet SCCs form a central class of methods in voting theory. Numerous Condorcet SCCs have been introduced and studied in the literature (e.g., Fishburn 1977; Zwicker 2016). For example, the Schulze SCC, the Simpson SCC, and the Kemeny SCC are Condorcet SCCs, whereas the Borda SCC is not. Recently, several new Condorcet SCCs have been proposed, including Split Cycle (Holliday and Pacuit 2023a), Stable Voting (Holliday and Pacuit 2023b), the maximum flow value SCC (Gori 2024), and the River voting method (Döring et al. 2026). The maximum-weight arborescence SCC is a new member of this family. As stated above, it satisfies further desirable properties. In particular, it satisfies independence of clones, a property shared by few SCCs, including Ranked Pairs, Schulze, Split Cycle, Stable Voting, and the River voting method.

We emphasize that arborescences also play a role in the construction of the River voting method. However, its underlying rationale differs from ours, and the two methods do not coincide. For example, consider $p \in \mathbf{L}^*$ such that $\mathrm{Al}(p) = \{1, 2, 3, 4\}$, $\mathrm{In}(p) = \{1, \ldots, 23\}$, and such that

[5] Tideman (1987).
[6] Fishburn (1977, p.479).
[7] Smith (1973, p.1038) and Fishburn (1977, p.478).
[8] Young (1974).

- for $i \in \{1, 2, 3, 4, 5, 6, 7\}$, $p(i) = [1, 3, 4, 2]$,
- for $i \in \{8\}$, $p(i) = [1, 4, 3, 2]$,
- for $i \in \{9, 10\}$, $p(i) = [2, 1, 4, 3]$,
- for $i \in \{11, 12, 13, 14\}$, $p(i) = [3, 2, 1, 4]$,
- for $i \in \{15, 16, 17, 18, 19\}$, $p(i) = [4, 2, 3, 1]$,
- for $i \in \{20, 21, 22, 23\}$, $p(i) = [4, 3, 2, 1]$.

A computation shows that $\mathsf{A}(p) = \{3\}$, whereas the River voting method determines the set $\{4\}$.

The following result establishes that, when the alternatives are at most three, the maximum-weight arborescence SCC coincides with both the Simpson and the Schulze SCCs.

Proposition 16. *Let $p \in \mathbf{L}^*$ with $|\mathrm{Al}(p)| \leq 3$. Then $\mathsf{A}(p) = \mathsf{Sch}(p) = \mathsf{Sim}(p)$.*

In general, the equality in Proposition 16 may fail for $p \in \mathbf{L}^*$ with $|\mathrm{Al}(p)| \geq 4$. As an example, we can consider $p \in \mathbf{L}^*$ such that $\mathrm{Al}(p) = \{1, 2, 3, 4\}$, $\mathrm{In}(p) = \{1, \dots, 32\}$, and such that

- for $i \in \{1, 2, 3\}$, $p(i) = [1, 2, 3, 4]$,
- for $i \in \{4, 5, 6, 7\}$, $p(i) = [1, 3, 2, 4]$,
- for $i \in \{8, 9, 10, 11, 12, 13\}$, $p(i) = [2, 1, 4, 3]$,
- for $i \in \{14, 15, 16\}$, $p(i) = [2, 4, 1, 3]$,
- for $i \in \{17, 18, 19, 20, 21, 22\}$, $p(i) = [3, 1, 2, 4]$,
- for $i \in \{23\}$, $p(i) = [4, 3, 1, 2]$,
- for $i \in \{24, 25, 26, 27, 28, 29, 30, 31, 32\}$, $p(i) = [4, 3, 2, 1]$.

We have that $N(p)$ coincides with the network N_3 in Figure 5, and then $\mathsf{A}(p) = \{3\}$, whereas $\mathsf{Sch}(p) = \mathsf{Sim}(p) = \{1\}$.

The following result establishes an additional property satisfied by A.

Proposition 17. *Let $p \in \mathbf{L}^*$ and let $x, y \in \mathrm{Al}(p)$ with $x \neq y$. Assume that, for every $z \in \mathrm{Al}(p) \setminus \{x\}$, $c^{N(p)}(x, y) > c^{N(p)}(z, x)$. Then, $y \notin \mathsf{A}(p)$.*

It can be verified that the Simpson and the Schulze SCCs satisfy this property, whereas other Condorcet SCCs, such as the Copeland SCC, do not.

A Proofs

A.1 Proofs of Propositions 1, 2, 3, 4, 7, 8, 9, 11, and 13

Proof of Proposition 1. Let $X = \mathrm{Ve}(N) = \mathrm{Ve}(N^\psi)$. Let $D \in \mathcal{D}(X)$. First, note that $N^\psi \circ D^\psi = N \circ D$. Indeed,

$$N^\psi \circ D^\psi = \sum_{(x,y) \in \mathrm{Ar}(D^\psi)} c^{N^\psi}(x, y) = \sum_{(x,y) \in \mathrm{Ar}(D)^\psi} c^N(\psi^{-1}(x), \psi^{-1}(y))$$
$$= \sum_{(u,v) \in \mathrm{Ar}(D)} c^N(\psi^{-1}(\psi(u)), \psi^{-1}(\psi(v))) = \sum_{(u,v) \in \mathrm{Ar}(D)} c^N(u, v) = N \circ D.$$

Consider now $x \in X$ and prove that $\mu^N(x) = \mu^{N^\psi}(\psi(x))$. Indeed, let $D \in \mathcal{A}_x(X)$ be such that $\mu^N(x) = N \circ D$. We know that $D^\psi \in \mathcal{A}_{\psi(x)}(X)$ and then

$$\mu^{N^\psi}(\psi(x)) \geq N^\psi \circ D^\psi = N \circ D = \mu^N(x).$$

Now let $D \in \mathcal{A}_{\psi(x)}(X)$ be such that $\mu^{N^\psi}(\psi(x)) = N^\psi \circ D$. We know that $D^{\psi^{-1}} \in \mathcal{A}_x(X)$ and then

$$\mu^N(x) \geq N \circ D^{\psi^{-1}} = N^\psi \circ (D^{\psi^{-1}})^\psi = N^\psi \circ D^{\psi\psi^{-1}} = N^\psi \circ D^{id_X} = N^\psi \circ D = \mu^{N^\psi}(\psi(x)).$$

The above inequalities imply $\mu^N(x) = \mu^{N^\psi}(\psi(x))$.

In order to prove that $\mathscr{A}(N^\psi) = \psi(\mathscr{A}(N))$, consider $x \in X$ and the following facts:

(a_1) $x \in \mathscr{A}(N^\psi)$;

(a_2) for every $y \in X$, $\mu^{N^\psi}(x) \geq \mu^{N^\psi}(y)$;

(a_3) for every $y \in X$, $\mu^{N^\psi}(\psi(\psi^{-1}(x))) \geq \mu^{N^\psi}(\psi(\psi^{-1}(y)))$;

(a_4) for every $y \in X$, $\mu^N(\psi^{-1}(x)) \geq \mu^N(\psi^{-1}(y))$;

(a_5) for every $y \in X$, $\mu^N(\psi^{-1}(x)) \geq \mu^N(y)$;

(a_6) $\psi^{-1}(x) \in \mathscr{A}(N)$;

(a_7) $x \in \psi(\mathscr{A}(N))$.

We have that (a_i) is equivalent to (a_{i+1}) for all $i \in \{1, \dots, 6\}$. Then (a_1) is equivalent to (a_7), as desired. □

Proof of Proposition 2. Let $X = \mathrm{Ve}(N)$. First, observe that, for every $D \in \mathcal{A}$ with $\mathrm{Ve}(D) = X$, $(N + k) \circ D = (N \circ D) + k(|X| - 1)$. Indeed, since $|\mathrm{Ar}(D)| = |X| - 1$, we have

$$(N + k) \circ D = \sum_{(x,y)\in \mathrm{Ar}(D)} (c^N(x, y) + k) = \sum_{(x,y)\in \mathrm{Ar}(D)} c^N(x, y) + \sum_{(x,y)\in \mathrm{Ar}(D)} k = (N \circ D) + k(|X| - 1).$$

Thus, for every $x \in X$, $\mu^{N+k}(x) = \mu^N(x) + k(|X| - 1)$. Indeed, consider $x \in X$. Let $D \in \mathcal{A}_x(X)$ be such that $\mu^{N+k}(x) = (N + k) \circ D$. Then

$$\mu^N(x) + k(|X| - 1) \geq (N \circ D) + k(|X| - 1) = (N + k) \circ D = \mu^{N+k}(x).$$

Now let $D \in \mathcal{A}_x(X)$ be such that $\mu^N(x) = N \circ D$. Then

$$\mu^{N+k}(x) \geq (N + k) \circ D = (N \circ D) + k(|X| - 1) = \mu^N(x) + k(|X| - 1).$$

The above inequalities imply then $\mu^{N+k}(x) = \mu^N(x) + k(|X| - 1)$.

In order to prove that $\mathscr{A}(N + k) = \mathscr{A}(N)$, consider $x \in X$ and the following facts:

(a_1) $x \in \mathscr{A}(N + k)$;

(a_2) for every $y \in X$, $\mu^{N+k}(x) \geq \mu^{N+k}(y)$;

(a_3) for every $y \in X$, $\mu^{N+k}(x) - k(|X| - 1) \geq \mu^{N+k}(y) - k(|X| - 1)$;

(a_4) for every $y \in X$, $\mu^N(x) \geq \mu^N(y)$;

(a_5) $x \in \mathscr{A}(N)$.

We have that (a_i) is equivalent to (a_{i+1}) for all $i \in \{1, \dots, 4\}$. Then (a_1) is equivalent to (a_5), as desired. □

Proof of Proposition 3. Let $X = \mathrm{Ve}(N)$. First, observe that, for every $D \in \mathcal{A}$ with $\mathrm{Ve}(D) = X$, $kN \circ D = k(N \circ D)$. Indeed,

$$kN \circ D = \sum_{(x,y)\in \mathrm{Ar}(D)} kc^N(x,y) = k \sum_{(x,y)\in \mathrm{Ar}(D)} c^N(x,y) = k(N \circ D).$$

Thus, for every $x \in X$, $\mu^{kN}(x) = k\mu^N(x)$. Indeed, consider $x \in X$. Let $D \in \mathcal{A}_x(X)$ be such that $\mu^{kN}(x) = kN \circ D$. Then, recalling that $k > 0$,

$$k\mu^N(x) \geq k(N \circ D) = kN \circ D = \mu^{kN}(x).$$

Now let $D \in \mathcal{A}_x(X)$ be such that $\mu^N(x) = N \circ D$. Then

$$\mu^{kN}(x) \geq (kN) \circ D = k(N \circ D) = k\mu^N(x).$$

The above inequalities imply then $\mu^{kN}(x) = k\mu^N(x)$.

In order to prove that $\mathscr{A}(kN) = \mathscr{A}(N)$, consider $x \in X$ and the following facts:

(a_1) $x \in \mathscr{A}(kN)$;

(a_2) for every $y \in X$, $\mu^{kN}(x) \geq \mu^{kN}(y)$;

(a_3) for every $y \in X$, $k\mu^N(x) \geq k\mu^N(y)$;

(a_4) for every $y \in X$, $\mu^N(x) \geq \mu^N(y)$;

(a_5) $x \in \mathscr{A}(N)$.

Recalling that $k > 0$, we have that (a_i) is equivalent to (a_{i+1}) for all $i \in \{1, \dots, 4\}$. Then (a_1) is equivalent to (a_5), as desired. □

Lemma 18. *Let $X \in P_f(\mathbb{N})$, $x_0, y_0 \in X$ with $x_0 \neq y_0$, and $D \in \mathcal{A}_{y_0}(X)$. Let $E \in \mathcal{D}(X)$ be such that*

$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(z_0, x_0)\}) \cup \{(x_0, y_0)\},$$

where $z_0 \in \pi^D(x_0)$. Then $E \in \mathcal{A}_{x_0}(X)$.

Proof. Since $D \in \mathcal{A}_{y_0}(X)$, we have that, for every $y \in X$, $(y, x_0) \in \mathrm{Ar}(D)$ if and only if $y = z_0$. Thus, for every $y \in X$, $(y, x_0) \notin \mathrm{Ar}(E)$. Consider now $y \in X \setminus \{x_0\}$. If $y = y_0$, then the unique $x \in X$ such that $(x, y) \in \mathrm{Ar}(E)$ is $x = x_0$. If instead $y \neq y_0$, then the unique $x \in X$ such that $(x, y) \in \mathrm{Ar}(E)$ is the unique element in $\pi^D(y)$.

We are then left with proving that, for every $x \in X \setminus \{x_0\}$, there exists a path in E from x_0 to x. The sequence (x_0, y_0) is a path from x_0 to y_0 in E. Consider now $x \in X \setminus \{x_0, y_0\}$. If $x \in \delta^D(x_0)$, then consider the path $\sigma = (x_j)_{j=1}^m$ in D from x_0 to x. We have that, for every $j \in \{1, \dots, m\}$, $x_j \neq z_0$ since, otherwise, we would have a path in D from x_0 to z_0 that, together with the arc (z_0, x_0), would form a cycle in D, a contradiction. Then, σ is also a path from x_0 to x in E, since, for every $j \in \{1, \dots, m-1\}$, $(x_j, x_{j+1}) \neq (z_0, x_0)$. If $x \notin \delta^D(x_0)$, consider a path $(x_j)_{j=1}^m$ in D from y_0 to x. Of course, for every $j \in \{1, \dots, m-1\}$, $x_j \neq x_0$, otherwise, x would be an element of $\delta^D(x_0)$. Moreover, we know that $x_0 \neq x = x_m$. Thus,

$$(x_0, x_1, \dots, x_m)$$

is a path in E from x_0 to x. □

Lemma 19. *Let $X \in P_f(\mathbb{N})$, $x_0, y_0 \in X$ with $x_0 \neq y_0$, and $D \in \mathcal{A}_{y_0}(X)$ with $(y_0, x_0) \notin \mathrm{Ar}(D)$. Let $(x_j^*)_{j=1}^m$ with $m \geq 3$ be the unique path from y_0 to x_0 in D. Let $E \in \mathcal{D}(X)$ be such that*

$$\mathrm{Ar}(E) = \Big(\mathrm{Ar}(D) \setminus \{(y_0, x_2^*), (x_{m-1}^*, x_0)\}\Big) \cup \{(x_0, x_2^*), (x_{m-1}^*, y_0)\}.$$

Then $E \in \mathcal{A}_{x_0}(X)$.

Proof. Since $D \in \mathcal{A}_{y_0}(X)$, we have that, for every $y \in X$, $(y, x_0) \in \mathrm{Ar}(D)$ if and only if $y = x^*_{m-1}$. Thus, for every $y \in X$, $(y, x_0) \notin \mathrm{Ar}(E)$.

Consider now $y \in X \setminus \{x_0\}$. If $y = y_0$, then the unique $x \in X$ such that $(x, y) \in \mathrm{Ar}(E)$ is $x = x^*_{m-1}$. If $y = x^*_2$, then the unique $x \in X$ such that $(x, y) \in \mathrm{Ar}(E)$ is $x = x_0$. If instead $y \neq y_0$ and $y \neq x^*_2$, then the unique $x \in X$ such that $(x, y) \in \mathrm{Ar}(E)$ is the unique element in $\pi^D(y)$.

We are then left with proving that, for every $x \in X \setminus \{x_0\}$, there exists a path in E from x_0 to x.

- If $x \in \delta^D(x_0)$, there is a path $(x_j)_{j=1}^t$ from x_0 to x in D. For every $j \in \{1, \ldots, t-1\}$, we have $(x_j, x_{j+1}) \notin \{(y_0, x^*_2), (x^*_{m-1}, x_0)\}$. Then $(x_j)_{j=1}^t$ is also a path in E from x_0 to x.
- If $x = x^*_2$, we have that (x_0, x^*_2) is a path in E from x_0 to x.
- If $x \in \delta^D(x^*_2) \setminus \delta^D(x_0)$, there is a path $(x_j)_{j=1}^t$ from x^*_2 to x in D. For every $j \in \{1, \ldots, t-1\}$, we have $(x_j, x_{j+1}) \notin \{(y_0, x^*_2), (x^*_{m-1}, x_0)\}$. Then $(x_j)_{j=1}^t$ is also a path from x^*_2 to x in E. We then conclude that
$$(x_0, x_1, \ldots, x_t)$$
is a path in E from x_0 to x.
- If $x \notin \delta^D(x^*_2) \cup \{x^*_2, y_0\}$, there is a path $(x_j)_{j=1}^t$ from y_0 to x in D. We have that
$$\{x_j : j \in \{1, \ldots, t-1\}\} \cap \{x^*_j : j \in \{2, \ldots, m\}\} = \varnothing,$$
and, in particular, $(x_j, x_{j+1}) \notin \{(y_0, x^*_2), (x^*_{m-1}, x_0)\}$. If $m = 3$, we conclude that the sequence
$$(x_0, x^*_2, x_1, \ldots, x_t)$$
is a path in E from x_0 to x. If $m > 3$, we conclude that the sequence
$$(x_0, x^*_2, \ldots, x^*_{m-1}, x_1, \ldots, x_t)$$
is a path in E from x_0 to x.
- If $x = y_0$, we have that the sequence
$$(x_0, x^*_2, \ldots, x^*_{m-1}, y_0)$$
is a path in E from x_0 to y_0.

