UNIVERSITÀ DEGLI STUDI FIRENZE

DISEI
DIPARTIMENTO DI SCIENZE PER L'ECONOMIA E L'IMPRESA

Working Papers - Economics

Investments in First-Price and Second-Price Procurement Auctions

Muhammed Ceesay, Nicola Doni, Domenico Menicucci

Working Paper N. 01/2025

DISEI, Università degli Studi di Firenze
Via delle Pandette 9, 50127 Firenze (Italia) www.disei.unifi.it

Investments in First-Price and Second-Price Procurement Auctions*

Muhammed Ceesay† Nicola Doni‡ Domenico Menicucci§

January 9, 2026

Abstract

This paper is about a procurement auction setting with two sellers in which before the auction seller i can make an investment which improves the ex ante probability distribution of his cost; seller j observes seller i's investment decision before bidding occurs. Under somewhat restrictive assumptions on the pre- and the post-investment cost distributions, Arozamena and Cantillon (2004) prove that in the first price auction seller i's investment induces seller j to bid more aggressively. This negative strategic effect contributes to AC's result that the investment incentive for seller i is stronger in the second price auction than in the first price auction.

We prove that under weaker but economically significant assumptions, and discretely distributed costs, an investment by seller i may actually induce seller j to bid less aggressively in the first price auction (i.e., the strategic effect may be positive), and the investment incentive may be stronger in the latter auction. Moreover, in some cases the buyer prefers the first price auction precisely because it provides a stronger investment incentive, even though the second price auction is preferable when no investment is possible. We prove that the two auctions are not equivalent in a setting in which each seller has the option to invest and the sellers are ex ante symmetric, and that the second price auction gives a stronger investment incentive to the initially stronger seller than to the other seller (this increases asymmetries), but such result does not necessarily hold in the first price auction.

Keywords: Procurement Auctions, First-Price Auction, Second-Price Auction, Pre-Auction Investment, Strategic Effect, Auction Ranking.

1 Introduction

This paper is about a procurement auction setting with two sellers competing to supply the good a buyer wants to procure. One seller, denoted seller i, before the auction has the opportunity to make an investment in cost reduction which improves the ex ante probability distribution of the cost c_i he incurs to supply the good. The other seller, seller j, observes whether seller i has made the investment or not and may condition his bidding strategy to this information. We are mainly interested in comparing the first price auction (FPA henceforth) and the second price auction (SPA henceforth) in terms of seller i's investment incentive and in terms of the buyer's expected payment to the seller who is selected to supply the good, taking into account

*This research has been supported by the Italian Ministry of University through the project Auctions: Theory and Applications (call Prin 2022, financed by the European Union – NextGenerationEU), project 20223BACPK, CUP B53D23009640001.

†Dipartimento di Scienze per l'Economia e l'Impresa, Università degli Studi di Firenze, Via delle Pandette 32, Firenze, I-50127, Italy; muhammed.ceesay@unifi.it

‡Dipartimento di Scienze per l'Economia e l'Impresa, Università degli Studi di Firenze, Via delle Pandette 32, Firenze, I-50127, Italy; nicola.doni@unifi.it

§Dipartimento di Scienze per l'Economia e l'Impresa, Università degli Studi di Firenze, Via delle Pandette 32, Firenze, I-50127, Italy; domenico.menicucci@unifi.it

seller i's possibility to invest. In this setting, Arozamena and Cantillon (2004) (AC henceforth) identify somewhat restrictive conditions on the initial and on the post-investment cost distributions under which the investment incentive is greater in the SPA. We impose less restrictive assumptions and identify settings such that the opposite result holds and such that the buyer prefers the FPA because of its superior investment incentive, although she would prefer the SPA if no investment were possible. We then consider a setting in which both sellers can make a cost-reducing investment.

More in detail, AC suppose the post-investment distribution of c_i is first order stochastically dominated by the pre-investment distribution conditional on costs no less than a certain c, for each possible c;[1] each post-investment distribution which satisfies this requirement is said to be an *upgrade* of the pre-investment distribution. Under this assumption, AC prove that in the FPA seller i's investment induces seller j to bid more aggressively, which reduces the profit of each type of seller i. This *strategic effect* affects negatively seller i's incentive to invest in the FPA. Conversely, in the SPA no such effect exists as bidding the own cost is a weakly dominant strategy for seller j regardless of the distributions of costs. Moreover, AC show that the investment incentive in the SPA is stronger than in the FPA if seller i's investment determines a *leadership change* in the sense that the distribution of c_j is an upgrade of the initial distribution of c_i and the post-investment distribution of c_i is an upgrade of the distribution of c_j.[2] In this case the SPA makes it more likely that seller i invests,[3] which ultimately reduces the buyer's payment. This conceivably makes the buyer prefer the SPA to the FPA in some circumstances.[4]

Our assumptions differ from those in AC, most importantly because we consider investments such that the post-investment distribution of c_i is first order stochastically dominated by the initial distribution; each post-investment distribution which satisfies this condition is called a *weak upgrade* of the initial distribution – it is immediate that any upgrade is a weak upgrade, but the reverse implication is not true. Moreover, we do not assume that seller i's investment determines a change in leadership as in AC. As a consequence, the set of possible changes in the distribution of c_i (and the set of possible initial distributions) we consider is significantly wider than in AC, but still economically meaningful and immediate to interpret as it is based on the standard principle of first order stochastic dominance.

Changes in the distribution of a seller's cost involve asymmetric distributions of costs between sellers, before and/or after the investment. This complicates identifying the equilibrium bidding strategies in the FPA, and then we consider discretely distributed costs such that the cost of each seller belongs to the set $\{c_L, c_M, c_H\}$ with $c_H - c_M = c_M - c_L > 0$. In this setting it is possible to derive the equilibrium strategies in the FPA in closed form, even though the distribution of c_i is different from the distribution of c_j, and to determine precisely the effect of any investment on seller i's profit.[5]

In such an environment we find that AC's results extend beyond their assumptions, but we also prove that under our weaker assumptions, in the FPA an investment of seller i may induce less aggressive bidding by seller j, which weakly increases the profit of each type of seller i – in this case the strategic effect is positive, something that never occurs when the post-investment distribution of c_i is an upgrade as in AC. Moreover, we identify a few settings such that the investment incentive is stronger in the FPA than in the SPA, sometimes because of the positive strategic effect mentioned above, but sometimes even if the strategic

[1] A cost distribution which is first order stochastically dominated by another is more likely to yield a lower cost than the dominating distribution.

[2] We are describing here AC's main results for the case of two sellers, but in fact AC allow for an arbitrary number of sellers.

[3] Grimm et al. (2009) perform an experimental analysis about the results in AC.

[4] AC do not examine the buyer's preference but remark that the property that the SPA provides a stronger investment incentive is not enough to conclude that the buyer prefers the SPA.

[5] Conversely, AC suppose that for each seller the support of the distribution of his cost is an interval. This makes a closed form for the equilibrium strategies unavailable, except in a few particular cases. Nevertheless, AC are able to use the properties of the system of differential equations which characterizes the equilibrium strategies to prove their results.

effect is negative. As an example, consider an investment which increases the probability that seller i has cost c_L and correspondingly reduces the probability that seller i has cost c_H. The strategic effect reduces the profits in the FPA of seller i with cost c_M – denoted type i_M – and of seller i with cost c_L – denoted type i_L. But there is another effect, called *direct effect*, on seller i's ex ante expected profit which arises as the new distribution of c_i gives higher (lower) probability to type i_L (to type i_H), who earns a higher profit than type i_H. This effect is positive in the FPA and in the SPA, but since the profit of type i_L in the FPA is (often) greater than in the SPA, the direct effect is stronger in the FPA. In some circumstances this dominates over the negative strategic effect – in particular when the initial probabilities of types i_M, i_L are small – and then the investment incentive is stronger in the FPA.[6] In this case, for a suitably intermediate investment cost the investment occurs only in the FPA and the ensuing change in the probability distribution of c_i decreases the buyer's payment in the FPA. We determine settings in which this makes the buyer prefer the FPA to the SPA even though the latter is preferable when no investment is possible. Therefore the FPA may favor an investment more than the SPA and because of this reason it becomes preferable for the buyer, overturning the latter's preference in a world without investments. We believe this gives a more complete viewpoint on how the comparison between the FPA and the SPA may be affected by pre-auction investments.

Since we allow for asymmetric sellers, it is natural to inquire whether the FPA or the SPA is more likely to induce an investment by the seller who is initially stronger in terms of cost distribution, or by the weaker seller. We show that in the SPA the stronger seller always has a greater incentive to make a small investment, whereas in the FPA the strategic effect generates the opposite result in some specific circumstances.

For the special case in which the sellers have the same initial cost distribution, we extend our analysis to post-investment distributions which are not weak upgrades. We prove that for any such distribution the investment incentive is greater in the FPA,[7] but the investment increases the buyer's payment in the FPA. As a result, the buyer weakly prefers the SPA whenever the post-investment distribution is not a weak upgrade.

Finally, we examine a setting in which each seller can make an investment – still for equal initial distributions – and the investment by seller 1 affects the distribution of c_1 just like the investment by seller 2 affects the distribution of c_2: sellers are ex ante symmetric in each respect. We prove that the SPA is more effective than the FPA in inducing investment by both sellers because once seller i has made the investment, in the FPA seller j is hurt by a strong negative strategic effect if he makes the investment, but no such effect exists in the SPA. However, in some cases the FPA is more effective in inducing investment by a single seller: these are essentially the cases mentioned above in which the strategic effect in the FPA is positive. As a result, payment equivalence between the FPA and the SPA breaks down for multiple reasons, although the sellers are ex ante symmetric. Conversely, Tan (1992) examines cost-reducing investments by sellers before they participate in the auction under the assumption that each seller's investment is not observed by the other sellers before the auction takes place. One main result in Tan (1992) is that the FPA and the SPA are equivalent in terms of sellers' investments and of buyer's payment under the assumption that the investment technology has decreasing returns to scale. Although the assumption of non-observable investments simplifies the analysis, in some circumstances a seller's investment is in fact observable by the other sellers, for instance if it determines the location of a plant or the capacity of a seller. A more recent literature allows the buyer to design an optimal mechanism a' la Myerson (1981), and examines the difference between the case in which the mechanism is designed before the sellers choose investments and the case in which the

[6]This result does not hold in AC because leadership change requires a large change in the probability distribution of c_i – a large increase in the probability that seller i has cost c_L, which reduces significantly the profit of types i_L and i_M in the FPA, and then the negative strategic effect dominates.

[7]In some cases the investment actually lowers seller i's ex ante expected profit in the FPA and in the SPA. In these cases seller i does not make the investment even when the investment cost is zero.

sellers first invest and then the buyer designs the mechanism.[8] The results of this literature do not apply to our setting as we suppose that a unique seller can invest, that sellers may be ex ante asymmetric, and we focus on standard auction, that is the FPA and the SPA.

The rest of the paper is organized as follows. Section 2 introduces the model and summarizes some results from Ceesay, Doni, Menicucci (2025) (CDM henceforth) which are useful for our analysis. Section 3 (4) examines the case in which the seller who may invest is the stronger (the weaker) seller in terms of his initial probability to have cost c_L. Section 5 is about a setting in which both sellers can make an investment. Section 6 concludes. Section 7 provides the proofs of our results.

2 The setting

A (female) buyer wants to procure a certain good (or service) that can be supplied by either of two different (male) sellers. If seller 1 (seller 2) is selected to supply the good, then he incurs a production cost c_1 (c_2) he privately observes which is equal either to c_L, or to c_M, or to c_H, with $c_L \geq 0$ and $c_M = c_L + \Delta$, $c_H = c_M + \Delta$ with $\Delta > 0$. For $i = 1, 2$, the ex ante distribution of c_i is identified by the two numbers h_i, m_i such that

$$h_i = \Pr\{c_i = c_H\}, \qquad m_i = \Pr\{c_i = c_M\}, \qquad l_i = 1 - h_i - m_i = \Pr\{c_i = c_L\}$$

The distribution of c_1 is stochastically independent of the distribution of c_2. Although the random variables c_1 and c_2 have the same support $\{c_L, c_M, c_H\}$, they are asymmetrically distributed unless $(h_1, m_1) = (h_2, m_2)$. Without loss of generality, we suppose that h_1, m_1, h_2, m_2 satisfy

$$h_1 + m_1 \leq h_2 + m_2 \tag{1}$$

which makes seller 1 ex ante (weakly) stronger than seller 2 in the sense that $\Pr\{c_1 = c_L\} \geq \Pr\{c_2 = c_L\}$. The expected profit of each seller is given by his expected revenue minus his cost times his probability to be selected to supply the good. The buyer wants to minimize her expected payment to the seller supplying the good, that is her expected cost to procure the good. Towards this purpose, she chooses between running a first price auction (FPA in the following) or a second price auction (SPA in the following) to determine the supplier and her payment to the latter. In this setting, CDM identify an equilibrium for the FPA and compare the buyer's expected payment in the FPA and in the SPA.[9]

In this paper, to the above environment we add the possibility for a seller i to change the ex ante distribution of his cost. Precisely, in Sections 3 and 4 we suppose that after the buyer has chosen the FPA or the SPA, seller i ($i = 1$ in Section 3, $i = 2$ in Section 4) decides whether to make a cost-reduction investment which improves the distribution of c_i in terms of first order stochastic dominance, but costs $k > 0$. We assume the investment is observable, i.e., the other seller observes whether seller i has made the investment or not. After seller i's investment decision, each seller privately learns his cost and the auction is played.[10]

We compare the FPA with the SPA in this environment, but since our results partially rely on the analysis by CDM, in the rest of this section we describe the results in CDM which are useful for us.

2.1 Equilibrium in the FPA

We use i_j to denote type j of seller i, for $i = 1, 2$ and $j = L, M, H$. Under the innocuous assumption (1), Proposition 1 in CDM proves that a unique Bayes-Nash Equilibrium (BNE in the following) exists in the

[8] See for instance Bag (1997), Cisternas and Figueroa (2015), Li and Wan (2017), Piccione and Tan (1996).

[9] Actually, CDM analyse auctions for the sale of an object, in which the bidders are prospective buyers, but their results can be easily extended to procurement auctions: See Subsections 2.1 and 2.2 below.

[10] In Section 5 we examine a setting in which each seller can make an investment which improves his cost distribution.

FPA and it is such that type 1_H, type 2_H both bid c_H, whereas types $1_M, 1_L, 2_M, 2_L$ play mixed strategies.[11] In particular, type 1_M or type 2_M (but not both) bids c_H with positive probability, except for a non-generic set of parameters, and as a result ρ_1 (ρ_2), defined as the probability that seller 1 (seller 2) bids c_H, may be greater than h_1 (than h_2). Moreover, $\rho_1\Delta$ coincides with the expected profit of type 2_M, $\rho_2\Delta$ is type 1_M's expected profit, whereas the expected profit of types 1_L, 2_L is $\underline{b}_L - c_L$, in which $\underline{b}_L = c_L + (\rho_1 + h_2 + m_2)\Delta$ is the lowest bid in the support of the mixed strategy of both type 1_L and type 2_L.

Three equilibrium regimes exist, which mainly differ because of the seller types who bid c_H with positive probability, in addition to types 1_H and 2_H. When

$$h_2(h_1 + m_1) < h_1(h_1 + m_2) \tag{2}$$

the BNE is denoted E_{2M}, with $\rho_1 = h_1$, $\rho_2 = h_1\frac{h_1+h_2+m_2}{2h_1+m_1} > h_2$, and only type 2_M bids c_H with positive probability. When $h_2(h_1 + m_1) > h_1(h_1 + m_2)$ (that is, when (2) is violated strictly) but

$$h_2 - m_2 \leq h_1 + m_1 \tag{3}$$

the BNE is denoted E_{1M}, with $\rho_1 = \sqrt{\frac{1}{4}m_2^2 + h_2(h_1 + m_1)} - \frac{1}{2}m_2 > h_1$, $\rho_2 = h_2$, and only type 1_M bids c_H with positive probability. When (3) is violated, the BNE is denoted E_{1ML}, with $\rho_1 = h_2 - m_2$, $\rho_2 = h_2$, and both type 1_M (with probability 1) and type 1_L bid c_H with positive probability.

