UNIVERSITÀ DEGLI STUDI FIRENZE

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DIPARTIMENTO DI SCIENZE PER L'ECONOMIA E L'IMPRESA

Working Papers
Quantitative Methods for Social Sciences

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Michele Gori

Working Paper N. 04/2023

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze (Italia) www.disei.unifi.it

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Michele Gori
Dipartimento di Scienze per l'Economia e l'Impresa
Università degli Studi di Firenze
via delle Pandette 9, 50127 Firenze, Italy
e-mail: michele.gori@unifi.it
https://orcid.org/0000-0003-3274-041X

May 30, 2023

Abstract

An abstract decision problem is an ordered pair where the first component is a nonempty and finite set of alternatives from which a society has to make a choice and the second component is an irreflexive relation on that set representing a dominance relation. A crucial problem is to find a reasonable solution that allows to select, for any given abstract decision problem, some of the alternatives. A variety of solutions have been proposed over the years. In this paper we propose a new solution, called justifiable set, that naturally stems from the work by Bubboloni and Gori (The flow network method, Social Choice and Welfare 51, p.621-656, 2018) and that is based on the concept of maximum flow value in a digraph. We analyse its properties and its relation with other solutions like the core, the admissible set, the Copeland set and the generalized stable set. We also show that the justifiable set allows to define a new Condorcet social choice correspondence strictly related to the Copeland social choice correspondence and fulfilling lots of desirable properties among which anonymity, neutrality, Pareto optimality and monotonicity.

Keywords: abstract decision problem; admissible set; generalized stable set; maximum flow; Copeland set; Copeland method.

JEL classification: D71.

1 Introduction

An abstract decision problem is a pair (X, R) where X is a nonempty and finite set and R is an irreflexive relation on X. The set X is interpreted as a set of alternatives from which a society has to make a choice. The relation R describes a dominance relation among alternatives and if $(x, y) \in R$ we say that x dominates y. The relation R can be interpreted by saying that if an alternative y is taken into consideration by the society, then the existence of alternatives dominating y implies that the society must reject y and start considering one of those alternatives. If instead there is no alternative dominating y, then the society will continue to consider y as there is no reason for the society to change its mind.[1]

The problem of finding a sensible method to select one or more sets of alternatives for any given abstract decision problem is a crucial issue. In the literature many methods, called solutions, are available, each of them based on a clear and sensible rationale. Among them there are the core,

[1]Such an interpretation is basically in line with the one by Kalai and Schmeidler (1977) and Shenoy (1980) but other interpretations are possible. Von Neumann and Morgenstern (1944, p.41) interpret the fact that $(x, y) \in R$ by saying that the alternative x, if taken into consideration, excludes acceptance of the alternative y (without forecasting what alternative will ultimately be accepted).

the stable set (Von Neumann and Morgenstern 1944), the Copeland set (Copeland, 1951; Fishburn, 1977), the admissible set (Kalai et al. 1976), the generalized stable set (Van Deemen 1991), the socially stable set (Delver and Monsuur 2001), the m-stable set (Peris and Subiza 2013), the w-stable set (Han and Van Deemen 2016). The core, the admissible set and the Copeland set are solutions meeting many desirable properties and, differently from the other mentioned solutions, they always select a unique set of alternatives. However, they also have some disadvantages. If fact, the core can be empty and the admissible set, that is never empty and always includes the core, is usually very large and then unable to carefully discriminate among alternatives. The Copeland set is never empty and it is definitely natural and robust when applied to abstract decision problems whose dominance relation is asymmetric and quasi-complete, that is, the so called tournaments. On the other hand, it may exhibit some drawbacks when applied to abstract decision problems that are not tournaments. In particular, the Copeland set does not necessarily includes the core and some of its elements may lie outside the admissible set. Similar considerations hold true also for the other versions of the Copeland set due to Henriet (1995) and Laslier (1997), versions that coincide with the original one on tournaments.

In this paper we propose a new solution for abstract decision problems, called justifiable set. The definition of justifiable set is based on the concept of maximum flow value in a digraph. Given two distinct vertices x and y in a digraph, the maximum flow value from x and y is the largest possible number of arc-disjoint paths from x to y. Let us observe that, under a mathematical viewpoint, any abstract decision problem is a digraph once the set of alternatives is identified with the set of vertices and the dominance relation is identified with the set of arcs. Given now an abstract decision problem (X, R), the justifiable set associated with (X, R) is then defined as the set of the alternatives x having the property that, for every other alternative y, the maximum flow value from x to y is greater than or equal to the maximum flow value from y to x. That definition is strongly inspired to the ideas in Gvozdik (1987), Belkin and Gvozdik (1989) and Bubboloni and Gori (2018). More precisely, the justifiable set coincides with the so called flow network solution (with parameter $k = 1$) restricted to the set of those networks that can be identified with digraphs, that is, the set of zero-one networks. For that reason, the analysis of the justifiable set greatly benefits from the numerous results about the flow network solution proved by Bubboloni and Gori (2018).

The justifiable set is proved to satisfy many properties some of which mitigating the disadvantages of the core, the admissible set and the Copeland set. In particular, the justifiable set always determines a unique set of alternatives and that set is nonempty and includes the core. Moreover, the justifiable set is a subset of the admissible set so that it is able to better discriminate among the alternatives than the admissible set. Further, it coincides with the Copeland set when computed on tournaments. That implies first that the justifiable set is certainly sound on tournaments. Moreover, it can be seen as a new way to extend the Copeland set from the set of tournaments to the whole set of abstract decision problems by means of a rationale different from the idea of associating a score with each alternative that underlies the Copeland set. It is also worth mentioning that the justifiable set is neutral, that is, it equally treats alternatives, satisfies a desirable monotonicity property and it is interestingly related to the generalized stable set, the w-stable set and the m-stable set.

We also propose a remarkable application of the justifiable set to social choice theory. Consider a society where individuals have to select some elements in a given set of alternatives by aggregating their preferences on those alternatives. Assume that individual preferences are represented by linear orders and call a preference profile any list of preferences, each of them associated with a specific individual in the society. Any function from the set of preference profiles to the set of nonempty subsets of alternatives is called social choice correspondence (SCC) and represents a particular method for aggregating individual preferences. The concept of SCC is crucial in social choice theory and a large number of SCCs have been proposed over the years. By means of the justifiable set we are able to define a new SCC, called justifiable majority SCC, simply associating with any preference profile the justifiable set applied the majority digraph determined by the preference profile. The justifiable majority SCC turns out to satisfies a variety of desirable properties. Indeed, the justifiable majority SCC is proved to be anonymous, neutral, Pareto optimal, monotonic and immune to the reversal bias. Moreover, it is a refinement of the Schwartz SCC and then, in particular, it satisfies the Condorcet principle. Further, it always selects all the weak Condorcet winners and never selects the Condorcet loser. Finally it coincides with the classic Copeland SCC (Copeland, 1951; Fishburn, 1977) on the set of the preference profiles whose majority digraph is a tournament and hence, in particular, on the set of preference profiles related to an odd number of individuals.

The paper is organized as follows. In Section 2 we introduce some preliminary notation. In Section 3 we give the definition of abstract decision problem and we recall some fundamental solutions, namely the core and the admissible set. In Section 4 we present the concept of maximum flow value between two alternatives in an abstract decision problem and in Section 5 the definition of justifiable set is given. Section 6 is devoted to the comparison of the justifiable set with the core and the admissible set and in Section 7 some general properties of the justifiable set are proved. In Sections 8 and 9 we compare the justifiable set with the Copeland set, the generalized stable set, the w-stable set and the m-stable set. Finally, in Sections 10, 11 and 12 we define the justifiable majority SCC, we study its properties and we describe its strong relation to the Copeland SCC.

2 Preliminary notation

We assume $0 \notin \mathbb{N}$ and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X be a finite set. We set $X^2_d = \{(x,y) \in X^2 : x = y\}$ and $X^2_* = \{(x,y) \in X^2 : x \neq y\}$. The size of X is denoted by $|X|$. The set of all the subsets of X is denoted by $P(X)$. The set of all the nonempty subsets of X is denoted by $P_*(X)$. The set of bijective functions from X to X is denoted by $\mathrm{Sym}(X)$ and the identity function is denoted by id.

A relation on X is a subset of X^2. Let R be a relation on X. The asymmetric part of R is the relation on X given by $\mathrm{as}(R) = \{(x,y) \in R : (y,x) \notin R\}$. We say that R is reflexive if, for every $x \in X$, $(x,x) \in R$; irreflexive if, for every $x \in X$, $(x,x) \notin R$; asymmetric if, for every $x, y \in X$, $(x,y) \in R$ implies $(y,x) \notin R$ (that is, $R = \mathrm{as}(R)$); transitive if, for every $x,y,z \in X$, $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$; quasi-transitive if $\mathrm{as}(R)$ is transitive; antisymmetric if, for every $x,y \in X$, $(x,y) \in R$ and $(y,x) \in R$ imply $x = y$; complete if, for every $x,y \in X$, $(x,y) \in R$ or $(y,x) \in R$; quasi-complete if, for every $x,y \in X$ with $x \neq y$, $(x,y) \in R$ or $(y,x) \in R$; a linear order if R is complete, transitive and antisymmetric. The set of maximal elements of R is defined as

$$\mathrm{Max}(R) = \{x \in X : \forall y \in X\ (y,x) \in R \text{ implies } (x,y) \in R\} = \{x \in X : \forall y \in X\ (y,x) \notin \mathrm{as}(R)\}.$$

Observe that $\mathrm{Max}(R) = \mathrm{Max}(\mathrm{as}(R))$. The reversal of R is the relation $R^r = \{(x,y) \in X^2_* : (y,x) \in R\}$. For every $\psi \in \mathrm{Sym}(X)$, we set $R^\psi = \{(x,y) \in X^2_* : (\psi^{-1}(x), \psi^{-1}(y)) \in R\}$. For every $x \in X$, we set $D^R(x) = \{y \in X : (x,y) \in R\}$, $\overline{D}^R(x) = \{y \in X : (y,x) \in R\}$, $D^R_*(x) = \{y \in X : (x,y) \in \mathrm{as}(R)\}$, $\overline{D}^R_*(x) = \{y \in X : (y,x) \in \mathrm{as}(R)\}$, $I^R(x) = \{y \in X : (x,y) \in R \text{ and } (y,x) \in R\}$ and $N^R(x) = \{y \in X : (x,y) \notin R \text{ and } (y,x) \notin R\}$.

In the rest of the paper X is a fixed nonempty and finite set.

3 Abstract decision problems and classic solutions

An abstract decision problem on X is an ordered pair (X,R), where R is an irreflexive relation on X called dominance relation. If $(x,y) \in R$, we say that x dominates y (according to R). The set of abstract decision problems on X is denoted by $\mathcal{A}(X)$. Given $(X,R) \in \mathcal{A}(X)$, the elements of X are interpreted as mutually exclusive alternatives from which a society has to make a choice. The dominance relation R is instead interpreted as the complete description of a specific criterion applied by the society to pairwise compare the alternatives. The idea is that, assuming that the alternative y is taken into consideration by the society, if $\overline{D}^R(y) \neq \varnothing$ then, on the basis of the criterion described by R, the society must reject y and start considering one of the alternatives in $\overline{D}^R(y)$, if instead $\overline{D}^R(y) = \varnothing$ the society has no reason to change its mind. Thus, the fact that x dominates y according to R means that if y is taken into consideration by the society then the society must reject y and might start considering x.

Abstract decision problems can model a variety of situations. Consider, for instance, a society whose purpose is to establish which teams must be considered the best at a certain stage of a round-robin competition where ties are not allowed (so that each match always determined a winner and a loser). In that case, the decision problem can be modelled by the abstract decision problem (X, R), where X is the set of teams and R is the relation on X such that $(x, y) \in R$ if x played against y and won. In other words, the society is comparing any pair of teams on the basis of the result of the their match, if any. Consider now a society whose purpose is to select some candidates among the ones running for a certain office. In that case, the decision problem can be modelled by the abstract decision problem (X, R), where X is the set of candidates and R is the relation on X such that $(x, y) \in R$ if the majority of individuals prefers x to y. The society is then comparing any pair of candidates on the basis of a majority criterion. Note that in both examples the dominance relation R is asymmetric.

The theory of abstract decision problems focuses on the problem of determining a subset of alternatives whose elements could be reasonably interpreted as potential outcomes. Any method for determining one or more subsets of alternatives is called a solution. Let us introduce now some important solutions that will be useful for our purposes. The first one is the set of Condorcet winners.