$\square$

Proof of Proposition 4. Let $X = \mathrm{Ve}(N)$.

(i) Assume $y_0 \in \mathscr{A}(N)$. We prove that $x_0 \in \mathscr{A}(N)$ showing that $\mu^N(x_0) \geq \mu^N(y_0)$. Let $D \in \mathcal{A}_{y_0}(X)$ be such that $\mu^N(y_0) = N \circ D$. Let $E \in \mathcal{D}(X)$ be such that
$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(z_0, x_0)\}) \cup \{(x_0, y_0)\},$$
where $z_0 \in \pi^D(x_0)$. By Lemma 18, $E \in \mathcal{A}_{x_0}(X)$. Thus, $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(z_0, x_0) + c^N(x_0, y_0) \geq N \circ D = \mu^N(y_0)$, as desired.

(ii) In order to prove that $y_0 \notin \mathscr{A}(N)$ we prove that $\mu^N(x_0) > \mu^N(y_0)$. Let $D \in \mathcal{A}_{y_0}(X)$ be such that $\mu^N(y_0) = N \circ D$. Let $E \in \mathcal{D}(X)$ be such that
$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(z_0, x_0)\}) \cup \{(x_0, y_0)\},$$
where $z_0 \in \pi^D(x_0)$. By Lemma 18, $E \in \mathcal{A}_{x_0}(X)$. Thus, $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(z_0, x_0) + c^N(x_0, y_0) > N \circ D = \mu^N(y_0)$, as desired. $\square$

Proof of Proposition 5. Let $X = \mathrm{Ve}(N)$.

(i) Assume $y_0 \in \mathscr{A}(N)$. We prove that $x_0 \in \mathscr{A}(N)$ showing that $\mu^N(x_0) \geq \mu^N(y_0)$. Let $D \in \mathcal{A}_{y_0}(X)$ be such that $\mu^N(y_0) = N \circ D$.

Assume first $(y_0, x_0) \in \mathrm{Ar}(D)$ and so $y_0 \in \pi^D(x_0)$. Let $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(y_0, x_0)\}) \cup \{(x_0, y_0)\}.$$

By Lemma 18, $E \in \mathcal{A}_{x_0}(X)$. We get $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(y_0, x_0) + c^N(x_0, y_0) \geq N \circ D = \mu^N(y_0)$, as desired.

Assume now $(y_0, x_0) \notin \mathrm{Ar}(D)$. Consider the path $(x_j^*)_{j=1}^m$ from y_0 to x_0 in D. Since $(y_0, x_0) \notin \mathrm{Ar}(D)$, it must be $m \geq 3$. Let $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = \big(\mathrm{Ar}(D) \setminus \{(y_0, x_2^*), (x_{m-1}^*, x_0)\}\big) \cup \{(x_0, x_2^*), (x_{m-1}^*, y_0)\}.$$

By Lemma 19, we know that $E \in \mathcal{A}_{x_0}(X)$. Thus, $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(y_0, x_2^*) - c^N(x_{m-1}^*, x_0) + c^N(x_0, x_2^*) + c^N(x_{m-1}^*, y_0) \geq N \circ D = \mu^N(y_0)$, as desired.

(ii) We prove that $y_0 \notin \mathscr{A}(N)$ showing that $\mu^N(x_0) > \mu^N(y_0)$. Let $D \in \mathcal{A}_{y_0}(X)$ be such that $\mu^N(y_0) = N \circ D$.

Assume first $(y_0, x_0) \in \mathrm{Ar}(D)$ and so $y_0 \in \pi^D(x_0)$. Let $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(y_0, x_0)\}) \cup \{(x_0, y_0)\}.$$

By Lemma 18, $E \in \mathcal{A}_{x_0}(X)$. We get $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(y_0, x_0) + c^N(x_0, y_0) > N \circ D = \mu^N(y_0)$, as desired.

Assume now $(y_0, x_0) \notin \mathrm{Ar}(D)$. Consider the path $(x_j^*)_{j=1}^m$ from y_0 to x_0 in D. Since $(y_0, x_0) \notin \mathrm{Ar}(D)$, it must be $m \geq 3$. Let $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = \big(\mathrm{Ar}(D) \setminus \{(y_0, x_2^*), (x_{m-1}^*, x_0)\}\big) \cup \{(x_0, x_2^*), (x_{m-1}^*, y_0)\}.$$

By Lemma 19, we know that $E \in \mathcal{A}_{x_0}(X)$. Thus, $\mu^N(x_0) \geq N \circ E = (N \circ D) - c^N(y_0, x_2^*) - c^N(x_{m-1}^*, x_0) + c^N(x_0, x_2^*) + c^N(x_{m-1}^*, y_0) > N \circ D = \mu^N(y_0)$, as desired.

(iii) The proof is analogous to that of (ii). $\square$

Proof of Proposition 6. Let $X = \mathrm{Ve}(N)$.

(i) Since N is balanced, we also have that, for every $z \in X \setminus \{x_0, y_0\}$, $c^N(z, y_0) \geq c^N(z, x_0)$. As a consequence, we can apply Proposition 5(i).

(ii) Since N is balanced, we also have that, for every $z \in X \setminus \{x_0, y_0\}$, $c^N(z, y_0) > c^N(z, x_0)$. As a consequence, we can apply Proposition 5(ii). $\square$

Proof of Proposition 7. Let $X = \mathrm{Ve}(N)$.

(i) Since $\mathscr{A}(N) \neq \varnothing$, there exists $y_0 \in \mathscr{A}(N)$. If $x_0 = y_0$, then we conclude. Assume then $x_0 \neq y_0$. By assumption, we know that $c^N(x_0, y_0) \geq c^N(z, x_0)$ for all $z \in X \setminus \{x_0\}$. By Proposition 4(i), we deduce $x_0 \in \mathscr{A}(N)$.

(ii) By (i), we know that $x_0 \in \mathscr{A}(N)$. Fix then $y_0 \in X$ with $y_0 \neq x_0$. By assumption, we know that $c^N(x_0, y_0) > c^N(z, x_0)$ for all $z \in X \setminus \{x_0\}$. By Proposition 4(ii), we then deduce $y_0 \notin \mathscr{A}(N)$. We then conclude that $\mathscr{A}(N) = \{x_0\}$.

(iii) Assume by contradiction that $x_0 \in \mathscr{A}(N)$. Thus, $c^N(z, x_0) > c^N(x_0, z)$ for all $z \in X \setminus \{x_0\}$. Consider $D \in \mathcal{A}_{x_0}(X)$ such that $\mu^N(x_0) = N \circ D$. There exists $y_0 \in X \setminus \{x_0\}$ such that $(x_0, y_0) \in \mathrm{Ar}(D)$ and so $x_0 \in \pi^D(y_0)$. Let $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(x_0, y_0)\}) \cup \{(y_0, x_0)\}.$$

By Lemma 18, $E \in \mathcal{A}_{y_0}(X)$. We have that

$$\mu^N(y_0) \geq N \circ E = N \circ D - c^N(x_0, y_0) + c^N(y_0, x_0) > N \circ D = \mu^N(x_0),$$

a contradiction. We then conclude that $x_0 \notin \mathscr{A}(N)$. $\square$

Proof of Proposition 8. Let $X = \mathrm{Ve}(N)$. Since N is balanced, there exists $k \in \mathbb{R}$ such that, for every $(x, y) \in X^2_*$, $c^N(x, y) + c^N(y, x) = k$.

(i) Let $y \in X$ with $y \neq x_0$. We know that $c^N(x_0, y) \geq c^N(y, x_0) = k - c^N(x_0, y)$, that is, $c^N(x_0, y) \geq \frac{k}{2}$. As a consequence, $c^N(y, x_0) \leq \frac{k}{2}$. We conclude that

$$\min\{c^N(x_0, y) : y \in X \setminus \{x_0\}\} \geq \tfrac{k}{2} \geq \max\{c^N(y, x_0) : y \in X \setminus \{x_0\}\}.$$

Thus, we can apply Proposition 7(i) and deduce that $x_0 \in \mathscr{A}(N)$.

(ii) Let $y \in X$ with $y \neq x_0$. We know that $c^N(x_0, y) > c^N(y, x_0) = k - c^N(x_0, y)$, that is, $c^N(x_0, y) > \frac{k}{2}$. As a consequence, $c^N(y, x_0) < \frac{k}{2}$. We conclude that

$$\min\{c^N(x_0, y) : y \in X \setminus \{x_0\}\} > \tfrac{k}{2} > \max\{c^N(y, x_0) : y \in X \setminus \{x_0\}\}.$$

Thus, we can apply Proposition 7(ii) and deduce that $\mathscr{A}(N) = \{x_0\}$.

(iii) Let $y \in X$ with $y \neq x_0$. We know that $c^N(x_0, y) < c^N(y, x_0) = k - c^N(x_0, y)$, that is, $c^N(x_0, y) < \frac{k}{2}$. As a consequence, $c^N(y, x_0) > \frac{k}{2}$. Thus,

$$\min\{c^N(y, x_0) : y \in X \setminus \{x_0\}\} > \tfrac{k}{2} > \max\{c^N(x_0, y) : y \in X \setminus \{x_0\}\}.$$

Thus, we can apply Proposition 7(iii) and deduce that $x_0 \notin \mathscr{A}(N)$. □

Proof of Proposition 9. Let $X = \mathrm{Ve}(N)$. Since N is balanced, there exists $k \in \mathbb{R}$ such that, for every $(x, y) \in X^2_*$, $c^N(x, y) + c^N(y, x) = k$.

Assume by contradiction that there exists $y_0 \in X \setminus S$ such that $y_0 \in \mathscr{A}(N)$. Let $D \in \mathcal{A}_{y_0}(X)$ be such that $N \circ D = \mu^N(y_0)$. Pick $u_0 \in S$. There exists a path $(x_j)_{j=1}^m$ in D from y_0 to u_0. Then there exists $j_0 \in \{1, \ldots, m-1\}$ such that $x_{j_0} \notin S$ and $x_{j_0+1} \in S$. Set $x_0 = x_{j_0+1}$ and $z_0 = x_{j_0} \in \pi^D(x_0)$. Observe that $x_0 \neq y_0$. We know that $c^N(x_0, z_0) > c^N(z_0, x_0)$ and $c^N(x_0, y_0) > c^N(y_0, x_0)$, that is, $c^N(z_0, x_0) < \frac{k}{2}$ and $c^N(x_0, y_0) > \frac{k}{2}$. Let then $E \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(E) = (\mathrm{Ar}(D) \setminus \{(z_0, x_0)\}) \cup \{(x_0, y_0)\}.$$

By Lemma 18, $E \in \mathcal{A}_{x_0}(X)$. We have that $N \circ E = N \circ D - c^N(z_0, x_0) + c^N(x_0, y_0) > N \circ D = \mu^N(y_0)$. Thus, $\mu^N(x_0) > \mu^N(y_0)$ and so $y_0 \notin \mathscr{A}(N)$, a contradiction. □

Proof of Proposition 11. Let $X = \mathrm{Ve}(N)$. Since $\mathscr{A}(N) \neq \varnothing$, there exist $y_0 \in \mathscr{A}(N)$ and $D \in \mathcal{A}^N$ such that $r(D) = y_0$.

Let $x_0 \in X$. We prove that $x_0 \in \mathscr{A}(N)$. If $x_0 = y_0$, then $x_0 \in \mathscr{A}(N)$. Assume that $x_0 \neq y_0$, and let $(v_j)_{j=1}^m$ be the unique path in D from $v_1 = y_0$ to $v_m = x_0$. In particular, $m \geq 2$. Let $D' \in \mathcal{D}(X)$ be such that

$$\mathrm{Ar}(D') = \Big(\mathrm{Ar}(D) \setminus \big\{(v_j, v_{j+1}) : j \in \{1, \ldots, m-1\}\big\}\Big) \cup \big\{(v_{j+1}, v_j) : j \in \{1, \ldots, m-1\}\big\}.$$

Let us prove that $D' \in \mathcal{A}_{x_0}(X)$. The unique arc entering x_0 in D is $(v_{m-1}, v_m) = (v_{m-1}, x_0)$. Thus, by construction, $\pi^{D'}(x_0) = \varnothing$. Consider now $y \in X \setminus \{x_0\}$. If $y = y_0$, then no arc enters y_0 in D, whereas $(v_2, v_1) = (v_2, y_0)$ is the unique arc entering y_0 in D'. Hence $\pi^{D'}(y_0) = \{v_2\}$. If $m \geq 3$ and $y = v_j$ for some $j \in \{2, \ldots, m-1\}$, then the unique arc entering v_j in D is (v_{j-1}, v_j), whereas (v_{j+1}, v_j) is the unique arc entering v_j in D'. Hence $\pi^{D'}(v_j) = \{v_{j+1}\}$. Finally, if $y \in X \setminus \{v_1, \ldots, v_m\}$, then $\pi^{D'}(y) = \pi^D(y)$. Therefore, every vertex of $X \setminus \{x_0\}$ has a unique entering arc in D'.

It remains to prove that every vertex of $X \setminus \{x_0\}$ is reachable from x_0 in D'. Assume first that $y \in \{v_1, \ldots, v_{m-1}\}$. Then $y = v_j$ for a unique $j \in \{1, \ldots, m-1\}$. For every $t \in \{j, \ldots, m-1\}$, the arc (v_{t+1}, v_t) belongs to $\mathrm{Ar}(D')$ by construction. Since $v_j, \ldots, v_m$ are distinct, it follows that

$$(v_m, v_{m-1}, \ldots, v_j)$$

is a path in D' from $x_0 = v_m$ to $y = v_j$. In particular, if $m = 2$, then necessarily $j = 1$, $y = v_1 = y_0$, and the displayed path is simply $(v_2, v_1) = (x_0, y_0)$.

Assume now that $y \in X \setminus \{v_1, \dots, v_m\}$, and let $(z_i)_{i=1}^{\ell}$ be the unique path in D from $y_0 = z_1$ to $y = z_\ell$. Since $z_1 = v_1$, the set

$$\big\{i \in \{1, \dots, \ell\} : z_i \in \{v_1, \dots, v_m\}\big\}$$

is nonempty. Set

$$i_0 = \max\big\{i \in \{1, \dots, \ell\} : z_i \in \{v_1, \dots, v_m\}\big\},$$

and let $k \in \{1, \dots, m\}$ be such that $z_{i_0} = v_k$. Since $y \notin \{v_1, \dots, v_m\}$, we have $i_0 < \ell$, so that z_{i_0+1} is well defined. By the definition of i_0, $z_{i_0+1}, \dots, z_\ell \notin \{v_1, \dots, v_m\}$. Since $(z_i)_{i=1}^{\ell}$ is a path in D, we have $(v_k, z_{i_0+1}) = (z_{i_0}, z_{i_0+1}) \in \mathrm{Ar}(D)$. This arc does not belong to

$$\big\{(v_t, v_{t+1}) : t \in \{1, \dots, m-1\}\big\},$$

because z_{i_0+1} does not belong to $\{v_1, \dots, v_m\}$. Hence $(v_k, z_{i_0+1}) \in \mathrm{Ar}(D')$. For the same reason, all the arcs between consecutive vertices among $z_{i_0+1}, \dots, z_\ell$ belong to $\mathrm{Ar}(D')$. If $i_0 + 1 = \ell$, there are no such additional arcs, and this assertion is vacuous. If $k < m$, then, by the construction of D',

$$(v_m, v_{m-1}, \dots, v_k, z_{i_0+1}, \dots, z_\ell)$$

is a path in D' from x_0 to y. If $k = m$, the corresponding path is

$$(v_m, z_{i_0+1}, \dots, z_\ell).$$

Indeed, in either case, the vertices in the part coming from $(v_j)_{j=1}^{m}$ are distinct, the vertices $z_{i_0+1}, \dots, z_\ell$ are distinct, and no vertex of the first part belongs to the second part.