Figure 1 below illustrates the equilibrium regimes by fixing h_2, m_2 and partitioning the set of (h_1, m_1) which satisfy (1) (that is, the triangle with bold edges) into three regions: Region R_{2M} is the set of (h_1, m_1) for which (2) holds, hence E_{2M} is the equilibrium when $(h_1, m_1) \in R_{2M}$; region R_{1M} (region R_{1ML}) is the region such that the equilibrium is E_{1M} (is E_{1ML}). The curve C consists of (h_1, m_1) such that (2) is an equality and is the set of (h_1, m_1) such that $\rho_1 = h_1$, $\rho_2 = h_2$. In Figure 1 we assume $h_2 > m_2$, hence there exist (h_1, m_1) close to $(0, 0)$ which violate (3) and R_{1ML} is non-empty. If instead $h_2 \leq m_2$, then (3) is satisfied for each (h_1, m_1), R_{1ML} is empty and the curve C connects $(0, 0)$ to (h_2, m_2).

Please insert here Figure 1, with the following caption:
The regions R_{1M}, R_{2M}, R_{1ML} when $h_2 > m_2$

In Sections 3, 4 we examine the effect of changes in (h_1, m_1) or in (h_2, m_2) on the buyer's expected payment in the FPA, which we denote P^F. Such changes affect P^F as they determine changes in $\rho_1, \rho_2, \underline{b}_L$ and Lemma 1 establishes that P^F depends on $\rho_1, \rho_2, \underline{b}_L$ in a monotone way, a very intuitive result.

Lemma 1 (Monotonicity of P^F with respect to ρ_1, ρ_2, $\underline{b}_L$) *P^F is strictly increasing in ρ_1, ρ_2, $\underline{b}_L$.*

We use π_{ij}^F (π_{ij}^S) to denote the equilibrium expected profit of type i_j in the FPA (in the SPA), for $i = 1, 2$, $j = L, M, H$, hence $\pi_{1H}^F = \pi_{2H}^F = \pi_{1H}^S = \pi_{2H}^S = 0$. For each equilibrium regime, $\pi_{1M}^F, \pi_{1L}^F, \pi_{2M}^F, \pi_{2L}^F$ and $\pi_{1M}^S, \pi_{1L}^S, \pi_{2M}^S, \pi_{2L}^S$ are reported in the following table (in which the common factor Δ is omitted),[12] and ρ_{1M} in the line relative to E_{1M} is equal to $\sqrt{\frac{1}{4}m_2^2 + h_2(h_1 + m_1)} - \frac{1}{2}m_2$:

equilibrium \ seller type	1_M	1_L	2_M	2_L
E_{2M}	$h_1\frac{h_1+h_2+m_2}{2h_1+m_1}$	$h_1 + h_2 + m_2$	h_1	$h_1 + h_2 + m_2$
E_{1M}	h_2	$\rho_{1M} + h_2 + m_2$	ρ_{1M}	$\rho_{1M} + h_2 + m_2$
E_{1ML}	h_2	$2h_2$	$h_2 - m_2$	$2h_2$
SPA	h_2	$2h_2 + m_2$	h_1	$2h_1 + m_1$

(4)

Table 1: Profits of seller types in the FPA (E_{2M}, E_{1M}, E_{1ML}), and in the SPA

[11] We consider a FPA with the "Vickrey tie-breaking rule" introduced by Maskin and Riley (2000) which has the consequence that if the two sellers submit the same bid, then the seller with the lower cost wins and receives a payment equal to the other seller's cost. See Subsection 7.1 for a description of the tie-breaking rule and of the mixed strategies of types $1_M, 1_L, 2_M, 2_L$.

[12] A unique equilibrium regime exists for the SPA as for each seller i it is weakly dominant to bid c_i for all parameter values.

2.2 Ranking the FPA and the SPA with fixed distributions

Proposition 2 in CDM allows to compare P^F, the buyer's payment in the FPA, with her payment in SPA, denoted P^S: There exists a set of (h_2, m_2), denoted S_2, such that if $(h_2, m_2) \in S_2$ then $P^F \geq P^S$ for each (h_1, m_1) which satisfies (1); S_2 is the white set in Figure 2 (see Subsection 7.1 for additional details). The complementary set, denoted F_2 (in gray in Figure 2), is such that for each $(h_2, m_2) \in F_2$ there exists a non-empty set of (h_1, m_1), which we denote F_1, such that $P^F < P^S$.

Please insert here Figure 2, with the following caption:
The sets F_2 (in grey) and S_2

When the set F_1 is non-empty, it includes $(h_1, m_1) = (\max\{h_2 - m_2, 0\}, 0)$, which is the distribution of c_1 inducing the most aggressive bidding in the FPA by both sellers given h_2, m_2. However, F_1 does not include any (h_1, m_1) such that $h_1 \geq h_2$, as this inequality implies that each type of seller 1 and each type of seller 2 prefers the FPA to the SPA, hence $P^F > P^S$.[13] Figure 3 shows the set F_1 for a case with $h_2 > m_2$.

Please insert here Figure 3, with the following caption:
The set F_1 (in grey) of (h_1, m_1) such that $P^F < P^S$ when $h_2 = 0.58, m_2 = 0.32$

3 Investment by Seller 1

Here we consider the environment described by Section 2 and suppose that the distribution of c_2 is fixed at (h_2, m_2), but seller 1 can make an investment – after learning the auction format but before observing c_1 – which changes the distribution of c_1 from (h_1, m_1) into $(\tilde{h}_1, \tilde{m}_1)$, with $(\tilde{h}_1, \tilde{m}_1)$ in the following set Σ_1:

$$\Sigma_1 = \{(\tilde{h}_1, \tilde{m}_1) : \tilde{h}_1 \leq h_1 \quad \text{and} \quad \tilde{h}_1 + \tilde{m}_1 \leq h_1 + m_1 \quad \text{with at least one strict inequality}\} \tag{5}$$

This is the set of distributions of c_1 which are first order stochastically dominated by the initial distribution, that is each distribution in Σ_1 is more likely to yield a lower cost than the distribution (h_1, m_1). We say that each $(\tilde{h}_1, \tilde{m}_1)$ in Σ_1 is a *weak upgrade* of (h_1, m_1).

AC consider a more restrictive notion of improvement in the distribution of c_1 which applies when the new distribution is first order stochastically dominated by the initial distribution conditional on considering costs not smaller than a given c, for arbitrary c. In our context, $(\tilde{h}_1, \tilde{m}_1)$ satisfies such a property if and only if $(\tilde{h}_1, \tilde{m}_1)$ is in the set Ψ_1 in (6), in which the inequality $\frac{m_1}{h_1} < \frac{\tilde{m}_1}{\tilde{h}_1}$ is equivalent to the property that $(\tilde{h}_1, \tilde{m}_1)$ is first order stochastically dominated by (h_1, m_1) conditional on $c_1 \geq c_M$:

$$\Psi_1 = \left\{ (\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 : \frac{m_1}{h_1} < \frac{\tilde{m}_1}{\tilde{h}_1} \right\} \tag{6}$$

Consistently with the terminology in AC, we say that each $(\tilde{h}_1, \tilde{m}_1)$ in Ψ_1 is an *upgrade* of (h_1, m_1).

Figure 4 represents graphically Σ_1 and Ψ_1 in the space $(\tilde{h}_1, \tilde{m}_1)$.[14] Precisely, Σ_1 is the trapezoid with vertices $(h_1, 0)$, $(0, 0)$, $(0, h_1 + m_1)$, (h_1, m_1), but the latter point does not belong to Σ_1. The set Ψ_1 is the triangular subset of Σ_1 with vertices $(0, 0)$, $(0, h_1 + m_1)$, (h_1, m_1), not including the segment connecting $(0, 0)$ to (h_1, m_1) because $\frac{m_1}{h_1} < \frac{\tilde{m}_1}{\tilde{h}_1}$ means that $(\tilde{h}_1, \tilde{m}_1)$ lies higher with respect to the mentioned segment.

Please insert here Figure 4, with the following caption:
The set Ψ_1 of $(\tilde{h}_1, \tilde{m}_1)$ which are upgrades of (h_1, m_1);
The set Σ_1 of $(\tilde{h}_1, \tilde{m}_1)$ which are weak upgrades of (h_1, m_1)

[13] When c_1, c_2 are asymmetrically distributed, in the FPA (but not in the SPA) it may occur that the seller with the higher cost wins, which is socially inefficient. If moreover the total sellers' profits are higher in the FPA, then $P^F > P^S$.

[14] The sets Σ_1 and Ψ_1 depend on h_1, m_1, hence a complete notation would be $\Sigma_1^{h_1, m_1}$ and $\Psi_1^{h_1, m_1}$, but for the sake of brevity we omit the superscript h_1, m_1. A similar remark applies to the function D_1 and to the set $\mathfrak{F}_1$ introduced in Subsection 3.1.

After seller 1's decision about making the investment or not, which seller 2 observes, each seller learns his own cost and the auction, FPA or SPA, is played.

In the rest of Section 3 we compare the FPA and the SPA in terms of seller 1's incentive to invest and then in terms of the buyer's expected payment.

3.1 Seller 1's incentive to invest

Whether the FPA or the SPA provides the greater investment incentive is determined by comparing the change in seller 1's expected profit due to the investment under the FPA with the analogous change under the SPA. Therefore, for the SPA we define

$$\Pi_1^S = h_1 \cdot 0 + m_1 \pi_{1M}^S + l_1 \pi_{1L}^S, \qquad \tilde{\Pi}_1^S = \tilde{h}_1 \cdot 0 + \tilde{m}_1 \pi_{1M}^S + \tilde{l}_1 \pi_{1L}^S \tag{7}$$

with π_{1M}^S, π_{1L}^S from the last line in (4). Hence Π_1^S is the ex ante expected profit of seller 1 in the SPA with the initial distribution of c_1, and $\tilde{\Pi}_1^S$ is seller 1's expected profit after he has made the investment. Notice that in the SPA the profit of type 1_M (of type 1_L) is π_{1M}^S (is π_{1L}^S) independently of the distribution of c_1: the new distribution of c_1 attaches different probabilities to $0, \pi_{1M}^S, \pi_{1L}^S$, the profits of the three types of seller 1, but does not change such profits. Since $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$ and $0 < \pi_{1M}^S < \pi_{1L}^S$, it follows $\tilde{\Pi}_1^S > \Pi_1^S$.

Likewise, for the FPA we set

$$\Pi_1^F = h_1 \cdot 0 + m_1 \pi_{1M}^F + l_1 \pi_{1L}^F, \qquad \tilde{\Pi}_1^F = \tilde{h}_1 \cdot 0 + \tilde{m}_1 \tilde{\pi}_{1M}^F + \tilde{l}_1 \tilde{\pi}_{1L}^F$$

in which π_{1M}^F, π_{1L}^F are the profits of types $1_M, 1_L$ in the FPA, respectively, under the initial distribution and Π_1^F is the expected profit of seller 1 in the FPA when the distribution of c_1 is (h_1, m_1). Notice that π_{1M}^F, π_{1L}^F depend on the equilibrium regime determined by whether (h_1, m_1) is in region R_{2M}, or in R_{1M}, or in R_{1ML}: see Figure 1 and (4). When the distribution of c_1 is $(\tilde{h}_1, \tilde{m}_1)$, the profits of types $1_M, 1_L$ are denoted $\tilde{\pi}_{1M}^F, \tilde{\pi}_{1L}^F$ and $\tilde{\Pi}_1^F$ is seller 1's resulting expected profit. Typically $\tilde{\pi}_{1M}^F \neq \pi_{1M}^F$ and/or $\tilde{\pi}_{1L}^F \neq \pi_{1L}^F$ because if (h_1, m_1) and $(\tilde{h}_1, \tilde{m}_1)$ are in different regions, then different lines in (4) apply; but even if (h_1, m_1) and $(\tilde{h}_1, \tilde{m}_1)$ are in the same region, then $(h_1, m_1) \neq (\tilde{h}_1, \tilde{m}_1)$ may imply $\tilde{\pi}_{1M}^F \neq \pi_{1M}^F$ and/or $\tilde{\pi}_{1L}^F \neq \pi_{1L}^F$.

As AC point out, the total effect of seller 1's investment on his ex ante profit is determined by two effects. The first is called *direct effect* and is due to the improvement of the distribution of c_1, for fixed bidding by seller 2, which attaches higher probability of low c_1 (since $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$) that is to higher profits. The second effect is called *strategic effect* and is due to the change in the equilibrium bidding by seller 2 as a result of the change in the distribution of c_1, which affects the profits of types $1_M, 1_L$. As we remarked above, in the SPA there is no strategic effect but in the FPA this effect plays a role. Proposition 1 in AC assumes that the post-investment distribution of c_1 is an upgrade of the initial distribution and shows that under the former distribution, in the FPA seller 2 bids more aggressively than under the latter distribution. This reduces the profits of all types of seller 1,[15] that is the strategic effect is negative in the FPA. Next subsection is about this effect in our setting.

3.1.1 The strategic effect in the FPA of a change in the distribution of c_1

The next lemma is about the effect of a change in the distribution of c_1 on the profits of types $1_M, 1_L$ in the FPA. It somewhat extends Proposition 1 in AC but also shows that different conclusions may emerge if the assumption of upgrade is replaced with weak upgrade.

[15] In some cases this reduces seller 1's ex ante expected profit as well, a result Thomas (1997) obtains in a related setting.

Lemma 2 (Profit changes in the FPA for types $1_M, 1_L$ when seller 1 makes the investment)
(i) The inequality $\tilde{\pi}^F_{1L} \leq \pi^F_{1L}$ *holds for each* $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$.
(ii) The inequality $\tilde{\pi}^F_{1M} \leq \pi^F_{1M}$ *holds for each* $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \cap (\Psi_1 \cup R_{1M} \cup R_{1ML})$*, but not necessarily otherwise. For instance,* $\tilde{\pi}^F_{1M} > \pi^F_{1M}$ *if* $(h_1, m_1) \in R_{2M}$ *and* $\tilde{h}_1 = h_1$*,* $\tilde{m}_1 < m_1$.

Lemma 2 generalizes Proposition 1 in AC, for the case in which the seller who may invest is the stronger seller in terms of (1), as it establishes $\tilde{\pi}^F_{1L} \leq \pi^F_{1L}$ not only when $(\tilde{h}_1, \tilde{m}_1) \in \Psi_1$ but for each $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$, and $\tilde{\pi}^F_{1M} \leq \pi^F_{1M}$ for each $(\tilde{h}_1, \tilde{m}_1)$ in the set $\Sigma_1 \cap (\Psi_1 \cup R_{1M} \cup R_{1ML})$, which is a superset of Ψ_1.