Definition 1. *Let $(X, R) \in \mathcal{A}(X)$. An alternative x is a Condorcet winner of (X, R) if, for every $y \in X\backslash\{x\}$, we have that $(x, y) \in \mathrm{as}(R)$. The set of Condorcet winners of (X, R) is denoted by $CW(X, R)$.*

Note that $|CW(X, R)| \leqslant 1$. Indeed, if there were $x^*, x^{**} \in CW(X, R)$ with $x^* \neq x^{**}$, then it should be $(x^*, x^{**}) \in \mathrm{as}(R)$ and $(x^{**}, x^*) \in \mathrm{as}(R)$, a contradiction. The set of Condorcet winners is often empty. The core is another solution selecting a larger set of alternatives, namely the ones that are dominated by no alternative.

Definition 2. *Let $(X, R) \in \mathcal{A}(X)$. The core of (X, R) is the set*

$$CO(X, R) = \{x \in X : \forall y \in X \ (y, x) \notin R\}.$$

Of course, $CW(X, R) \subseteq CO(X, R) \subseteq \mathrm{Max}(R)$. Moreover, if R quasi-complete then we have that $CW(X, R) = CO(X, R)$. Note also that the core may be empty.

A further remarkable solution for abstract decision problems is the admissible set introduced by Kalai et al. (1976) and Kalai and Schmeidler (1977).[2] Let us present its definition. Let (X, R) be an abstract decision problem. A path in (X, R) is a sequence $(x_j)_{j=1}^m$, where $m \geqslant 2$, $x_1, \ldots, x_m$ are distinct elements of X and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in R$. If $x, y \in X$ are distinct, a path from x to y in (X, R) is a path $(x_j)_{j=1}^m$ in (X, R) such that $x_1 = x$ and $x_m = y$. If there is a path $(x_j)_{j=1}^m$ from x to y in (X, R) we say that x directly or indirectly dominates y (according to R). Indeed, if $m = 2$ the path exactly describes the fact that x dominates y. If instead $m \geqslant 3$, the fact that, for every $j \in \{1, \ldots, m-1\}$, x_j dominates x_{j+1}, suggests that there is a sort of indirect domination of x over y, meaning that if y is taken into consideration by the society then y must be rejected and the society might start considering x after reviewing some alternatives.

Let us denote by R^τ the reflexive and transitive closure of R, that is, the smallest reflexive and transitive relation on X containing R. It is easily seen that

$$R^\tau = \left\{(x, y) \in X_*^2 : \text{ there exists a path in } (X, R) \text{ from } x \text{ to } y\right\} \cup X_d^2.$$

Thus, if $(x, y) \in R^\tau$ we have that $x = y$ or x directly or indirectly dominates y, while if $(x, y) \notin R^\tau$ we have that x neither directly nor indirectly dominates y.

Definition 3. *Let $(X, R) \in \mathcal{A}(X)$. The admissible set of (X, R) is the set $AD(X, R) = \mathrm{Max}(R^\tau)$.*

Thus, an alternative x is in the admissible set if the fact that y directly or indirectly dominates x implies that in turn x directly or indirectly dominates y. It is worth noting that when (X, R) is

[2]Equivalent definitions of that concept are given by Schwartz (1972, 1986) and Shenoy (1979, 1980). See also Van Deemen (1997) and Gori (2022).

a tournament, that is, R is asymmetric and quasi-complete, then $AD(X,R)$ coincides with the top cycle associated with (X,R). Moreover, for every abstract decision problem (X,R), $AD(X,R) \neq \varnothing$.[3]

Finally note that $|X| = 1$ implies that, for every $(X,R) \in \mathcal{A}(X)$, $CW(X,R) = CO(X,R) = AD(X,R) = X$.

4 Abstract decision problems and maximum flow value

Let $(X,R) \in \mathcal{A}(X)$ and let $x, y \in X$ with $x \neq y$. We denote by $\Gamma(X,R;x,y)$ the set of paths from x to y in (X,R). Given $\gamma = (x_j)_{j=1}^m \in \Gamma(X,R;x,y)$, we set $A(\gamma) = \{(x_1,x_2),\ldots,(x_{m-1},x_m)\} \subseteq R$. Let $k \in \mathbb{N}$. A sequence of k arc-disjoint paths in $\Gamma(X,R;x,y)$ is a sequence $(\gamma_i)_{i=1}^k$ of k paths in $\Gamma(X,R;x,y)$ such that, for every $i,j \in \{1,\ldots,k\}$ with $i \neq j$, $A(\gamma_i) \cap A(\gamma_j) = \varnothing$. We denote by $\Gamma_k(X,R;x,y)$ the set of sequences of k arc-disjoint paths in $\Gamma(X,R;x,y)$. Of course, $\Gamma(X,R;x,y) = \varnothing$ implies $\Gamma_k(X,R;x,y) = \varnothing$ for all $k \in \mathbb{N}$ and $\Gamma(X,R;x,y) \neq \varnothing$ implies $\Gamma_1(X,R;x,y) \neq \varnothing$. Moreover $\Gamma_k(X,R;x,y) = \varnothing$ for some $k \in \mathbb{N}$.

Any path in $\Gamma(X,R;x,y)$ justifies the claim that x directly or indirectly dominates y and it can be identified with an argument supporting that claim. As a consequence, any sequence $(\gamma_i)_{i=1}^k \in \Gamma_k(X,R;x,y)$ can be interpreted as a family of k independent arguments supporting the claim that x directly or indirectly dominates y. Here independent means that any two distinct arguments involve different elements of the dominance relation so that they are substantially different from each other. The maximum flow value from x to y in (X,R) is defined as

$$\varphi_{xy}^{(X,R)} = \max\Big(\{0\} \cup \{k \in \mathbb{N} : \Gamma_k(X,R;x,y) \neq \varnothing\}\Big). \tag{1}$$

Thus, $\varphi_{xy}^{(X,R)}$ can be interpreted as a measure of the strength of the claim that x directly or indirectly dominates y as it represents the maximum number of independent arguments supporting that claim.

4.1 Networks

A network on X is a triple $N = (X, X_*^2, c)$, where c is a function from X_*^2 to $\mathbb{N}_0$. Let $N = (X, X_*^2, c)$ be a network on X. For every $x \in X$, the outdegree and the indegree of x in N are respectively defined by

$$o^N(x) = \sum_{y \in X \setminus \{x\}} c(x,y), \qquad i^N(x) = \sum_{y \in X \setminus \{x\}} c(y,x).$$

The reversal of N is the network $N^r = (X, X_*^2, c^r)$ where, for every $(x,y) \in X_*^2$, $c^r(x,y) = c(y,x)$. Given $\psi \in \mathrm{Sym}(X)$, we set $N^\psi = (X, X_*^2, c^\psi)$, where c^ψ is defined, for every $(x,y) \in X_*^2$, by $c^\psi(x,y) = c(\psi^{-1}(x), \psi^{-1}(y))$. We say that N is a balanced network if there exists $k \in \mathbb{N}_0$ such that, for every $(x,y) \in X_*^2$, $c(x,y) + c(y,x) = k$. Let $x, y \in X$ with $x \neq y$. A flow from x to y in N is a function $f : X_*^2 \to \mathbb{N}_0$ such that, for every $(u,v) \in X_*^2$, $f(u,v) \leqslant c(u,v)$ and, for every $z \in X \setminus \{x,y\}$,

$$\sum_{v \in X \setminus \{z\}} f(z,v) = \sum_{u \in X \setminus \{z\}} f(u,z).$$

The set of flows from x to y in N is nonempty and finite and it is denoted by $\mathcal{F}(N;x,y)$. Given $f \in \mathcal{F}(N;x,y)$, the value of f is the integer

$$\varphi^N(f) = \sum_{v \in X \setminus \{x\}} f(x,v) - \sum_{u \in X \setminus \{x\}} f(u,x).$$

The number

$$\varphi_{xy}^N = \max_{f \in \mathcal{F}(N;x,y)} \varphi^N(f), \tag{2}$$

[3]See, for instance, Theorem 4.1 in Shenoy (1980) or Theorem 9 in Gori (2022).

which is well defined and belongs to $\mathbb{N}_0$, is called the maximum flow value from x to y in N. If $f \in \mathcal{F}(N; x, y)$ is such that $\varphi^N(f) = \varphi^N_{xy}$, then f is called a maximum flow from x to y in N.

Let $(X, R) \in \mathcal{A}(X)$. We have that (X, R) formally corresponds to a digraph (without loops) and can be identified with the network $N(X, R) = (X, X^2_*, c^{(X,R)})$, where $c^{(X,R)} : X^2_* \to \mathbb{N}_0$ is defined, for every $(x, y) \in X^2_*$, by $c^{(X,R)}(x, y) = 1$ if $(x, y) \in R$, and $c^{(X,R)}(x, y) = 0$ if $(x, y) \notin R$. Observe that, for every $x \in X$, the outdegree of x in $N(X, R)$ is $|D^R(x)|$ and the indegree of x in $N(X, R)$ is $|\overline{D}^R(x)|$. Moreover, it can be proved that, for every $x, y \in X$ with $x \neq y$, the maximum flow value from x to y in (X, R) defined in (1) coincides with the maximum flow value from x to y in $N(X, R)$ defined in (2) (see Bang-Jensen and Gutin, 2008, Lemma 7.1.5). It can be also verified that $N(X, R)^r = N(X, R^r)$ and that, for every $\psi \in \mathrm{Sym}(X)$, $N(X, R)^\psi = N(X, R^\psi)$. Finally, $N(X, R)$ is balanced if and only if (X, R) is a tournament or $(X, R) = (X, \varnothing)$ or $(X, R) = (X, X^2_*)$.

The identification of the abstract decision problem (X, R) with the network $N(X, R)$ will turn out to be crucial to apply to (X, R) some of the results proved in Bubboloni and Gori (2018).

5 The justifiable set

Definition 4. *Let $(X, R) \in \mathcal{A}(X)$. The flow relation associated with (X, R) is the relation on X given by*

$$\mathfrak{F}(X, R) = \left\{(x, y) \in X^2_* : \varphi^{(X,R)}_{xy} \geqslant \varphi^{(X,R)}_{yx}\right\} \cup X^2_d.$$

The flow relation was first proposed by Gvozdik (1987) for balanced networks and later generalized on the whole set of networks by Belkin and Gvozdik (1989) and Bubboloni and Gori (2018). Here we are basically considering the flow relation associated with those networks corresponding to abstract decision problems. We are now ready to introduce the main object of the paper.

Definition 5. *Let $(X, R) \in \mathcal{A}(X)$. The justifiable set associated with (X, R) is the set $J(X, R) = \mathrm{Max}(\mathfrak{F}(X, R))$.*

It is useful to observe that

$$J(X, R) = \left\{x \in X : \forall y \in X \backslash \{x\}\ \varphi^{(X,R)}_{xy} \geqslant \varphi^{(X,R)}_{yx}\right\} = \{x \in X : \forall y \in X\ (x, y) \in \mathfrak{F}(X, R)\}.$$

Thus, an alternative x belongs to the set $J(X, R)$ if and only if, for any other alternative y, the maximum number of independent arguments supporting the claim that x directly or indirectly dominates y is greater than or equal to the maximum number of independent arguments supporting the claim that y directly or indirectly dominates x.

It is important to note that the justifiable set is not properly a novel object as it corresponds to the restriction of the so-called flow network solution (with parameter $k = 1$) defined in Bubboloni and Gori (2018) to the set of networks corresponding to abstract decision problems. As a consequence, some important properties of the justifiable set can be directly deduced by the results proved by Bubboloni and Gori (2018). Nevertheless, the analysis of the justifiable set can be further deepened looking at those issues that are important to the theory of abstract decision problems, a framework different from the one considered by Bubboloni and Gori (2018). In particular, we are focusing on the comparison of the justifiable set with other solutions.

Let us begin with establishing an important fact, namely that the justifiable set is always nonempty.

Proposition 6. *Let $(X, R) \in \mathcal{A}(X)$. Then $J(X, R) \neq \varnothing$.*

Proof. By Theorem 3 in Bubboloni and Gori (2018) applied to $N(X, R)$, we deduce that $\mathfrak{F}(X, R)$ is complete and quasi-transitive. As a consequence, the set of the maximal elements of $\mathfrak{F}(X, R)$ is nonempty, that is, $J(X, R) \neq \varnothing$. □

Note also that $|X| = 1$ implies that, for every $(X, R) \in \mathcal{A}(X)$, $J(X, R) = X$.