Since $N = N^r$, we have that, for every $j \in \{1, \dots, m-1\}$, $c^N(v_j, v_{j+1}) = c^N(v_{j+1}, v_j)$. Therefore,

$$N \circ D' = N \circ D - \sum_{j=1}^{m-1} c^N(v_j, v_{j+1}) + \sum_{j=1}^{m-1} c^N(v_{j+1}, v_j) = N \circ D.$$

Since $D \in \mathcal{A}^N$, it follows that $D' \in \mathcal{A}^N$. Since $r(D') = x_0$, we conclude that $x_0 \in \mathscr{A}(N)$. As x_0 was arbitrary, $\mathscr{A}(N) = X$. $\square$

Proof of Proposition 13. Let $N \in \mathcal{N}$ be balanced with constant $k \in \mathbb{R}$ and let $X = \mathrm{Ve}(N)$. We know that $\mathscr{M}^*(N) = \mathscr{M}_*(N)$.

Let $a = \max\{c^N(x, y) : (x, y) \in X_*^2\}$ and let $(x_0, y_0) \in X_*^2$ such that $c^N(x_0, y_0) = a$. Let $z_0 \in \mathbb{N}$ be the unique element of $X \setminus \{x_0, y_0\}$.

Of course, $a \geq \frac{k}{2}$. If $a = \frac{k}{2}$, then $c^N(x, y) = \frac{k}{2}$ for all $(x, y) \in X_*^2$. Then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0, y_0, z_0\}$. Assume then $a > \frac{k}{2}$. There are several cases to discuss. They are all listed below. The computations related to each case are simple and then omitted.

- If $c^N(y_0, z_0) = \frac{k}{2}$ and $c^N(z_0, x_0) = \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0, z_0\}$.
- If $c^N(y_0, z_0) = \frac{k}{2}$ and $c^N(z_0, x_0) > \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{z_0\}$.
- If $c^N(y_0, z_0) = \frac{k}{2}$ and $c^N(z_0, x_0) < \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$.
- If $c^N(y_0, z_0) > \frac{k}{2}$ and $c^N(z_0, x_0) = \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$.
- If $c^N(y_0, z_0) > \frac{k}{2}$ and $c^N(z_0, x_0) > \frac{k}{2}$, then there are six subcases:
  - if $a = c^N(y_0, z_0) = c^N(z_0, x_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0, y_0, z_0\}$,
  - if $a = c^N(y_0, z_0) > c^N(z_0, x_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$,
  - if $a = c^N(z_0, x_0) > c^N(y_0, z_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{z_0\}$,
  - if $a > c^N(y_0, z_0) = c^N(z_0, x_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0, z_0\}$,
  - if $a > c^N(y_0, z_0) > c^N(z_0, x_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$,

– if $a > c^N(z_0, x_0) > c^N(y_0, z_0)$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{z_0\}$.

- If $c^N(y_0, z_0) > \frac{k}{2}$ and $c^N(z_0, x_0) < \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$.
- If $c^N(y_0, z_0) < \frac{k}{2}$ and $c^N(z_0, x_0) = \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0, z_0\}$.
- If $c^N(y_0, z_0) < \frac{k}{2}$ and $c^N(z_0, x_0) > \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{z_0\}$.
- If $c^N(y_0, z_0) < \frac{k}{2}$ and $c^N(z_0, x_0) < \frac{k}{2}$, then $\mathscr{A}(N) = \mathscr{M}^*(N) = \mathscr{S}(N) = \{x_0\}$.

□

A.2 Proof of Theorem 10

Lemma 20. *Let $X \in P_f(\mathbb{N})$, $x_0, y_0, z_0 \in X$, $D \in \mathcal{A}_{x_0}(X)$, and $E \in \mathcal{A}_{y_0}(X)$ with $(x_0, z_0) \in \mathrm{Ar}(E)$. Then there exist $D' \in \mathcal{A}_{x_0}(X)$ and $E' \in \mathcal{A}_{y_0}(X)$ such that $(x_0, z_0) \in \mathrm{Ar}(D')$ and, for every $N \in \mathcal{N}$ with* $\mathrm{Ve}(N) = X$*,*

$$N \circ D' + N \circ E' = N \circ D + N \circ E.$$

Proof. If $(x_0, z_0) \in \mathrm{Ar}(D)$, it is sufficient to set $D' = D$ and $E' = E$. If $y_0 = x_0$, it is sufficient to set $D' = E$ and $E' = D$. Assume then that $y_0 \neq x_0$ and $(x_0, z_0) \notin \mathrm{Ar}(D)$. Since $(x_0, z_0) \in \mathrm{Ar}(E)$, we have that $x_0 \neq z_0$. Moreover, $y_0 \neq z_0$, since $y_0 = r(E)$ and no arc enters $r(E)$. Thus, x_0, y_0, z_0 are distinct.

Let

$$S = \{z_0\} \cup \delta^E(z_0).$$

We have $x_0 \notin S$. Indeed, assume, by contradiction, that $x_0 \in S$. Since $x_0 \neq z_0$, there exists a path in E from z_0 to x_0. This path, together with the arc (x_0, z_0), determines a cycle in E, a contradiction. We also have $y_0 \notin S$. Indeed, assume, by contradiction, that $y_0 \in S$. Since $y_0 \neq z_0$, there exists a path in E from z_0 to y_0. In particular, there exists an arc of E entering y_0. This is a contradiction since y_0 is the root of E.

Consider $(u, v) \in \mathrm{Ar}(E)$ with $u \notin S$ and $v \in S$. If $v \neq z_0$, then the unique arc entering v in E belongs to the path from z_0 to v, and hence $u \in S$, a contradiction. Thus, $v = z_0$ and, since (x_0, z_0) is the unique arc entering z_0 in E, we have that $(u, v) = (x_0, z_0)$. Furthermore, if $(u, v) \in \mathrm{Ar}(E)$ and $u \in S$, then $v \in S$, by the definition of S. We have therefore proved that

$$\begin{aligned} &\{(u, v) \in \mathrm{Ar}(E) : u \notin S,\ v \in S\} = \{(x_0, z_0)\}, \\ &\{(u, v) \in \mathrm{Ar}(E) : u \in S,\ v \notin S\} = \varnothing. \end{aligned} \tag{5}$$

Let $D', E' \in \mathcal{D}(X)$ be defined by

$$\begin{aligned} \mathrm{Ar}(D') &= \{(u, v) \in \mathrm{Ar}(E) : v \in S\} \cup \{(u, v) \in \mathrm{Ar}(D) : v \notin S\}, \\ \mathrm{Ar}(E') &= \{(u, v) \in \mathrm{Ar}(D) : v \in S\} \cup \{(u, v) \in \mathrm{Ar}(E) : v \notin S\}. \end{aligned} \tag{6}$$

Let us prove that $D' \in \mathcal{A}_{x_0}(X)$. Since $x_0 \notin S$ and $D \in \mathcal{A}_{x_0}(X)$, we have that $\pi^{D'}(x_0) = \varnothing$. Moreover, for every $v \in X \setminus \{x_0\}$,

$$\pi^{D'}(v) = \begin{cases} \pi^E(v) & \text{if } v \in S, \\ \pi^D(v) & \text{if } v \notin S. \end{cases}$$

Therefore, $|\pi^{D'}(v)| = 1$ for every $v \in X \setminus \{x_0\}$. Observe also that, from (5) and (6), we have $(x_0, z_0) \in \mathrm{Ar}(D')$.

Let us prove that, for every $v \in X \setminus \{x_0\}$, there exists a path in D' from x_0 to v.

Assume first that $v \in S$. There exist $t \in \mathbb{N}$ and distinct vertices $w_1, \ldots, w_t \in S$ such that $w_1 = z_0$, $w_t = v$, and $(w_j, w_{j+1}) \in \mathrm{Ar}(E)$ for every $j \in \mathbb{N}$ with $j < t$. If $v = z_0$, take $t = 1$. For every $j \in \mathbb{N}$ with $j < t$, we have that $w_{j+1} \in S$, and hence (6) implies that $(w_j, w_{j+1}) \in \mathrm{Ar}(D')$. Since $x_0 \notin S$ and $(x_0, z_0) \in \mathrm{Ar}(D')$, it follows that

$$(x_0, w_1, \ldots, w_t)$$

is a path in D' from x_0 to v.

Assume now that $v \notin S$, and let $(v_j)_{j=1}^m$ be a path in D from $v_1 = x_0$ to $v_m = v$. If

$$\{v_1, \dots, v_m\} \cap S = \varnothing,$$

then $v_{j+1} \notin S$ for every $j \in \{1, \dots, m-1\}$, and (6) implies that $(v_j)_{j=1}^m$ is a path in D' from x_0 to v. Assume instead that $\{v_1, \dots, v_m\} \cap S \neq \varnothing$. Since $v_1 = x_0 \notin S$ and $v_m = v \notin S$, we have that $m \geq 3$. Set

$$k = \max\{j \in \{1, \dots, m\} : v_j \in S\},$$

and note that $1 < k < m$. There exist $t \in \mathbb{N}$ and distinct vertices $w_1, \dots, w_t \in S$ such that $w_1 = z_0$, $w_t = v_k$, and $(w_j, w_{j+1}) \in \mathrm{Ar}(E)$ for every $j \in \mathbb{N}$ with $j < t$. If $v_k = z_0$, take $t = 1$. By the definition of k, $v_{k+1}, \dots, v_m \notin S$. Therefore,

$$(x_0, w_1, \dots, w_t, v_{k+1}, \dots, v_m)$$

is a sequence of distinct vertices. Indeed, $w_1, \dots, w_t \in S$, whereas $x_0, v_{k+1}, \dots, v_m \notin S$; moreover, $x_0, v_{k+1}, \dots, v_m$ are distinct because they belong to the path $(v_j)_{j=1}^m$ in D.

We have that $(x_0, w_1) = (x_0, z_0) \in \mathrm{Ar}(D')$. Moreover, $(w_j, w_{j+1}) \in \mathrm{Ar}(D')$ for every $j \in \mathbb{N}$ with $j < t$, since $(w_j, w_{j+1}) \in \mathrm{Ar}(E)$ and $w_{j+1} \in S$. Furthermore,

$$(w_t, v_{k+1}) = (v_k, v_{k+1}) \in \mathrm{Ar}(D'),$$

since $(v_k, v_{k+1}) \in \mathrm{Ar}(D)$ and $v_{k+1} \notin S$. Finally, $(v_j, v_{j+1}) \in \mathrm{Ar}(D')$ for every $j \in \mathbb{N}$ with $k+1 \leq j < m$, since $(v_j, v_{j+1}) \in \mathrm{Ar}(D)$ and $v_{j+1} \notin S$. Thus,

$$(x_0, w_1, \dots, w_t, v_{k+1}, \dots, v_m)$$

is a path in D' from x_0 to v.

That completes the proof that $D' \in \mathcal{A}_{x_0}(X)$. As already observed, $(x_0, z_0) \in \mathrm{Ar}(D')$.

Let us now prove that $E' \in \mathcal{A}_{y_0}(X)$. Since $y_0 \notin S$ and $E \in \mathcal{A}_{y_0}(X)$, we have that $\pi^{E'}(y_0) = \varnothing$. Moreover, for every $v \in X \setminus \{y_0\}$,

$$\pi^{E'}(v) = \begin{cases} \pi^D(v) & \text{if } v \in S, \\ \pi^E(v) & \text{if } v \notin S. \end{cases}$$

Therefore, $|\pi^{E'}(v)| = 1$ for every $v \in X \setminus \{y_0\}$.

Let us prove that, for every $v \in X \setminus \{y_0\}$, there exists a path in E' from y_0 to v. Assume first that $v \notin S$, and let $(v_j)_{j=1}^m$ be a path in E from $v_1 = y_0$ to $v_m = v$. We have that $v_j \notin S$ for every $j \in \{1, \dots, m\}$. Indeed, if $v_j \in S$ for some j, the second equality in (5) would imply that $v_m \in S$, a contradiction. Thus, $v_{j+1} \notin S$ for every $j \in \{1, \dots, m-1\}$, and (6) implies that $(v_j)_{j=1}^m$ is a path in E' from y_0 to v.

Assume now that $v \in S$, and let $(v_j)_{j=1}^m$ be a path in D from $v_1 = x_0$ to $v_m = v$. Set

$$k = \max\{j \in \{1, \dots, m\} : v_j \notin S\}.$$

This maximum is well defined, since $v_1 = x_0 \notin S$, and $k < m$, since $v_m = v \in S$. By the definition of k, $v_{k+1}, \dots, v_m \in S$. There exist $t \in \mathbb{N}$ and distinct vertices $w_1, \dots, w_t \in X \setminus S$ such that $w_1 = y_0$, $w_t = v_k$, and $(w_j, w_{j+1}) \in \mathrm{Ar}(E)$ for every $j \in \mathbb{N}$ with $j < t$. If $v_k = y_0$, take $t = 1$. Indeed, a path in E from y_0 to $v_k \notin S$ cannot meet S, by the second equality in (5). Therefore,

$$(w_1, \dots, w_t, v_{k+1}, \dots, v_m)$$

is a sequence of distinct vertices. Indeed, $w_1, \dots, w_t \in X \setminus S$, whereas $v_{k+1}, \dots, v_m \in S$; moreover, the vertices in each of the two sequences are distinct, the first by construction and the second because $(v_j)_{j=1}^m$ is a path in D.

For every $j \in \mathbb{N}$ with $j < t$, we have that $(w_j, w_{j+1}) \in \mathrm{Ar}(E')$, since $(w_j, w_{j+1}) \in \mathrm{Ar}(E)$ and $w_{j+1} \notin S$. Moreover,

$$(w_t, v_{k+1}) = (v_k, v_{k+1}) \in \mathrm{Ar}(E'),$$

since $(v_k, v_{k+1}) \in \mathrm{Ar}(D)$ and $v_{k+1} \in S$. Finally, $(v_j, v_{j+1}) \in \mathrm{Ar}(E')$ for every $j \in \mathbb{N}$ with $k+1 \leq j < m$, since $(v_j, v_{j+1}) \in \mathrm{Ar}(D)$ and $v_{j+1} \in S$. Thus,

$$(w_1, \dots, w_t, v_{k+1}, \dots, v_m)$$

is a path in E' from y_0 to v.

This completes the proof that $E' \in \mathcal{A}_{y_0}(X)$.

It remains to prove the last part of the statement. Let $N \in \mathcal{N}$ be such that $\mathrm{Ve}(N) = X$. By (6) and the definition of $N \circ D$, we have

$$\begin{aligned} N \circ D' + N \circ E' &= \sum_{\substack{(u,v)\in \mathrm{Ar}(E)\\ v\in S}} c^N(u,v) + \sum_{\substack{(u,v)\in \mathrm{Ar}(D)\\ v\notin S}} c^N(u,v) + \sum_{\substack{(u,v)\in \mathrm{Ar}(D)\\ v\in S}} c^N(u,v) + \sum_{\substack{(u,v)\in \mathrm{Ar}(E)\\ v\notin S}} c^N(u,v) \\ &= \sum_{(u,v)\in \mathrm{Ar}(D)} c^N(u,v) + \sum_{(u,v)\in \mathrm{Ar}(E)} c^N(u,v) = N \circ D + N \circ E. \end{aligned}$$

□

Proposition 21. *Let $N, M \in \mathcal{N}$ be such that $\mathrm{Ve}(N) = \mathrm{Ve}(M)$. Let $x_0 \in \mathscr{A}(N)$ and $z_0 \in \mathrm{Ve}(N) \setminus \{x_0\}$. Assume that $c^M(x_0, z_0) \geq c^N(x_0, z_0)$, and that $c^M(x, y) = c^N(x, y)$ for every $(x, y) \in (\mathrm{Ve}(N))^2_* \setminus \{(x_0, z_0)\}$. Then $x_0 \in \mathscr{A}(M)$.*

Proof. Set $X = \mathrm{Ve}(N)$ and $\Delta = c^M(x_0, z_0) - c^N(x_0, z_0)$. Of course, $\Delta \geq 0$. Let

$$Z = \max\{N \circ F : F \in \mathcal{A}(X)\}$$

and

$$\nu = \max\{N \circ F : F \in \mathcal{A}(X),\ (x_0, z_0) \in \mathrm{Ar}(F)\}.$$

The second maximum is well defined since $F_0 \in \mathcal{D}(X)$ such that

$$\mathrm{Ar}(F_0) = \{(x_0, x) : x \in X \setminus \{x_0\}\}$$

belongs to $\mathcal{A}_{x_0}(X)$ and contains (x_0, z_0).