Conversely, $\tilde{\pi}^F_{1M} > \pi^F_{1M}$ for some $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$ because when $(\tilde{h}_1, \tilde{m}_1) \in R_{2M}$, (4) reveals that $\tilde{\pi}^F_{1M}$ is equal to $\tilde{h}_1 \frac{\tilde{h}_1 + h_2 + m_2}{2\tilde{h}_1 + \tilde{m}_1}$, which is decreasing in $\tilde{m}_1$. In particular, given $(h_1, m_1) \in R_{2M}$, for each $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{h}_1 = h_1$ and $\tilde{m}_1 < m_1$ (i.e., each $(\tilde{h}_1, \tilde{m}_1)$ lying on the right vertical edge of Σ_1) we have $\tilde{\pi}^F_{1M} > \pi^F_{1M}$ and $\tilde{\pi}^F_{1L} = \pi^F_{1L}$. Therefore each such $(\tilde{h}_1, \tilde{m}_1)$ is a weak upgrade of (h_1, m_1) and the types of seller 1 Pareto prefer $(\tilde{h}_1, \tilde{m}_1)$ to (h_1, m_1). This shows that when the new distribution of c_1 is not an upgrade but just a weak upgrade of the initial distribution, then the result in Proposition 1 in AC that each type of seller 1 is worse off with the new distribution of c_1 does not hold because the strategic effect may be non-negative.[16]

The root of this result is that $\tilde{\pi}^F_{1M}$ is strictly decreasing in $\tilde{m}_1$, which is a consequence of the features of the equilibrium in the FPA when (h_1, m_1) is in region R_{2M}. Precisely, if $\tilde{h}_1 = h_1$ and $\tilde{m}_1 < m_1$ then type 2_M faces less frequently type 1_M (and more frequently type 1_L) but from (4) it follows that the profit of type 2_M is unchanged: $\tilde{\pi}^F_{2M} = \pi^F_{2M} = h_1$. Hence, to compensate for $\tilde{m}_1 < m_1$, type 1_M needs to bid less aggressively and the lower bound of the support of his mixed strategy increases. Then also the upper bound of the support of the mixed strategy of type 1_L increases (there cannot be a gap between the two supports). Hence it must be optimal for type 1_L to submit some relatively high bids which were suboptimal for him before the decrease in the probability that seller 1 has type M. In order for such bids to become optimal for type 1_L, type 2_M needs to bid less aggressively, as this increases the chances of type 1_L to win with the newly submitted bids. A consequence of type 2_M bidding less aggressively is that the profit of type 1_M increases, that is $\tilde{\pi}^F_{1M} > \pi^F_{1M}$.

3.1.2 Comparing the FPA and the SPA in terms of incentive to invest for seller 1

Here we compare the investment incentives in the two auctions through the sign of

$$D_1(\tilde{h}_1, \tilde{m}_1) = \tilde{\Pi}^F_1 - \Pi^F_1 - (\tilde{\Pi}^S_1 - \Pi^S_1)$$

In order to describe our results, we denote with $\mathfrak{F}_1$ the set of $(\tilde{h}_1, \tilde{m}_1)$ for which the investment incentive is stronger in the FPA:

$$\mathfrak{F}_1 = \{(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 : D_1(\tilde{h}_1, \tilde{m}_1) > 0\}$$

Conversely, the incentive to invest is weakly greater in the SPA if $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \backslash \mathfrak{F}_1$, as then $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$.[17]

Below, we first illustrate the result of AC in our setting, then we indicate cases in which more general assumptions lead to a different result. To this purpose, it is useful to write $D_1(\tilde{h}_1, \tilde{m}_1)$ as follows:

$$D_1(\tilde{h}_1, \tilde{m}_1) = \Delta m_1(\pi^F_{1M} - \pi^S_{1M}) + \tilde{m}_1 \Delta\pi^F_{1M} + \Delta l_1(\pi^F_{1L} - \pi^S_{1L}) + \tilde{l}_1 \Delta\pi^F_{1L} \tag{8}$$

in which $\Delta m_1 = \tilde{m}_1 - m_1$, $\Delta\pi^F_{1M} = \tilde{\pi}^F_{1M} - \pi^F_{1M}$ and $\Delta l_1 = \tilde{l}_1 - l_1$, $\Delta\pi^F_{1L} = \tilde{\pi}^F_{1L} - \pi^F_{1L}$. From (8) we see that $D_1(\tilde{h}_1, \tilde{m}_1)$ is determined by the size of the change in the probability that $c_1 = v_M$ (that $c_1 = c_L$), by the

[16] More in general, a small $\tilde{m}_1$ and $\tilde{h}_1$ close to h_1 make it likely that $\tilde{\pi}^F_{1M}$ is larger than π^F_{1M}. This is established by a more general version of Lemma 2(ii) which is found in Subsection 7.3.

[17] We have noticed in footnote 15 that sometimes $\tilde{\Pi}^F_1 < \Pi^F_1$. But when $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ we can conclude that $\tilde{\Pi}^F_1 > \Pi^F_1$ because $\tilde{\Pi}^S_1 > \Pi^S_1$.

preferences of types $1_M, 1_L$ over the two auctions under the initial distribution, and by the profit change for types $1_M, 1_L$ in the FPA due to the change in the distribution of c_1.

Proposition 3 in AC proves that the SPA provides seller 1 with a stronger incentive to invest under the following two assumptions:

- AC1: The distribution of c_2 is an upgrade of the initial distribution of c_1 (or the two distributions coincide);
- AC2: The post-investment distribution of c_1 is an upgrade of the distribution of c_2 (or the two distributions coincide).

Conditions AC1 and AC2 require a "leadership change" which imposes significant restrictions on the relation among (h_1, m_1), (h_2, m_2), $(\tilde{h}_1, \tilde{m}_1)$. In particular, given (1), the only way AC1 and AC2 can hold in our ternary environment is as follows:

$$h_1 + m_1 = h_2 + m_2, \quad h_2 < h_1 \qquad \text{and} \qquad \tilde{h}_1 + \tilde{m}_1 \le h_2 + m_2, \quad \frac{m_2}{h_2} < \frac{\tilde{m}_1}{\tilde{h}_1} \tag{9}$$

From (9) it follows that $(h_1, m_1) \in R_{2M}$, $(\tilde{h}_1, \tilde{m}_1) \in R_{1M} \cup R_{1ML}$, $\Delta\pi^F_{1M} < 0$, $\Delta\pi^F_{1L} < 0$ and eventually $D_1(\tilde{h}_1, \tilde{m}_1) < 0$, consistently with Proposition 3 in AC. More in general, Lemma 4 below establishes that $D_1(\tilde{h}_1, \tilde{m}_1)$ is negative for each $(\tilde{h}_1, \tilde{m}_1) \in R_{1M} \cup R_{1ML}$.

Conversely, we do not assume leadership change but just $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$ and we prove that the main message of AC – the investment incentive is weaker in FPA than the SPA – does not necessarily apply under such assumptions. This may happen because of the strategic effect or because of the direct effect.

1. Strategic effect: We have seen in Subsection 3.1.1 that $\Delta\pi^F_{1M} > 0$, $\Delta\pi^F_{1L} = 0$ for some $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$; from this, Lemma 3 below identifies $(\tilde{h}_1, \tilde{m}_1)$ such that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$.
2. Direct effect: $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ may hold even though $\Delta\pi^F_{1M} < 0$, $\Delta\pi^F_{1L} < 0$ as $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$ implies $\Delta l_1 \ge 0$, but generically $\Delta l_1 > 0$, and if $\pi^F_{1L} > \pi^S_{1L}$ then the third term in the right hand side of (8) is positive; in some cases this makes $D_1(\tilde{h}_1, \tilde{m}_1)$ positive albeit $\Delta\pi^F_{1M} < 0$, $\Delta\pi^F_{1L} < 0$. That is, the direct effect in the FPA is stronger than in the SPA and may dominate the negative strategic effect. This cannot hold under a leadership change since the latter requires a (relatively) significant change in the distribution of c_1, which generates a (relatively) significant decrease in the profits of types $1_M, 1_L$ in the FPA; then the strategic effect dominates and $D_1(\tilde{h}_1, \tilde{m}_1) < 0$. Conversely, the set Σ_1 includes $(\tilde{h}_1, \tilde{m}_1)$ close to (h_1, m_1) and then it is possible that $D_1(\tilde{h}_1, \tilde{m}_1)$ is positive: see Lemma 5 below.

We use *status quo* to describe the initial distributions (h_1, m_1) and (h_2, m_2) and begin with the simple case of symmetric status quo.

1: Symmetric status quo When $(h_1, m_1) = (h_2, m_2)$ we have $\pi^F_{1M} = \pi^S_{1M}$, $\pi^F_{1L} = \pi^S_{1L}$, which nullifies the difference in the direct effect between the two auctions, hence (8) reduces to

$$D_1(\tilde{h}_1, \tilde{m}_1) = \tilde{m}_1 \Delta\pi^F_{1M} + \tilde{l}_1 \Delta\pi^F_{1L} \tag{10}$$

From Lemma 2 it follows that for each $(\tilde{h}_1, \tilde{m}_1) \in R_{1M} \cup R_{1ML}$ we have $\Delta\pi^F_{1M} \le 0$, $\Delta\pi^F_{1L} \le 0$, thus $D_1(\tilde{h}_1, \tilde{m}_1) \le 0$; this generalizes Proposition 3 in AC because $\Psi_1 \subseteq R_{1M} \cup R_{1ML}$ when the status quo is symmetric.[18] But as we remarked shortly after Lemma 2, the strategic effect may be non-negative and in

[18]Furthermore, $(h_1, m_1) = (h_2, m_2)$ implies $(R_{1M} \cup R_{1ML}) \subseteq \Sigma_1$.

particular if $\tilde{h}_1 = h_1$, $\tilde{m}_1 \in (0, m_1)$, then $\Delta\pi^F_{1M} > 0$, $\Delta\pi^F_{1L} = 0$, hence $D_1(\tilde{h}_1, \tilde{m}_1) > 0$. Lemma 3 completes the picture by establishing that given any $\tilde{m}_1 \in [0, m_1)$, the sign of $D_1(\tilde{h}_1, \tilde{m}_1)$ is positive if and only if $\tilde{h}_1$ is close enough to h_1.

Lemma 3 (Investment incentives under symmetric status quo) *Suppose that* $(h_1, m_1) = (h_2, m_2)$. *Then there exists a function* $\varphi : [0, m_1) \to [0, h_1]$ *with* $\varphi(0) = h_1$, $\varphi(\tilde{m}_1) < h_1$ *for each* $\tilde{m}_1 \in (0, m_1)$ *such that* $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$ *if and only if* $(\tilde{h}_1, \tilde{m}_1) \in R_{1M} \cup R_{1ML}$ *or* $\tilde{m}_1 \in [0, m_1)$ *and* $\tilde{h}_1 \leq \varphi(\tilde{m}_1)$.

Lemma 3 states that there is a function of $\tilde{m}_1 \in [0, m_1)$, denoted φ and defined uniquely through $D_1(\varphi(\tilde{m}_1), \tilde{m}_1) = 0$, such that

$$\mathfrak{F}_1 = \{(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 : \tilde{m}_1 \in (0, m_1) \quad \text{and} \quad \varphi(\tilde{m}_1) < \tilde{h}_1 \leq h_1\}$$

that is the graph of φ separates the set $\mathfrak{F}_1$ from its complementary set in Σ_1 as Figure 5 illustrates.

Please insert here Figure 5 with the following caption:
The set $\mathfrak{F}_1$ (in grey) when $(h_1, m_1) = (h_2, m_2) = (0.58, 0.32)$

2: Asymmetric status quo When the initial distributions are asymmetric, matters are more complicated because typically $\pi^F_{1M} \neq \pi^S_{1M}$ and/or $\pi^F_{1L} \neq \pi^S_{1L}$ and $D_1(\tilde{h}_1, \tilde{m}_1)$ does not boil down to the simple expression in (10). We begin with a sufficient condition for $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$.

Lemma 4 (Investment incentives when $(\tilde{h}_1, \tilde{m}_1) \in R_{1M} \cup R_{1ML}$**)** *If* $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \cap (R_{1M} \cup R_{1ML})$, *then* $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$.

Lemma 4 is immediate if we write $D_1(\tilde{h}_1, \tilde{m}_1)$ in a slightly different way with respect to (8) as follows:

$$D_1(\tilde{h}_1, \tilde{m}_1) = \Delta m_1(\tilde{\pi}^F_{1M} - \pi^S_{1M}) + m_1\Delta\pi^F_{1M} + \Delta l_1(\tilde{\pi}^F_{1L} - \pi^S_{1L}) + l_1\Delta\pi^F_{1L} \tag{11}$$

When $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \cap (R_{1M} \cup R_{1ML})$, Lemma 2 proves $\Delta\pi^F_{1M} \leq 0$, $\Delta\pi^F_{1L} \leq 0$ and (4) reveals $\tilde{\pi}^F_{1M} - \pi^S_{1M} = 0$, $\tilde{\pi}^F_{1L} - \pi^S_{1L} \leq 0$; hence $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$. Conversely, $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$ does not necessarily hold if $(\tilde{h}_1, \tilde{m}_1) \in R_{2M}$: Lemma 3 provides an example based on a reduction in the probability of type 1_M and next lemma shows that also a reduction in the probability of type 1_H may result in $D_1(\tilde{h}_1, \tilde{m}_1) > 0$.

Lemma 5 (Sufficient conditions for $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ **under asymmetric status quo)** *Suppose that* $h_1 > \frac{1}{2} + \frac{1}{2}h_2$. *Then* $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ *if* $\tilde{m}_1 = m_1$ *and* $\tilde{h}_1$ *is slightly smaller than* h_1.

In order to gain an intuition for Lemma 5, we notice that $h_1 > \frac{1}{2} + \frac{1}{2}h_2$ implies $(h_1, m_1) \in R_{2M}$ and then (4) makes clear that the inequality $\tilde{h}_1 < h_1$ reduces the profits of types $1_M, 1_L$, that is the strategic effect is negative, but it makes Δl_1 positive, that is increases the probability that seller 1 has type L. Moreover, $h_1 > \frac{1}{2} + \frac{1}{2}h_2$ implies $\pi^F_{1L} > \pi^S_{1L}$, hence the direct effect is greater in the FPA than in the SPA ($\pi^F_{1M} \neq \pi^S_{1M}$ is irrelevant as $\Delta m_1 = 0$) and the sign of $D_1(\tilde{h}_1, \tilde{m}_1)$ in (8) is ambiguous. Then it is convenient to notice that the term $\Delta l_1(\tilde{\pi}^F_{1L} - \pi^S_{1L})$ in (11) is equal to $(h_1 - \tilde{h}_1)(\tilde{h}_1 - h_2)$, hence (11) reduces to

$$D_1(\tilde{h}_1, \tilde{m}_1) = m_1\Delta\pi^F_{1M} + (h_1 - \tilde{h}_1)(\tilde{h}_1 - h_2) + l_1\Delta\pi^F_{1L} \tag{12}$$

The profit of type 1_L is greater in the FPA under the initial distribution and remains so after the investment if $\tilde{h}_1$ is not too smaller than h_1 such that $\tilde{h}_1 > h_2$. Then $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ as long as h_1 is large since then m_1 and l_1 are small and $\Delta\pi^F_{1M} < 0$, $\Delta\pi^F_{1L} < 0$ in the right hand side of (12) receive little weights. In particular, Lemma 5 establishes that $h_1 > \frac{1}{2} + \frac{1}{2}h_2$ and $\tilde{h}_1$ slightly smaller than h_1 suffice to make $D_1(\tilde{h}_1, \tilde{m}_1)$ positive, hence $\mathfrak{F}_1$ includes some $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{h}_1 < h_1$ and $\tilde{m}_1 = m_1$,[19] although $D_1(\tilde{h}_1, m_1) < 0$ for each $\tilde{h}_1 < h_1$ under a symmetric status quo. Essentially, the strategic effect has little relevance when m_1, l_1 are small and since the direct effect is greater in the FPA, it is intuitive that $D_1(\tilde{h}_1, \tilde{m}_1)$ is positive.

In the appendix we provide some more general results about the set $\mathfrak{F}_1$ when $(h_1, m_1) \in R_{2M}$. Broadly speaking, they indicate that $\mathfrak{F}_1$ is non-empty when h_1 and/or m_1 is large such that (1) is an approximate equality. For the sake of brevity we do not report these results here, except for Lemma 6.

Lemma 6 (Sufficient conditions for $\mathfrak{F}_1 = \varnothing$) *If $(h_1, m_1) \in R_{2M}$ satisfies $h_1 \leq h_2$ and $2h_1(h_2 + m_2 - h_1) \geq 4h_1 m_1 + m_1^2$, then $\mathfrak{F}_1 = \varnothing$.*

3.2 Ranking the FPA and the SPA given seller 1's possibility to invest

The comparison between the FPA and the SPA in terms of buyer's expected payment depends on her payment in the two auctions without the investment, P^F, P^S, and on her payment after the investment, denoted $\tilde{P}^F$, $\tilde{P}^S$. It also depends on the investment cost k and on which auction provides the stronger investment incentive, as for intermediate values of k the investment occurs only in one of the two auctions.