6 Comparison with the core and the admissible set

The next propositions show that the justifiable set is core inclusive and it is also a refinement of the admissible set.

Proposition 7. *Let $(X, R) \in \mathcal{A}(X)$. Then $CO(X, R) \subseteq J(X, R)$.*

Proof. Let $x \in CO(X, R)$. Thus, for every $y \in X$, $(y, x) \notin R$. In order to prove that $x \in J(X, R)$, consider $y^* \in X$ with $y^* \neq x$ and prove that $\varphi_{xy^*}^{(X,R)} \geqslant \varphi_{y^*x}^{(X,R)}$. We claim that there is no path from y^* to x in (X, R). Indeed, assume by contradiction that there exists a path $(x_j)_{j=1}^m$ from y^* to x in (X, R). Thus, in particular, we have that $(x_{m-1}, x) \in R$ and that is a contradiction. We deduce then that $\varphi_{y^*x}^{(X,R)} = 0$. Since $\varphi_{xy^*}^{(X,R)} \geqslant 0$, we get $\varphi_{xy^*}^{(X,R)} \geqslant \varphi_{y^*x}^{(X,R)}$. ☐

Note that it can be $CO(X, R) \neq J(X, R)$ when $CO(X, R) \neq \varnothing$. Consider, for instance, the abstract decision problem (X, R), where $X = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1)\}$. In that case we have $CO(X, R) = \{3\}$ and $J(X, R) = \{1, 2, 3\}$.

Proposition 8. *Let $(X, R) \in \mathcal{A}(X)$. Then $J(X, R) \subseteq AD(X, R)$.*

Proof. Let us prove the inclusion $J(X, R) \subseteq AD(X, R)$ showing that $X \backslash AD(X, R) \subseteq X \backslash J(X, R)$. Consider $x \in X \backslash AD(X, R)$. Thus, there exists $y \in X$ such that $(y, x) \in R^\tau$ and $(x, y) \notin R^\tau$. Then $y \neq x$, there is a path from y to x in (X, R) and there is no path from x to y in (X, R). We deduce then that $\varphi_{yx}^{(X,R)} > 0$ and $\varphi_{xy}^{(X,R)} = 0$. That implies that $x \notin J(X, R)$. ☐

Note that it can be $J(X, R) \neq AD(X, R)$. Consider, for instance, the abstract decision problem (X, R), where $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (1, 3), (2, 4), (3, 4), (4, 1)\}$. In that case we have $J(X, R) = \{1, 2, 3\}$ and $AD(X, R) = \{1, 2, 3, 4\}$.

Proposition 8 has several interesting consequences. Let $D \subseteq X$. We say that D is a dominating set for (X, R) if $D \neq \varnothing$ and, for every $x \in D$ and $y \in X \backslash D$, $(x, y) \in \mathrm{as}(R)$.

Proposition 9. *Let $(X, R) \in \mathcal{A}(X)$ and let $D \subseteq X$ be a dominating set for (X, R). Then $AD(X, R) \subseteq D$.*

Proof. Consider $x \in AD(X, R)$ and assume by contradiction that $x \notin D$. Since $D \neq \varnothing$, we can pick $y \in D$. Thus, $(y, x) \in \mathrm{as}(R)$ so that, in particular, $x \neq y$ and $(y, x) \in R^\tau$. Since $x \in AD(X, R)$, we have that $(x, y) \in R^\tau$ so that there is a path $(x_j)_{j=1}^m$ in (X, R) from x to y. In particular, there exists $i^* \in \{1, \ldots, m-1\}$ such that $x_{i*} \notin D$ and $x_{i*+1} \in D$. Since $(x_{i*}, x_{i*+1}) \in R$, we deduce that $(x_{i*+1}, x_{i*}) \notin \mathrm{as}(R)$ and that contradicts the fact that D is a dominating set for (X, R). ☐

Corollary 10. *Let $(X, R) \in \mathcal{A}(X)$ and let $D \subseteq X$ be a dominating set for (X, R). Then $J(X, R) \subseteq D$.*

Proof. Simply note that by Propositions 8 and 9 we get $J(X, R) \subseteq AD(X, R) \subseteq D$. ☐

Proposition 11. *Let $(X, R) \in \mathcal{A}(X)$. If $CW(X, R) \neq \varnothing$, then $CW(X, R) = CO(X, R) = J(X, R) = AD(X, R)$.*

Proof. Assume that $CW(X, R) \neq \varnothing$. Thus $CW(X, R)$ is a dominating set for (X, R). By Propositions 7, 8 and 9 and by the fact that $CW(X, R) \subseteq CO(X, R)$, we have that $CW(X, R) \subseteq CO(X, R) \subseteq J(X, R) \subseteq AD(X, R) \subseteq CW(X, R)$. Thus, we conclude that in fact $CW(X, R) = CO(X, R) = J(X, R) = AD(X, R)$. ☐

A cycle in (X, R) is a sequence $(x_j)_{j=1}^m$, where $m \geqslant 3$, $x_1, \ldots, x_{m-1}$ are distinct elements of X, $x_1 = x_m$ and, for every $j \in \{1, \ldots, m-1\}$, $(x_j, x_{j+1}) \in R$. We say that a relation R on X is acyclic if there is no cycle in (X, R).

Proposition 12. *Let $(X, R) \in \mathcal{A}(X)$ and assume that R is acyclic. Then*

$$CO(X, R) = J(X, R) = AD(X, R) = \mathrm{Max}(R).$$

Proof. If R is acyclic, then R is asymmetric. As a consequence, $CO(X,R) = \mathrm{Max}(R)$. Since by Propositions 7 and 8, we have that $CO(X,R) \subseteq J(X,R) \subseteq AD(X,R)$, we conclude the proof showing that $AD(X,R) \subseteq CO(X,R)$. Consider $x \in AD(X,R)$. Assume by contradiction that $x \notin CO(X,R)$. Then there exists $y \in X\backslash\{x\}$ such that $(y,x) \in R$. Thus, $(y,x) \in R^\tau$ and since $x \in AD(X,R)$, it must be $(x,y) \in R^\tau$. There is then a path $(x_j)_{j=1}^m$ from x to y in (X,R). Thus, the sequence $(x'_j)_{j=1}^{m+1}$ in X such that, for every $j \in \{1,\dots,m\}$, $x'_j = x_j$ and $x'_{m+1} = x$ is a cycle in R and that contradicts the fact that R is acyclic. $\square$

7 Some properties of the justifiable set

The next proposition shows that the justifiable set is a neutral solution, that is, it equally treats the alternatives.

Proposition 13. *Let $(X,R) \in \mathcal{A}(X)$ and $\psi \in \mathrm{Sym}(X)$. Then $J(X,R^\psi) = \psi(J(X,R))$.*

Proof. From Proposition 8 in Bubboloni and Gori (2018) applied to $N(X,R)$, we know that $\mathfrak{F}(X,R^\psi) = \mathfrak{F}(X,R)^\psi$. Consider now $x \in J(X,R^\psi)$. Thus, for every $y \in X$, we have that $(x,y) \in \mathfrak{F}(X,R^\psi) = \mathfrak{F}(X,R)^\psi$. Thus, for every $y \in X$, we have that $(\psi^{-1}(x),\psi^{-1}(y)) \in \mathfrak{F}(X,R)$. Since $\psi \in \mathrm{Sym}(X)$, for every $y \in X$, we have that $(\psi^{-1}(x),y) \in \mathfrak{F}(X,R)$. That means that $\psi^{-1}(x) \in J(X,R)$ and then $x = \psi(\psi^{-1}(x)) \in \psi(J(X,R))$. That proves that $J(X,R^\psi) \subseteq \psi(J(X,R))$. Consider now $x \in \psi(J(X,R))$. Thus, there exists $z \in J(X,R)$ such that $x = \psi(z)$. We know that, for every $y \in X$, $(z,y) \in \mathfrak{F}(X,R)$. Thus, for every $y \in X$, $(\psi(z),\psi(y)) \in \mathfrak{F}(X,R)^\psi$. Since $\psi \in \mathrm{Sym}(X)$, we have that, for every $y \in X$, $(\psi(z),y) \in \mathfrak{F}(X,R)^\psi = \mathfrak{F}(X,R^\psi)$. Thus $x = \psi(z) \in J(X,R^\psi)$. That proves that $\psi(J(X,R)) \subseteq J(X,R^\psi)$. Then, we can conclude that $J(X,R^\psi) = \psi(J(X,R))$. $\square$

The next two propositions state conditions that prevent an alternative to be an element of the justifiable set.

Proposition 14. *Let $(X,R) \in \mathcal{A}(X)$ and $y \in X$. Assume that there exists $x \in X\backslash\{y\}$ such that $(x,y) \in \mathrm{as}(R)$, $D^R(y) \subseteq D^R(x)$ and $\overline{D}^R(x) \subseteq \overline{D}^R(y)$. Then $y \notin J(X,R)$.*

Proof. From Proposition 10 in Bubboloni and Gori (2018) applied to $N(X,R)$, we deduce that $(x,y) \in \mathfrak{F}(X,R)$ and $(y,x) \notin \mathfrak{F}(X,R)$. In particular, we get $y \notin J(X,R)$. $\square$

Proposition 15. *Let $(X,R) \in \mathcal{A}(X)$ and $y \in X$. If $|D^R(y)| < |\overline{D}^R(y)|$, then $y \notin J(X,R)$.*

Proof. From Proposition 13 in Bubboloni and Gori (2018) applied to $N(X,R)$, we have that if $|D^R(y)| < |\overline{D}^R(y)|$, then there exists $x \in X\backslash\{y\}$ such that $(x,y) \in \mathrm{as}(\mathfrak{F}(X,R))$. As a consequence, $y \notin J(X,R)$. $\square$

The next proposition shows that the justifiable set satisfies a monotonicity criterion that generalizes the standard concept of monotonicity for tournament solutions to solutions for abstract decision problems (Laslier, 1997, Definition 2.3.1).

Proposition 16. *Let $(X,R),(X,R') \in \mathcal{A}(X)$ and $x \in X$. Assume that $D^R(x) \subseteq D^{R'}(x)$, $\overline{D}^{R'}(x) \subseteq \overline{D}^R(x)$ and*

$$\{(y_1,y_2) \in R : y_1 \neq x, y_2 \neq x\} = \{(y_1,y_2) \in R' : y_1 \neq x, y_2 \neq x\}.$$

Then $x \in J(X,R)$ implies $x \in J(X,R')$.

Proof. From Proposition 11 in Bubboloni and Gori (2018), we deduce that, for every $y \in X\backslash\{x\}$, $(x,y) \in \mathfrak{F}(X,R)$ implies $(x,y) \in \mathfrak{F}(X,R')$. As a consequence, $x \in J(X,R)$ implies $x \in J(X,R')$. $\square$

The next proposition shows that the justifiable set satisfies a property that generalizes the concept of regularity for tournament solutions to solutions for abstract decision problems (Laslier, 1997, Definition 2.4.6). Note that, given $(X,R) \in \mathcal{A}(X)$, we have that Proposition 26 (for $k = 1$) in Bubboloni and Gori (2018) applied to $\mathfrak{F}(X,R)$ assures that $\mathfrak{F}(X,R) = X^2$ if and only if $J(X,R) = X$. We are going to use that fact several times in the rest of the section.

Proposition 17. *Let $(X, R) \in \mathcal{A}(X)$. Then $J(X, R) = X$ if and only if, for every $x \in X$, $|D^R(x)| = |\overline{D}^R(x)|$.*

Proof. We know that $J(X, R) = X$ if and only if $\mathfrak{F}(X, R) = X^2$. By Proposition 14 in Bubboloni and Gori (2018) applied to $N(X, R)$, we also have that $\mathfrak{F}(X, R) = X^2$ if and only if, for every $x \in X$, $|D^R(x)| = |\overline{D}^R(x)|$. That completes the proof. □

The next proposition explores the effect on the justifiable set of reversing the dominance relation. Proposition 18 implies, in particular, that if $|X| \geqslant 2$ and the justifiable set of (X, R) consists of a single alternative, then such an alternative cannot be an element of the justifiable set associated with (X, R^r).