Since $x_0 \in \mathscr{A}(N)$, there exists $D \in \mathcal{A}_{x_0}(X)$ such that $N \circ D = Z$. Let $E \in \mathcal{A}(X)$ be such that $(x_0, z_0) \in \mathrm{Ar}(E)$ and $N \circ E = \nu$.

By Lemma 20, there exist $D' \in \mathcal{A}_{x_0}(X)$ and $E' \in \mathcal{A}_{r(E)}(X)$ such that $(x_0, z_0) \in \mathrm{Ar}(D')$ and

$$N \circ D' + N \circ E' = N \circ D + N \circ E. \tag{7}$$

Since D is a maximum-weight arborescence in N, we have that $N \circ D \geq N \circ E'$. By (7),

$$N \circ D' = (N \circ D - N \circ E') + N \circ E \geq N \circ E = \nu.$$

On the other hand, $(x_0, z_0) \in \mathrm{Ar}(D')$, and hence the definition of ν implies that $N \circ D' \leq \nu$. We conclude that $N \circ D' = \nu$.

For every $F \in \mathcal{A}(X)$, we have

$$M \circ F = \begin{cases} N \circ F + \Delta & \text{if } (x_0, z_0) \in \mathrm{Ar}(F), \\ N \circ F & \text{if } (x_0, z_0) \notin \mathrm{Ar}(F). \end{cases} \tag{8}$$

From the definitions of Z and ν, we deduce that, for every $F \in \mathcal{A}(X)$,

$$M \circ F \leq \max\{Z, \nu + \Delta\}. \tag{9}$$

Moreover, since $N \circ D = Z$, $N \circ D' = \nu$ and (8), we get $M \circ D \geq Z$ and $M \circ D' = \nu + \Delta$. Consequently,

$$\max\{M \circ D, M \circ D'\} \geq \max\{Z, \nu + \Delta\}.$$

Combining this inequality with (9), we deduce that either D or D' is a maximum-weight arborescence in M. Since both D and D' are rooted at x_0, we conclude that $x_0 \in \mathscr{A}(M)$. □

Proposition 22. *Let $N, M \in \mathcal{N}$ be such that* $\mathrm{Ve}(N) = \mathrm{Ve}(M)$*. Let* $x_0 \in \mathscr{A}(N)$ *and* $z_0 \in \mathrm{Ve}(N) \setminus \{x_0\}$*. Assume that* $c^N(z_0, x_0) \geq c^M(z_0, x_0)$*, and that* $c^M(x, y) = c^N(x, y)$ *for every* $(x, y) \in (\mathrm{Ve}(N))^2_* \setminus \{(z_0, x_0)\}$*. Then* $x_0 \in \mathscr{A}(M)$*.*

Proof. Let $X = \mathrm{Ve}(N)$. For every $D \in \mathcal{A}(X)$, we have that $N \circ D \geq M \circ D$, with equality if $(z_0, x_0) \notin \mathrm{Ar}(D)$.

For every $x \in X$, $\mu^N(x) \geq \mu^M(x)$. Indeed, fix $x \in X$ and let $D \in \mathcal{A}_x(X)$ be such that $M \circ D = \mu^M(x)$. Thus,

$$\mu^N(x) \geq N \circ D \geq M \circ D = \mu^M(x).$$

Consider now $D \in \mathcal{A}_{x_0}(X)$ such that $N \circ D = \mu^N(x_0)$. Since no arc enters the root x_0 in D, we have that $(z_0, x_0) \notin \mathrm{Ar}(D)$. Thus,

$$\mu^M(x_0) \geq M \circ D = N \circ D = \mu^N(x_0).$$

Since $x_0 \in \mathscr{A}(N)$, we have that, for every $x \in X$, $\mu^N(x_0) \geq \mu^N(x)$. Therefore, for every $x \in X$,

$$\mu^M(x_0) \geq \mu^N(x_0) \geq \mu^N(x) \geq \mu^M(x).$$

We then conclude that $x_0 \in \mathscr{A}(M)$. □

Proposition 23. *Let $N, M \in \mathcal{N}$ be such that* $\mathrm{Ve}(N) = \mathrm{Ve}(M)$*. Let* $x_0 \in \mathscr{A}(N)$ *and* $z_0 \in \mathrm{Ve}(N) \setminus \{x_0\}$*. Assume that* $c^M(x_0, z_0) \geq c^N(x_0, z_0)$*,* $c^N(z_0, x_0) \geq c^M(z_0, x_0)$*, and that* $c^N(x, y) = c^M(x, y)$ *for all* $(x, y) \in (\mathrm{Ve}(N))^2_* \setminus \{(x_0, z_0), (z_0, x_0)\}$*. Then* $x_0 \in \mathscr{A}(M)$*.*

Proof. Let $N' \in \mathcal{N}$ be such that $\mathrm{Ve}(N') = \mathrm{Ve}(N)$, $c^{N'}(x_0, z_0) = c^M(x_0, z_0)$, and, for every $(x, y) \in (\mathrm{Ve}(N))^2_* \setminus \{(x_0, z_0)\}$, $c^{N'}(x, y) = c^N(x, y)$. Since $c^{N'}(x_0, z_0) \geq c^N(x_0, z_0)$, Proposition 21, applied to N and N', implies that $x_0 \in \mathscr{A}(N')$.

Moreover, N' and M differ only in the capacity of (z_0, x_0), and $c^{N'}(z_0, x_0) = c^N(z_0, x_0) \geq c^M(z_0, x_0)$. Proposition 22, applied to N' and M, implies that $x_0 \in \mathscr{A}(M)$. □

Proof of Theorem 10. If $\mathrm{Ve}(N) = \{x_0\}$, then $N = M$, and the conclusion is immediate. Assume therefore that $\mathrm{Ve}(N) \setminus \{x_0\} \neq \varnothing$, and let $z_1, \ldots, z_k$ be an enumeration of its elements, where $k \geq 1$. Consider the sequence $(N_j)_{j=0}^k$ of networks in $\mathcal{N}(\mathrm{Ve}(N))$ recursively defined as follows: $N_0 = N$ and, for every $j \in \{1, \ldots, k\}$,

- $c^{N_j}(x_0, z_j) = c^M(x_0, z_j)$ and $c^{N_j}(z_j, x_0) = c^M(z_j, x_0)$,
- for every $(x, y) \in (\mathrm{Ve}(N))^2_* \setminus \{(x_0, z_j), (z_j, x_0)\}$, $c^{N_j}(x, y) = c^{N_{j-1}}(x, y)$.

For every $j \in \{0, \ldots, k-1\}$, we can apply Proposition 23 to the networks N_j and N_{j+1} and deduce that, if $x_0 \in \mathscr{A}(N_j)$, then $x_0 \in \mathscr{A}(N_{j+1})$. Since $x_0 \in \mathscr{A}(N_0) = \mathscr{A}(N)$, we therefore conclude that $x_0 \in \mathscr{A}(N_k) = \mathscr{A}(M)$. □

A.3 Proof of Theorem 12

Proof of Theorem 12. Set $O = \mathrm{Ve}(N) \setminus \{u_0\}$ and $K = C \cup \{u_0\}$. Thus, we have that

$$\begin{aligned} &\text{for every } (x, y) \in O^2_*,\ c^M(x, y) = c^N(x, y), \\ &\text{for every } d \in K \text{ and } x \in O,\ c^M(d, x) = c^N(u_0, x), \\ &\text{for every } d \in K \text{ and } x \in O,\ c^M(x, d) = c^N(x, u_0). \end{aligned} \tag{10}$$

If $O = \varnothing$, then $\mathrm{Ve}(N) = \{u_0\}$ and hence $\mathscr{A}(N) = \{u_0\}$. Since $\mathrm{Ve}(M) = K$ and $\mathscr{A}(M) \neq \varnothing$, part (ii) of Theorem 12 follows, while part (i) of Theorem 12 is vacuous.

Assume henceforth that $O \neq \varnothing$. Set

$$h = \max\{c^N(x, u_0) : x \in O\}.$$

Let $s \in \mathbb{N} \setminus \mathrm{Ve}(M)$ and set

$$\kappa = \max_{B \in \mathcal{A}_s(K \cup \{s\})} \left\{ \sum_{(c,d) \in \mathrm{Ar}(B) \cap K_*^2} c^M(c,d) + \big(|\gamma^B(s)| - 1\big) h \right\}. \tag{11}$$

First part. We first prove that

$$\text{for every } x \in O,\ \mu^M(x) \geq \mu^N(x) + \kappa, \tag{12}$$

and that

$$\max_{x \in K} \mu^M(x) \geq \mu^N(u_0) + \kappa. \tag{13}$$

Let us introduce some objects that will be used in both proofs.

Let $B \in \mathcal{A}_s(K \cup \{s\})$ attain the maximum in (11), and write

$$\gamma^B(s) = \{r_1, \dots, r_t\},$$

where $t \geq 1$ and $r_1, \dots, r_t$ are distinct elements of K. In what follows, a set or a sum or a union indexed by $j \in \{2, \dots, t\}$ is understood to be empty if $t = 1$. Number these vertices so that r_1 is the second vertex of the unique path in B from s to u_0. For every $j \in \{1, \dots, t\}$, set

$$K_j = \{r_j\} \cup \delta^B(r_j).$$

We prove that $K_1, \dots, K_t$ form a partition of K.

- For every $j \in \{1, \dots, t\}$, $r_j \in K_j$ and so $K_j \neq \varnothing$.
- Let $d \in K$, and consider the unique path in B from s to d. Its second vertex is an element r_j of $\gamma^B(s)$. If $d = r_j$, then $d \in K_j$. If $d \neq r_j$, the part of this path beginning at r_j is a path in B from r_j to d, and hence $d \in \delta^B(r_j) \subseteq K_j$. Therefore,

$$K = \bigcup_{j=1}^{t} K_j.$$

- Suppose now by contradiction that $K_i \cap K_j \neq \varnothing$ for some distinct $i, j \in \{1, \dots, t\}$, and let $d \in K_i \cap K_j$. If $d \notin \{r_i, r_j\}$, a path in B from r_i to d, together with the arc (s, r_i), determines a path in B from s to d. Similarly, a path in B from r_j to d, together with the arc (s, r_j), determines another path in B from s to d. These paths are distinct, since their second vertices are respectively r_i and r_j, a contradiction. If $d = r_i$, then $d \neq r_j$. Since $d \in K_j$, there exists a path in B from r_j to d. This path, together with the arc (s, r_j), determines a path in B from s to d distinct from the path (s, r_i), a contradiction. The case $d = r_j$ is symmetric. In every case, this contradicts the uniqueness of the path in B from s to d. Therefore, $K_i \cap K_j = \varnothing$ whenever $i \neq j$.

We record a further useful fact, namely,

$$\mathrm{Ar}(B) = \{(s, r_j) : j \in \{1, \dots, t\}\} \cup \bigcup_{j=1}^{t} \big(\mathrm{Ar}(B) \cap (K_j)_*^2\big). \tag{14}$$

The inclusion from right to left in (14) is immediate. To prove the opposite inclusion, let $(c, d) \in \mathrm{Ar}(B)$. If $c = s$, then $d \in \gamma^B(s)$, and hence $d = r_j$ for some $j \in \{1, \dots, t\}$. Thus,

$$(c, d) \in \{(s, r_j) : j \in \{1, \dots, t\}\}.$$

Assume now that $c \neq s$. Since no arc enters s in B, we have $c, d \in K$. Let $j \in \{1, \dots, t\}$ be the unique index such that $d \in K_j$. We have $d \neq r_j$, since the unique arc entering r_j in B is (s, r_j) and $c \neq s$. Hence there exists a path in B from r_j to d. Together with the arc (s, r_j), this path determines the unique path in B from s to d. Since (c, d) is the unique arc entering d in B, the last arc of this path is (c, d). Therefore, either $c = r_j$ or the part of the path from r_j to c is a path in B. In both cases,

$$c \in \{r_j\} \cup \delta^B(r_j) = K_j.$$

Consequently, $(c, d) \in \mathrm{Ar}(B) \cap (K_j)^2_*$, which proves the opposite inclusion in (14).

Let $z \in O$ be such that

$$c^N(z, u_0) = h.$$

Fix $x \in O$. We prove that $\mu^M(x) \geq \mu^N(x) + \kappa$ by constructing an arborescence $T \in \mathcal{A}_x(\mathrm{Ve}(M))$ such that $M \circ T = \mu^N(x) + \kappa$.

Let $H \in \mathcal{A}_x(\mathrm{Ve}(N))$ be such that $N \circ H = \mu^N(x)$. Since $x \neq u_0$, the vertex u_0 has a unique parent in H. Let $v_0 \in O$ be the unique element of $\pi^H(u_0)$. Define $T \in \mathcal{D}(\mathrm{Ve}(M))$ by

$$\begin{aligned} \mathrm{Ar}(T) = {} & \big(\mathrm{Ar}(H) \cap O^2_*\big) \cup \{(r_1, y) : (u_0, y) \in \mathrm{Ar}(H)\} \cup \{(v_0, r_1)\} \\ & \cup \big(\mathrm{Ar}(B) \cap K^2_*\big) \cup \{(z, r_j) : j \in \{2, \ldots, t\}\}. \end{aligned} \tag{15}$$

We prove that $T \in \mathcal{A}_x(\mathrm{Ve}(M))$. We verify separately the three conditions in the definition of an arborescence.

First, no arc enters x in T. Indeed, no arc of H enters x. Therefore, neither of the first two sets on the right-hand side of (15) contains an arc entering x. All the arcs in the remaining three sets have their terminal vertices in K, whereas $x \in O$ and $O \cap K = \varnothing$. Hence no arc enters x in T.

We now prove that every $y \in \mathrm{Ve}(M) \setminus \{x\}$ has a unique parent in T.

- Assume that $y \in O \setminus \{x\}$. Let q be the unique element of $\pi^H(y)$. Only the first two sets on the right-hand side of (15) contain arcs whose terminal vertices belong to O. If $q \in O$, then (q, y) belongs to $\mathrm{Ar}(H) \cap O^2_*$ and hence to $\mathrm{Ar}(T)$. In this case, no other arc enters y in T. If $q = u_0$, then $(r_1, y) \in \mathrm{Ar}(T)$, and no other arc enters y in T. Therefore, y has a unique parent in T, namely q if $q \in O$ and r_1 if $q = u_0$.

- Assume that $y = r_1$. The arc (v_0, r_1) belongs to $\mathrm{Ar}(T)$. The first two sets on the right-hand side of (15) contain only arcs whose terminal vertices belong to O. Moreover, the unique arc entering r_1 in B is (s, r_1), which does not belong to $\mathrm{Ar}(B) \cap K^2_*$. Finally, the last set in (15) contains only arcs entering $r_2, \ldots, r_t$. Therefore, $\pi^T(r_1) = \{v_0\}$.

- Assume that $y \in K \setminus \{r_1, \ldots, r_t\}$. Since $K_1, \ldots, K_t$ form a partition of K, there exists a unique $j \in \{1, \ldots, t\}$ such that $y \in K_j \setminus \{r_j\}$. The unique arc entering y in B has both vertices in K_j, by (14), and hence belongs to $\mathrm{Ar}(B) \cap K^2_* \subseteq \mathrm{Ar}(T)$. The first two sets on the right-hand side of (15) contain only arcs whose terminal vertices belong to O, while the third and fifth sets contain only arcs entering, respectively, r_1 and the vertices $r_2, \ldots, r_t$. Therefore, the unique arc entering y in B is also the unique arc entering y in T.

- Finally, assume that $y \in \{r_2, \ldots, r_t\}$. There exists a unique $j \in \{2, \ldots, t\}$ such that $y = r_j$, and $(z, r_j) \in \mathrm{Ar}(T)$. The first two sets on the right-hand side of (15) contain only arcs whose terminal vertices belong to O, while the third set contains only the arc (v_0, r_1). Moreover, the unique arc entering r_j in B is (s, r_j), which does not belong to $\mathrm{Ar}(B) \cap K^2_*$. The only arc in the last set of (15) entering r_j is (z, r_j). Therefore, $\pi^T(r_j) = \{z\}$.