Proposition 3 in AC establishes that under assumptions AC1 and AC2, the investment incentive is greater under the SPA. Our analysis extends this result significantly beyond these assumptions: see Lemmas 4, 6, which provide sufficient conditions for $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$.

This is relevant because it is immediate that the SPA is superior to the FPA for each k, in the sense that it generates a lower payment for the buyer, if $(\tilde{h}_1, \tilde{m}_1)$ satisfies $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$ and $(h_2, m_2) \in S_2$ – the set S_2 has been introduced in Subsection 2.2. Indeed, then $\tilde{\Pi}_1^F - \Pi_1^F \leq \tilde{\Pi}_1^S - \Pi_1^S$ and

- when k is large, the investment occurs in no auction and $P^F \geq P^S$ as $(h_2, m_2) \in S_2$;
- when k is intermediate between $\tilde{\Pi}_1^F - \Pi_1^F$ and $\tilde{\Pi}_1^S - \Pi_1^S$, the investment occurs only in the SPA and $P^F \geq P^S > \tilde{P}^S$, hence $P^F > \tilde{P}^S$;
- when k is small, the investment occurs in both auctions and $\tilde{P}^F \geq \tilde{P}^S$ as $(h_2, m_2) \in S_2$.

Proposition 1 records this result and allows the condition $(h_2, m_2) \in S_2$ to be replaced by $(h_2, m_2) \in F_2$ and $(h_1, m_1) \notin F_1$, $(\tilde{h}_1, \tilde{m}_1) \notin F_1$ since $P^F \geq P^S$ and $\tilde{P}^F \geq \tilde{P}^S$ hold also in the latter case.

Proposition 1 (Sufficient conditions for the buyer to prefer the SPA for each k) *Suppose that $(h_2, m_2) \in S_2$, or that $(h_2, m_2) \in F_2$ and $(h_1, m_1) \notin F_1$, $(\tilde{h}_1, \tilde{m}_1) \notin F_1$. Then the buyer weakly prefers the SPA to the FPA for each k if $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$.*

The inequality $D_1(\tilde{h}_1, \tilde{m}_1) < 0$ favors the SPA not only when $P^F \geq P^S$ and $\tilde{P}^F \geq \tilde{P}^S$ but also, for instance, when $P^F < P^S$. In such a case the buyer prefers the FPA to the SPA when the investment occurs in neither auction, but $D_1(\tilde{h}_1, \tilde{m}_1) < 0$ makes the investment occur only in the SPA if k is intermediate. As a result, the SPA is superior as long as P^F is just slightly less than P^S and $(\tilde{h}_1, \tilde{m}_1)$ is stronger enough than (h_1, m_1), so that $\tilde{P}^S$ is significantly less than P^S, thus less than P^F. Conversely, $P^F < \tilde{P}^S$ if $(\tilde{h}_1, \tilde{m}_1)$ is just a little bit stronger than (h_1, m_1); then the FPA is preferable to the SPA for intermediate k even though $D_1(\tilde{h}_1, \tilde{m}_1) < 0$, and is actually preferable for each k if furthermore $\tilde{P}^F < \tilde{P}^S$.[20] In this case the result stems from the superiority of the FPA under asymmetric distributions, independently of the investment incentives.

In next subsection we rely on Lemmas 3 and 5 to identify settings in which $P^F > P^S$ but the investment incentive is stronger in the FPA than in the SPA and this makes the buyer prefer the FPA.

[19] But $D_1(\tilde{h}_1, \tilde{m}_1)$ is negative if $\tilde{h}_1$ is small enough such that $\tilde{h}_1 - h_2 < 0$.

[20] For instance, this is the case if $(h_2, m_2) \in F_2$ and (h_1, m_1), $(\tilde{h}_1, \tilde{m}_1)$ are both close to $(\max\{h_2 - m_2, 0\}, 0)$, hence they are both in F_1.

3.2.1 Two examples such that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $P^F > P^S > \tilde{P}^F$

Here we describe two settings such that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $P^F > P^S > \tilde{P}^F$,[21] that is the investment occurs only in the FPA when k is between $\tilde{\Pi}_1^S - \Pi_1^S$ and $\tilde{\Pi}_1^F - \Pi_1^F$ and this makes the FPA superior to the SPA as $P^S > \tilde{P}^F$. Hence the buyer prefers the FPA exactly because it gives a stronger investment incentive, unlike under the assumptions in AC, even though the opposite ranking holds when no investment occurs.

Example 1: $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{h}_1 < h_1$, $\tilde{m}_1 = m_1$ Here we consider $(h_1, m_1) \in R_{2M}$ and provide sufficient conditions for $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{h}_1 < h_1$, $\tilde{m}_1 = m_1$ to satisfy $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $P^F > P^S > \tilde{P}^F$.

Proposition 2 (Sufficient conditions for the FPA to be superior to the SPA for intermediate k) *Let $h_1 = h_2 + m_2 > \frac{2}{3} + \frac{1}{3}h_2$ and $m_1 = 0$. Suppose $\tilde{m}_1 = 0$ and that $\tilde{h}_1$ is slightly greater than $1 + h_2 - h_1$. Then $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $P^F > P^S > \tilde{P}^F$.*

Proposition 2 considers a case in which $h_1 > h_2$, hence $P^F > P^S$: see Subsection 2.2. Moreover, $m_1 = \tilde{m}_1 = 0$ and the investment of seller 1 improves the distribution of c_1 by reducing (increasing) the probability that seller 1 has type H (has type L). If h_1 is large, then Lemma 5 applies and $\tilde{h}_1$ slightly smaller than h_1 increases seller 1's profit in the FPA more than in the SPA. As a result, the investment occurs only in the FPA if k is intermediate. Moreover, $\tilde{h}_1 < h_1$ reduces the buyer's payment in the FPA by Lemma 1 since it reduces $\rho_1, \rho_2, \underline{b}_L$. More in detail, (12) yields $D_1(\tilde{h}_1, \tilde{m}_1) = (h_1 - \tilde{h}_1)(\tilde{h}_1 - (1 + h_2 - h_1))$, hence $1 + h_2 - h_1$ is the smallest value of $\tilde{h}_1$ which is consistent with $D_1(\tilde{h}_1, \tilde{m}_1) \geq 0$. Furthermore, at $\tilde{h}_1 = 1 + h_2 - h_1$ the inequality $P^S > \tilde{P}^F$ holds if h_1 is large, which is intuitive as a large h_1 (i) reduces $\tilde{h}_1 = 1 + h_2 - h_1$, which increases the payment reduction in the FPA due to the investment; (ii) increases P^S.

Example 2: $(\tilde{h}_1, \tilde{m}_1)$ with $\tilde{m}_1 < m_1$ We begin by considering a symmetric status quo, that is $(h_1, m_1) = (h_2, m_2)$. Then Lemma 3 reveals that $D_1(h_1, \tilde{m}_1) > 0$ for each $\tilde{m}_1 \in (0, m_1)$, hence the investment occurs only under the FPA when k is intermediate. However, $\tilde{m}_1 < m_1$ increases $\tilde{P}^F$ above $P^F = P^S$ by Lemma 1 since it increases ρ_2 (does not change $\rho_1, \underline{b}_L$). Then we consider $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{m}_1 < m_1$ and $\tilde{h}_1 < h_1$ and $(\tilde{h}_1, \tilde{m}_1) \in R_{2M}$: The first inequality increases $\tilde{P}^F$ but the second inequality decreases $\tilde{P}^F$ through a decrease in $\rho_1, \rho_2, \underline{b}_L$; as a result, it is generally unclear whether $\tilde{P}^F$ is greater or smaller than P^F. Proposition 3(ia) below identifies $(\tilde{h}_1, \tilde{m}_1)$ such that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $P^F = P^S > \tilde{P}^F$, that is such that the decrease in $\tilde{P}^F$ due to $\tilde{h}_1 < h_1$ dominates the increase due to $\tilde{m}_1 < m_1$.

Proposition 3 (Ranking the FPA and the SPA under symmetric status quo, First part) *Suppose that $(h_1, m_1) = (h_2, m_2)$. Then $\mathfrak{F}_1 \neq \varnothing$ by Lemma 3 and*

(i) the FPA is superior to the SPA if

(ia) $(\tilde{h}_1, \tilde{m}_1)$ is in a suitable subset of $\mathfrak{F}_1$ which includes $\tilde{h}_1 = h_2 - \varepsilon$, $\tilde{m}_1 = m_2 - \frac{3h_2 - 3h_2^2 + m_2 - 2h_2 m_2}{h_2 m_2}\varepsilon$ with $\varepsilon > 0$ close to 0, and k is between $\tilde{\Pi}_1^S - \Pi_1^S$ and $\tilde{\Pi}_1^F - \Pi_1^F$;

(ib) $(h_2, m_2) \in F_2$, $(\tilde{h}_1, \tilde{m}_1) \in F_1$, and $k < \tilde{\Pi}_1^F - \Pi_1^F$.

(ii) In each circumstance not covered by (ia) or (ib) but such that $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$, the buyer weakly prefers the SPA.

Proposition 3(ia) assumes that the status quo is symmetric and identifies suitable $(\tilde{h}_1, \tilde{m}_1)$, with $\tilde{h}_1$ slightly smaller than h_1, $\tilde{m}_1$ slightly smaller than m_1, such that the FPA yields a greater investment incentive and this makes the buyer prefer the FPA to the SPA for intermediate k. Proposition 3(ib) indicates another

[21] In some cases, in order to simplify the description and the interpretation of the example, we make some assumptions about the parameters even though the result holds more generally than under our statements.

circumstance in which the FPA is superior to the SPA, such that $(h_2, m_2) \in F_2$ and $(\tilde{h}_1, \tilde{m}_1) \in F_1$. Then $D_1(\tilde{h}_1, \tilde{m}_1) < 0$, that is $(\tilde{h}_1, \tilde{m}_1)$ is not in $\mathfrak{F}_1$ in Figure 5, but if k is small then the investment occurs in both auctions[22] and the resulting asymmetry favors the FPA because $(h_2, m_2) \in F_2$, $(\tilde{h}_1, \tilde{m}_1) \in F_1$ imply $\tilde{P}^F < \tilde{P}^S$. Here the FPA is preferable to the SPA because of its superior performance in the asymmetric setting arising when seller 1 makes the investment, but it ceases to be preferable if k is large enough to deter the investment under the FPA. Finally, Proposition 3(ii) establishes that the two mentioned circumstances are the only ones such that the buyer prefers the FPA given a symmetric status quo and $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$. In next subsection we extend the analysis by removing the restriction $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$.

Although Proposition 3(ia) assumes a symmetric status quo, by continuity its result holds even if (h_1, m_1) is slightly different from (h_2, m_2) and such that $P^F > P^S$. That is, there exist $(h_1, m_1) \neq (h_2, m_2)$ and $(\tilde{h}_1, \tilde{m}_1) \in \mathfrak{F}_1$ such that $\tilde{h}_1 < h_1$, $\tilde{m}_1 < m_1$ and $P^F > P^S > \tilde{P}^F$.

3.2.2 Symmetric status quo and arbitrary $(\tilde{h}_1, \tilde{m}_1)$

We have assumed up to now that the post-investment distribution of c_1 is a weak upgrade of the initial distribution, that is $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$. This is a way to represent an improvement in the distribution of c_1, but it is somewhat restrictive. Under the assumption of symmetric status quo, next lemma deals with the case of $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$.

Lemma 7 *Suppose that* $(h_1, m_1) = (h_2, m_2)$ *and* $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$. *Then* $D_1(\tilde{h}_1, \tilde{m}_1) \geq 0$ *and* $\tilde{P}^F > P^S = P^F$.

In a sense, this lemma's result is the opposite of Proposition 3 in AC, that is the incentive to invest is invariably greater with the FPA when $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$. However, this conclusion must take into account that $\tilde{\Pi}_1^F$ may be smaller than Π_1^F, and in such a case seller 1 does not make the investment, even though $D_1(\tilde{h}_1, \tilde{m}_1) > 0$, because that would make him worse off even when $k = 0$.[23] Precisely, $\tilde{\Pi}_1^F > \Pi_1^F$ if $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$ is close to Σ_1 but, as it is intuitive, $\tilde{\Pi}_1^F < \Pi_1^F$ if $(\tilde{h}_1, \tilde{m}_1)$ is "sufficiently far" from Σ_1.

The other result in Lemma 7 is that any new distribution of c_1 which is not a weak upgrade of the initial distribution leads to an increase in the buyer's expected payment under the FPA with respect to the status quo. This plays a significant role in the proof of the following proposition.

Proposition 4 (Ranking the FPA and the SPA under symmetric status quo, Second part) *Suppose that* $(h_1, m_1) = (h_2, m_2)$. *Then the SPA is weakly superior to the FPA for each* $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$.

Propositions 3 and 4 jointly establish that starting from a symmetric status quo, the buyer prefers the SPA except in the specific circumstances described by Proposition 3(i).

4 Investment by Seller 2

In this section we suppose that the distribution of c_1 is fixed at (h_1, m_1) but seller 2 can make an investment which changes the distribution of c_2 from (h_2, m_2) into $(\tilde{h}_2, \tilde{m}_2)$, with $(\tilde{h}_2, \tilde{m}_2)$ in the set $\Sigma_2 = \{(\tilde{h}_2, \tilde{m}_2) : \tilde{h}_2 \leq h_2 \text{ and } \tilde{h}_2 + \tilde{m}_2 \leq h_2 + m_2\}$; each $(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2$ is said to be a *weak upgrade* of (h_2, m_2). Moreover, we say that $(\tilde{h}_2, \tilde{m}_2)$ is an *upgrade* of (h_2, m_2) if and only if $(\tilde{h}_2, \tilde{m}_2)$ is in the set

$$\Psi_2 = \left\{ (\tilde{h}_2, \tilde{m}_2) \in \Sigma_2 : \frac{m_2}{h_2} < \frac{\tilde{m}_2}{\tilde{h}_2} \right\} \tag{13}$$

[22] The proof of Proposition 3(ib) shows $\tilde{\Pi}_1^F - \Pi_1^F > 0$, thus there exist $k > 0$ such that $k < \tilde{\Pi}_1^F - \Pi_1^F$.

[23] Conversely, if $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$ then $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ implies $\tilde{\Pi}_1^F > \Pi_1^F$: see footnote 17.

Since we are considering distributions of c_2, a graphical representation relies on the space $(\tilde{h}_2, \tilde{m}_2)$ and distinguishes the set of $(\tilde{h}_2, \tilde{m}_2)$ such that $h_1 + m_1 \leq \tilde{h}_2 + \tilde{m}_2$ (that is, $(\tilde{h}_2, \tilde{m}_2)$ lies on or above the segment $\tilde{h}_2 + \tilde{m}_2 = h_1 + m_1$ in Figure 6 (or, equivalently, (1) is satisfied with $(h_2, m_2) = (\tilde{h}_2, \tilde{m}_2)$), from the set of $(\tilde{h}_2, \tilde{m}_2)$ such that $\tilde{h}_2 + \tilde{m}_2 < h_1 + m_1$.

Given (h_1, m_1), we partition the set of $(\tilde{h}_2, \tilde{m}_2)$ such that $h_1 + m_1 \leq \tilde{h}_2 + \tilde{m}_2$ into three regions depending on the equilibrium regime determined by $(\tilde{h}_2, \tilde{m}_2)$. In particular, we use $\mathcal{R}_{2M}$ to denote the set of $(\tilde{h}_2, \tilde{m}_2)$ such that E_{2M} in Subsection 2.1 is the equilibrium, that is such that (2) is satisfied with $(\tilde{h}_2, \tilde{m}_2)$: $\mathcal{R}_{2M} = \{(\tilde{h}_2, \tilde{m}_2) : \tilde{h}_2(h_1 + m_1) < h_1(h_1 + \tilde{m}_2)\}$. Likewise, $\mathcal{R}_{1M}$ ($\mathcal{R}_{1ML}$) is the set of $(\tilde{h}_2, \tilde{m}_2)$ such that E_{1M} (E_{1ML}) in Subsection 2.1 is the equilibrium: $\mathcal{R}_{1M} = \{(\tilde{h}_2, \tilde{m}_2) : \tilde{h}_2(h_1 + m_1) \geq h_1(h_1 + \tilde{m}_2)$ and $\tilde{h}_2 - \tilde{m}_2 \leq h_1 + m_1\}$, ($\mathcal{R}_{1ML} = \{(\tilde{h}_2, \tilde{m}_2) : h_1 + m_1 < \tilde{h}_2 - \tilde{m}_2\}$).