Proposition 18. *Let $(X, R) \in \mathcal{A}(X)$. Then $J(X, R) \neq X$ implies $J(X, R) \nsubseteq J(X, R^r)$.*

Proof. Assume $J(X, R) \neq X$. Thus, $\mathfrak{F}(X, R) \neq X^2$ and by Proposition 34 (for $k = 1$) in Bubboloni and Gori (2018) applied to $\mathfrak{F}(X, R)$, we have that $J(X, R) \nsubseteq J(X, R^r)$. □

Proposition 19. *Let $(X, R) \in \mathcal{A}(X)$. Then $R = R^r$ implies $J(X, R) = X$.*

Proof. By Proposition 12 in Bubboloni and Gori (2018) applied to $N(X, R)$, we know that $\mathfrak{F}(X, R^r) = \mathfrak{F}(X, R)^r$. If $R = R^r$ we have then that $\mathfrak{F}(X, R) = \mathfrak{F}(X, R)^r$. Since $\mathfrak{F}(X, R)$ is complete we deduce that $\mathfrak{F}(X, R) = X^2$ and that implies $J(X, R) = X$. □

Proposition 20. *Let $(X, R) \in \mathcal{A}(X)$. Then, $J(X, R) = X$ if and only if $J(X, R^r) = X$.*

Proof. By Proposition 12 in Bubboloni and Gori (2018) applied to $N(X, R)$, we know that $\mathfrak{F}(X, R^r) = \mathfrak{F}(X, R)^r$. Thus we get that $\mathfrak{F}(X, R^r) = X^2$ is equivalent to $\mathfrak{F}(X, R)^r = X^2$ that in turn is equivalent to $\mathfrak{F}(X, R) = X^2$. Since we know that $\mathfrak{F}(X, R) = X^2$ is equivalent to $J(X, R) = X$ and $\mathfrak{F}(X, R^r) = X^2$ is equivalent to $J(X, R^r) = X$, the proof is completed. □

We conclude by presenting a further property of the justifiable set.

Proposition 21. *Let $(X, R) \in \mathcal{A}(X)$. For every $y \in X \backslash J(X, R)$, there exists $x \in J(X, R)$ such that $\varphi_{xy}^{(X,R)} > \varphi_{yx}^{(X,R)}$.*

Proof. First of all note that, for every $x, y \in X$ with $x \neq y$, we have that $\varphi_{xy}^{(X,R)} > \varphi_{yx}^{(X,R)}$ is equivalent to $(x, y) \in \mathrm{as}(\mathfrak{F}(X, R))$. For every $y \in X \backslash J(X, R)$, consider the set $E(y) = \{z \in X \backslash \{y\} : (z, y) \in \mathrm{as}(\mathfrak{F}(X, R))\}$ and note that $y \notin E(y)$ and $E(y) \neq \varnothing$.

Assume by contradiction that there exists $y^* \in X \backslash J(X, R)$ such that, for every $x \in J(X, R)$, we have $(x, y^*) \notin \mathrm{as}(\mathfrak{F}(X, R))$. Set $y_1 = y^*$ and note that $E(y_1) \subseteq X \backslash J(X, R)$. Pick then an element in $E(y_1)$ and call it y_2. Observe that $E(y_2) \subseteq X \backslash J(X, R)$. Indeed, if there existed $x^* \in E(y_2) \cap J(X, R)$, from $(x^*, y_2) \in \mathrm{as}(\mathfrak{F}(X, R))$, $(y_2, y_1) \in \mathrm{as}(\mathfrak{F}(X, R))$ and the fact that $\mathfrak{F}(X, R)$ is quasi-transitive, we would deduce $(x^*, y_1) \in \mathrm{as}(\mathfrak{F}(X, R))$, that is, $x^* \in E(y_1) \cap J(X, R)$, a contradiction. Pick then an element in $E(y_2)$ and call it y_3. Using the same argument as before we can prove that $E(y_3) \subseteq X \backslash J(X, R)$. Thus, we can recursively define a sequence $(y_j)_{j=1}^{\infty} \subseteq X \backslash J(X, R)$ such that, for every $j \in \mathbb{N}$, $(y_{j+1}, y_j) \in \mathrm{as}(\mathfrak{F}(X, R))$. Since $X \backslash J(X, R)$ is finite we can find $j_1 < j_2$ such that $y_{j_1} = y_{j_2}$ and $y_{j_1}, \ldots, y_{j_2-1}$ are distinct. Due to quasi-transitivity $(y_{j_1}, y_{j_2-1}) \in \mathrm{as}(\mathfrak{F}(X, R))$. Since $(y_{j_2-1}, y_{j_2}) = (y_{j_2-1}, y_{j_1}) \in \mathrm{as}(\mathfrak{F}(X, R))$, we get a contradiction. □

Corollary 22. *Let $(X, R) \in \mathcal{A}(X)$ and $x \in X$. If $J(X, R) = \{x\}$, then, for every $y \in X \backslash \{x\}$, we have that $\varphi_{xy}^{(X,R)} > \varphi_{yx}^{(X,R)}$.*

8 Comparison with the Copeland set

In what follows we set

$$\mathcal{A}_{wt}(X) = \{(X,R) \in \mathcal{A}(X) : R \text{ is quasi-complete}\}, \quad \mathcal{A}_{pt}(X) = \{(X,R) \in \mathcal{A}(X) : R \text{ is asymmetric}\},$$

and $\mathcal{A}_t(X) = \mathcal{A}_{wt}(X) \cap \mathcal{A}_{pt}(X)$. The set $\mathcal{A}_{wt}(X)$ is called set of weak tournaments; the set $\mathcal{A}_{pt}(X)$ is called set of partial tournaments; the set $\mathcal{A}_t(X)$ is called set of tournaments.

The Copeland set is an important solution naturally defined on the set of tournaments (see Laslier, 1997, Definition 3.1.1).

Definition 23. *Let $(X,R) \in \mathcal{A}_t(X)$. The Copeland set associated with (X,R) is the set*

$$COP(X,R) = \operatorname*{argmax}_{x \in X} |D^R(x)|.$$

Thus, the alternatives in the Copeland set are the ones that dominate the largest number of alternatives. The problem of extending the definition of the Copeland set to an environment larger than $\mathcal{A}_t(X)$ has been considered in the literature. In particular, extensions to $\mathcal{A}_{wt}(X)$ and $\mathcal{A}_{pt}(X)$ are proposed. In order to comment such extensions it is convenient to introduce the set

$$\Sigma = \{(\alpha,\beta,\gamma,\delta) \in \mathbb{R}^4 : \alpha \geqslant 0,\ \beta \geqslant 0,\ \alpha+\beta > 0,\ \gamma \in [-\beta,\alpha],\ \delta \in [-\beta,\alpha]\}. \tag{3}$$

Definition 24. *Let $(X,R) \in \mathcal{A}(X)$ and $(\alpha,\beta,\gamma,\delta) \in \Sigma$. The generalized Copeland set with parameters $(\alpha,\beta,\gamma,\delta)$ associated with (X,R) is the set*

$$COP^{(\alpha,\beta,\gamma,\delta)}(X,R) = \operatorname*{argmax}_{x \in X} \left(\alpha|D^R_*(x)| - \beta|\overline{D}^R_*(x)| + \gamma|I^R(x)| + \delta|N^R(x)|\right).$$

The generalized Copeland set with parameters $(\alpha,\beta,\gamma,\delta) \in \Sigma$ selects the alternatives that maximize a score that takes into account, for every alternative x, not only the number of alternatives dominated by x but a linear combination of the four numbers $|D^R_*(x)|$, $|\overline{D}^R_*(x)|$, $|I^R(x)|$ and $|N^R(x)|$. Such a linear combination obeys to some very natural criteria: the larger is $|D^R_*(x)|$ the higher is the quality of x (unless $\alpha = 0$); the larger is $|\overline{D}^R_*(x)|$ the lower is the quality of x (unless $\beta = 0$); elements in $I^R(x)$ and $N^R(x)$ cannot have an impact stronger than the ones in $D^R_*(x)$ and $\overline{D}^R_*(x)$ to determine the quality of x.

As the following simple proposition shows, for every $(\alpha,\beta,\gamma,\delta) \in \Sigma$, $COP^{(\alpha,\beta,\gamma,\delta)}$ is actually an extension of COP to $\mathcal{A}(X)$.

Proposition 25. *Let $(\alpha,\beta,\gamma,\delta) \in \Sigma$. Then, for every $(X,R) \in \mathcal{A}_t(X)$, $COP^{(\alpha,\beta,\gamma,\delta)}(X,R) = COP(X,R)$.*

Proof. Let $(X,R) \in \mathcal{A}_t(X)$ and $x \in X$. Observe first that, since R is quasi-complete and asymmetric, $I^R(x) = \varnothing$, $N^R(x) = \{x\}$, $D^R(x) = D^R_*(x)$, $\overline{D}^R(x) = \overline{D}^R_*(x)$ and $|D^R_*(x)| + |\overline{D}^R_*(x)| = |X|-1$. Thus, we have that

$$\begin{aligned} \alpha|D^R_*(x)| - \beta|\overline{D}^R_*(x)| + \gamma|I^R(x)| + \delta|N^R(x)| &= \alpha|D^R_*(x)| - \beta|\overline{D}^R_*(x)| + \delta \\ = \alpha|D^R_*(x)| - \beta(|X|-1-|D^R_*(x)|) + \delta &= (\alpha+\beta)|D^R_*(x)| - \beta(|X|-1) + \delta \\ &= (\alpha+\beta)|D^R(x)| - \beta(|X|-1) + \delta. \end{aligned}$$

Since $\alpha+\beta > 0$, we conclude that $COP^{(\alpha,\beta,\gamma,\delta)}(X,R) = COP(X,R)$. □

Proposition 25 has an important consequence. The definition of the Copeland set takes into account, for any alternative x, only the number of alternatives dominated by x (the larger is that number the higher is the quality of alternative x). However, in order to evaluate the quality of an alternative x it may be reasonable to consider also the number of alternatives dominating x and possibly consider in a special manner the number of alternatives that both are dominated by x and dominate x as well as the number of alternatives that are not dominated and do not dominate x. Proposition 25 explains that if those numbers are combined as described in the formula defining $COP^{(\alpha,\beta,\gamma,\delta)}$, we still get COP. That makes COP a very convincing solution on $\mathcal{A}_t(X)$.

For every $(\alpha,\beta,\gamma,\delta)\in\Sigma$, $COP^{(\alpha,\beta,\gamma,\delta)}$ is a reasonable extension of COP to the set $\mathcal{A}(X)$ that is based on a clear rationale. Some extensions of that type are considered in the literature, especially their restriction to the set $\mathcal{A}_{wt}(X)$ or to the set $\mathcal{A}_{pt}(X)$. The most used extensions are $COP^{(1,0,0,0)}$, $COP^{(1,0,1,0)}$, $COP^{(1,0,\frac{1}{2},0)}$, $COP^{(1,1,0,0)}$ (Copeland, 1951; Fishburn, 1977; Klamer, 2005; Henriet, 1985).[4]

Note that $COP^{(1,1,0,0)}$ and $COP^{(1,0,\frac{1}{2},0)}$ coincide on $\mathcal{A}_{wt}(X)$. Indeed, consider $(X,R)\in\mathcal{A}_{wt}(X)$. Then, for every $x\in X$, we have that $|D^R_*(x)|+|\overline{D}^R_*(x)|+|I^R(x)|=|X|-1$ so that

$$|D^R_*(x)|-|\overline{D}^R_*(x)|=|D^R_*(x)|-\big(|X|-1-|D^R_*(x)|-|I^R(x)|\big)$$

$$=2|D^R_*(x)|+|I^R(x)|-(|X|-1)=2\left(|D^R_*(x)|+\frac{1}{2}|I^R(x)|\right)-(|X|-1),$$

and that implies $COP^{(1,1,0,0)}(X,R)=COP^{(1,0,\frac{1}{2},0)}(X,R)$. On the other hand, we have that $COP^{(1,0,0,0)}$, $COP^{(1,0,1,0)}$ and $COP^{(1,0,\frac{1}{2},0)}$ are in general different on $\mathcal{A}_{wt}(X)$. Indeed, if $X=\{1,2,3,4\}$ and $R=\{(1,3),(1,4),(2,1),(2,3),(2,4),(3,2),(3,4),(4,1),(4,2),(4,3)\}$, then

$$COP^{(1,0,0,0)}(X,R)=\{1,2\},\quad COP^{(1,0,1,0)}(X,R)=\{2,4\},\quad COP^{(1,0,\frac{1}{2},0)}(X,R)=\{2\}.$$

Of course, $COP^{(1,0,0,0)}$, $COP^{(1,0,1,0)}$ and $COP^{(1,0,\frac{1}{2},0)}$ coincide on $\mathcal{A}_{pt}(X)$ while $COP^{(1,1,0,0)}$ and $COP^{(1,0,\frac{1}{2},0)}$ are in general different on $\mathcal{A}_{pt}(X)$. In order to show that fact, note that if $X=\{1,2,3,4\}$ and $R=\{(3,2),(3,1),(4,3)\}$, then $COP^{(1,1,0,0)}(X,R)=\{3,4\}$ and $COP^{(1,0,\frac{1}{2},0)}(X,R)=\{3\}$. Finally, the considered solutions are in general different on the whole set $\mathcal{A}(X)$.