The preceding cases exhaust $\mathrm{Ve}(M) \setminus \{x\}$. Hence every vertex distinct from x has a unique parent in T.

We next prove that, for every $y \in \mathrm{Ve}(M) \setminus \{x\}$, there exists a path in T from x to y.

- Let $y \in O \setminus \{x\}$. Consider a path $(x_i)_{i=1}^m$ in H from $x_1 = x$ to $x_m = y$. If this path does not contain u_0, then all its vertices belong to O and all its arcs belong to the first set on the right-hand side of (15). Hence it is a path in T.

Suppose now that the path contains u_0, and let k be the unique index such that $x_k = u_0$. Since $x, y \in O$, we have $1 < k < m$. Moreover, $x_{k-1} = v_0$, and

$$(x_1, \dots, x_{k-1}, r_1, x_{k+1}, \dots, x_m) \tag{16}$$

is a path in T from x to y. Indeed, $(x_{k-1}, r_1) = (v_0, r_1)$ belongs to the third set on the right-hand side of (15), (r_1, x_{k+1}) belongs to the second set, and all the remaining arcs belong to the first set. Furthermore, the vertices in (16) are distinct: the vertices x_i, with $i \neq k$, are distinct elements of O, whereas $r_1 \in K$ and $O \cap K = \varnothing$.

- Let $y = r_1$. Consider a path $(x_i)_{i=1}^m$ in H from $x_1 = x$ to $x_m = u_0$. Then $x_{m-1} = v_0$, and

$$(x_1, \dots, x_{m-1}, r_1) \tag{17}$$

is a path in T from x to r_1. Indeed, $x_1, \dots, x_{m-1} \in O$, the arcs between consecutive vertices among them belong to the first set on the right-hand side of (15), and the last arc (v_0, r_1) belongs to the third set. The vertices in (17) are distinct, since $r_1 \in K$ and $O \cap K = \varnothing$.

- Let $y \in K_1 \setminus \{r_1\}$. By the definition of K_1, there exists a path in B from r_1 to y. All the arcs of this path belong to $\mathrm{Ar}(B) \cap (K_1)^2_*$ and hence to $\mathrm{Ar}(T)$. Concatenating the path in (17) with the path in B from r_1 to y gives a path in T from x to y. Indeed, the first path has all its vertices in $O \cup \{r_1\}$, while the second has all its vertices in K_1, and these two sets intersect only at r_1.

- Let $y \in K_j$ for some $j \in \{2, \dots, t\}$. If $z \neq x$, consider the path in T from x to z constructed in the first case, and concatenate it with the arc (z, r_j). If $z = x$, consider the path (z, r_j). In either case, we obtain a path in T from x to r_j whose vertices different from r_j belong to $O \cup \{r_1\}$.

  If $y = r_j$, the required path has been obtained. If $y \neq r_j$, append the path in B from r_j to y. All the arcs of this latter path belong to $\mathrm{Ar}(B) \cap (K_j)^2_* \subseteq \mathrm{Ar}(T)$. Moreover, its vertices belong to K_j, whereas all the preceding vertices, except r_j, belong to $O \cup \{r_1\}$. Since $K_j \cap O = \varnothing$ and $r_1 \notin K_j$, the resulting vertices are distinct. Thus, the resulting sequence is a path in T from x to y.

The preceding cases exhaust $\mathrm{Ve}(M) \setminus \{x\}$. Therefore, every vertex distinct from x is reachable from x in T. This completes the proof that $T \in \mathcal{A}_x(\mathrm{Ve}(M))$.

We now compute the weight of T. Since $u_0 \neq x$ and v_0 is the unique parent of u_0 in H, we have

$$\mathrm{Ar}(H) = \big(\mathrm{Ar}(H) \cap O^2_*\big) \cup \{(u_0, y) \in \mathrm{Ar}(H) : y \in O\} \cup \{(v_0, u_0)\}.$$

The three sets on the right-hand side are pairwise disjoint.

In the construction of T, every arc in $\mathrm{Ar}(H) \cap O^2_*$ is retained, every arc (u_0, y) of H is replaced by (r_1, y), and the arc (v_0, u_0) is replaced by (v_0, r_1). By (10),

$$\begin{aligned}
&\text{for every } (a,b) \in \mathrm{Ar}(H) \cap O^2_*,\ c^M(a,b) = c^N(a,b),\\
&\text{for every } (u_0, y) \in \mathrm{Ar}(H),\ c^M(r_1, y) = c^N(u_0, y),\\
&\ c^M(v_0, r_1) = c^N(v_0, u_0).
\end{aligned}$$

Consequently,

$$\sum_{(a,b) \in \mathrm{Ar}(H) \cap O^2_*} c^M(a,b) + \sum_{\substack{y \in O \\ (u_0, y) \in \mathrm{Ar}(H)}} c^M(r_1, y) + c^M(v_0, r_1) = N \circ H.$$

The remaining arcs of T are the arcs in $\mathrm{Ar}(B) \cap K^2_*$ and the arcs (z, r_j), with $j \in \{2, \dots, t\}$. These sets of arcs, together with the three sets considered above, are pairwise disjoint. Moreover, for every $j \in \{2, \dots, t\}$, $c^M(z, r_j) = c^N(z, u_0) = h$. Therefore,

$$\sum_{j=2}^{t} c^M(z, r_j) = (t-1)h.$$

Since $|\gamma^B(s)| = t$ and B attains the maximum in (11), we also have

$$\kappa = \sum_{(c,d)\in \mathrm{Ar}(B)\cap K_*^2} c^M(c,d) + (t-1)h.$$

It follows that

$$\begin{aligned} M \circ T &= \sum_{(a,b)\in \mathrm{Ar}(H)\cap O_*^2} c^M(a,b) \;+\; \sum_{\substack{y\in O\\ (u_0,y)\in \mathrm{Ar}(H)}} c^M(r_1,y) \;+\; c^M(v_0,r_1) \;+\; \sum_{(c,d)\in \mathrm{Ar}(B)\cap K_*^2} c^M(c,d) \;+\; \sum_{j=2}^{t} c^M(z,r_j) \\ &= N\circ H \;+\; \sum_{(c,d)\in \mathrm{Ar}(B)\cap K_*^2} c^M(c,d) \;+\; (t-1)h = \mu^N(x) + \kappa. \end{aligned}$$

Since $T \in \mathcal{A}_x(\mathrm{Ve}(M))$, the definition of $\mu^M(x)$ and equality just proved yield

$$\mu^M(x) \geq M\circ T = \mu^N(x) + \kappa.$$

Thus, (12) is finally proved.

We now prove (13). Let $H \in \mathcal{A}_{u_0}(\mathrm{Ve}(N))$ be such that $N \circ H = \mu^N(u_0)$. Define $T \in \mathcal{D}(\mathrm{Ve}(M))$ by

$$\mathrm{Ar}(T) = \big(\mathrm{Ar}(H)\cap O_*^2\big) \cup \{(r_1,y) : (u_0,y)\in \mathrm{Ar}(H)\} \cup \big(\mathrm{Ar}(B)\cap K_*^2\big) \cup \{(z,r_j) : j\in\{2,\dots,t\}\}. \tag{18}$$

We prove that $T \in \mathcal{A}_{r_1}(\mathrm{Ve}(M))$. We verify separately the three conditions in the definition of an arborescence.

First, no arc enters r_1 in T. Indeed, the first two sets on the right-hand side of (18) contain only arcs whose terminal vertices belong to O. Moreover, the unique arc entering r_1 in B is (s, r_1), which does not belong to $\mathrm{Ar}(B) \cap K_*^2$. Finally, the last set in (18) contains only arcs entering $r_2, \dots, r_t$. Hence no arc enters r_1 in T.

We next prove that every $y \in \mathrm{Ve}(M) \setminus \{r_1\}$ has a unique parent in T.

- Assume that $y \in O$. Let q be the unique element of $\pi^H(y)$. Only the first two sets on the right-hand side of (18) contain arcs whose terminal vertices belong to O. If $q \in O$, then (q, y) belongs to $\mathrm{Ar}(H) \cap O_*^2$ and hence to $\mathrm{Ar}(T)$; in this case, no other arc enters y in T. If $q = u_0$, then $(r_1, y) \in \mathrm{Ar}(T)$, and no other arc enters y in T. Therefore, y has a unique parent in T, namely q if $q \in O$ and r_1 if $q = u_0$.
- Assume that $y \in K \setminus \{r_1,\dots,r_t\}$. Since $K_1,\dots,K_t$ form a partition of K, there exists a unique $j \in \{1,\dots,t\}$ such that $y \in K_j \setminus \{r_j\}$. The unique arc entering y in B has both vertices in K_j, by (14), and hence belongs to $\mathrm{Ar}(B) \cap K_*^2 \subseteq \mathrm{Ar}(T)$. The first two sets on the right-hand side of (18) contain only arcs whose terminal vertices belong to O, while the last set contains only arcs entering $r_2,\dots,r_t$. Therefore, the unique arc entering y in B is also the unique arc entering y in T.
- Finally, assume that $y \in \{r_2,\dots,r_t\}$. There exists a unique $j \in \{2,\dots,t\}$ such that $y = r_j$. The first two sets on the right-hand side of (18) contain only arcs whose terminal vertices belong to O. Moreover, the unique arc entering r_j in B is (s, r_j), which does not belong to $\mathrm{Ar}(B)\cap K_*^2$. The only arc in the last set of (18) entering r_j is (z, r_j). Hence z is the unique parent of r_j in T.

The preceding cases exhaust $\mathrm{Ve}(M)\setminus\{r_1\}$. Hence every vertex distinct from r_1 has a unique parent in T.

We now prove that, for every $y \in \mathrm{Ve}(M)\setminus\{r_1\}$, there exists a path in T from r_1 to y.

- Let $y \in O$, and consider a path
$$(u_0, x_2, \dots, x_m)$$
in H from u_0 to $x_m = y$. Then
$$(r_1, x_2, \dots, x_m) \tag{19}$$
is a path in T from r_1 to y. Indeed, (r_1, x_2) belongs to the second set on the right-hand side of (18), while all the remaining arcs belong to the first set. Moreover, the vertices in (19) are distinct, since $x_2,\dots,x_m$ are distinct elements of O, whereas $r_1 \in K$ and $O \cap K = \varnothing$.

- Let $y \in K_1 \setminus \{r_1\}$. By the definition of K_1, there exists a path in B from r_1 to y. All its arcs belong to $\mathrm{Ar}(B) \cap (K_1)^2_*$ and hence to $\mathrm{Ar}(T)$. Therefore, this is also a path in T from r_1 to y.
- Let $y \in K_j$ for some $j \in \{2, \dots, t\}$. Consider the path in T from r_1 to z constructed in the first case, and concatenate it with the arc (z, r_j). This gives a path in T from r_1 to r_j. Indeed, all its vertices preceding r_j belong to $O \cup \{r_1\}$, whereas $r_j \in K_j$; hence its vertices are distinct.

  If $y = r_j$, the required path has been obtained. If $y \neq r_j$, append the path in B from r_j to y. All the arcs of this latter path belong to $\mathrm{Ar}(B) \cap (K_j)^2_* \subseteq \mathrm{Ar}(T)$. Moreover, its vertices belong to K_j, whereas all the preceding vertices, except r_j, belong to $O \cup \{r_1\}$. Since $K_j \cap O = \varnothing$ and $r_1 \notin K_j$, the resulting vertices are distinct. Thus, the resulting sequence is a path in T from r_1 to y.

The preceding cases exhaust $\mathrm{Ve}(M) \setminus \{r_1\}$. Therefore, every vertex distinct from r_1 is reachable from r_1 in T. This completes the proof that $T \in \mathcal{A}_{r_1}(\mathrm{Ve}(M))$.

We finally compute the weight of T. Since no arc enters the root u_0 in H, we have

$$\mathrm{Ar}(H) = \big(\mathrm{Ar}(H) \cap O^2_*\big) \cup \{(u_0, y) \in \mathrm{Ar}(H) : y \in O\},$$

where the two sets on the right-hand side are disjoint.

In the construction of T, every arc in $\mathrm{Ar}(H) \cap O^2_*$ is retained, while every arc (u_0, y) of H is replaced by (r_1, y). By (10), we have that

$$\text{for every } (a,b) \in \mathrm{Ar}(H) \cap O^2_*,\ c^M(a,b) = c^N(a,b)$$
$$\text{for every } (u_0,y) \in \mathrm{Ar}(H),\ c^M(r_1,y) = c^N(u_0,y)$$

Consequently,

$$\sum_{(a,b) \in \mathrm{Ar}(H) \cap O^2_*} c^M(a,b) + \sum_{\substack{y \in O \\ (u_0,y) \in \mathrm{Ar}(H)}} c^M(r_1,y) = N \circ H.$$

The remaining arcs of T are the arcs in $\mathrm{Ar}(B) \cap K^2_*$ and the arcs (z, r_j), with $j \in \{2, \dots, t\}$. These sets of arcs, together with the two sets considered above, are pairwise disjoint. As proved above,

$$\sum_{j=2}^{t} c^M(z, r_j) = (t-1)h$$

and

$$\kappa = \sum_{(c,d) \in \mathrm{Ar}(B) \cap K^2_*} c^M(c,d) + (t-1)h.$$

It follows that

$$\begin{aligned} M \circ T &= \sum_{(a,b) \in \mathrm{Ar}(H) \cap O^2_*} c^M(a,b) + \sum_{\substack{y \in O \\ (u_0,y) \in \mathrm{Ar}(H)}} c^M(r_1,y) + \sum_{(c,d) \in \mathrm{Ar}(B) \cap K^2_*} c^M(c,d) + \sum_{j=2}^{t} c^M(z,r_j) \\ &= N \circ H + \sum_{(c,d) \in \mathrm{Ar}(B) \cap K^2_*} c^M(c,d) + (t-1)h = \mu^N(u_0) + \kappa. \end{aligned}$$

Since $r_1 \in K$ and $T \in \mathcal{A}_{r_1}(\mathrm{Ve}(M))$, the definition of $\mu^M(r_1)$ and the above equality yield

$$\max_{x \in K} \mu^M(x) \geq \mu^M(r_1) \geq M \circ T = \mu^N(u_0) + \kappa.$$

Thus, (13) is proved.

Second part. We now prove that

$$\text{for every } x \in O,\ \mu^M(x) \leq \mu^N(x) + \kappa, \tag{20}$$

and that

$$\max_{x \in K} \mu^M(x) \le \mu^N(u_0) + \kappa. \tag{21}$$

Let $x \in \mathrm{Ve}(M)$ and $T \in \mathcal{A}_x(\mathrm{Ve}(M))$. Define

$$F = \big(K, \mathrm{Ar}(T) \cap K_*^2\big) \in \mathcal{D}(K)$$

and set

$$\mathcal{R} = \left\{r \in K : \pi^F(r) = \varnothing\right\}, \quad m = |\mathcal{R}|, \quad w = \sum_{(c,d) \in \mathrm{Ar}(F)} c^M(c,d).$$

By the definition of F, $\mathrm{Ar}(F) \subseteq \mathrm{Ar}(T)$. Since at most one arc enters each vertex in T, it follows that, for every $d \in K$, $|\pi^F(d)| \le 1$. Moreover, F is acyclic.

We now prove the following property, which will be used to construct an arborescence on $K \cup \{s\}$:

$$\text{for every } d \in K, \text{ there exists } r \in \mathcal{R} \text{ such that } d = r \text{ or } d \in \delta^F(r). \tag{22}$$

Let $d \in K$. If $d \in \mathcal{R}$, the conclusion follows by taking $r = d$. Suppose then that $d \notin \mathcal{R}$. Recall that every vertex in $K \setminus \mathcal{R}$ has a unique parent in F, and that this parent belongs to K. Define recursively $d_0 = d$ and, whenever $d_i \notin \mathcal{R}$, let $d_{i+1} \in K$ be the unique vertex such that $(d_{i+1}, d_i) \in \mathrm{Ar}(F)$. We claim that there exists $\ell \ge 1$ such that $d_\ell \in \mathcal{R}$. Otherwise, the recursion would define an infinite sequence of vertices in the finite set K. Hence, there would exist indices $0 \le i < j$ such that $d_i = d_j$ and such that $d_i, d_{i+1}, \ldots, d_{j-1}$ are distinct. Since, for every $k \in \{i, \ldots, j-1\}$, $(d_{k+1}, d_k) \in \mathrm{Ar}(F)$, the sequence

$$(d_j, d_{j-1}, \ldots, d_i)$$

would determine a cycle in F, contradicting the acyclicity of F. Therefore, let ℓ be the least index such that $d_\ell \in \mathcal{R}$. The vertices $d_0, \ldots, d_\ell$ are distinct, and

$$(d_\ell, d_{\ell-1}, \ldots, d_0)$$

is a path in F from d_ℓ to d. Taking $r = d_\ell$ proves (22).