In a similar way we partition the set of $(\tilde{h}_2, \tilde{m}_2)$ such that $\tilde{h}_2 + \tilde{m}_2 < h_1 + m_1$ into three regions, $\mathcal{R}^*_{1M}, \mathcal{R}^*_{2M}, \mathcal{R}^*_{2ML}$, which are analogous to the three regions identified in the paragraph above, after switching the roles of seller 1 and seller 2, hence $\mathcal{R}^*_{2ML} = \{(\tilde{h}_2, \tilde{m}_2) : \tilde{h}_2 + \tilde{m}_2 \leq h_1 - m_1\}$, $\mathcal{R}^*_{2M} = \{(\tilde{h}_2, \tilde{m}_2) : h_1(\tilde{h}_2 + \tilde{m}_2) \geq \tilde{h}_2(\tilde{h}_2 + m_1)$ and $h_1 - m_1 < \tilde{h}_2 + \tilde{m}_2\}$, $\mathcal{R}^*_{1M} = \{(\tilde{h}_2, \tilde{m}_2) : h_1(\tilde{h}_2 + \tilde{m}_2) < \tilde{h}_2(\tilde{h}_2 + m_1)\}$. For instance, when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2ML}$ (which requires $h_1 > m_1$) the equilibrium is such that type 2_M bids c_H with probability 1 and also type 2_L bids c_H with positive probability, but less than 1. This is analogous to equilibrium E_{1ML}, in which types $1_M, 1_L$ bid c_H with positive probability; while the latter equilibrium regime applies if $\tilde{h}_2 - \tilde{m}_2 \geq h_1 + m_1$, the former applies if $h_1 - m_1 \geq \tilde{h}_2 + \tilde{m}_2$. Likewise, when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M}$ (when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$), the equilibrium is analogous to E_{1M} (to E_{2M}).

Please insert here Figure 6, with the following caption:
The regions $\mathcal{R}_{2M}, \mathcal{R}_{1M}, \mathcal{R}_{1ML}, \mathcal{R}^*_{1M}, \mathcal{R}^*_{2M}, \mathcal{R}^*_{2ML}$ in the space $(\tilde{h}_2, \tilde{m}_2)$

4.1 Seller 2's incentive to invest

4.1.1 The strategic effect in the FPA of a change in the distribution of c_2

Next lemma is about the effect of a change in the distribution of c_2 on the profits of types $2_M, 2_L$ in the FPA.

Lemma 8 (Profit changes in the FPA for types $2_M, 2_L$ when seller 2 makes the investment)
(i) The inequality $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$ holds for each $(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2$.
(ii) The inequality $\tilde{\pi}^F_{2M} \leq \pi^F_{2M}$ holds for each $(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2 \cap (\Psi_2 \cup \mathcal{R}_{2M} \cup \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML})$, but not necessarily otherwise. For instance, $\tilde{\pi}^F_{2M} > \pi^F_{2M}$ if $(h_2, m_2) \in \mathcal{R}_{1M}$ and $\tilde{h}_2 = h_2$, $\tilde{m}_2 < m_2$.

Lemma 8 is analogous to Lemma 2 as it generalizes Proposition 1 in AC, for the case in which the seller who may invest is weaker in terms of (1), by establishing $\Delta\pi^F_{2L} \leq 0$ and $\Delta\pi^F_{2M} \leq 0$ for each $(\tilde{h}_2, \tilde{m}_2)$ in a set which is a superset of Ψ_2. But in some cases the investment increases the profit of type 2_M, as $\tilde{\pi}^F_{2M}$ is decreasing in $\tilde{m}_2$ when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M} \cup \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$. This result is due to the features of the equilibrium in the FPA like the result $\tilde{\pi}^F_{1M} > \pi^F_{1M}$ in Lemma 2(ii).

4.1.2 Incentive comparison for seller 2

Arguing as in Subsection 3.1, we define Π^F_2 ($\tilde{\Pi}^F_2$) as the ex ante expected profit of seller 2 in the FPA with the initial distribution (with the post-investment distribution) for c_2; Π^S_2, $\tilde{\Pi}^S_2$ are defined likewise for the

SPA. Then we let[24]

$$
\begin{aligned}
D_2(\tilde{h}_2, \tilde{m}_2) &= \tilde{\Pi}_2^F - \Pi_2^F - (\tilde{\Pi}_2^S - \Pi_2^S) \\
&= \Delta m_2(\tilde{\pi}_{2M}^F - \pi_{2M}^S) + m_2 \Delta\pi_{2M}^F + \Delta l_2(\tilde{\pi}_{2L}^F - \pi_{2L}^S) + l_2 \Delta\pi_{2L}^F \quad (14)
\end{aligned}
$$

and define $\mathfrak{F}_2$ (analogous to $\mathfrak{F}_1$) as the set of $(\tilde{h}_2, \tilde{m}_2)$ for which the investment incentive is greater in the FPA:

$$\mathfrak{F}_2 = \{(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2 : D_2(\tilde{h}_2, \tilde{m}_2) > 0\}$$

Proposition 3 in AC shows that the investment incentive is greater in the SPA if seller 2's investment determines a leadership change in terms of upgrades. In our setting this leadership change requires $(h_2, m_2) \in \mathcal{R}_{2M} \cup \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$ (see Figure 6) and $\tilde{h}_2 + \tilde{m}_2 \leq h_1 + m_1$, $\frac{m_1}{h_1} \leq \frac{\tilde{m}_2}{\tilde{h}_2}$, hence $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}^* \cup \mathcal{R}_{2ML}^*$. Then Corollary 2 below shows $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$, consistently with Proposition 3 in AC. But under weaker assumptions, different results may emerge. We consider first the case in which $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M} \cup \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$, and then the case in which $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}^* \cup \mathcal{R}_{1M}^* \cup \mathcal{R}_{2ML}^*$.

Case 1: $(\tilde{h}_2, \tilde{m}_2) \in R_{2M} \cup R_{1M} \cup R_{1ML}$ We characterize in Lemma 9 below the set of $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ such that $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ and then use this result to shed some light on the set $\mathfrak{F}_2 \cap (\mathcal{R}_{1M} \cup \mathcal{R}_{1ML})$.

When $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$, (4) shows that the probabilities $\tilde{h}_2, \tilde{m}_2$ matter only through the sum $\tilde{h}_2 + \tilde{m}_2$, which we denote $\tilde{s}_2$, and to simplify the expressions we write s_2 instead of $h_2 + m_2$, we write s_1 instead of $h_1 + m_1$; hence $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M} \cap \Sigma_2$ implies $s_1 \leq \tilde{s}_2 \leq s_2$. By relying on (4) it follows that (14) reduces to

$$D_2(\tilde{h}_2, \tilde{m}_2) = (1 - h_2)(h_1 - \rho_1) + (1 - s_2)(\tilde{s}_2 - s_2) + (s_2 - \tilde{s}_2)(\tilde{s}_2 - s_1) \quad (15)$$

in which the first two terms in the right hand side are negative or zero and represent the profit reduction in the FPA for types 2_M, 2_L due to the investment; the third term is due to the increase in the probability, $s_2 - \tilde{s}_2$, that seller 2 has type L, a state of the world in which seller 2's profit is by $\tilde{s}_2 - s_1$ higher in the FPA than in the SPA. Hence $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ if the latter term dominates the profit decreases for types $2_M, 2_L$. Notice that if $\tilde{s}_2 = s_2$, then there is no increase in the probability that seller 2 has type L, and if $\tilde{s}_2 = s_1$ then the reduction in the profit of type 2_L in the FPA is sufficiently large to make him indifferent between the FPA and the SPA. In both these cases the third term in (15) is 0 and $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$. In particular, $\tilde{s}_2$ is equal to s_1 if $(\tilde{h}_2, \tilde{m}_2) = (h_1, m_1)$, that is when the post-investment distribution of c_2 coincides with the distribution of c_1; thus the following corollary is obtained.

Corollary 1 (Investment incentives when $(\tilde{h}_2, \tilde{m}_2) = (h_1, m_1)$**)** *The investment incentive is weakly stronger in the SPA when seller 2's investment makes the sellers symmetric, that is* $D_2(h_1, m_1) \leq 0$.

On the other hand, if s_2 is sufficiently larger than s_1 then there exist $\tilde{s}_2 \in (s_1, s_2)$ such that the sum between the second and third term in (15) is positive. Then $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ for some $(\tilde{h}_2, \tilde{m}_2)$ if and only if h_2 is sufficiently large and/or ρ_1 is not much larger than h_1, in such a way to satisfy (16) below.

Lemma 9 (Necessary and sufficient condition for $\mathfrak{F}_2 \cap \mathcal{R}_{2M} \neq \varnothing$**)** *There exist* $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ *such that* $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ *if and only if* $1 + s_1 - s_2 < s_2$ *and*

$$(s_2 - \frac{1}{2} - \frac{1}{2}s_1)^2 > (1 - h_2)(\rho_1 - h_1) \quad (16)$$

[24]Notice that (14) is analogous to (11).

When $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$, no simple characterization like the one in Lemma 9 is available because the expressions of $\tilde{\pi}^F_{2M}, \tilde{\pi}^F_{2L}$ are more complicated than when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$. However, a simple remark yields Lemma 10(i).

Lemma 10 *(i) Given a particular $(\tilde{h}'_2, \tilde{m}'_2) \in \Sigma_2 \cap \mathcal{R}_{2M}$, suppose that $(\tilde{h}''_2, \tilde{m}''_2) \in \Sigma_2 \cap (\mathcal{R}_{1M} \cup \mathcal{R}_{1ML})$ and $\tilde{h}''_2 + \tilde{m}''_2 = \tilde{h}'_2 + \tilde{m}'_2$. Then $D_2(\tilde{h}''_2, \tilde{m}''_2) \geq D_2(\tilde{h}'_2, \tilde{m}'_2)$, hence $D_2(\tilde{h}'_2, \tilde{m}'_2) > 0$ implies $D_2(\tilde{h}''_2, \tilde{m}''_2) > 0$.*
(ii) Suppose that $(h_2, m_2) \in \mathcal{R}_{1ML}$. Then $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ for each $(\tilde{h}_2, \tilde{m}_2)$ such that $\tilde{h}_2 = h_2$ and $\tilde{m}_2 < m_2$, that is for each $(\tilde{h}_2, \tilde{m}_2)$ along the right edge of Σ_2.

Lemma 10(i) says that if $(\tilde{h}'_2, \tilde{m}'_2) \in \mathcal{R}_{2M}$ is such that $D_2(\tilde{h}'_2, \tilde{m}'_2) > 0$, then also $(\tilde{h}''_2, \tilde{m}''_2) \in \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$ satisfies $D_2(\tilde{h}''_2 + \tilde{m}''_2) > 0$ as long as the sum $\tilde{h}''_2 + \tilde{m}''_2$ is equal to the sum $\tilde{h}'_2 + \tilde{m}'_2$. See for instance Figure 7, in which $\mathfrak{F}_2 \cap \mathcal{R}_{2M}$ is non-empty and therefore also $(\mathcal{R}_{1M} \cup \mathcal{R}_{1ML}) \cap \Sigma_2$ includes some points of $\mathfrak{F}_2$. Lemma 10(i) relies on $\tilde{\pi}^F_{2M} = \tilde{\rho}_1, \tilde{\pi}^F_{2L} = \tilde{\rho}_1 + \tilde{s}_2$, and $\tilde{\rho}_1 = h_1$ when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ but $\tilde{\rho}_1 \geq h_1$ when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$. Hence the post-investment profits of types 2_M, 2_L in the FPA are higher in the latter case and it follows that $D_2(\tilde{h}''_2, \tilde{m}''_2) > 0$ if $D_2(\tilde{h}'_2, \tilde{m}'_2) > 0$.

Please insert here Figure 7 with the following caption:
The set of $\mathfrak{F}_2$ (in grey) for a case with $(h_2, m_2) \in \mathcal{R}_{1M}$

This effect applies also when (16) is violated, that is it is possible that $\mathfrak{F}_2$ includes points in $\mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$ even though $\mathfrak{F}_2 \cap \mathcal{R}_{2M} = \varnothing$ – but it may also occur that $\mathfrak{F}_2$ is empty. However, Lemma 10(ii) shows that $(h_2, m_2) \in \mathcal{R}_{1ML}$ is a sufficient condition for $\mathfrak{F}_2 \neq \varnothing$. In detail, $(\tilde{h}_2, \tilde{m}_2)$ such that $\tilde{h}_2 = h_2$, $\tilde{m}_2 < m_2$ imply $\Delta\pi^F_{2M} > 0 = \Delta\pi^F_{2L}$ and $\Delta l_2 = -\Delta m_2 > 0$, that is the probability that seller 2 has type L (has type M) increases (decreases). Since $\tilde{\pi}^F_{2L} - \pi^S_{2L} > \tilde{\pi}^F_{2M} - \pi^S_{2M}$, (14) shows that $D_2(\tilde{h}_2, \tilde{m}_2) > 0$.

Case 2: $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{1M} \cup \mathcal{R}^*_{2ML}$ Here we first show that $\mathfrak{F}_2 \cap (\mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML})$ is empty. To this purpose, we employ $(\hat{h}_2, \hat{m}_2)$ to denote the distribution of c_2 such that the two sellers are symmetric, that is $(\hat{h}_2, \hat{m}_2) = (h_1, m_1)$. When the distribution of c_2 changes from (h_2, m_2) to $(\tilde{h}_2, \tilde{m}_2)$, the profit change for seller 2, in the FPA and in the SPA, can be decomposed in the profit change when moving from the initial distribution to $(\hat{h}_2, \hat{m}_2)$, plus the profit change when the distribution moves from $(\hat{h}_2, \hat{m}_2)$ to the final distribution. By Corollary 1, the first profit change in the SPA is at least as large as in the FPA. Then, starting from a symmetric situation, Lemma 3 establishes that the investment incentive is greater in the SPA as $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML}$. Since both effects favor the SPA, a clear-cut result emerges.

Corollary 2 (Investment incentives when $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML}$) *For each $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML}$ we have $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$. In particular, $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$ if (h_1, m_1) is an upgrade of (h_2, m_2) and $(\tilde{h}_2, \tilde{m}_2)$ is an upgrade of (h_1, m_1).*

By assumption, at the beginning of the game seller 2 is weaker than seller 1 in the sense of (1), hence Corollary 2 establishes that the SPA provides a stronger incentive than the FPA for each investment which makes seller 2 stronger than seller 1 as long as $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M} \cup \mathcal{R}^*_{2ML}$. These are weaker assumptions than those in Proposition 3 in AC, that is Corollary 2 generalizes Proposition 3 in AC.

However, some $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$ may satisfy $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ because the argument for Corollary 2 does not apply if $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$. Indeed, starting from a symmetric status quo Lemma 3 shows that the investment incentive is greater in the FPA than in the SPA for some $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$. Hence the second profit change described just before Corollary 2 does not favor the SPA and in some cases it leads to $D_2(\tilde{h}_2, \tilde{m}_2) > 0$. For instance, in Figure 7 the set $\mathfrak{F}_2$ has non-empty intersection with $\mathcal{R}^*_{1M}$.