Interestingly, as proved in the next proposition, all the generalized Copeland sets applied to a weak tournament always lead to a subset of the admissible set.

Proposition 26. *Let $(\alpha,\beta,\gamma,\delta)\in\Sigma$. Then, for every $(X,R)\in\mathcal{A}_{wt}(X)$, $COP^{(\alpha,\beta,\gamma,\delta)}\subseteq AD(X,R)$.*

Proof. Let $(X,R)\in\mathcal{A}_{wt}(X)$. Observe first that $COP^{(\alpha,\beta,\gamma,\delta)}(X,R)=COP^{(1,0,\rho,0)}(X,R)$, where $\rho=\frac{\gamma+\beta}{\alpha+\beta}\in[0,1]$. Indeed, we have that $N^R(x)=\{x\}$ and $|D^R_*(x)|+|\overline{D}^R_*(x)|+|I^R(x)|=|X|-1$. Thus, for every $x\in X$, we have that

$$\begin{aligned}&\alpha|D^R_*(x)|-\beta|\overline{D}^R_*(x)|+\gamma|I^R(x)|+\delta|N^R(x)|\\&=\alpha|D^R_*(x)|+\gamma|I^R(x)|-\beta\left(|X|-1-|D^R_*(x)|-|I^R(x)|\right)+\delta\\&=(\alpha+\beta)|D^R_*(x)|+(\gamma+\beta)|I^R(x)|-\beta(|X|-1)+\delta\\&=(\alpha+\beta)\left(|D^R_*(x)|+\frac{\gamma+\beta}{\alpha+\beta}|I^R(x)|\right)-\beta(|X|-1)+\delta,\end{aligned}$$

and that implies $COP^{(\alpha,\beta,\gamma,\delta)}(X,R)=COP^{(1,0,\rho,0)}(X,R)$. Thus, we are left with proving that $COP^{(1,0,\rho,0)}(X,R)\subseteq AD(X,R)$. Let $x^*\in COP^{(1,0,\rho,0)}(X,R)$. Assume by contradiction that $x^*\notin AD(X,R)$. Thus, there exists $y\in X$ with $y\neq x^*$ such that $(y,x^*)\in R^\tau$ and $(x^*,y)\notin R^\tau$. In particular, $(x^*,y)\notin R$ and being R quasi-complete we get $(y,x^*)\in \mathrm{as}(R)$. Consider now $z\in D^R_*(x^*)\cup I^R(x^*)$. We know we have $(x^*,z)\in R$ so that $z\notin\{x^*,y\}$. Thus, it cannot be $(z,y)\in R$ otherwise we would have that (x^*,z,y) is a path from x^* to y in (X,R) and that would imply

[4] It is worth mentioning that two further extensions of COP to $\mathcal{A}_{pt}(X)$ can be obtained by means of the concepts of possible winners and necessary winners proposed by Aziz et al (2015) while a further extension of COP to $\mathcal{A}_{wt}(X)$ can be obtained by considering its conservative extension as proposed by Brandt et al. (2018).

$(x^*, y) \in R^\tau$, a contradiction. Thus, $(z, y) \notin R$ and since R is quasi-complete, we get $(y, z) \in \mathrm{as}(R)$. We conclude then that $D_*^R(x^*) \cup I^R(x^*) \cup \{x^*\} \subseteq D_*^R(y)$. Thus,

$$|D_*^R(x^*)| + \rho|I^R(x^*)| \leqslant |D_*^R(x^*)| + |I^R(x^*)| < |D_*^R(x^*)| + |I^R(x^*)| + 1$$
$$\leqslant |D_*^R(y)| \leqslant |D_*^R(y)| + \rho|I^R(y)|.$$

That proves that $x^* \notin COP^{(1,0,\rho,0)}(X, R)$, a contradiction. □

We observe that, as solutions on $\mathcal{A}(X)$, $COP^{(1,0,0,0)}$, $COP^{(1,0,1,0)}$, $COP^{(1,0,\frac{1}{2},0)}$ and $COP^{(1,1,0,0)}$ can select alternatives outside the admissible set. Considering, for instance,

$$(X, R) = (\{1, 2, 3, 4, 5\}, \{(1, 2), (2, 3), (2, 4), (2, 5)\}) \in \mathcal{A}_{pt}(X), \tag{4}$$

we have that all the considered solutions select $\{2\}$ while $AD(X, R) = \{1\}$.

The next proposition shows that the justifiable set is another possible extension of the Copeland set to the whole set $\mathcal{A}(X)$. That fact is important for two main reasons. First, it provides a new interpretation of the Copeland set on $\mathcal{A}_t(X)$ in terms of maximum flow value that certainly makes the Copeland set an even more convincing solution on that environment. Secondarily, it shows that the Copeland set can be extended to $\mathcal{A}(X)$ by means of a rationale that does not depend on a quite arbitrary choice of some parameters and that guarantees to prevent selecting alternatives outside the admissible set.

Proposition 27. *For every $(X, R) \in \mathcal{A}_t(X)$, $J(X, R) = COP(X, R)$.*

Proof. Since $(X, R) \in \mathcal{A}_t(X)$, we know that $N(X, R)$ is balanced. By Proposition 16 in Bubboloni and Gori (2018) applied to $N(X, R)$, we deduce that, for every $x, y \in X_*^2$, $\varphi_{xy}^{(X,R)} - \varphi_{yx}^{(X,R)} = |D^R(x)| - |D^R(y)|$. That immediately implies that $J(X, R) = COP(X, R)$. □

Note that, considering (X, R) as in (4), we have that $J(X, R) = AD(X, R) = \{1\}$. In particular, $J(X, R)$ is in general different from $COP^{(1,0,0,0)}$, $COP^{(1,0,1,0)}$, $COP^{(1,0,\frac{1}{2},0)}$ and $COP^{(1,1,0,0)}$ when restricted to partial tournaments. The following examples finally show that the justifiable set is in general different from those generalized Condorcet sets when restricted to weak tournaments, as well. Indeed, let $X = \{1, 2, 3, 4\}$ and

$$R_1 = \{(1, 3), (4, 1), (3, 4), (4, 2), (3, 2), (1, 4), (2, 1), (4, 3), (3, 1)\},$$
$$R_2 = \{(1, 4), (4, 1), (3, 2), (1, 2), (2, 4), (1, 3), (2, 1), (4, 2), (3, 4)\},$$
$$R_3 = \{(1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 3)\}.$$

A computation shows that

$$COP^{(1,0,0,0)}(X, R_1) = \{2, 3, 4\}, \quad J(X, R_1) = \{3, 4\},$$
$$COP^{(1,0,\frac{1}{2},0)}(X, R_2) = COP^{(1,1,0,0)}(X, R_2) = \{1, 3\}, \quad J(X, R_2) = \{3\},$$
$$COP^{(1,0,1,0)}(X, R_3) = \{2, 3\}, \quad J(X, R_3) = \{2\}.$$

9 Comparison with the generalized stable set

Let (X, R) be an abstract decision problem. Let E_R be the equivalence relation on X defined as

$$E_R = \left\{(x, y) \in X^2 : (x, y) \in R^\tau \text{ and } (y, x) \in R^\tau\right\}.$$

We denote by $\mathscr{S}(X, R)$ the quotient set of X by E_R. Recall that $\mathscr{S}(X, R)$ is a partition of X. The elements of $\mathscr{S}(X, R)$ are called strong components of (X, R). Note that, given $Y \in \mathscr{S}(X, R)$

and $x \in X$, we have that there exists $y \in Y$ such that $(x, y) \in R^\tau$ if and only if, for every $y \in Y$, $(x, y) \in R^\tau$. We set

$$\mathscr{A}(X, R) = \{Y \in \mathscr{S}(X, R) : \forall x \in X \backslash Y \ \forall y \in Y, (x, y) \notin R^\tau\}.$$

Theorem 28 below states a well-known fact.[5]

Theorem 28. *Let (X, R) be an abstract decision problem. Then $\mathscr{A}(X, R) \neq \varnothing$ and $AD(X, R) = \bigcup_{Y \in \mathscr{A}(X,R)} Y$.*

Consider now an abstract decision problem (X, R) and let Y be a nonempty subset of X. The relation $R_{|Y} = R \cap Y^2$ is an irreflexive relation on Y so that $(Y, R_{|Y})$ is an abstract decision problem on Y. As a consequence, we can compute $J(Y, R_{|Y})$. The following result shows that $J(X, R)$ is made up by taking some specific elements from each set in $\mathscr{A}(X, R)$.

Theorem 29. *Let (X, R) be an abstract decision problem. Then*

$$J(X, R) = \bigcup_{Y \in \mathscr{A}(X,R)} J(Y, R_{|Y}) \tag{5}$$

Proof. In order to simplify the notation, given $Y \subseteq X$ with $Y \neq \varnothing$ and $x, y \in Y$ with $x \neq y$, we write φ_{xy} instead of $\varphi_{xy}^{(X,R)}$ and φ_{xy}^{Y} instead of $\varphi_{xy}^{(Y,R_{|Y})}$.

We start claiming that, for every $Y \in \mathscr{A}(X, R)$ and $x, y \in Y$ with $x \neq y$, we have $\varphi_{xy} = \varphi_{xy}^{Y}$. Indeed, let $Y \in \mathscr{A}(X, R)$ and $x, y \in Y$ with $x \neq y$.

- Let us prove first that $\varphi_{xy} \geqslant \varphi_{xy}^{Y}$. Let $k = \varphi_{xy}^{Y}$. If $k = 0$ the inequality is trivially satisfied so that we can assume $k \geqslant 1$. We know then that there exists a sequence $(\gamma_j)_{j=1}^{k}$ of k arc-disjoint paths from x to y in $(Y, R_{|Y})$. Of course, $(\gamma_j)_{j=1}^{k}$ is a also sequence of k arc-disjoint paths from x to y in (X, R) so that $\varphi_{xy} \geqslant k = \varphi_{xy}^{Y}$.

- Let us prove now that $\varphi_{xy}^{Y} \geqslant \varphi_{xy}$. Let $k = \varphi_{xy}$. If $k = 0$ the inequality is trivially satisfied so that we can assume $k \geqslant 1$. We know then that there exists a sequence $(\gamma_j)_{j=1}^{k}$ of k arc-disjoint paths from x to y in (X, R). Let $j \in \{1, \dots, k\}$ and let $\gamma_j = (x_i^j)_{i=1}^{m}$, where $m \geqslant 2$ and $x_1^j, \dots, x_m^j \in X$ are distinct, $x_1^j = x$ and $x_m^j = y$. We are going to prove that γ_j is a path from x to y in $(Y, R_{|Y})$ showing that $x_1^j, \dots, x_m^j \in Y$. Assume by contradiction that there exists $i^* \in \{1, \dots, m\}$ such that $x_{i^*}^j \notin Y$. Since $x, y \in Y$, we have that $m \geqslant 3$ and $i^* \notin \{1, m\}$. Observe now that $\sigma = (x_{i+i^*-1}^j)_{i=1}^{m-i^*+1}$ is a path from $x_{i^*}^j$ to y in (X, R). Thus, $(x_{i^*}^j, y) \in R^\tau$. Since $y \in Y \subseteq AD(X, R)$, we also have that $(y, x_{i^*}^j) \in R^\tau$ so that $(x_{i^*}^j, y) \in E_R$. That implies $x_{i^*}^j \in Y$, a contradiction. Thus, each element of the sequence $(\gamma_j)_{j=1}^{k}$ is in fact a path from x to y in $(Y, R_{|Y})$. We also have that $(\gamma_j)_{j=1}^{k}$ is a sequence of k arc-disjoint paths from x to y in $(Y, R_{|Y})$. We conclude then that $\varphi_{xy}^{Y} \geqslant k = \varphi_{xy}$.