Since $K \neq \varnothing$, (22) implies that $\mathcal{R} \neq \varnothing$ and hence $m \ge 1$. In what follows, a set or a sum indexed by $j \in \{2, \ldots, m\}$ is understood to be empty if $m = 1$.

Let $E \in \mathcal{D}(K \cup \{s\})$ be such that

$$\mathrm{Ar}(E) = \mathrm{Ar}(F) \cup \{(s, r) : r \in \mathcal{R}\}.$$

We prove that $E \in \mathcal{A}_s(K \cup \{s\})$.

No arc enters s in E. Every $r \in \mathcal{R}$ has s as its unique parent in E, while every $d \in K \setminus \mathcal{R}$ has in E the same unique parent as in F. Finally, let $d \in K$, and let $r \in \mathcal{R}$ be such that either $d = r$ or there exists a path in F from r to d. If $d = r$, then (s, r) is a path in E from s to d. If $d \neq r$, the arc (s, r), together with the path in F from r to d, determines a path in E from s to d. It follows from the definition of an arborescence that $E \in \mathcal{A}_s(K \cup \{s\})$.

Since $E \in \mathcal{A}_s(K \cup \{s\})$, the value associated with E in the maximization problem defining κ cannot exceed κ. Therefore,

$$\sum_{(c,d) \in \mathrm{Ar}(E) \cap K_*^2} c^M(c,d) + \big(|\gamma^E(s)| - 1\big)h \le \kappa.$$

Now,

$$\mathrm{Ar}(E) \cap K_*^2 = \mathrm{Ar}(F), \qquad \gamma^E(s) = \mathcal{R}, \qquad |\mathcal{R}| = m.$$

Moreover, by the definition of w,

$$\sum_{(c,d) \in \mathrm{Ar}(F)} c^M(c,d) = w.$$

Substituting these equalities into the preceding inequality, we obtain

$$w + (m-1)h \le \kappa. \tag{23}$$

We now distinguish the two possible locations of the root of T.

Assume first that $x \in O$. Every element of $\mathcal{R}$ is distinct from the root x and hence has a unique parent in T. This parent belongs to O. Indeed, if an element of $\mathcal{R}$ had a parent in K, an arc of F would enter it, contrary to the definition of $\mathcal{R}$.

Consider a path $(x_i)_{i=1}^{\ell_0}$ in T from $x_1 = x$ to $x_{\ell_0} = u_0$, and set

$$i_0 = \min\{i \in \{1, \dots, \ell_0\} : x_i \in K\}.$$

Since $x_1 = x \in O$ and $x_{\ell_0} = u_0 \in K$, the integer i_0 is well defined and satisfies $2 \leq i_0 \leq \ell_0$. By its definition, $x_1, \dots, x_{i_0-1} \in O$ and $x_{i_0} \in K$. Moreover, $x_{i_0} \in \mathcal{R}$ since the unique parent of x_{i_0} in T is $x_{i_0-1} \in O$. Consequently, no arc of F enters x_{i_0}.

Since $x_{i_0} \in \mathcal{R}$, we may number the elements of $\mathcal{R}$ so that

$$\mathcal{R} = \{\rho_1, \dots, \rho_m\} \quad \text{and} \quad \rho_1 = x_{i_0}.$$

If $m = 1$, this notation simply means that $\mathcal{R} = \{\rho_1\}$. As said, a set or a sum or a union indexed by $j \in \{2, \dots, m\}$ is understood to be empty if $m = 1$, while a statement concerning every $j \in \{2, \dots, m\}$ is understood to be vacuous. For every $j \in \{1, \dots, m\}$, let p_j be the unique element of $\pi^T(\rho_j)$. As observed above, for every $j \in \{1, \dots, m\}$, $p_j \in O$ and, in particular, $p_1 = x_{i_0-1}$.

We observe now that

$$\{(p, d) \in \mathrm{Ar}(T) : p \in O,\ d \in K\} = \{(p_j, \rho_j) : j \in \{1, \dots, m\}\}. \tag{24}$$

Indeed, from what already proved about p_j, the right-hand side is included in the left-hand side. To prove the opposite inclusion, let $(p, d) \in \mathrm{Ar}(T)$ with $p \in O$ and $d \in K$. Since (p, d) is the unique arc entering d in T, no arc with both vertices in K enters d. Hence $\pi^F(d) = \varnothing$ and $d \in \mathcal{R}$. Therefore, $d = \rho_j$ for a unique $j \in \{1, \dots, m\}$; the uniqueness of the parent of ρ_j in T then implies that $p = p_j$. Thus, $(p, d) = (p_j, \rho_j)$. That completes the proof of (24).

Let $G \in \mathcal{D}(\mathrm{Ve}(N))$ be such that

$$\mathrm{Ar}(G) = \big(\mathrm{Ar}(T) \cap O_*^2\big) \cup \{(u_0, y) : y \in O,\ (d, y) \in \mathrm{Ar}(T) \text{ for some } d \in K\} \cup \{(p_1, u_0)\}. \tag{25}$$

Thus, the arcs of T having both vertices in O are retained, every arc of T from a vertex in K to a vertex y in O is replaced by an arc from u_0 to y, and the arc (p_1, ρ_1) is replaced by (p_1, u_0). The other arcs entering K from O are not included in G.

We prove that $G \in \mathcal{A}_x(\mathrm{Ve}(N))$.

First, no arc enters x in G. Indeed, an arc in either of the first two sets on the right-hand side of (25) entering x would correspond to an arc of T entering x. This is impossible because x is the root of T. The arc in the third set enters u_0, which is distinct from x.

We now prove that every $y \in \mathrm{Ve}(N) \setminus \{x\}$ has a unique parent in G.

- Let $y = u_0$. The arc (p_1, u_0) belongs to $\mathrm{Ar}(G)$, and neither of the first two sets on the right-hand side of (25) contains an arc entering u_0. Hence p_1 is the unique parent of u_0 in G.
- Let $y \in O \setminus \{x\}$, and let p be the unique parent of y in T. If $p \in O$, then (p, y) belongs to $\mathrm{Ar}(T) \cap O_*^2$ and hence to $\mathrm{Ar}(G)$. In this case, no other arc enters y in G. If $p \in K$, then $(u_0, y) \in \mathrm{Ar}(G)$, and no other arc enters y in G. Therefore, y has a unique parent in G, namely p if $p \in O$ and u_0 if $p \in K$.

The preceding cases exhaust $\mathrm{Ve}(N) \setminus \{x\}$. Hence every vertex distinct from x has a unique parent in G. We next prove that, for every $y \in \mathrm{Ve}(N) \setminus \{x\}$, there exists a path in G from x to y.

- Let $y = u_0$. Recall that $(x_i)_{i=1}^{\ell_0}$ is the path in T from $x_1 = x$ to $x_{\ell_0} = u_0$, and that x_{i_0} is its first vertex belonging to K. The sequence

$$(x_1, \dots, x_{i_0-1}, u_0) \tag{26}$$

is a path in G from x to u_0. Indeed, $x_1, \dots, x_{i_0-1} \in O$, and hence the arcs between consecutive vertices among them belong to $\mathrm{Ar}(T) \cap O_*^2 \subseteq \mathrm{Ar}(G)$. Moreover, $(x_{i_0-1}, u_0) = (p_1, u_0) \in \mathrm{Ar}(G)$. The vertices in (26) are distinct, since $x_1, \dots, x_{i_0-1}$ are distinct elements of O, whereas $u_0 \notin O$.

- Let $y \in O \setminus \{x\}$, and consider a path $(y_i)_{i=1}^{\ell}$ in T from $y_1 = x$ to $y_\ell = y$. If this path does not meet K, then all its vertices belong to O and all its arcs belong to $\mathrm{Ar}(T) \cap O_*^2 \subseteq \mathrm{Ar}(G)$. Hence it is a path in G from x to y.

  Suppose now that this path meets K, and set
  $$i_1 = \max\{i \in \{1, \dots, \ell\} : y_i \in K\}.$$
  Since $y_1 = x \in O$ and $y_\ell = y \in O$, we have $1 < i_1 < \ell$ and $y_{i_1+1}, \dots, y_\ell \in O$.

  We prove that
  $$\{x_1, \dots, x_{i_0-1}\} \cap \{y_{i_1+1}, \dots, y_\ell\} = \varnothing. \tag{27}$$
  Suppose, by contradiction, that d belongs to the intersection in (27). The part of the path $(x_i)_{i=1}^{\ell_0}$ from x to d has all its vertices in O, because d occurs before x_{i_0}. On the other hand, the part of the path $(y_i)_{i=1}^{\ell}$ from x to d contains $y_{i_1} \in K$, because d occurs after y_{i_1}. Thus, these parts determine two distinct paths in T from x to d. This contradicts the uniqueness of the path from the root x to d in T, and proves (27).

  It follows that
  $$(x_1, \dots, x_{i_0-1}, u_0, y_{i_1+1}, \dots, y_\ell)$$
  is a path in G from x to y. Indeed, its vertices are distinct by (27) and because $u_0 \notin O$. The arcs between consecutive vertices among $x_1, \dots, x_{i_0-1}$ belong to the first set on the right-hand side of (25); the arc $(x_{i_0-1}, u_0) = (p_1, u_0)$ belongs to the third set; the arc (u_0, y_{i_1+1}) belongs to the second set because $(y_{i_1}, y_{i_1+1}) \in \mathrm{Ar}(T)$, $y_{i_1} \in K$, $y_{i_1+1} \in O$; and all the remaining arcs belong to the first set.

The preceding cases exhaust $\mathrm{Ve}(N) \setminus \{x\}$. Hence every vertex distinct from x is reachable from x in G. This completes the proof that $G \in \mathcal{A}_x(\mathrm{Ve}(N))$.

We now compare the weights of T and G. By (24), the following four sets form a partition of $\mathrm{Ar}(T)$:
$$\mathrm{Ar}(F), \quad \{(p_j, \rho_j) : j \in \{1, \dots, m\}\}, \quad \mathrm{Ar}(T) \cap O_*^2, \quad \{(d, y) \in \mathrm{Ar}(T) : d \in K,\ y \in O\}.$$
The arcs in $\mathrm{Ar}(T) \cap O_*^2$ are retained in G. Every arc (d, y) of T with $d \in K$ and $y \in O$ is replaced by (u_0, y), while (p_1, ρ_1) is replaced by (p_1, u_0). By (10), corresponding arcs have the same capacities. Therefore,
$$M \circ T = N \circ G + w + \sum_{j=2}^{m} c^M(p_j, \rho_j).$$
For every $j \in \{2, \dots, m\}$, we have $c^M(p_j, \rho_j) = c^N(p_j, u_0) \le h$. Consequently, by (23),
$$M \circ T \le N \circ G + w + (m-1)h \le N \circ G + \kappa \le \mu^N(x) + \kappa.$$
Since $T \in \mathcal{A}_x(\mathrm{Ve}(M))$ was arbitrary, it follows that $\mu^M(x) \le \mu^N(x) + \kappa$. Thus, (20) is proved.

Assume now that $x \in K$. Since no arc enters the root x in T, we have $x \in \mathcal{R}$. Thus,
$$\mathcal{R} = \{\rho_1, \dots, \rho_m\} \quad \text{and} \quad \rho_1 = x.$$
For every $j \in \{2, \dots, m\}$, the vertex ρ_j is distinct from the root x and hence has a unique parent p_j in T. Since $\rho_j \in \mathcal{R}$, this parent belongs to O. Moreover,
$$\{(p, d) \in \mathrm{Ar}(T) : p \in O,\ d \in K\} = \{(p_j, \rho_j) : j \in \{2, \dots, m\}\}. \tag{28}$$
Indeed, every arc on the right-hand side belongs to the set on the left-hand side. Conversely, if $(p, d) \in \mathrm{Ar}(T)$ with $p \in O$ and $d \in K$, then $d \neq x$ and no arc of F enters d. Hence $d \in \mathcal{R} \setminus \{x\}$, which proves the opposite inclusion.

Define $G \in \mathcal{D}(\mathrm{Ve}(N))$ by
$$\mathrm{Ar}(G) = \big(\mathrm{Ar}(T) \cap O_*^2\big) \cup \{(u_0, y) : y \in O,\ (d, y) \in \mathrm{Ar}(T) \text{ for some } d \in K\}. \tag{29}$$

Thus, the arcs of T having both vertices in O are retained in G and every arc of T from an element of K to y in O is replaced by the arc from u_0 to y.

We prove that $G \in \mathcal{A}_{u_0}(\mathrm{Ve}(N))$.

First, no arc enters u_0 in G, since both sets on the right-hand side of (29) contain only arcs whose terminal vertices belong to O.

Let $y \in O$, and let p be the unique parent of y in T. If $p \in O$, then (p, y) belongs to $\mathrm{Ar}(T) \cap O_*^2$ and hence to $\mathrm{Ar}(G)$; in this case, no other arc enters y in G. If $p \in K$, then $(u_0, y) \in \mathrm{Ar}(G)$, and no other arc enters y in G. Therefore, every vertex of O has a unique parent in G.

We next prove that every $y \in O$ is reachable from u_0 in G. Consider a path $(y_i)_{i=1}^{\ell}$ in T from $y_1 = x$ to $y_\ell = y$, and set

$$a = \max\{i \in \{1, \ldots, \ell\} : y_i \in K\}.$$

Since $y_1 = x \in K$ and $y_\ell = y \in O$, the integer a is well defined and satisfies $1 \leq a < \ell$. Moreover, $y_{a+1}, \ldots, y_\ell \in O$. Therefore,

$$(u_0, y_{a+1}, \ldots, y_\ell)$$

is a path in G from u_0 to y. Indeed, its vertices are distinct, since $u_0 \notin O$ and $y_{a+1}, \ldots, y_\ell$ are distinct elements of O. Its first arc belongs to the second set on the right-hand side of (29), because $(y_a, y_{a+1}) \in \mathrm{Ar}(T)$ and $y_a \in K$, while all its remaining arcs belong to the first set. This completes the proof that $G \in \mathcal{A}_{u_0}(\mathrm{Ve}(N))$.

We now compare the weights of T and G. By (28), the following four sets form a partition of $\mathrm{Ar}(T)$:

$$\mathrm{Ar}(F), \quad \{(p_j, \rho_j) : j \in \{2, \ldots, m\}\}, \quad \mathrm{Ar}(T) \cap O_*^2, \quad \{(d, y) \in \mathrm{Ar}(T) : d \in K,\ y \in O\}.$$

The arcs in the last two sets correspond, by (29), to arcs of G having the same capacities. Therefore,

$$M \circ T = N \circ G + w + \sum_{j=2}^{m} c^M(p_j, \rho_j).$$

For every $j \in \{2, \ldots, m\}$, $c^M(p_j, \rho_j) = c^N(p_j, u_0) \leq h$. Thus, by (23),

$$M \circ T \leq N \circ G + w + (m-1)h \leq N \circ G + \kappa \leq \mu^N(u_0) + \kappa.$$

Since $T \in \mathcal{A}_x(\mathrm{Ve}(M))$ was arbitrary, we obtain $\mu^M(x) \leq \mu^N(u_0) + \kappa$. Since this relation holds for every $x \in K$, (21) follows.

Third part. Combining (12) with (20), and (13) with (21), we deduce that

$$\text{for every } x \in O,\ \mu^M(x) = \mu^N(x) + \kappa \tag{30}$$

and

$$\max_{x \in K} \mu^M(x) = \mu^N(u_0) + \kappa. \tag{31}$$

In order to complete the proof of the theorem, define now

$$\lambda = \max_{x \in \mathrm{Ve}(N)} \mu^N(x).$$

Since $\mathrm{Ve}(M) = O \cup K$ and $O \cap K = \varnothing$, (30) and (31) imply that

$$\max_{x \in \mathrm{Ve}(M)} \mu^M(x) = \max\left\{\max_{x \in O} \mu^M(x), \max_{x \in K} \mu^M(x)\right\} = \max\left\{\max_{x \in O} \mu^N(x), \mu^N(u_0)\right\} + \kappa = \lambda + \kappa. \tag{32}$$

Let $x_0 \in O$. Consider the following facts:

(a_1) $x_0 \in \mathscr{A}(M)$,

(a_2) $\mu^M(x_0) = \max_{x \in \mathrm{Ve}(M)} \mu^M(x)$, (a$_3$) $\mu^M(x_0) = \lambda + \kappa$,

(a$_4$) $\mu^N(x_0) + \kappa = \lambda + \kappa$,

(a$_5$) $\mu^N(x_0) = \lambda$,

(a$_6$) $x_0 \in \mathscr{A}(N)$.