4.2 Ranking the FPA and the SPA given seller 2's possibility to invest

In the comparison between the FPA and the SPA, a result analogous to Proposition 1 holds, establishing that the buyer weakly prefers the SPA to the FPA for each k if $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$ and, for instance, $(h_2, m_2) \in S_2$, $(\tilde{h}_2, \tilde{m}_2) \in S_2$, (1) holds with $(\tilde{h}_2, \tilde{m}_2)$, or if $(h_2, m_2) \in S_2$ and $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$ with $h_1 \leq \tilde{h}_2$.

In the following we provide a few examples such that $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ and $P^F > P^S > \tilde{P}^F$, that is without the investment the buyer's payment is higher in the FPA, but the investment incentive is greater in the FPA and the post-investment payment in the FPA is lower than the initial payment in the SPA.

4.2.1 Two examples such that $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ and $P^F > P^S > \tilde{P}^F$

Example 3: $(h_2, m_2) \in \mathcal{R}_{2M}$, $(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2 \cap \mathcal{R}_{2M}$ Here we consider (h_2, m_2) in $\mathcal{R}_{2M}$ and give sufficient conditions for $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ and $\tilde{P}^F < P^S$, while $P^F - P^S$ may have either sign.

Proposition 5 *Let $s_1 = h_1 + m_1$, $s_2 = h_2 + m_2$, $\tilde{s}_2 = \tilde{h}_2 + \tilde{m}_2$. Suppose $(h_2, m_2) \in \mathcal{R}_{2M}$ with s_2 close to 1, $h_2 > h_1 - \frac{(1-s_1)^2}{1-h_1}$, and $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ is such that $\tilde{s}_2$ is close to s_1. Then $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ and $\tilde{P}^F < P^S$; if moreover $h_2 \leq h_1$, then $P^F > P^S > \tilde{P}^F$.*

The intuition for this example is that (h_2, m_2) and $(\tilde{h}_2, \tilde{m}_2)$ in $\mathcal{R}_{2M}$ relies on $\tilde{\pi}^F_{2M} = \pi^F_{2M} = \pi^S_{2M}$, hence $D_2(\tilde{h}_2, \tilde{m}_2) = \Delta l_2(\tilde{\pi}^F_{2L} - \pi^S_{2L}) + l_2 \Delta \pi^F_{2L}$ is positive because of the argument given just after (15): (i) the probability that seller 2 has type L increases; (ii) type 2_L's profit in the FPA is greater than in the SPA; (iii) the profit reduction of type 2_L in the FPA has little weight since s_2 is about 1 and l_2 is about zero. Moreover, $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ implies that $\tilde{P}^F$ depends on $(\tilde{h}_2, \tilde{m}_2)$ only through $\tilde{s}_2$, and $\tilde{s}_2 < s_2$ reduces $\tilde{\rho}_2$ and $\tilde{\underline{b}}_L$ (there is no effect on $\tilde{\rho}_1$), hence $\tilde{P}^F < P^F$ by Lemma 1 and $\tilde{P}^F$ is minimized when $\tilde{s}_2$ is close to s_1 – that is when $(\tilde{h}_2, \tilde{m}_2)$ is close to the border between $\mathcal{R}_{2M}$ and $\mathcal{R}^*_{2M}$. The proof of Proposition 5 then establishes $\tilde{P}^F < P^S$ if $h_2 > h_1 - \frac{(1-s_1)^2}{1-h_1}$, that is unless h_2 is too small as a small h_2 decreases P^S and makes it more difficult to satisfy $\tilde{P}^F < P^S$. Finally, $P^F - P^S$ may be positive or negative without the investment, but $P^F > P^S$ when $h_2 \leq h_1$: see Subsection 2.2.

Example 4: A two-type setting with $(h_2, m_2) \in \mathcal{R}_{1ML}$ In this example we consider a two-type setting in which neither seller 1 nor seller 2 may have type M and $(h_2, m_2) \in \mathcal{R}_{1ML}$, $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1ML}$.

Proposition 6 *Suppose that $m_1 = 0$, $m_2 = 0$, $\tilde{m}_2 = 0$ and $h_2 > \frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{2} h_1$. Then $D_2(\tilde{h}_2, 0) > 0$ and $P^F > P^S > \tilde{P}^F$ if $\tilde{h}_2 < h_2$ and $\tilde{h}_2$ is slightly greater than $1 + h_1 - h_2$.*

From (14) it follows that $D_2(\tilde{h}_2, 0) = (h_2 - \tilde{h}_2)(2\tilde{h}_2 - 2h_1) + (1 - h_2)(2\tilde{h}_2 - 2h_2)$, in which the first term is positive as $\tilde{h}_2 < h_2$ increases the probability that seller 2 has type L, and type 2_L has a higher profit under the FPA. But $\tilde{h}_2 < h_2$ also reduces the profit of type 2_L in the FPA from $2h_2$ to $2\tilde{h}_2$: see the negative second term. If h_2 is sufficiently larger than h_1, then there exist $\tilde{h}_2 \in (h_1, h_2)$ such that the former effect dominates the latter effect and $D_2(\tilde{h}_2, 0) > 0$. Since $\tilde{P}^F$ is increasing in $\tilde{h}_2$ by Lemma 1, we consider $\tilde{h}_2$ slightly greater than the smallest $\tilde{h}_2$ satisfying $D_2(\tilde{h}_2, 0) \geq 0$, which is $\tilde{h}_2 = 1 + h_1 - h_2$, and then the resulting decrease in the buyer's payment in the FPA implies $P^S > \tilde{P}^F$ if h_2 is larger than $\frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{2} h_1$ as stated by Proposition 6. This occurs as the larger is h_2, the lower $\tilde{h}_2$ can be while satisfying $D_2(\tilde{h}_2, 0) \geq 0$, which decreases $\tilde{P}^F$ the most. The lower bound on h_2 given in Proposition 6 is increasing with respect to h_1 because a larger h_1 increases type 2_L's profit in the SPA. This makes it more difficult to satisfy $D_2(\tilde{h}_2, 0) \geq 0$ and then a greater h_2 is needed to satisfy such inequality and $P^S > \tilde{P}^F$.

5 Investment by both sellers

In this section we consider a setting in which each seller can make an investment to improve the probability distribution of his own cost. We begin with the case of symmetric sellers.

5.1 Symmetric sellers

We suppose that the status quo is symmetric and use (h, m) to denote both (h_1, m_1) and (h_2, m_2). The distribution of c_1 (of c_2) if seller 1 (seller 2) makes the investment is $(\tilde{h}, \tilde{m})$, which is first order stochastically dominated by (h, m), that is $(\tilde{h}, \tilde{m}) \in \Sigma_1 = \Sigma_2$. The sellers' investment decisions are simultaneous and commonly observed before the auction is played.

We use $\pi_{ij}^F(I, N)$ to denote the expected profit of type i_j in the FPA if seller 1 (seller 2) makes (does not make) the investment, for $i = 1, 2$ and $j = L, M.H$, and define $\pi_{ij}^F(I, I)$, $\pi_{ij}^F(N, I)$, $\pi_{ij}^F(N, N)$ likewise. In a similar way, $\pi_{ij}^S(I, N)$, $\pi_{ij}^S(I, I)$, $\pi_{ij}^S(N, I)$, $\pi_{ij}^S(N, N)$ denote the profit of type i_j in the SPA as a function of the investment decisions.

The ex ante expected profit in the FPA of a seller who makes the investment when his opponent does not is denoted Π_{IN}^F and is defined as $\tilde{m}\pi_{1M}^F(I, N) + \tilde{l}\pi_{1L}^F(I, N)$, or equivalently as $\tilde{m}\pi_{2M}^F(N, I) + \tilde{l}\pi_{2L}^F(N, I)$ since sellers are ex ante symmetric. The terms $\Pi_{II}^F, \Pi_{NI}^F, \Pi_{NN}^F$ are defined likewise; for instance, $\Pi_{NI}^F = m\pi_{1M}^F(N, I) + l\pi_{1L}^F(N, I)$. For the SPA we use $\Pi_{IN}^S, \Pi_{II}^S, \Pi_{NI}^S, \Pi_{NN}^S$.

The normal form of the investment game under the FPA (the SPA) is denoted G^F (G^S) and is

G^F :

$1\backslash 2$	I	N
I	$\Pi_{II}^F - k, \Pi_{II}^F - k$	$\Pi_{IN}^F - k, \Pi_{NI}^F$
N	$\Pi_{NI}^F, \Pi_{IN}^F - k$	Π_{NN}^F, Π_{NN}^F

G^S :

$1\backslash 2$	I	N
I	$\Pi_{II}^S - k, \Pi_{II}^S - k$	$\Pi_{IN}^S - k, \Pi_{NI}^S$
N	$\Pi_{NI}^S, \Pi_{IN}^S - k$	Π_{NN}^S, Π_{NN}^S

(17)

Next lemma establishes three inequalities which shed some light on G^F and G^S.

Lemma 11 (Inequalities for G^F and G^S) *The profits in (17) satisfy*

$$\Pi_{II}^F - \Pi_{NI}^F < \Pi_{IN}^F - \Pi_{NN}^F, \qquad \Pi_{II}^S - \Pi_{NI}^S < \Pi_{IN}^S - \Pi_{NN}^S \tag{18}$$

$$\Pi_{II}^F - \Pi_{NI}^F \leq \Pi_{II}^S - \Pi_{NI}^S \tag{19}$$

The inequalities in (18) reveal that a seller's incentive to invest is higher if the other seller has not invested, both in the FPA and in the SPA. The intuition is quite simple in the SPA, as the profits of types M and L of seller 1 (to fix the ideas) are h and $2h+m$, respectively, if seller 2 does not make the investment, but decrease to $\tilde{h}$ and $2\tilde{h} + \tilde{m}$ if seller 2 invests (the inequalities $h \leq \tilde{h}$ and $2h + m \leq 2\tilde{h} + \tilde{m}$ hold as $(\tilde{h}, \tilde{m}) \in \Sigma_2$). In the latter case, the improvement in the distribution of c_1 which occurs if seller 1 invests acts on lower prizes for seller 1, which reduces his incentive to invest. A similar intuition applies to the FPA, which shares the property that the investment of seller 2 reduces the profits of types 1_M and 1_L. According to a standard terminology, in both G^F and G^S investments are strategic substitutes.

Inequalities (18) have consequences for the Nash Equilibria (NE henceforth) in G^F and G^S, and in particular create the possibility of existence of asymmetric NE, in which seller i invests but seller j does not because seller i's investment reduces seller j's gain from investing. Precisely, focussing on G^S to begin with, it follows that its (pure-strategy) NE are only (I, I) (if $k \leq \Pi_{II}^S - \Pi_{NI}^S$), or both (I, N) and (N, I) (if $\Pi_{II}^S - \Pi_{NI}^S < k \leq \Pi_{IN}^S - \Pi_{NN}^S$), or only (N, N) (if $\Pi_{IN}^S - \Pi_{NN}^S < k$). A similar result holds for G^F, but with different thresholds for k, that is $\Pi_{II}^F - \Pi_{NI}^F$ replaces $\Pi_{II}^S - \Pi_{NI}^S$ and $\Pi_{IN}^F - \Pi_{NN}^F$ replaces $\Pi_{IN}^S - \Pi_{NN}^S$.

The different thresholds reveal that the investment incentives in the two auctions are not the same. In detail, the SPA is more effective in promoting investments by both sellers because (19) shows that a seller's incentive to invest, given that the other seller has made the investment, is weakly greater in the SPA, hence the range of k such that (I, I) is a NE in G^S is a superset of the range of k such that (I, I) is a NE in G^F – this is just the content of Corollary 1. When (I, I) is a NE in G^S, the SPA is weakly superior to the FPA. When instead k is such that (N, I) and (I, N) are NE in the SPA, then in the FPA either (N, N) is the unique NE, or both (N, I) and (I, N) are NE. In the first case the buyer definitely prefers the SPA, but in the second case the buyer's preference is determined by the standard comparison under asymmetrically distributed costs summarized in Subsection 2.2. Finally, when (N, N) is the unique NE of G^S it is possible that one seller has an incentive to invest in the FPA, and then (I, N) and (N, I) are NE in the FPA – this occurs when $\Pi^S_{IN} - \Pi^S_{NN} < k < \Pi^F_{IN} - \Pi^F_{NN}$. The inequality $\Pi^S_{IN} - \Pi^S_{NN} < \Pi^F_{IN} - \Pi^F_{NN}$ is equivalent to $D_1(\tilde{h}, \tilde{m}) > 0$ and Lemma 3, which characterizes the set of $(\tilde{h}, \tilde{m})$ such that $D_1(\tilde{h}, \tilde{m}) > 0$ for a symmetric status quo, shows that $\Pi^S_{IN} - \Pi^S_{NN} < \Pi^F_{IN} - \Pi^F_{NN}$ if and only if $(\tilde{h}, \tilde{m})$ is in the set $\mathfrak{F}_1$ in Figure 5. In such case the FPA is more effective in promoting investment by a single seller because the strategic effect for the FPA is positive when $(\tilde{h}, \tilde{m})$ is such that $\tilde{h} = h$ and $\tilde{m} \in (0, m)$, or close.[25] If moreover k is intermediate and $(\tilde{h}, \tilde{m})$ is in the suitable subset of $\mathfrak{F}_1$ identified by Proposition 3(ia), then (N, N) is the unique NE in G^S, (N, I) and (I, N) are NE in G^F and the buyer's payment in the FPA is lower than in the SPA.

Proposition 7 *Suppose that the status quo is symmetric, with $(h_1, m_1) = (h_2, m_2) = (h, m)$, and that each seller can make an investment such that $(\tilde{h}_1, \tilde{m}_1) = (\tilde{h}_2, \tilde{m}_2) = (\tilde{h}, \tilde{m}) \in \Sigma_1$. Then the SPA is weakly preferable to the FPA for each k except if*

(i) $(\tilde{h}, \tilde{m})$ is in the set identified by Proposition 3(ia) and k is between $\Pi^S_{IN} - \Pi^S_{NN}$ and $\Pi^F_{IN} - \Pi^F_{NN}$, so that no seller invests in the SPA and just one seller invests in the FPA;

(ii) $(h, m) \in F_2$, $(\tilde{h}, \tilde{m}) \in F_1$, and k is between $\Pi^S_{II} - \Pi^S_{NI}$ and $\min\{\Pi^S_{IN} - \Pi^S_{NN}, \Pi^F_{IN} - \Pi^F_{NN}\}$, so that both in the FPA and in the SPA a single seller invests and the FPA is superior to the SPA in the resulting asymmetric setting.

In Proposition 7, sellers are assumed to be symmetric both in terms of ex ante cost distribution and of investment opportunity; in addition, each seller privately observes his own cost, costs are independently distributed, sellers are risk neutral. Nevertheless, there are circumstances in which the FPA and the SPA are not equivalent from the buyer's perspective. In particular, depending on the initial distribution (h, m) and on the post-investment distribution $(\tilde{h}, \tilde{m})$, either auction may provide a stronger incentive towards investment by a single seller and end up as the superior auction. Or both auctions may induce the investment by a single seller, and the resulting asymmetry may favor either auction. One unambiguous conclusion is that the SPA is more likely to induce both sellers to invest, and then the buyer weakly prefers the SPA.

5.2 Asymmetric sellers

Here we assume an asymmetric status quo but suppose that each seller can make a same small investment. We inquire, for both the FPA and the SPA, whether the ex ante stronger seller or the ex ante weaker seller has a higher incentive to make the investment, in order to find out for either auction whether it tends to increase or to reduce the asymmetry between sellers.

[25] In such a case, if each seller's investment were not observable to the other seller then (N, N) would be more likely to be an equilibrium in the FPA than in the case of observable investment. The reason is that with non-observable investments, a seller's deviation to I is not observed by the other seller and then the deviating seller fails to benefit from the positive strategic effect.