Let us move on now to prove (5). Let us denote by C the set on the right hand side of (5). Let us prove first that $J(X, R) \subseteq C$. Let $x \in J(X, R)$. Since $J(X, R) \subseteq AD(X, R)$, there exists $Y \in \mathscr{A}(X, R)$ such that $x \in Y$. Let us show that $x \in J(Y, R_{|Y})$, that is, that $\varphi_{xy}^{Y} \geqslant \varphi_{yx}^{Y}$ for all $y \in Y \backslash \{x\}$. Consider $y \in Y \backslash \{x\}$. Since $x \in J(X, R)$, we know that $\varphi_{xy} \geqslant \varphi_{yx}$. By the claim we immediately conclude that $\varphi_{xy}^{Y} \geqslant \varphi_{yx}^{Y}$.

Let us prove now that $C \subseteq J(X, R)$. Let $x \in C$. Then there exists $Y \in \mathscr{A}(X, R)$ such that $x \in J(Y, R_{|Y})$. Note that, in particular, $x \in Y$ and then $x \in AD(X, R)$. Let us show that $x \in J(X, R)$, that is, that $\varphi_{xy} \geqslant \varphi_{yx}$ for all $y \in X \backslash \{x\}$. Consider $y \in X \backslash \{x\}$. Assume first that $y \in Y$. Thus, $\varphi_{xy}^{Y} \geqslant \varphi_{yx}^{Y}$ and by the claim $\varphi_{xy} \geqslant \varphi_{yx}$. Assume now that $y \notin Y$. Thus $\varphi_{yx} = 0$. Indeed, if it were $\varphi_{yx} > 0$ we would get the contradiction $(y, x) \in R^\tau$. From $\varphi_{yx} = 0$, we immediately deduce that $\varphi_{xy} \geqslant \varphi_{yx} = 0$. ◻

[5] See, for instance, Theorem 5 in Kalai and Schmeidler (1977) or Theorem 9 in Gori (2022).

Corollary 30. *Let (X, R) be an abstract decision problem and $Y \in \mathscr{A}(X, R)$. Then $J(X, R) \cap Y = J(Y, R_{|Y})$.*

Proof. We know that the sets in $\mathscr{A}(X, R)$ are pairwise disjoint. Moreover, for every $Y' \in \mathscr{A}(X, R)$, $J(Y', R_{|Y'}) \subseteq Y'$. Thus, using Theorem 29, we deduce that

$$J(X, R) \cap Y = \Big(\bigcup_{Y' \in \mathscr{A}(X,R)} J(Y', R_{|Y'}) \Big) \cap Y = \bigcup_{Y' \in \mathscr{A}(X,R)} \big(J(Y', R_{|Y'}) \cap Y \big)$$

$$= J(Y, R_{|Y}) \cap Y = J(Y, R_{|Y}).$$

□

Theorem 29 allows to get an interesting link between the justifiable set and the generalized stable set by Van Deemen (1991). We stress that such a solution is defined by the author for partial tournaments only. In what follows, given $A \subseteq P_*(X)$ nonempty, we denote by $G(A)$ the nonempty set of all the possible subsets of X built by picking a single element by each element in A. Thus, for example, if $X = \{1, 2, 3, 4\}$ and $A = \{\{1\}, \{2, 3\}, \{4\}\}$, $G(A) = \{\{1, 2, 4\}, \{1, 3, 4\}\}$.

Consider $(X, R) \in \mathcal{A}_{pt}(X)$. A subset V of X is called a generalized stable set of (X, R) if

- for every $x, y \in V$ with $x \neq y$, $(x, y) \notin R^\tau$,
- for every $y \in X \backslash V$, there exists $x \in V$ such that $(x, y) \in R^\tau$.

By Theorem 2 in Van Deemen (1991) we know that set of the generalized stable sets coincides with the set $G(\mathscr{A}(X, R))$. That fact along with Theorem 29 allows to get the following proposition.

Proposition 31. *Let $(X, R) \in \mathcal{A}_{pt}(X)$ and let*

$$k = \max\{|J(Y, R_{|Y})| : Y \in \mathscr{A}(X, R)\}.$$

Then $J(X, R)$ can be expressed as union of k distinct generalized stable sets but it cannot be expressed as union of less than k generalized stable sets.

Proof. Let $s = |\mathscr{A}(X, R)|$ and let $Y_1, \ldots, Y_s$ be the elements of $\mathscr{A}(X, R)$. For every $i \in \{1, \ldots, s\}$, let $c_i = |J(Y_i, R_{|Y_i})|$ and let $x_i^1, \ldots, x_i^{c_i}$ be the elements of $J(Y_i, R_{|Y_i})$. By assumption we have that $\max\{c_1, \ldots, c_s\} = k$. Without loss of generality we can assume that $c_1 = k$.

For every $t \in \{1, \ldots, k\}$, let

$$V_t = \left\{ x_i^{\min\{t, c_i\}} : i \in \{1, \ldots, s\} \right\}.$$

Of course, for every $t \in \{1, \ldots, k\}$, $V_t \in G(\mathscr{A}(X, R))$ so that V_t is a generalized stable set. Moreover, for every $t \in \{1, \ldots, k\}$, we have that V_t is the unique set among $V_1, \ldots, V_k$ having x_1^t as element. That proves that $V_1, \ldots, V_k$ are distinct. Finally, recalling Theorem 29, it is immediate to show that

$$J(X, R) = \bigcup_{Y \in \mathscr{A}(X,R)} J(Y, R_{|Y}) = \bigcup_{t=1}^{k} V_t.$$

That proves that $J(X, R)$ is union of k distinct generalized stable sets.

Consider now $V_1, \ldots, V_h$ generalized stable sets with $h < k$. Since $V_1, \ldots, V_h \in G(\mathscr{A}(X, R))$, each V_t contains at most an element of the set Y_1 and then at most an element of $J(Y_1, R_{|Y_1})$. As a consequence, since $|J(Y_1, R_{|Y_1})| = k$, we have that $J(Y_1, R_{|Y_1}) \nsubseteq \bigcup_{t=1}^h V_t$. In particular, applying Theorem 29, we get $J(X, R) \nsubseteq \bigcup_{t=1}^h V_t$. That proves that $J(X, R)$ cannot be expressed as union of less than k generalized stable sets. □

By Proposition 31 we can understand that some generalized stable sets are subsets of $J(X, R)$. That provides a sensible method to refine the concept of generalized stable set.

Definition 32. *Let $(X,R) \in \mathcal{A}_{pt}(X)$. We say that V is a justifiable generalized stable set of (X,R) if V is a generalized stable set and $V \subseteq J(X,R)$.*

The following proposition is immediately proved.

Proposition 33. *Let $(X,R) \in \mathcal{A}_{pt}(X)$, $A = \{J(Y,R_{|Y}) : Y \in \mathscr{A}(X,R)\}$ and $V \subseteq X$. Then V is a justifiable generalized stable set (X,R) if and only if $V \in G(A)$.*

Proof. Consider the following facts:

(a_1) V is a justifiable generalized stable set of (X,R),

(a_2) $V \in G(\mathscr{A}(X,R))$ and $V \subseteq J(X,R)$,

(a_3) $V \in G(\{Y \cap J(X,R) : Y \in \mathscr{A}(X,R)\})$,

(a_4) $V \in G(A)$.

Using Theorem 2 in Van Deemen (1991) and Definition 32, we get that (a_1) is equivalent to (a_2). A simple set theoretical argument shows that (a_2) is equivalent to (a_3). Applying Corollary 30, we get that (a_3) is equivalent to (a_4). Thus, we conclude that (a_1) is equivalent to (a_4), as desired. □

By means of the previous results, we can also easily highlight some links between the justifiable set and the w-stable set defined by Han and Van Deemen (2016). Note that that solution is defined by the authors for partial tournaments only. Consider $(X,R) \in \mathcal{A}_{pt}(X)$. A nonempty subset V of X is called a w-stable set of (X,R) if[6]

- for every $x, y \in V$ with $x \neq y$, $(x,y) \notin R^\tau$,
- for every $x \in V$ and $y \in X \backslash V$, $(y,x) \in R^\tau$ implies $(x,y) \in R^\tau$.

By Theorem 2 in Van Deemen (1991) and Theorem 4.1 in Han and Van Deemen (2016) we have that a nonempty subset W of X is a w-stable set of (X,R) if and only if $W \subseteq V$, where V is a generalized stable set of (X,R). In particular, each generalized stable set is a w-stable set. By Proposition 31 we have then that, for every $(X,R) \in \mathcal{A}_{pt}(X)$, $J(X,R)$ is union of w-stable sets. In particular, there are w-stable sets that are included in $J(X,R)$. As done for generalized stable sets, we can naturally define a refinement of the concept of w-stable set as described in Definition 34. Our definition provides an alternative approach to Definition 5.1 in Han and Van Deemen (2016) where the authors propose a refinement based on Copeland scores.

Definition 34. *Let $(X,R) \in \mathcal{A}_{pt}(X)$. We say that W is a justifiable w-stable set of (X,R) if W is a w-stable set and $W \subseteq J(X,R)$.*

Finally, we propose a simple remark about the link between the justifiable set and the m-stable set proposed by Peris and Subiza (2013). Also in this case, that solution is defined by the authors for partial tournaments only. Consider $(X,R) \in \mathcal{A}_{pt}(X)$. A nonempty subset V of X is called a m-stable set of (X,R) if

- for every $x, y \in V$, if $(x,y) \in R^\tau$ then $(y,x) \in R^\tau$,
- for every $x \in V$ and $y \in X \backslash V$, $(y,z) \notin R^\tau$.

By Lemma 1(f) in Peris and Subiza (2013) we have that V is a m-stable set of (X,R) if and only if there exists $T \subseteq \mathscr{A}(X,R)$ with $T \neq \varnothing$ such that

$$V = \bigcup_{Y \in T} Y.$$

As an immediate consequence of Theorem 29, we have that if V is an m-stable set of (X,R), then $V \cap J(X,R) \neq \varnothing$.

[6] We add the condition $V \neq \varnothing$ to the original definition of w-stable set by Han and Van Deemen (2016, Definition 4.1). Indeed, the authors implicitly exclude the case $V = \varnothing$ in their reasoning, even though the empty set formally satisfies the conditions of their definition.

10 The justifiable majority SCC

Let us denote by $\mathcal{L}(X)$ the set of linear orders on X. Given $q \in \mathcal{L}(X)$ and $x, y \in X$, we write $x \succeq_q y$ instead of $(x,y) \in q$ and we write $x \succ_q y$ instead of $(x,y) \in \mathrm{as}(q)$.

We interpret X as the set of alternatives. Consider a countably infinite set V whose elements are to be interpreted as potential voters. For simplicity, we assume $V = \mathbb{N}$. Let us consider the set

$$\mathcal{L}(X)^* = \bigcup_{\substack{I \subseteq V \\ I \neq \varnothing \text{ finite}}} \mathcal{L}(X)^I.$$

An element of $\mathcal{L}(X)^*$ is called preference profile. Thus, a preference profile is a function from a finite and nonempty subset of V to $\mathcal{L}(X)$. Given $p \in \mathcal{L}(X)^*$, we denote by $\mathrm{Dom}(p)$ the domain of p and, for every $i \in \mathrm{Dom}(p)$, $p(i) \in \mathcal{L}(X)$ is interpreted as the preference relation of voter i on the set of alternatives X.

Let $p \in \mathcal{L}(X)^*$. We denote by p^r the element of $\mathcal{L}(X)^*$ such that $\mathrm{Dom}(p^r) = \mathrm{Dom}(p)$ and such that, for every $i \in \mathrm{Dom}(p)$, $p^r(i) = p(i)^r$. If $\varphi \in \mathrm{Sym}(\mathrm{Dom}(p))$ and $\psi \in \mathrm{Sym}(X)$, we denote by $p^{(\varphi,\psi)}$ the element of $\mathcal{L}(X)^*$ such that $\mathrm{Dom}(p^{(\varphi,\psi)}) = \mathrm{Dom}(p)$ and such that, for every $i \in \mathrm{Dom}(p^{(\varphi,\psi)})$,

$$p^{(\varphi,\psi)}(i) = p(\varphi^{-1}(i))^{\psi}.$$

For every $(x,y) \in X^2_*$, we set $c_p(x,y) = |\{i \in \mathrm{Dom}(p) : x \succ_{p(i)} y\}|$. Note that, given $(x,y) \in X^2_*$, we have that $c_{p^r}(x,y) = c_p(y,x)$ and, for every $\varphi \in \mathrm{Sym}(\mathrm{Dom}(p))$ and $\psi \in \mathrm{Sym}(X)$, we have that $c_{p^{(\varphi,\psi)}}(x,y) = c_p(\psi^{-1}(x), \psi^{-1}(y))$. The majority relation associated with p is the relation on X defined by

$$\Gamma(p) = \{(x,y) \in X^2_* : c_p(x,y) > c_p(y,x)\}.$$

Thus, $(x,y) \in \Gamma(p)$ if, according to p, the number of individuals preferring x to y is larger than the number of individuals preferring y to x. Of course, we also have that

$$\Gamma(p) = \left\{(x,y) \in X^2_* : c_p(x,y) > \tfrac{|\mathrm{Dom}(p)|}{2}\right\},$$

so that $(x,y) \in \Gamma(p)$ if and only if the majority of the voters prefers x to y. It is immediate to prove that $\Gamma(p)$ is asymmetric and in general it is not quasi-complete. The pair $(X, \Gamma(p))$ is called the majority digraph associated with p and $(X, \Gamma(p)) \in \mathcal{A}_{pt}(X)$. The next result is a useful proposition whose simple proof is omitted.