By the definition of $\mathscr{A}(M)$ and μ^M, (a$_1$) is equivalent to (a$_2$); by (32), (a$_2$) is equivalent to (a$_3$); by (30), (a$_3$) is equivalent to (a$_4$); by an obvious algebraic fact, (a$_4$) is equivalent to (a$_5$); by the definition of $\mathscr{A}(N)$, μ^N, and λ, (a$_5$) is equivalent to (a$_6$). We deduce then that (a$_1$) is equivalent to (a$_6$) which proves part (i) of Theorem 12.

Finally, consider the following facts:

(b$_1$) $K \cap \mathscr{A}(M) \neq \varnothing$,

(b$_2$) $\max\limits_{y \in K} \mu^M(y) = \max\limits_{y \in \mathrm{Ve}(M)} \mu^M(y)$,

(b$_3$) $\max\limits_{x \in K} \mu^M(x) = \lambda + \kappa$,

(b$_4$) $\mu^N(u_0) + \kappa = \lambda + \kappa$,

(b$_5$) $\mu^N(u_0) = \lambda$,

(b$_6$) $u_0 \in \mathscr{A}(N)$.

By the definition of $\mathscr{A}(M)$ and μ^M, (b$_1$) is equivalent to (b$_2$); by (32), (b$_2$) is equivalent to (b$_3$); by (31), (b$_3$) is equivalent to (b$_4$); by an obvious algebraic fact, (b$_4$) is equivalent to (b$_5$); by the definition of $\mathscr{A}(N)$, μ^N, and λ, (b$_5$) is equivalent to (b$_6$). We deduce then that (b$_1$) is equivalent to (b$_6$) which proves part (ii) of Theorem 12. □

B Proof of Theorem 14

Let $X \in P_f(\mathbb{N})$ and $(X, R) \in \mathcal{D}(X)$. Let R^τ be the relation on X defined as

$$R^\tau = X_d^2 \cup \left\{(x, y) \in X_*^2 : y \in \delta^{(X,R)}(x)\right\}.$$

We have that R^τ corresponds to the transitive and reflexive closure of R. The relation

$$E_R = \{(x, y) \in X^2 : (x, y) \in R^\tau \text{ and } (y, x) \in R^\tau\},$$

is an equivalence relation on X. We denote by $SC(X, R)$ the quotient set of X by E_R. The elements of $SC(X, R)$ are called strong components of (X, R) and, as well known, they form a partition of X. We set

$$ISC(X, R) = \{Y \in SC(X, R) : \forall x \in X \setminus Y\ \forall y \in Y, (x, y) \notin R^\tau\}.$$

The elements of $ISC(X, R)$ are called initial strong components.

The proof of Theorem 14 is based on several preliminary results. Theorem 24 and Lemma 25 are well known.[9]

Theorem 24. *Let $X \in P_f(\mathbb{N})$ and $(X, R) \in \mathcal{D}(X)$. Then $ISC(X, R) \neq \varnothing$ and*

$$\mathscr{A}_*(X, R) = \bigcup_{Y \in ISC(X,R)} Y.$$

[9]Theorem 24 corresponds to Theorem 5 in Kalai and Schmeidler (1977) (see also Theorem 9 in Gori, 2023); Lemma 25 corresponds to Lemma 12 in Gori (2023).

Lemma 25. *Let $X \in P_f(\mathbb{N})$, $(X,R) \in \mathcal{D}(X)$, and $x \in X$. Then there exists $y \in \mathscr{A}_*(X,R)$ such that $(y,x) \in R^\tau$.*

Lemma 26. *Let $X \in P_f(\mathbb{N})$ and $(X,R) \in \mathcal{D}(X)$. The following two facts are equivalent:*

(a) *$|ISC(X,R)| = 1$;*

(b) *there exists $y \in X$ such that, for every $x \in X$, $(y,x) \in R^\tau$.*

Proof. (a)$\Rightarrow$(b). Assume that $|ISC(X,R)| = 1$. Then, there exists $Y \subseteq X$ nonempty such that $ISC(X,R) = \{Y\}$ and so, by Theorem 24, $\mathscr{A}_*(X,R) = Y$. Pick any $y \in Y$ and prove that, for every $x \in X$, $(y,x) \in R^\tau$. Consider then $x \in X$. By Lemma 25, there exists $y^* \in Y$ such that $(y^*,x) \in R^\tau$. Since $y, y^* \in Y$, we have $(y,y^*) \in R^\tau$. By transitivity of R^τ, we conclude $(y,x) \in R^\tau$.

(b)$\Rightarrow$(a). Assume that there exists $y \in X$ such that, for every $x \in X$, $(y,x) \in R^\tau$. Let Y be the unique element in $SC(X,R)$ such that $y \in Y$. Of course, $y \in \mathscr{A}_*(X,R)$ and then, by Theorem 24, $Y \subseteq \mathscr{A}_*(X,R)$ and $Y \in ISC(X,R)$. Suppose by contradiction that $|ISC(X,R)| > 1$. Then there exists $Z \in ISC(X,R)$ with $Y \neq Z$. Given now $x \in Z$, we know that $(y,x) \in R^\tau$. Since $Y \cap Z = \varnothing$, we have $y \notin Z$ and that contradicts the fact that $Z \in ISC(X,R)$. $\square$

Proposition 27. *Let $X \in P_f(\mathbb{N})$ and $(X,R) \in \mathcal{D}(X)$. Assume that $ISC(X,R) = \{Y\}$. Then, for every $x \in Y$, there exists $(X,T) \in \mathcal{A}_x(X)$ such that $T \subseteq R$.*

Proof. Since $ISC(X,R) = \{Y\}$, by Lemma 26, we deduce that (X,R) is connected, that is, $SC(X, R \cup R^r) = \{X\}$. The second part of the proof of Proposition 1.6.1 in Bang-Jensen and Gutin (2007) proves the desired result. $\square$

Lemma 28. *Let $X \in P_f(\mathbb{N})$, $(X,R) \in \mathcal{D}(X)$, $k \in \mathbb{N}$ with $k \geq 2$. Assume that $|ISC(X,R)| = k$ and $ISC(X,R) = \{Y_1,\dots,Y_k\}$, where $Y_1,\dots,Y_k \subseteq X$. Let $y_1,\dots,y_k \in X$ be such that $y_j \in Y_j$ for all $j \in \{1,\dots,k\}$, and*

$$R_\diamond = R \cup \left\{(y_1,y_j) \in X_*^2 : j \in \{2,\dots,k\}\right\}.$$

Then $|ISC(X,R_\diamond)| = 1$ and $y_1 \in \mathscr{A}_*(X,R_\diamond)$.

Proof. Let us prove that, for every $z \in X$, $(y_1,z) \in R_\diamond^\tau$. Consider $z \in X$. By Lemma 25, we know that there exists $y^* \in \mathscr{A}_*(X,R)$ such that $(y^*,z) \in R^\tau$ and so $(y^*,z) \in R_\diamond^\tau$. If $y^* \in Y_1$, then $(y_1,y^*) \in R^\tau$ and so $(y_1,y^*) \in R_\diamond^\tau$. By transitivity of $R_\diamond^\tau$, we conclude that $(y_1,z) \in R_\diamond^\tau$. If $y^* \in Y_j$ for some $j \in \{2,\dots,k\}$, then $(y_j,y^*) \in R^\tau$ and so $(y_j,y^*) \in R_\diamond^\tau$; $(y_1,y_j) \in R_\diamond$ and so $(y_1,y_j) \in R_\diamond^\tau$. By transitivity of $R_\diamond^\tau$, we conclude that $(y_1,z) \in R_\diamond^\tau$. By Lemma 26, we deduce $|ISC(X,R_\diamond)| = 1$. Of course, what has been proved about y_1 guarantees that $y_1 \in \mathscr{A}_*(X,R_\diamond)$. $\square$

Proposition 29. *Let $X \in P_f(\mathbb{N})$ and $(X,R) \in \mathcal{D}(X)$. Assume that $|ISC(X,R)| = k$ with $k \in \mathbb{N}$. Then,*

(i) *for every $x_0 \in \mathscr{A}_*(X,R)$, there exists $(X,T) \in \mathcal{A}_{x_0}(X)$ such that $|T \setminus R| = k-1$,*

(ii) *for every $x_0 \in \mathscr{A}_*(X,R)$ and $(X,T) \in \mathcal{A}_{x_0}(X)$, we have $|T \setminus R| \geq k-1$,*

(iii) *for every $x_0 \in X \setminus \mathscr{A}_*(X,R)$ and $(X,T) \in \mathcal{A}_{x_0}(X)$, we have $|T \setminus R| \geq k$.*

Proof. (i) Consider $x_0 \in \mathscr{A}_*(X,R)$. If $k = 1$, simply apply Proposition 27. Assume then that $k \geq 2$. Let $Y_1,\dots,Y_k \subseteq X$ be such that $ISC(X,R) = \{Y_1,\dots,Y_k\}$. Without loss of generality, we can suppose that $x_0 \in Y_1$. Consider then $y_1,\dots,y_k$ such that $y_1 = x_0$ and, for every $j \in \{2,\dots,k\}$, $y_j \in Y_j$. Let

$$R_\diamond = R \cup \left\{(y_1,y_j) \in X_*^2 : j \in \{2,\dots,k\}\right\}.$$

Observe that $R \cap \left\{(y_1,y_j) \in X_*^2 : j \in \{2,\dots,k\}\right\} = \varnothing$ and $\left|\left\{(y_1,y_j) \in X_*^2 : j \in \{2,\dots,k\}\right\}\right| = k-1$. By Lemma 28, $|ISC(X,R_\diamond)| = 1$ and $y_1 \in \mathscr{A}_*(X,R_\diamond)$. By Proposition 27, there exists $(X,T) \in \mathcal{A}_{y_1}(X)$ such that $T \subseteq R_\diamond$. Thus,

$$T \setminus R \subseteq R_\diamond \setminus R = \left\{(y_1,y_j) \in X_*^2 : j \in \{2,\dots,k\}\right\}.$$

We conclude proving that also the opposite inclusion holds true. Assume by contradiction that

$$\{(y_1, y_j) \in X_*^2 : j \in \{2,\dots,k\}\} \not\subseteq T \setminus R.$$

Since $\{(y_1, y_j) \in X_*^2 : j \in \{2,\dots,k\}\} \cap R = \varnothing$, that is equivalent to

$$\{(y_1, y_j) \in X_*^2 : j \in \{2,\dots,k\}\} \not\subseteq T.$$

Thus, there exists $j^* \in \{2,\dots,k\}$ such that $(y_1, y_{j^*}) \notin T$. We know that there exists a path $(z_i)_{i=1}^r$ from y_1 to y_{j^*} in (X,T). There exists then $i^* \in \{1,\dots,r-1\}$ such that $(z_{i^*}, z_{i^*+1}) \in T$, $z_{i^*} \notin Y_{j^*}$ and $z_{i^*+1} \in Y_{j^*}$. Since $Y_{j^*} \in ISC(X,R)$, we have that $(z_{i^*}, z_{i^*+1}) \in T \setminus R$ and so, recalling that $(y_1, y_{j^*}) \notin T$, we deduce

$$(z_{i^*}, z_{i^*+1}) \in \{(y_1, y_j) \in X_*^2 : j \in \{2,\dots,k\} \setminus \{j^*\}\}.$$

If $k = 2$, then $j^* = 2$ and the set $\{(y_1, y_j) \in X_*^2 : j \in \{2,\dots,k\} \setminus \{j^*\}\}$ is empty, which leads to a contradiction. If $k \geq 3$, since $Y_1, \dots, Y_k$ are pairwise disjoint, we have that, for every $j \in \{2,\dots,k\} \setminus \{j^*\}$, $y_j \notin Y_{j^*}$, which is a contradiction.

(ii) Consider $x_0 \in \mathscr{A}_*(X,R)$ and $(X,T) \in \mathcal{A}_{x_0}(X)$. We have to show that $|T \setminus R| \geq k-1$. Consider $Y_1, \dots, Y_k \subseteq X$ such that $ISC(X,R) = \{Y_1, \dots, Y_k\}$ and without loss of generality assume that $x_0 \in Y_1$. Moreover let $y_2, \dots, y_k$ be such that $y_j \in Y_j$ for all $j \in \{2,\dots,k\}$. Let $j \in \{2,\dots,k\}$. Since $(X,T) \in \mathcal{A}_{x_0}(X)$, there exists a path $(z_i^j)_{i=1}^{r(j)}$ from x_0 to y_j in (X,T). There exists then $i(j) \in \{1,\dots,r(j)-1\}$ such that $z_{i(j)}^j \notin Y_j$ and $z_{i(j)+1}^j \in Y_j$. Since $Y_j \in ISC(X,R)$, we have that $(z_{i(j)}^j, z_{i(j)+1}^j) \in T \setminus R$. Thus

$$T \setminus R \supseteq \left\{(z_{i(j)}^j, z_{i(j)+1}^j) \in X_*^2 : j \in \{2,\dots,k\}\right\}.$$

Note also that if $j, j' \in \{2,\dots,k\}$ with $j \neq j'$, then $(z_{i(j)}^j, z_{i(j)+1}^j) \neq (z_{i(j')}^{j'}, z_{i(j')+1}^{j'})$ since $z_{i(j)+1}^j \in Y_j$, $z_{i(j')+1}^{j'} \in Y_{j'}$ and $Y_j \cap Y_{j'} = \varnothing$. Thus, we conclude $|T \setminus R| \geq k-1$.

(iii) Consider $x_0 \in X \setminus \mathscr{A}_*(X,R)$ and $(X,T) \in \mathcal{A}_{x_0}(X)$. We have to show that $|T \setminus R| \geq k$. Consider $Y_1, \dots, Y_k \subseteq X$ such that $ISC(X,R) = \{Y_1, \dots, Y_k\}$ and let $y_1, \dots, y_k$ be such that $y_j \in Y_j$ for all $j \in \{1,\dots,k\}$. Let $j \in \{1,\dots,k\}$. Since $(X,T) \in \mathcal{A}_{x_0}(X)$, there exists a path $(z_i^j)_{i=1}^{r(j)}$ from x_0 to y_j in (X,T). There exists then $i(j) \in \{1,\dots,r(j)-1\}$ such that $z_{i(j)}^j \notin Y_j$ and $z_{i(j)+1}^j \in Y_j$. Since $Y_j \in ISC(X,R)$, we have that $(z_{i(j)}^j, z_{i(j)+1}^j) \in T \setminus R$. Thus

$$T \setminus R \supseteq \left\{(z_{i(j)}^j, z_{i(j)+1}^j) \in X_*^2 : j \in \{1,\dots,k\}\right\}.$$

Note also that if $j, j' \in \{1,\dots,k\}$ with $j \neq j'$, then $(z_{i(j)}^j, z_{i(j)+1}^j) \neq (z_{i(j')}^{j'}, z_{i(j')+1}^{j'})$ since $z_{i(j)+1}^j \in Y_j$, $z_{i(j')+1}^{j'} \in Y_{j'}$ and $Y_j \cap Y_{j'} = \varnothing$. Thus, we deduce $|T \setminus R| \geq k$. $\square$

Proof of Theorem 14. Let $D = (X,R) \in \mathcal{D}$ and set $k = |ISC(X,R)|$. Observe that, for every $x \in X$ and $(X,T) \in \mathcal{A}_x(X)$, we have that

$$D \circ (X,T) = |R \cap T| = |T \setminus (T \setminus R)| = |T| - |T \setminus R| = |X| - 1 - |T \setminus R|.$$

Consider $x \in \mathscr{A}_*(X,R)$. By Proposition 29(i)-(ii), we know that there exists $(X,T^*) \in \mathcal{A}_x(X)$ such that $|T^* \setminus R| = k-1$, and that, for every $(X,T) \in \mathcal{A}_x(X)$, $|T \setminus R| \geq k-1$. As a consequence, $\mu^D(x) = |X| - 1 - (k-1) = |X| - k$. Consider now $x \in X \setminus \mathscr{A}_*(X,R)$. By Proposition 29(iii), we know that, for every $(X,T) \in \mathcal{A}_x(X)$, $|T \setminus R| \geq k$. As a consequence, $\mu^D(x) \leq |X| - 1 - k$. By (3), we finally conclude that $\mathscr{A}(D) = \mathscr{A}_*(D)$. $\square$

C Proofs of Propositions 15, 16, and 17

Proof of Proposition 15. The fact that A satisfies anonymity simply follows from the fact that A is a SCC of type C2.