Precisely, we suppose that the distribution of c_1 if seller 1 makes the investment is $\tilde{h}_1 = h_1 - \varepsilon_1$, $\tilde{m}_1 = m_1 - \alpha\varepsilon_1$ with $\alpha > -1$; ε_1 is chosen by seller 1 in $[0, \varepsilon]$ with $\varepsilon > 0$ close to zero. Hence the investment changes slightly the distribution of c_1 and the post-investment distribution is first order stochastically dominated by the initial distribution as $\alpha > -1$; the cost of the investment is $k\varepsilon_1$. Likewise, the post-investment distribution of c_2 is $\tilde{h}_2 = h_2 - \varepsilon_2$, $\tilde{m}_2 = m_2 - \alpha\varepsilon_2$ with $\alpha > -1$; ε_2 is chosen by seller 2 in $[0, \varepsilon]$ and the investment cost is $k\varepsilon_2$. We suppose that (h_1, m_1) is first order stochastically dominated by (h_2, m_2), hence seller 1 is ex ante stronger than seller 2 and can be seen as the leader – seller 2 can be seen as the follower.

In the FPA, seller 1's ex ante expected profit is $\tilde{\Pi}_1^F(\varepsilon_1, \varepsilon_2) = \tilde{m}_1 \tilde{\pi}_{1M}^F + (1 - \tilde{h}_1 - \tilde{m}_1)\tilde{\pi}_{1L}^F$. We use $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ as a measure of seller 1's gross investment incentive; likewise, $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2}$ measures seller 2's gross investment incentive. We compare $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ with $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2}$ to see whether the leader or the follower has the stronger investment incentive in the FPA. Of course, evaluating $\frac{\partial \tilde{\Pi}_1^F}{\partial \varepsilon_1}$ at $(\varepsilon_1, \varepsilon_2) = (0, 0)$ identifies seller 1's incentive when seller 2 sets $\varepsilon_2 = 0$. If instead seller 2 chooses $\varepsilon_2 \in (0, \varepsilon]$, then $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ should be replaced by $\frac{\partial \tilde{\Pi}_1^F(0,\varepsilon_2)}{\partial \varepsilon_1}$ (a similar remark holds for $\frac{\partial \tilde{\Pi}_2^F}{\partial \varepsilon_2}$ evaluated at $(\varepsilon_1, \varepsilon_2) = (0, 0)$). However, we are considering small investments and $\frac{\partial \tilde{\Pi}_1^F}{\partial \varepsilon_1}$, $\frac{\partial \tilde{\Pi}_2^F}{\partial \varepsilon_2}$ are continuous functions. Hence if for instance we find $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1} > \frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2}$, then $\frac{\partial \tilde{\Pi}_1^F(0,\varepsilon_2)}{\partial \varepsilon_1} > \frac{\partial \tilde{\Pi}_2^F(\varepsilon_1,0)}{\partial \varepsilon_2}$ holds for each $\varepsilon_2 \in (0, \varepsilon]$ and each $\varepsilon_1 \in (0, \varepsilon]$ as long as ε is close enough to zero. Likewise, for the SPA we use $\tilde{\Pi}_1^S(\varepsilon_1, \varepsilon_2) = \tilde{m}_1 \tilde{\pi}_{1M}^S + (1 - \tilde{h}_1 - \tilde{m}_1)\tilde{\pi}_{1L}^S$, $\tilde{\Pi}_2^S(\varepsilon_1, \varepsilon_2) = \tilde{m}_2 \tilde{\pi}_{2M}^S + (1 - \tilde{h}_2 - \tilde{m}_2)\tilde{\pi}_{2L}^S$ to derive and compare $\frac{\partial \tilde{\Pi}_1^S(0,0)}{\partial \varepsilon_1}$, $\frac{\partial \tilde{\Pi}_2^S(0,0)}{\partial \varepsilon_2}$.[26]

Next proposition establishes that in the SPA the leader's investment incentive is greater than the follower's for each $\alpha > -1$. However, a different result emerges for the FPA.

Proposition 8 *(i) For each (h_2, m_2), $(h_1, m_1) \in \Sigma_2$, $\alpha > -1$ we have $\frac{\partial \tilde{\Pi}_1^S(0,0)}{\partial \varepsilon_1} > \frac{\partial \tilde{\Pi}_2^S(0,0)}{\partial \varepsilon_2}$, that is in the SPA the leader has a greater incentive to make a small investment than the follower.*
(ii) In the FPA, either of the following sets of conditions is sufficient for $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > \max\{0, \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}\}$:

(iia) $(h_1, m_1) \in R_{2M}$, $h_2 + m_2 > \max\{\frac{1}{2}, h_1 + m_1\}$, $\alpha = 0$;

(iib) $(h_1, m_1) \in R_{1M}$, $h_2 + m_2$ close to 1, $\alpha > 0$ and large;

(iic) $(h_1, m_1) \in R_{1ML}$, $\alpha > 0$ and large.

The intuition for Proposition 8(i) is similar to the intuition for the inequality $\Pi_{II}^S - \Pi_{NI}^S < \Pi_{IN}^S - \Pi_{NN}^S$ in (18). In the SPA, the profits of types 1_M and 1_L are h_2 and $2h_2 + m_2$; the profits of types 2_M and 2_L are h_1 and $2h_1 + m_1$. Since $(h_1, m_1) \in \Sigma_2$, it follows that the profit of type 1_M (of type 1_L) is weakly greater than the profit of type 2_M (of type 2_L). Therefore the improvement in the distribution of c_1 due to the investment acts on higher profits with respect to the improvement in the distribution of c_2 and the profit of seller 1 increases weakly more than the profit of seller 2.

For the FPA, the profit comparison between the types of seller 1 and the types of seller 2 continues to hold (weakly), but because of the strategic effect a change in the distribution of c_i affects the profits of type i_M and of type i_L and in some cases this provides seller 2 with a stronger investment incentive, as Proposition 8(ii) establishes.

For instance, $\alpha = 0$ in Proposition 8(iia), which means that the investment of seller i reduces (increases) the probability of type i_H (of type i_L). From (4) it follows that this reduces the profit of type i_L (for seller 1 it reduces also the profit of type 1_M) and the key aspect is that the reduction in the profit of type 1_L has

[26] We are not addressing here the question about whether the FPA or the SPA gives a higher incentive to invest to seller 1 or to seller 2, as that question has been examined by Subsections 3.1 and 4.1 (for instance, Lemma 5 reveals that $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1} > \frac{\partial \tilde{\Pi}_1^S(0,0)}{\partial \varepsilon_1}$ if $\alpha = 0$ and $h_1 > \frac{1}{2} + \frac{1}{2}h_2$, Lemma 4 implies $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1} \le \frac{\partial \tilde{\Pi}_1^S(0,0)}{\partial \varepsilon_1}$ if $(h_1, m_1) \in R_{1M} \cup R_{1ML}$). Morever, small investments cannot alter the initial ranking between the FPA and the SPA based on the buyer's payment. That is, if for instance the buyer initially prefers the SPA to the FPA, then her preferences remain the same for each $\varepsilon_1 \in (0, \varepsilon]$, $\varepsilon_2 \in (0, \varepsilon]$.

a weight l_1, that is $1 - h_1 - m_1$, which is greater than the weight $1 - h_2 - m_2$ of the profit reduction of type 2_L. That is, the negative strategic effect is softer for seller 2 and this makes $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2}$ greater than $\frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$. In addition, the inequality $h_2 + m_2 > \frac{1}{2}$ guarantees $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > 0$, hence there exists a suitable $k > 0$ such that $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > k > \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$.

The results of Proposition 8(iib,iic) rely on similar arguments, but notice that $\alpha > 0$ and large means that the investment mainly reduces (increases) the probability that a seller has type M (has type L).

6 Conclusions

In this paper we have compared the FPA and the SPA in a procurement setting when one of the sellers may make an investment which improves his cost distribution. AC identify assumptions such that the SPA provides a greater investment incentive, but we prove that under less restrictive assumptions the opposite result may hold, and this may be key to make the buyer prefer the FPA. We also show that when ex ante symmetric sellers can both make an investment, the FPA and the SPA are not equivalent.

A topic for future research consists of dropping the assumption that there is a unique new distribution of c_i seller i can achieve through an investment, and rather allowing seller i to choose among multiple new distributions while incurring a cost which is greater the stronger the new distribution. It would be even more significant to allow each seller to choose simultaneously a new distribution, while starting from an asymmetric status quo and try to find out (i) which auction induces a more symmetric/asymmetric market;[27] (ii) how the buyer's preference between auctions depends on the status quo and on the cost functions.[28]

Furthermore, auction formats different from the FPA and from the SPA may be considered. For instance, Loertscher, Marx, Rey (2025) introduce the all-receive procurement auction, which is an analogue of the all-pay auction. An interesting question is how it compares to the FPA and to the SPA in terms of investment incentives and of the buyer's preference.

References

[1] Arozamena, L, Cantillon, E., 2004. Investment Incentives in Procurement Auctions. Review of Economic Studies 71, 1–18. https://doi.org/10.1111/0034-6527.00273

[2] Bag, P.K., 1997. Optimal auction design and R & D. European Economic Review 41, 1665-1674. https://doi.org/10.1016/S0014-2921(96)00033-5

[3] Cantillon, E., 2008. The Effect of Bidders' Asymmetries on Expected Revenue in Auctions. Games and Economic Behavior 62, 1-25. https://doi.org/10.1016/j.geb.2006.11.005

[4] Ceesay, M., Doni, N., Menicucci, D., 2025. Asymmetric Auctions with Discretely Distributed Valuations, The B.E. Journal of Theoretical Economics 25, 99-118. https://doi.org/10.1515/bejte-2024-0109

[5] Cisternas, G., Figueroa, N., 2015. Sequential Procurement Auctions and Theory Effect on Investment Decisions. Rand Journal of Economics 46, 824-843. https://doi.org/10.1111/1756-2171.12112

[6] Grimm, V., Mengel, F., Ponti, G., Viianto, L.A., 2009. Investment Incentives in Auctions: An Experiment", in Hinlopen A., Norman T. (eds.), Experiments and Competition Policy, Cambridge University Press, 267-300. https://doi.org/10.1017/cbo9780511576201.011

[27] Proposition 8 offers a preliminary result in this respect.

[28] The results in Cantillon (2008) roughly suggest that the buyer prefers a more symmetric market.

[7] Li, C., Wan, Z., 2017. Supplier Competition and Cost Improvement. Management Science 63, 2460-2477. https://doi.org/10.1287/mnsc.2016.2458

[8] Loertscher, S., Marx, L.M., Rey, P., 2025. All-Receive Procurement Auctions. TSE working paper 1637.

[9] Myerson, R.B., 1981. Optimal Auction Design. Mathematics of Operations Research 6, 58–73. https://doi.org/10.1287/moor.6.1.58

[10] Piccione, M, Tan, G., 1996. Cost-Reducing Investment, Optimal Procurement and Implementation by Auctions. International Economic Review 37, 663-685. https://doi.org/10.2307/2527446

[11] Tan, G. 1992. Entry and R & D in Procurement Contracting. Journal of Economic Theory 58, 41–60. https://doi.org/10.1016/0022-0531(92)90100-V

[12] Thomas, C., 1997. Disincentives For Cost-Reducing Investment. Economics Letters 57, 359-363. https://doi.org/10.1016/S0165-1765(97)00237-1

7 Appendix

7.1 Some features of the BNE for the FPA

We consider a FPA with the so-called "Vickrey tie-breaking rule" introduced by Maskin and Riley (2000), according to which in the FPA each seller i is required to submit both an ordinary bid b_i and a tie-breaker discount bid $d_i \geq 0$. The bids d_1, d_2 are relevant only when $b_1 = b_2$, in which case seller i wins if $d_i > d_j$ and then is paid $b_i - d_i$ by the buyer. The tie-breaking rule implies that for each seller i it is weakly dominant to set $d_i = b_i - c_i$, hence if $b_1 = b_2$ then the seller with the lowest cost wins and the payment he receives from the buyer is equal to the other seller's cost. In the following, to each b_i we implicitly associate $d_i = b_i - c_i$.

Let G_{ij} denote the c.d.f. of the bid submitted by type i_j in the FPA, for $i = 1, 2$ and $j = L, M, H$. Arguing as in Subsection 3.1 in CDM we deduce that in each BNE type i_H bids c_H with probability 1 (a pure strategy), the set of possible realizations of G_{iM} is an interval $[\underline{b}_{iM}, c_H]$ (in which $\underline{b}_{iM}$ may be equal to c_H) and the set of possible realizations of G_{iL} is an interval $[\underline{b}_L, \underline{b}_{iM}]$, in which $\underline{b}_L < \underline{b}_{iM}$.

Defining $G_i(b) = l_i G_{iL}(b) + m_i G_{iM}(b) + h_i G_{iH}(b)$ as the c.d.f. of the bid submitted by seller i, the indifference conditions for type $1_M, 1_L, 2_M, 2_L$, respectively, can be written as follows:

$$(b - c_M)(1 - G_2(b)) = \rho_2 \Delta \text{ for each } b \in [\underline{b}_{1M}, c_H) \tag{20}$$

$$(b - c_L)(1 - G_2(b)) = \underline{b}_L - c_L \text{ for each } b \in [\underline{b}_L, \underline{b}_{1M}] \tag{21}$$

$$(b - c_M)(1 - G_1(b)) = \rho_1 \Delta \text{ for each } b \in [\underline{b}_{2M}, c_H) \tag{22}$$

$$(b - c_L)(1 - G_1(b)) = \underline{b}_L - c_L \text{ for each } b \in [\underline{b}_L, \underline{b}_{2M}] \tag{23}$$

in which, for $i = 1, 2$, $\rho_i \geq h_i$ is the probability that seller i bids c_H, that is $\rho_i = \lim_{b \uparrow c_H}(1 - G_i(b))$. From (20)-(23) and (1) it follows that

$$\underline{b}_{2M} \leq \underline{b}_{1M}, \text{ with equality if and only if (1) is an equality, and } l_2 = G_1(\underline{b}_{2M}) \text{ (Lemma 1 in CDM)} \tag{24}$$

An equilibrium is identified by $\underline{b}_{1M}$, $\underline{b}_{2M}$, $\underline{b}_L$, ρ_1, ρ_2, and when (3) is violated the equilibrium is such that $\underline{b}_{1M} = c_H$, $\underline{b}_{2M} = c_M + \frac{h_2 - m_2}{m_2 + h_2}\Delta$, $\underline{b}_L = c_L + 2h_2\Delta$, $\rho_1 = h_2 - m_2$, $\rho_2 = h_2$. When instead (3) holds, the equilibrium satisfies

$$\underline{b}_{1M} = c_M + \frac{\rho_1 \Delta}{m_1 + h_1}, \quad b_{2M} = c_M + \frac{\rho_1 \Delta}{m_2 + h_2}, \quad \underline{b}_L = c_L + (\rho_1 + m_2 + h_2)\,\Delta$$

with

$$\rho_1 = h_1, \rho_2 = h_1\frac{h_1+h_2+m_2}{2h_1+m_1} \text{ if (2) holds;} \quad \rho_1 = \sqrt{\frac{1}{4}m_2^2 + h_2(h_1+m_1)} - \frac{1}{2}m_2, \rho_2 = h_2 \text{ if (2) is violated}$$

The buyer's expected payment is the expectation of the lowest bid submitted in the FPA, which has c.d.f. $H(b) = 1-(1-G_1(b))(1-G_2(b))$. Hence $P^F = \rho_1\rho_2 c_H + \int_{\underline{b}_L}^{c_H} b dH(b) = \underline{b}_L + \int_{\underline{b}_L}^{c_H}(1-G_1(b))(1-G_2(b))db$, which reduces to the following expressions for the equilibrium E_{2M}, E_{1M}, E_{1ML}, respectively

$$\begin{aligned}
P^F_{2M} &= c_H - \left(2 - \rho_2 m_1 - (2-h_2-m_2)(h_1+h_2+m_2) - h_1(h_1+h_2+m_2)\ln\frac{h_1+h_2+m_2}{2h_1+m_1}\right)\Delta \\
P^F_{1M} &= \underline{b}_L + \frac{(\underline{b}_L - c_L)(\underline{b}_{2M}-\underline{b}_L)}{\underline{b}_{2M}-c_L} + \rho_1(\underline{b}_L - c_L)\ln\left(\frac{(\underline{b}_{1M}-c_M)(\underline{b}_{2M}-c_L)}{(\underline{b}_{1M}-c_L)(\underline{b}_{2M}-c_M)}\right) + \rho_1\rho_2\left(\frac{c_H-\underline{b}_{1M}}{\underline{b}_{1M}-c_M}\right)\Delta \\
P^F_{1ML} &= c_H - \left(2 - 2h_2(2-h_2-m_2) - 2h_2(h_2-m_2)\ln\frac{h_2}{h_2-m_2}\right)\Delta
\end{aligned}$$

The buyer's expected payment in the SPA, P^S, is the expectation of the highest cost, which is equal to

$$P^S = c_H - ((2-2h_2-m_2)(1-h_1) - (1-h_2-m_2)m_1)\Delta$$

Figure 2 in Subsection 2.2 describes a set F_2 such that for each $(h_2, m_2) \in F_2$ there exists a non-empty set of (h_1, m_1) such that $P^F < P^S$ (see Figure 3)[29] The set F_2 is the set of (h_2, m_2) such that $P^F < P^S$ when $(h_1, m_1) = (\max\{h_2 - m_2\}, 0)$, and consists of (h_2, m_2) such that $(h_2+m_2)^2 \le m_2$ if $h_2 \le m_2$, consists of (h_2, m_2) such that $3h_2 + m_2 - 1 \le 2\frac{h_2}{m_2}(h_2-m_2)\ln\left(\frac{h_2}{h_2-m_2}\right)$ if $h_2 > m_2$.