Proposition 35. *Let $p \in \mathcal{L}(X)^*$, $\varphi \in \mathrm{Sym}(\mathrm{Dom}(p))$ and $\psi \in \mathrm{Sym}(X)$. Then $\Gamma(p^r) = \Gamma(p)^r$ and $\Gamma(p^{(\varphi,\psi)}) = \Gamma(p)^{\psi}$.*

A social choice correspondence (SCC) is a function from $\mathcal{L}(X)^*$ to $P_*(X)$. Thus, a SCC is a procedure that allows to select a nonempty set of alternatives for any conceivable preference profile. Let us define now the main object of the section.

Definition 36. *The justifiable majority* SCC *is the* SCC *defined, for every $p \in \mathcal{L}(A)^*$, by*

$$JM(p) = J(X, \Gamma(p)).$$

The justifiable majority SCC satisfies a variety of interesting properties that we are going to discuss in the next sections. We preliminary observe that, because of Proposition 6, the justifiable majority SCC is well defined, that is, it is always nonempty valued. Moreover, the justifiable majority SCC is a C1 function in the sense of Fishburn (1977) as it only depends on $(X, \Gamma(p))$. As a consequence it satisfies the property of homogeneity (see Fishburn, 1977, pp.476-477).

11 Properties of the justifiable-majority SCC

A SCC F is anonymous if, for every $p \in \mathcal{L}(X)^*$ and $\varphi \in \mathrm{Sym}(\mathrm{Dom}(p))$, $F(p^{(\varphi,id)}) = F(p)$; neutral if for every $p \in \mathcal{L}(X)^*$ and $\psi \in \mathrm{Sym}(X)$, $F(p^{(id,\psi)}) = \psi(F(p))$.

Proposition 37. *JM is anonymous.*

Proof. Let $p \in \mathcal{L}(X)^*$ and $\varphi \in \mathrm{Sym}(\mathrm{Dom}(p))$. By Proposition 35, we know that $\Gamma(p^{(\varphi,id)}) = \Gamma(p)$. Thus, we have that

$$JM(p^{(\varphi,id)}) = J(X, \Gamma(p^{(\varphi,id)})) = J(X, \Gamma(p)) = JM(p),$$

as desired. □

Proposition 38. *JM is neutral.*

Proof. Let $p \in \mathcal{L}(X)^*$ and $\psi \in \mathrm{Sym}(X)$. By Proposition 35, we know that $\Gamma(p^{(id,\psi)}) = \Gamma(p)^\psi$. Thus, applying Proposition 13, we have that

$$JM(p^{(id,\psi)}) = J(X, \Gamma(p^{(id,\psi)})) = J(X, \Gamma(p)^\psi) = \psi(J(X, \Gamma(p))) = \psi(JM(p)),$$

as desired. □

Let F be a SCC. We say that F satisfies the Schwartz principle if, for every $p \in \mathcal{L}(X)^*$, we have that $F(p) \subseteq AD(X, \Gamma(p))$.[7] Given $p \in \mathcal{L}(X)^*$, we say that $D \subseteq X$ is a dominating set for p if D is a dominating set for $(X, \Gamma(p))$. We say that F satisfies the Smith principle if, for every $p \in \mathcal{L}(X)^*$ and for every $D \subseteq X$ dominating set for p, we have that $F(p) \subseteq D$.[8] By Proposition 9 we have that if F satisfies the Schwartz principle, then F satisfies the Smith principle.

Proposition 39. *JM satisfies the Schwartz principle. In particular, JM satisfies the Smith principle.*

Proof. Let $p \in \mathcal{L}(X)^*$. By Proposition 8, we have that that $JM(p) = J(X, \Gamma(p)) \subseteq AD(X, \Gamma(p))$. □

Let $p \in \mathcal{L}(X)^*$ and $x \in X$. The alternative x is called Condorcet winner for p if, for every $y \in X \setminus \{x\}$, $(x, y) \in \Gamma(p)$; weak Condorcet winner for p if, for every $y \in X \setminus \{x\}$, $(y, x) \notin \Gamma(p)$. Thus, the set of Condorcet winners for p coincides with the set $CW(X, \Gamma(p))$; the set of weak Condorcet winners for p coincides with $CO(X, \Gamma(p))$. A SCC F is said to satisfy the Condorcet principle if $CW(X, \Gamma(p)) \neq \varnothing$ implies $F(p) = CW(X, \Gamma(p))$.

Proposition 40. *JM satisfies the Condorcet principle.*

Proof. Let $p \in \mathcal{L}(X)^*$ and assume that $CW(X, \Gamma(p)) \neq \varnothing$. By Proposition 11, we have that $CW(X, \Gamma(p)) = J(X, \Gamma(p)) = JM(p)$. □

The next proposition shows that JM always selects all the weak Condorcet winners.

Proposition 41. *Let $p \in \mathcal{L}(X)^*$. Then $CO(X, \Gamma(p)) \subseteq JM(p)$.*

Proof. By Proposition 7, we have that $CO(X, \Gamma(p)) \subseteq J(X, \Gamma(p)) = JM(p)$. □

A SCC F is said Pareto optimal if, for every $p \in \mathcal{L}(X)^*$ and $x, y \in X$, we have that $\{i \in \mathrm{Dom}(p) : x \succ_{p(i)} y\} = \mathrm{Dom}(p)$ implies $y \notin F(p)$.

Proposition 42. *JM is Pareto optimal.*

[7] Note that F satisfies the Schwartz principle if and only if F is a refinement of the Schwartz SCC defined in Fishburn (1977, p.473).

[8] Note that the definition of Smith principle here considered coincides with the one of strong Smith winner consistency in Barberà and Bossert (2023).

Proof. Let $p \in \mathcal{L}(X)^*$, $x, y \in X$ and set $\mathrm{Dom}(p) = I$. Assume that $\{i \in I : x \succ_{p(i)} y\} = I$ and prove that $y \notin JM(p)$. Note first that, $c_p(x, y) = |I|$ and $c_p(y, x) = 0$ so that $(x, y) \in \Gamma(p) = \mathrm{as}(\Gamma(p))$.

Consider now $z \in D^{\Gamma(p)}(y)$. Thus, $z \notin \{x, y\}$ and $(y, z) \in \Gamma(p)$. Considering then the set $A = \{i \in I : y \succ_{p(i)} z\}$, we know that $|A| > \frac{|I|}{2}$. Consider now $A' = \{i \in I : x \succ_{p(i)} z\}$. Note that if $i \in A$, then $y \succ_{p(i)} z$ and, since $x \succ_{p(i)} y$ and $p(i)$ is a linear order, we conclude that $x \succ_{p(i)} z$ so that $i \in A'$. Thus, $A \subseteq A'$. We deduce then that $|A'| > \frac{|I|}{2}$ so that $(x, z) \in \Gamma(p)$, that is, $z \in D^{\Gamma(p)}(x)$. Thus, $D^{\Gamma(p)}(y) \subseteq D^{\Gamma(p)}(x)$.

Consider next $z \in \overline{D}^{\Gamma(p)}(x)$. Thus, $z \notin \{x, y\}$ and $(z, x) \in \Gamma(p)$. Considering then the set $A = \{i \in I : z \succ_{p(i)} x\}$, we know that $|A| > \frac{|I|}{2}$. Consider now $A' = \{i \in I : z \succ_{p(i)} y\}$. Note that if $i \in A$, then $z \succ_{p(i)} x$ and, since $x \succ_{p(i)} y$ and $p(i)$ is a linear order, we conclude that $z \succ_{p(i)} y$ so that $i \in A'$. Thus, $A \subseteq A'$. We deduce then that $|A'| > \frac{|I|}{2}$ so that $(z, y) \in \Gamma(p)$, that is, $z \in \overline{D}^{\Gamma(p)}(y)$. Thus, $\overline{D}^{\Gamma(p)}(x) \subseteq \overline{D}^{\Gamma(p)}(y)$.

As a consequence we can apply Proposition 14 and deduce that $y \notin J(X, \Gamma(p)) = JM(p)$. $\square$

Let $p, p' \in \mathcal{L}(X)^*$ with $\mathrm{Dom}(p) = \mathrm{Dom}(p')$ and $x \in X$. We say that x improves its position from p to p' if

- for every $i \in \mathrm{Dom}(p)$ and $y \in X \backslash \{x\}$, $x \succ_{p(i)} y$ implies $x \succ_{p'(i)} y$;
- for every $i \in \mathrm{Dom}(p)$ and $y_1, y_2 \in X \backslash \{x\}$, $y_1 \succ_{p(i)} y_2$ if and only if $y_1 \succ_{p'(i)} y_2$.

A SCC F is said monotonic if, for every $p, p' \in \mathcal{L}(X)^*$ with $\mathrm{Dom}(p) = \mathrm{Dom}(p')$ and $x \in F(p)$, the fact that x improves its position from p to p' implies that $x \in F(p')$.

Proposition 43. *JM is monotonic.*

Proof. Let $p, p' \in \mathcal{L}(X)^*$ with $\mathrm{Dom}(p) = \mathrm{Dom}(p') = I$, $x \in JM(p) = J(X, \Gamma(p))$ and suppose that x improves its position from p to p'. We have to show that $x \in JM(p')$.

Consider $z \in D^{\Gamma(p)}(x)$. Thus $z \neq x$ and $(x, z) \in \Gamma(p)$. Considering then the set $A = \{i \in I : x \succ_{p(i)} z\}$, we know that $|A| > \frac{|I|}{2}$. Consider now $A' = \{i \in I : x \succ_{p'(i)} z\}$. Note that if $i \in A$, then $x \succ_{p(i)} z$ and, since x improves its position from p to p', we conclude that $x \succ_{p'(i)} z$ so that $i \in A'$. Thus, $A \subseteq A'$. We deduce then that $|A'| > \frac{|I|}{2}$ so that $(x, z) \in \Gamma(p')$ that is $z \in D^{\Gamma(p')}(x)$. Thus, $D^{\Gamma(p)}(x) \subseteq D^{\Gamma(p')}(x)$.

Consider next $z \in \overline{D}^{\Gamma(p')}(x)$. Thus $z \neq x$ and $(z, x) \in \Gamma(p')$. Considering the set $A' = \{i \in I : z \succ_{p'(i)} x\}$, we know that $|A'| > \frac{|I|}{2}$. Consider now $A = \{i \in I : z \succ_{p(i)} x\}$. Note that if $i \in A'$, then $z \succ_{p'(i)} x$ and, since x improves its position from p to p', we conclude that $z \succ_{p(i)} x$ so that $i \in A$. Thus, $A' \subseteq A$. We deduce then that $|A| > \frac{|I|}{2}$ so that $(z, x) \in \Gamma(p)$, that is, $z \in \overline{D}^{\Gamma(p)}(x)$. Thus, $\overline{D}^{\Gamma(p')}(x) \subseteq \overline{D}^{\Gamma(p)}(x)$.

Finally note that, since, for every $i \in \mathrm{Dom}(p)$ and $y_1, y_2 \in X \backslash \{x\}$, $y_1 \succ_{p(i)} y_2$ if and only if $y_1 \succ_{p'(i)} y_2$, we have that $(y_1, y_2) \in \Gamma(p)$ if and only if $(y_1, y_2) \in \Gamma(p')$. Thus,

$$\{(y_1, y_2) \in \Gamma(p) : y_1 \neq x, y_2 \neq x\} = \{(y_1, y_2) \in \Gamma(p') : y_1 \neq x, y_2 \neq x\}.$$

Since $x \in J(X, \Gamma(p))$, by Proposition 16, we conclude that $x \in J(X, \Gamma(p')) = JM(p')$. $\square$

A SCC F is said immune to the reversal bias if, for every $p \in \mathcal{L}(X)^*$, $F(p) \neq X$ implies $F(p) \nsubseteq F(p^r)$.