Let us prove that A satisfies neutrality. Let $p \in \mathbf{L}^*$ and $\psi \in \mathrm{Sym}(\mathrm{Al}(p))$. We must show that $\mathsf{A}(p^\psi) = \psi(\mathsf{A}(p))$. By Proposition 1, we have that $\mathsf{A}(p^\psi) = \mathscr{A}(N(p^\psi)) = \mathscr{A}(N(p)^\psi) = \psi(\mathscr{A}(N(p))) = \psi(\mathsf{A}(p))$, as desired.

Let us prove that A satisfies homogeneity. Let $p, q \in \mathbf{L}^*$ and $n \in \mathbb{N}$, and assume that q is an n-fold replication of p. We must show that $\mathsf{A}(q) = \mathsf{A}(p)$. We have that $N(q) = nN(p)$ and therefore, by Proposition 3, we have $\mathsf{A}(q) = \mathscr{A}(N(q)) = \mathscr{A}(nN(p)) = \mathscr{A}(N(p)) = \mathsf{A}(p)$, as desired.

Let us prove that A satisfies Pareto efficiency. Let $p \in \mathbf{L}^*$ and $x_0, y_0 \in \mathrm{Al}(p)$ be distinct and such that, for every $i \in \mathrm{In}(p)$, $x_0 \succ_{p(i)} y_0$. We must show that $y_0 \notin \mathsf{A}(p)$. We know that $c^{N(p)}(x_0, y_0) = |\mathrm{In}(p)|$ and then $c^{N(p)}(y_0, x_0) = 0$. Consider the set

$$S = \{x \in \mathrm{Al}(p) : x \succ_{p(i)} x_0 \text{ for all } i \in \mathrm{In}(p)\} \cup \{x_0\}.$$

- Assume first that $S = \{x_0\}$. We claim that $c^{N(p)}(x_0, y_0) > c^{N(p)}(z, x_0)$ for all $z \in \mathrm{Al}(p) \setminus \{x_0\}$. Indeed, if by contradiction there existed $z^* \in \mathrm{Al}(p) \setminus \{x_0\}$ such that $c^{N(p)}(x_0, y_0) \leq c^{N(p)}(z^*, x_0)$, then it would be $c^{N(p)}(z^*, x_0) = |\mathrm{In}(p)|$. Thus, for every $i \in \mathrm{In}(p)$, $z^* \succ_{p(i)} x_0$ and then $z^* \in S$, a contradiction. Then, by Proposition 4(ii), we conclude that $y_0 \notin \mathscr{A}(N(p)) = \mathsf{A}(p)$.

- Assume now that $S \neq \{x_0\}$. Consider the relation R on S defined as

$$R = \Big\{(x, y) \in S^2_* : x \succ_{p(i)} y \text{ for all } i \in \mathrm{In}(p)\Big\}.$$

  It is simple to show that R is transitive and asymmetric. Then there exists $x^* \in S$ that is a maximal element of R. Note that, since $S \neq \{x_0\}$, x_0 is not a maximal element of R and then $x^* \neq x_0$. Since $x^* \in S$ and $x^* \neq x_0$, we know that, for every $i \in \mathrm{In}(p)$, $x^* \succ_{p(i)} x_0$. Then, for every $i \in \mathrm{In}(p)$, $x^* \succ_{p(i)} y_0$. We claim that $c^{N(p)}(x^*, y_0) > c^{N(p)}(z, x^*)$ for all $z \in \mathrm{Al}(p) \setminus \{x^*\}$. Indeed, if by contradiction there existed $z^* \in \mathrm{Al}(p) \setminus \{x^*\}$ such that $c^{N(p)}(x^*, y_0) \leq c^{N(p)}(z^*, x^*)$, then it would be $c^{N(p)}(z^*, x^*) = |\mathrm{In}(p)|$. Thus, for every $i \in \mathrm{In}(p)$, $z^* \succ_{p(i)} x^*$ and then $z^* \succ_{p(i)} x_0$. Thus, $z^* \in S$ and $(z^*, x^*) \in R$, which contradicts the fact that x^* is maximal. Then, by Proposition 4(ii), we conclude that $y_0 \notin \mathscr{A}(N(p)) = \mathsf{A}(p)$.

Let us prove that A satisfies monotonicity. Let $p, q \in \mathbf{L}^*$ with $\mathrm{In}(p) = \mathrm{In}(q)$ and $\mathrm{Al}(p) = \mathrm{Al}(q)$. Assume that $x_0 \in \mathsf{A}(p)$ and that x_0 improves its position from p to q. We have to show that $x_0 \in \mathsf{A}(q)$. Observe now that

- $N(p)$ and $N(q)$ are such that $\mathrm{Ve}(N(p)) = \mathrm{Ve}(N(q)) = \mathrm{Al}(p)$,
- for every $y \in \mathrm{Al}(p) \setminus \{x_0\}$, $c^{N(q)}(x_0, y) \geq c^{N(p)}(x_0, y)$ and $c^{N(p)}(y, x_0) \geq c^{N(q)}(y, x_0)$,
- for every $(x, y) \in (\mathrm{Al}(p) \setminus \{x_0\})^2_*$, $c^{N(q)}(x, y) = c^{N(p)}(x, y)$.

Since $x_0 \in \mathsf{A}(p) = \mathscr{A}(N(p))$, by Theorem 10, we finally get $x_0 \in \mathscr{A}(N(q)) = \mathsf{A}(q)$, as desired.

Let us prove that A is independent of clones. Let $p \in \mathbf{L}^*$, $u_0 \in \mathrm{Al}(p)$, $C \in P_f(\mathbb{N})$ with $C \cap \mathrm{Al}(p) = \varnothing$, and $q \in \mathbf{L}^*$ obtained from p by adding the set of clones C of u_0. We must prove that

- for every $x_0 \in \mathrm{Al}(p) \setminus \{u_0\}$, $x_0 \in \mathsf{A}(p)$ if and only if $x_0 \in \mathsf{A}(q)$,
- $u_0 \in \mathsf{A}(p)$ if and only if $(C \cup \{u_0\}) \cap \mathsf{A}(q) \neq \varnothing$.

It is simple to check that $N(q)$ is obtained from $N(p)$ by adding the set of clones C of u_0. Thus, by Theorem 12(i), we deduce that, for every $x_0 \in \mathrm{Al}(p) \setminus \{u_0\} = \mathrm{Ve}(N(p)) \setminus \{u_0\}$, $x_0 \in \mathsf{A}(p) = \mathscr{A}(N(p))$ if and only if $x_0 \in \mathsf{A}(q) = \mathscr{A}(N(q))$; by Theorem 12(ii), we deduce that $u_0 \in \mathsf{A}(p) = \mathscr{A}(N(p))$ if and only if $(C \cup \{u_0\}) \cap \mathsf{A}(q) = (C \cup \{u_0\}) \cap \mathscr{A}(N(q)) \neq \varnothing$.

Let us prove that A satisfies the Smith principle, and so also the Condorcet principle. Let $p \in \mathbf{L}^*$ and $S \subseteq \mathrm{Al}(p)$ with $S \neq \varnothing$ and $c^{N(p)}(x,y) > \frac{|\mathrm{In}(p)|}{2}$ for all $x \in S$ and $y \in \mathrm{Al}(p) \setminus S$. We must prove that $\mathsf{A}(p) \subseteq S$. We know that $N(p)$ is balanced with constant $|\mathrm{In}(p)|$. Thus, $c^{N(p)}(x,y) > \frac{|\mathrm{In}(p)|}{2} > c^{N(p)}(y,x)$ for all $x \in S$ and $y \in \mathrm{Al}(p) \setminus S$. By Proposition 9, we conclude that $\mathsf{A}(p) = \mathscr{A}(N(p)) \subseteq S$.

Let us prove that A satisfies the inclusive Condorcet principle. Let $p \in \mathbf{L}^*$. If $|\mathrm{Al}(p)| = 1$, the property is trivially satisfied. Assume then that $|\mathrm{Al}(p)| \geq 2$ and let $x \in \mathrm{Al}(p)$ be such that $c^{N(p)}(x,y) \geq \frac{|\mathrm{In}(p)|}{2}$ for all $y \in \mathrm{Al}(p) \setminus \{x\}$. We must show that $x \in \mathsf{A}(p)$. We know that $N(p)$ is balanced with constant $|\mathrm{In}(p)|$. Thus, $c^{N(p)}(x,y) \geq \frac{|\mathrm{In}(p)|}{2} \geq c^{N(p)}(y,x)$ for all $y \in \mathrm{Al}(p) \setminus \{x\}$. By Proposition 8(i), we conclude that $x \in \mathscr{A}(N(p)) = \mathsf{A}(p)$.

Let us prove that A satisfies the Condorcet loser principle. Let $p \in \mathbf{L}^*$ with $|\mathrm{Al}(p)| \geq 2$ and $x \in \mathrm{Al}(p)$ be such that $c^{N(p)}(x,y) < \frac{|\mathrm{In}(p)|}{2}$ for all $y \in \mathrm{Al}(p) \setminus \{x\}$. We must prove that $x \notin \mathsf{A}(p)$. We know that $N(p)$ is balanced with constant $|\mathrm{In}(p)|$. Thus, $c^{N(p)}(x,y) < \frac{|\mathrm{In}(p)|}{2} < c^{N(p)}(y,x)$ for all $y \in \mathrm{Al}(p) \setminus \{x\}$. By Proposition 8(iii), we conclude that $x \notin \mathscr{A}(N(p)) = \mathsf{A}(p)$.

Finally, let us prove that A satisfies cancellation. Let $p \in \mathbf{L}^*$ satisfy $N(p) = N(p)^r$. Proposition 11 applied to $N(p)$ yields $\mathsf{A}(p) = \mathscr{A}(N(p)) = \mathrm{Al}(p)$. □

Proof of Proposition 16. If $|\mathrm{Al}(p)| \leq 2$, the proof is immediate. If $|\mathrm{Al}(p)| = 3$, then we have that $N(p) \in \mathcal{N}$ is balanced and $|\mathrm{Ve}(N(p))| = 3$. Therefore, we immediately conclude by applying Proposition 13 and recalling the definitions of A, Sch, and Sim. □

Proof of Proposition 17. Apply Proposition 4(ii) to $N(p)$, and the definition of A. □

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References

Bang-Jensen, J., Gutin, G., 2007. *Digraphs: theory, algorithms and applications.* Springer, London.

Berge, C., 1962. *The theory of graphs and its applications.* Methuen, London.

Bouyssou, D., 1992. Ranking methods based on valued preference relations: a characterization of the net flow method. *European Journal of Operational Research* 60, 61-67.

Bubboloni, D., Gori, M., 2025. A generalization to networks of Young's characterization of the Borda rule. *Annals of Operations Research* 349, 1501-1552.

Chu, Y.-J., Liu, T.-H., 1965. On the shortest arborescence of a directed graph. *Scientia Sinica* XIV, 1396-1400.

Delver, R., Monsuur, H., 2001. Stable sets and standards of behaviour. *Social Choice and Welfare* 18, 555-570.

Deo, N., 1974. *Graph theory with applications to engineering and computer science.* Prentice-Hall Series in Automatic Computation, Prentice-Hall, Englewood Cliffs, NJ.

Döring, M., Brill, M., Heitzig, J., 2026. The River voting method. *Proceedings of the AAAI Conference on Artificial Intelligence* 40(20), 16846-16854.

Duggan, J., 2013. Uncovered sets. *Social Choice and Welfare* 41, 489-535.

Edmonds, J., 1967. Optimum branchings. *Journal of Research of the National Bureau of Standards Section B* 71B, 233-240.

Fischer, F., Hudry, O., Niedermeier, R., 2016. Weighted tournament solutions. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D. (Eds.), *Handbook of computational social choice*, Chapter 4. Cambridge University Press, Cambridge.

Fishburn, P.C., 1977. Condorcet social choice functions. *SIAM Journal on Applied Mathematics* 33, 469-489.

González-Díaz, J., Hendrickx, R., Lohmann, E., 2014. Paired comparisons analysis: an axiomatic approach of ranking methods. *Social Choice and Welfare* 42, 139-169.

Gori, M., 2023. Families of abstract decision problems whose admissible sets intersect in a singleton. *Social Choice and Welfare* 61, 131-154.

Gori, M., 2024. A solution for abstract decision problems based on maximum flow value. *Mathematical Social Sciences* 130, 24-37.

Han, W., van Deemen, A., 2016. On the solution of w-stable sets. *Mathematical Social Sciences* 84, 87-92.

Harary, F., 1969. *Graph theory.* Addison-Wesley, Reading, MA.

Holliday, W.H., Pacuit, E., 2023a. Split cycle: a new Condorcet-consistent voting method independent of clones and immune to spoilers. *Public Choice* 197, 1-62.

Holliday, W.H., Pacuit, E., 2023b. Stable voting. *Constitutional Political Economy* 34, 421-433.

Kalai, E., Pazner, E.A., Schmeidler, D., 1976. Collective choice correspondences as admissible outcomes of social bargaining processes. *Econometrica* 44, 233-240.

Kalai, E., Schmeidler, D., 1977. An admissible set occurring in various bargaining situations. *Journal of Economic Theory* 14, 402-411.

Kemeny, J.G., 1959. Mathematics without numbers. *Daedalus* 88(4), 577-591.

Langville, A.N., Meyer, C.D., 2012. *Who is #1? The science of rating and ranking.* Princeton University Press, Princeton.

Laslier, J.-F., 1997. *Tournament solutions and majority voting.* Studies in Economic Theory, Volume 7. Springer.

Miller, N., 1980. A new solution set for tournaments and majority voting: further graph-theoretical approaches to the theory of voting. *American Journal of Political Science* 24, 68-96.

Peris, J.E., Subiza, B., 2013. A reformulation of von Neumann-Morgenstern stability: m-stability. *Mathematical Social Sciences* 66, 51-55.

Schulze, M., 2011. A new monotonic, clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method. *Social Choice and Welfare* 36, 267-303.

Schulze, M., 2018. The Schulze method of voting. *arXiv preprint* arXiv:1804.02973.

Schwartz, T., 1972. Rationality and the myth of the maximum. *Noûs* 6, 97-117.

Schwartz, T., 1986. *The logic of collective choice.* Columbia University Press, New York.

Shenoy, P.P., 1979. On coalition formation: a game theoretical approach. *International Journal of Game Theory* 8, 133-164.

Shenoy, P.P., 1980. A dynamic solution concept for abstract games. *Journal of Optimization Theory and Applications* 32, 151-169.

Simpson, P.B., 1969. On defining areas of voter choice: Professor Tullock on stable voting. *Quarterly Journal of Economics* 83, 478-490.

Smith, J.H., 1973. Aggregation of preferences with variable electorate. *Econometrica* 41, 1027-1041.

Szpilrajn, E., 1930. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae* 16, 386-389.

Tideman, T.N., 1987. Independence of clones as a criterion for voting rules. *Social Choice and Welfare* 4, 185-206.

van den Brink, R., Gilles, R.P., 2009. The outflow ranking method for weighted directed graphs. *European Journal of Operational Research* 193, 484-491.

van Deemen, A., 1991. A note on generalized stable sets. *Social Choice and Welfare* 8, 255-260.

Vaziri, B., Dabadghao, S., Yih, Y., Morin, T.L., 2018. Properties of sports ranking methods. *Journal of the Operational Research Society* 69(5), 776-787.

von Neumann, J., Morgenstern, O., 1944. *Theory of games and economic behavior*. Princeton University Press, Princeton.

Young, H.P., 1974. An axiomatization of Borda's rule. *Journal of Economic Theory* 9, 43-52.

Zwicker, W.S., 2016. Introduction to the theory of voting. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D. (Eds.), *Handbook of computational social choice*, Chapter 2. Cambridge University Press, Cambridge.