Proposition 44. *JM is immune to the reversal bias.*

Proof. Let $p \in \mathcal{L}(X)^*$ and assume that $JM(p) \neq X$. Thus $JM(p) = J(X, \Gamma(p)) \neq X$. By Proposition 18, we have that $J(X, \Gamma(p)) \nsubseteq J(X, \Gamma(p)^r)$. Since by Proposition 35 we know that $\Gamma(p)^r = \Gamma(p^r)$, we conclude that $JM(p) = J(X, \Gamma(p)) \nsubseteq J(X, \Gamma(p^r)) = JM(p^r)$. $\square$

Proposition 45. *Let $p \in \mathcal{L}(X)^*$. Then $JM(p) = X$ if and only if $JM(p^r) = X$.*

Proof. By Proposition 20 we know that $J(X, \Gamma(p)) = X$ if and only if $J(X, \Gamma(p)^r) = X$. By Proposition 35 we also know that $J(X, \Gamma(p)^r) = J(X, \Gamma(p^r))$. Thus, $JM(p) = J(X, \Gamma(p)) = X$ if and only if $JM(p^r) = J(X, \Gamma(p)^r) = X$. □

Let $p \in \mathcal{L}(X)^*$ and $x \in X$. The alternative x is called Condorcet loser for p if, for every $y \in X \backslash \{x\}$, $(y, x) \in \Gamma(p)$; intermediate Condorcet loser for p if, for every $y \in X \backslash \{x\}$, $(x, y) \notin \Gamma(p)$ and there exists $y^* \in X \backslash \{x\}$ such that $(y^*, x) \in \Gamma(p)$. The set of Condorcet losers for p is denoted by $CL(p)$; the set of intermediate Condorcet losers for p is denoted by $ICL(p)$. A scc F is said to satisfy the Condorcet loser principle if, for every $p \in \mathcal{L}(X)^*$, $F(p) \cap CL(p) = \varnothing$. A scc F is said to satisfy the intermediate Condorcet loser principle if, for every $p \in \mathcal{L}(X)^*$, $F(p) \cap ICL(p) = \varnothing$. Since, for every $p \in \mathcal{L}(X)^*$, $CL(p) \subseteq ICL(p)$ we have that if F satisfies the intermediate Condorcet loser principle, then it satisfies the Condorcet loser principle. Note that the definitions of intermediate Condorcet loser and intermediate Condorcet loser principle are introduced and studied in Barberà and Bossert (2023).

Proposition 46. *JM satisfies the intermediate Condorcet loser principle.*

Proof. Let $p \in \mathcal{L}(X)^*$. We have to show that $JM(p) \cap ICL(p) = \varnothing$. If $ICL(p) = \varnothing$ we immediately get the desired equality. Assume then that $ICL(p) \neq \varnothing$ and let $x \in ICL(p)$. We know that there exists $y^* \in X \backslash \{x\}$ such that $(y^*, x) \in \Gamma(p)$. Thus $\varphi_{y^* x}^{(X,\Gamma(p))} \geqslant 1$. Since, for every $y \in X \backslash \{x\}$, $(x, y) \notin \Gamma(p)$ we have also that $\varphi_{x y^*}^{(X,\Gamma(p))} = 0$. As a consequence, $x \notin J(X, \Gamma(p)) = JM(p)$. We conclude then that $JM(p) \cap ICL(p) = \varnothing$. □

12 Comparison with the Copeland scc

For every $p \in \mathcal{L}(X)^*$ and $x \in X$, let

$$w_p(x) = |\{y \in X \backslash \{x\} : c_p(x, y) > c_p(y, x)\}|,$$

$$l_p(x) = |\{y \in X \backslash \{x\} : c_p(x, y) < c_p(y, x)\}|,$$

$$t_p(x) = |\{y \in X \backslash \{x\} : c_p(x, y) = c_p(y, x)\}|.$$

The Copeland scc is a classic scc whose definition dates back to Copeland (1951) (see also Fishburn, 1977).

Definition 47. *The Copeland* scc *is the* scc *defined, for every $p \in \mathcal{L}(X)^*$, by*

$$\overline{COP}(p) = \underset{x \in X}{\operatorname{argmax}} \, (w_p(x) - l_p(x)) \, .$$

Other versions of the Copeland scc can be found in the literature. They can be seen as special instances of the following general definition. In what follows, let

$$\Sigma^* = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha \geqslant 0, \, \beta \geqslant 0, \, \alpha + \beta > 0, \, \gamma \in [-\beta, \alpha]\},$$

and note that if $(\alpha, \beta, \gamma) \in \Sigma^*$, then $(\alpha, \beta, \gamma, 0) \in \Sigma$, where Σ is defined in (3).

Definition 48. *Let $(\alpha, \beta, \gamma) \in \Sigma^*$. The generalized Copeland* scc *with parameters (α, β, γ) is the* scc *defined, for every $p \in \mathcal{L}(X)^*$, by*

$$\overline{COP}^{(\alpha,\beta,\gamma)}(p) = \underset{x \in X}{\operatorname{argmax}} \, (\alpha \, w_p(x) - \beta \, l_p(x) + \gamma \, t_p(x)) \, ,$$

Of course, $\overline{COP}^{(1,1,0)}$ coincides with $\overline{COP}$. Note also that $\overline{COP}^{(1,0,\frac{1}{2})}$ is considered by Saari and Merlin (1996), $\overline{COP}^{(1,0,\gamma)}$ with $\gamma \in [0,1]$ by Faliszewski et al. (2009), $\overline{COP}^{(1,0,1)}$ by Moulin (1983). Let us prove some simple facts about the generalized Copeland SCCs. For every $p \in \mathcal{L}(X)^*$, let

$$\overline{\Gamma}(p) = \{(x,y) \in X_*^2 : c_p(x,y) \geqslant c_p(y,x)\}.$$

Note that $\overline{\Gamma}(p) \in \mathcal{A}_{wt}(X)$ and $\mathrm{as}(\overline{\Gamma}(p)) = \Gamma(p)$.

Proposition 49. *Let $(\alpha,\beta,\gamma) \in \Sigma^*$ and $p \in \mathcal{L}(X)^*$. Then $\overline{COP}^{(\alpha,\beta,\gamma)}(p) = COP^{(\alpha,\beta,\gamma,0)}(X,\overline{\Gamma}(p))$.*

Proof. Simply note that, for every $x \in X$,

$$w_p(x) = |D_*^{\overline{\Gamma}(p)}(x)|, \quad l_p(x) = |\overline{D}_*^{\overline{\Gamma}(p)}(x)|, \quad t_p(x) = |I^{\overline{\Gamma}(p)}(x)|, \quad |N^{\overline{\Gamma}(p)}(x)| = 1.$$

That immediately implies the desired equality. $\square$

Since we proved that $COP^{(1,1,0,0)}$ coincides with $COP^{(1,0,\frac{1}{2},0)}$ on $\mathcal{A}_{wt}(X)$ and since $\overline{\Gamma}(p) \in \mathcal{A}_{wt}(X)$ for all $p \in \mathcal{L}(X)^*$, by Proposition 49 we deduce that $\overline{COP} = \overline{COP}^{(1,1,0)} = \overline{COP}^{(1,0,\frac{1}{2})}$. However, we usually have that $\overline{COP}^{(\alpha,\beta,\gamma)} \neq \overline{COP}^{(\alpha',\beta',\gamma')}$ when $(\alpha,\beta,\gamma) \neq (\alpha',\beta',\gamma')$. Define now

$$\mathcal{D} = \{p \in \mathcal{L}(X)^* : \overline{\Gamma}(p) \in \mathcal{A}_t(X)\},$$

and note that $\mathcal{D}$ is a quite large set since

$$\{p \in \mathcal{L}(X)^* : |\mathrm{Dom}(p)| \text{ is odd}\} \subseteq \mathcal{D}.$$

The following proposition holds true.

Proposition 50. *Let $(\alpha,\beta,\gamma) \in \Sigma^*$ and $p \in \mathcal{D}$. Then $\overline{COP}^{(\alpha,\beta,\gamma)}(p) = \overline{COP}(p)$.*

Proof. By Propositions 25 and 49 and the fact that $\overline{\Gamma}(p) \in \mathcal{A}_t(X)$, we have that

$$\overline{COP}^{(\alpha,\beta,\gamma)}(p) = COP^{(\alpha,\beta,\gamma,0)}(X,\overline{\Gamma}(p)) = COP(X,\overline{\Gamma}(p)) = COP^{(1,1,0,0)}(X,\overline{\Gamma}(p)) = \overline{COP}(p).$$

$\square$

Proposition 50 shows that all the generalized Copeland SCCs coincide on $\mathcal{D}$. Denoting by $\overline{COP}_{\mathcal{D}}$ the restriction of all those SCCs to $\mathcal{D}$, we have that $\overline{COP}_{\mathcal{D}}$ fulfils a very remarkable property, namely that, for every $p \in \mathcal{D}$, it selects the alternatives that maximize any score of the type $\alpha w_p(x) - \beta l_p(x) + \gamma t_p(x)$, where $(\alpha,\beta,\gamma) \in \Sigma^*$. That makes $\overline{COP}_{\mathcal{D}}$ a very convincing method to select alternatives when preference profiles are in $\mathcal{D}$. Extending $\overline{COP}_{\mathcal{D}}$ to the whole set $\mathcal{L}(X)^*$ is important and in fact any generalized Copeland SCC represents a possible reasonable extension of $\overline{COP}_{\mathcal{D}}$. However, it is not completely clear which among the generalized Copeland SCCs better serves on purpose.

The next proposition shows that JM is another possible way to extend $\overline{COP}_{\mathcal{D}}$ to the set $\mathcal{L}(X)^*$. Interestingly, JM is an extension of $\overline{COP}_{\mathcal{D}}$ to the set $\mathcal{L}(X)^*$ based on a rationale different from the one underlying all the generalized Copeland SCCs, namely the maximization of a suitable score.

Proposition 51. *Let $p \in \mathcal{D}$. Then $JM(p) = \overline{COP}(p)$.*

Proof. Since $p \in \mathcal{D}$, we have that $(X,\overline{\Gamma}(p)) \in \mathcal{A}_t(X)$ so that $\Gamma(p) = \overline{\Gamma}(p)$. By Propositions 25, 27 and 49 we have that

$$JM(p) = J(X,\Gamma(p)) = J(X,\overline{\Gamma}(p)) = COP(X,\overline{\Gamma}(p)) = COP^{(1,1,0,0)}(X,\overline{\Gamma}(p)) = \overline{COP}(p).$$

$\square$

It is important to note that JM is an extension of $\overline{COP}_{\mathcal{D}}$ satisfying the Schwartz principle. The fact that the generalized Copeland SCCs do not satisfy in general the Schwartz principle makes JM a very interesting extension of $\overline{COP}_{\mathcal{D}}$.

Let us complete the section by showing that $\overline{COP}$, $\overline{COP}^{(1,0,0)}$ and $\overline{COP}^{(1,0,1)}$ do not to satisfy in general the Schwartz principle. Consider $X = \{1,2,3,4,5\}$ and let $p \in \mathcal{L}(X)^*$ be such that $p : \{1,2,3,4,5,6\} \to \mathcal{L}(X)$ and[9]

$$p(1) = 12345,\ p(2) = 12345,\ p(3) = 12345,\ p(4) = 54312,\ p(5) = 25431,\ p(6) = 25431.$$

Then, we have that $\overline{COP}(p) = \overline{COP}^{(1,0,0)}(p) = \{2\}$ and $AD(X, \Gamma(p)) = JM(p) = \{1\}$. Consider now $X = \{1,2,3,4\}$ and let $p \in \mathcal{L}(X)^*$ be such that $p : \{1,2,3,4,5,6\} \to \mathcal{L}(X)$ and

$$p(1) = 1423,\ p(2) = 1423,\ p(3) = 3214,\ p(4) = 3214,\ p(5) = 4231,\ p(6) = 4312.$$

Then, we have that $\overline{COP}^{(1,0,1)}(p) = \{1,2,3,4\}$ and $AD(X, \Gamma(p)) = JM(p) = \{1,3,4\}$.

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[9]The notation for linear orders is standard. For instance, $p(1) = 12345$ means that $p(1)$ is the unique linear order on $X = \{1,2,3,4,5\}$ such that $1 \succ_{p(1)} 2 \succ_{p(1)} 3 \succ_{p(1)} 4 \succ_{p(1)} 5$.

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