7.2 Proof of Lemma 1

From (20), (21) it is immediate that $G_2(b)$ is decreasing in $\rho_2, \underline{b}_L$, hence seller 2 becomes less aggressive as ρ_2, $\underline{b}_L$ increase. Likewise, (22), (23) show that $G_1(b)$ is decreasing in $\rho_1, \underline{b}_L$.

7.3 Proof of Lemma 2

Here we prove Lemma 2(i) and the following more general version of Lemma 2(ii), which establishes in more detail the set of $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{\pi}^F_{1M} > \pi^F_{1M}$.

Lemma 2(ii) *The inequality* $\tilde{\pi}^F_{1M} \le \pi^F_{1M}$ *holds for each* $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \cap (\Psi_1 \cup R_{1M} \cup R_{1ML})$*, but not necessarily otherwise. In particular, if* $h_1 > \max\{0, h_2 - m_2\}$ *then there exists* h_1^* *between* $\max\{0, h_2-m_2\}$ *and* h_1*, and a strictly increasing function* $\gamma_1 : (h_1^*, h_1] \to (0, m_1]$ *such that* $\lim_{\tilde{h}_1 \downarrow h_1^*}\gamma_1(\tilde{h}_1) = 0$ *and* $\tilde{\pi}^F_{1M} > \pi^F_{1M}$ *if and only if* $\tilde{h}_1 \in (h_1^*, h_1]$ and $\tilde{m}_1 < \gamma_1(\tilde{h}_1)$.

In order to prove Lemma 2, we notice that $\pi^F_{1M} = \rho_2$ and $\pi^F_{1L} = \rho_1 + h_2 + m_2$, hence $\tilde{\pi}^F_{1M} \le \pi^F_{1M}$ is equivalent to $\tilde{\rho}_2 \le \rho_2$ and $\tilde{\pi}^F_{1L} \le \pi^F_{1L}$ is equivalent to $\tilde{\rho}_1 \le \rho_1$. We prove below that $\tilde{\rho}_2 \le \rho_2$ for each $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1 \cap (\Psi_1 \cup R_{1M} \cup R_{1ML})$ and $\tilde{\rho}_1 \le \rho_1$ for each $(\tilde{h}_1, \tilde{m}_1) \in \Sigma_1$.

Case of $(\tilde{h}_1, \tilde{m}_1) \in R_{1ML}$ If $(h_1, m_1) \in R_{1ML}$, then $\tilde{\rho}_2 = \rho_2 = h_2$, $\tilde{\rho}_1 = \rho_1 = h_2 - m_2$. If $(h_1, m_1) \in R_{1M}$, then $\tilde{\rho}_1 = h_2 - m_2 \le \rho_1 = \sqrt{\frac{1}{4}m_2^2 + h_2(h_1+m_1)} - \frac{1}{2}m_2$ is equivalent to $h_2 - m_2 \le h_1 + m_1$, which holds since $(h_1, m_1) \in R_{1M}$, and $\tilde{\rho}_2 = \rho_2 = h_2$. If $(h_1, m_1) \in R_{2M}$, then $\tilde{\rho}_1 = h_2 - m_2 \le h_1 = \rho_1$ because $(h_1, m_1) \in R_{2M}$, and $\tilde{\rho}_2 = h_2 \le \rho_2$.

[29] The complementary set, denoted S_2, is such that if $(h_2, m_2) \in S_2$ then $P^F \ge P^S$ for each (h_1, m_1) which satisfies (1)

Case of $(\tilde h_1,\tilde m_1)\in R_{1M}$ If $(h_1,m_1)\in R_{1M}$, then $\tilde\rho_1=\sqrt{\frac14 m_2^2+h_2(\tilde h_1+\tilde m_1)}-\frac12 m_2$ is not larger than $\rho_1=\sqrt{\frac14 m_2^2+h_2(h_1+m_1)}-\frac12 m_2$ since $\tilde h_1+\tilde m_1\le h_1+m_1$, and $\tilde\rho_2=\rho_2=h_2$. If $(h_1,m_1)\in R_{2M}$, then $\tilde\rho_1=\sqrt{\frac14 m_2^2+h_2(\tilde h_1+\tilde m_1)}-\frac12 m_2\le\rho_1=h_1$ is equivalent to $h_2(\tilde h_1+\tilde m_1)\le h_1(h_1+m_2)$, which holds since $(h_1,m_1)\in R_{2M}$ and $\tilde h_1+\tilde m_1\le h_1+m_1$, and $\tilde\rho_2=h_2\le\rho_2$.

Case of $(\tilde h_1,\tilde m_1)\in R_{2M}$ If $(h_1,m_1)\in R_{1M}$, then $\tilde\rho_1=\tilde h_1\le h_1\le\rho_1$. But $\tilde\rho_2\ge h_2=\rho_2$, thus $\tilde\pi^F_{1M}\ge\pi^F_{1M}$. In particular, $h_1^*=\max\{0,h_2-m_2\}$ and $\gamma_1(\tilde h_1)=\frac{1}{h_2}\tilde h_1^2+\frac{m_2-h_2}{h_2}\tilde h_1$.

If $(h_1,m_1)\in R_{2M}$, then $\tilde\rho_1=\tilde h_1\le h_1=\rho_1$. Moreover, if $(\tilde h_1,\tilde m_1)\in\Psi_1$ then $\frac{m_1}{h_1}\tilde h_1<\tilde m_1$, hence $\tilde\rho_2=\tilde h_1\frac{\tilde h_1+h_2+m_2}{2\tilde h_1+\tilde m_1}<\tilde h_1\frac{\tilde h_1+h_2+m_2}{2\tilde h_1+\frac{m_1}{h_1}\tilde h_1}=h_1\frac{\tilde h_1+h_2+m_2}{2h_1+m_1}\le\rho_2$. But $\tilde\pi^F_{1M}>\pi^F_{1M}$ if $(\tilde h_1,\tilde m_1)$ satisfies $\tilde m_1<\gamma_1(\tilde h_1)$, with $h_1^*=\max\{0,2\rho_2-h_2-m_2\}$ and $\gamma_1(\tilde h_1)=\frac{1}{\rho_2}\left(\tilde h_1^2+(h_2+m_2-2\rho_2)\tilde h_1\right)$.

7.4 Proof of Lemma 3

Lemma 2 implies $D_1(\tilde h_1,\tilde m_1)\le 0$ if $(\tilde h_1,\tilde m_1)\in\Sigma_1\cap(R_{1M}\cup R_{1ML})$, hence we consider now $(\tilde h_1,\tilde m_1)\in\Sigma_1\cap R_{2M}$. Then (10) yields (set $h=h_2$, $m=m_2$)

$$D_1(\tilde h_1,\tilde m_1)=\tilde m_1(\tilde h_1\frac{\tilde h_1+h+m}{2\tilde h_1+\tilde m_1}-h)+(1-\tilde h_1-\tilde m_1)(\tilde h_1-h) \tag{25}$$

After fixing $\tilde m_1\in(0,m)$, notice that (i) $D_1(\tilde h_1,\tilde m_1)<0$ if $\tilde h_1$ is such that $(\tilde h_1,\tilde m_1)\in C$ because $\tilde\rho_2-h=0$ and $\tilde h_1-h<0$ (ii) $D_1(\tilde h_1,\tilde m_1)>0$ if $\tilde h_1=h$; (iii) $D_1(\tilde h_1,\tilde m_1)$ is strictly increasing in $\tilde h_1$ since $\tilde h_1\le h$. Therefore there is a unique $\tilde h_1\le h$ such that $D_1(\tilde h_1,\tilde m_1)=0$ and we set $\varphi(\tilde m_1)$ equal to such $\tilde h_1$. Moreover, $\varphi(0)=h$ since $D_1(h,0)=0$.

7.5 Proof of Lemma 4

Given $(\tilde h_1,\tilde m_1)\in R_{1M}\cup R_{1ML}$, Lemma 2 implies $\Delta\pi^F_{1M}\le 0$, $\Delta\pi^F_{1L}\le 0$. Moreover, $\tilde\pi^F_{1M}-\pi^S_{1M}=0$, $\tilde\pi^F_{1L}-\pi^S_{1L}\le 0$. Hence $D_1(\tilde h_1,\tilde m_1)$ in (8) is negative or zero.

7.6 Proofs of Lemmas 5 and 6

We prove below some results about the set $\mathfrak{F}_1$. In particular, we rely on

$$\alpha_h = 4h_1^2(1+h_2-2h_1)+\left(4h_1+4h_1h_2-10h_1^2\right)m_1+(m_2-4h_1+2h_2+1)\,m_1^2-m_1^3 \tag{26}$$

$$\alpha_m = h_1\left(2h_1(h_2+m_2-h_1)-4h_1m_1-m_1^2\right) \tag{27}$$

to characterize precisely the set of (h_1,m_1) in R_{2M} such that $\mathfrak{F}_1=\varnothing$, and the complementary set in R_{2M}; α_h,α_m coincide, up to a common positive factor, with $\partial D_1(h_1,m_1)/\partial\tilde h_1$ and with $\partial D_1(h_1,m_1)/\partial\tilde m_1$, respectively.

Lemma 12 (Some features of $\mathfrak{F}_1$) *Suppose $(h_1,m_1)\in R_{2M}$, and let α_h,α_m be defined as in (26)-(27).*
(i) The set $\mathfrak{F}_1$ is empty if and only if

$$0\le\alpha_m\le\alpha_h \tag{28}$$

(ii) Suppose (28) is violated. Then $\mathfrak{F}_1\ne\varnothing$ and includes $(\tilde h_1,\tilde m_1)\in\Sigma_1\cap R_{2M}$ close to (h_1,m_1) such that

(iia) $\tilde h_1=h_1$, $\tilde m_1<m_1$ if $\alpha_m<0$;

(iib) $\tilde h_1<h_1$, $\tilde m_1=m_1$ if $\alpha_h<0$;

(iic) $\tilde h_1+\tilde m_1=h_1+m_1$ and $\tilde h_1<h_1$ if $\alpha_h<\alpha_m$.

Lemma 12(iia) is based on the equality $D_1(h_1, m_1) = 0$, since $(\tilde{h}_1, \tilde{m}_1) = (h_1, m_1)$ implies $\tilde{\Pi}_1^F - \Pi_1^F = 0$, $\tilde{\Pi}_1^S - \Pi_1^S = 0$, hence if $\alpha_m < 0$ (equivalently, if $\partial D_1(h_1, m_1)/\partial \tilde{m}_1 < 0$) then D_1 is locally decreasing with respect to $\tilde{m}_1$ and $D_1(h_1, \tilde{m}_1) > 0$ if $\tilde{m}_1$ is slightly smaller than m_1. A similar principle applies to Lemma 12(iib, iic). Figure 8 below refers to the case of $(h_1, m_1) = (0.66, 0.09)$, $(h_2, m_2) = (0.3, 0.5)$, and then $\alpha_h = -0.123$, $\alpha_m = -0.04$. Consistently with Lemma 12(iia, iib, iic), the set $\mathfrak{F}_1$ includes $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{h}_1 = h_1$, $\tilde{m}_1 < m_1$, such that $\tilde{h}_1 < h_1$, $\tilde{m}_1 = m_1$, and such that $\tilde{h}_1 + \tilde{m}_1 = h_1 + m_1$.

Please insert here Figure 8 with the following caption:
The set $\mathfrak{F}_1$ when $(h_1, m_1) = (0.66, 0.09)$, $(h_2, m_2) = (0.3, 0.5)$

In more general terms, from (27) it follows that $\alpha_m > 0$ if m_1 is about zero, whereas $\alpha_m < 0$ if m_1 is close to m_2, its maximum value in R_{2M}, that is if $(h_1, m_1) \in R_{2M}$ is close to (h_2, m_2).

From (26) it follows that $\alpha_h < 0$ if m_1 is close to 0 and h_1 is sufficiently large, which makes l_1 small Conversely, $\alpha_h > 0$ if h_1 is close to 0. In particular, in the proof of Lemma we show that $\alpha_h > \max\{0, \alpha_m\}$ if $h_1 \leq h_2$:

Figure 9 below illustrates the two curves $\alpha_h = 0$ and $\alpha_m = 0$ in the space (h_1, m_1) .

Please insert here Figure 9 with the following caption:
The set of (h_1, m_1) such that $\alpha_h = 0$ and the set of (h_1, m_1) such that $\alpha_m = 0$ when $(h_2, m_2) = (0.3, 0.5)$

7.6.1 Proof of Lemma 12

We are considering $(\tilde{h}_1, \tilde{m}_1) \in R_{2M}$ and $(h_1, m_1) \in R_{2M}$. In order to simplify notation, we use (x, y) instead of $(\tilde{h}_1, \tilde{m}_1)$. Then

$$\begin{aligned}\tilde{\Pi}_1^F - \Pi_1^F &= yx\frac{x+s_2}{2x+y} + (1-x-y)(x+s_2) - m_1 h_1 \frac{h_1+s_2}{2h_1+m_1} - (1-h_1-m_1)(h_1+s_2) \\ \tilde{\Pi}_1^S - \Pi_1^S &= yh_2 + (1-x-y)(h_2+s_2) - m_1 h_2 - (1-h_1-m_1)(h_2+s_2)\end{aligned}$$

hence

$$\begin{aligned}D_1(x,y) &= yx\frac{x+s_2}{2x+y} - yh_2 + (1-x-y)(x-h_2) \\ &\quad -m_1 h_1 \frac{h_1+s_2}{2h_1+m_1} - (1-h_1-m_1)(h_1+s_2) + m_1 h_2 + (1-h_1-m_1)(h_2+s_2)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial D_1}{\partial x} &= \frac{(s_2 - 4x + h_2 + 1)\, y^2 - y^3 + \left(4x + 4xh_2 - 10x^2\right) y + 4x^2 h_2 + 4x^2 - 8x^3}{(2x+y)^2} \\ \frac{\partial D_1(h_1,m_1)}{\partial x} &= \frac{(s_2 - 4h_1 + h_2 + 1)\, m_1^2 - m_1^3 + h_1 (4 + 4h_2 - 10h_1)\, m_1 + 4h_1^2 h_2 + 4h_1^2 - 8h_1^3}{(2h_1+m_1)^2} \\ \frac{\partial^2 D_1}{\partial x^2} &= -4\frac{s_2 y^2 + 3xy^2 + 6x^2 y + 4x^3}{(2x+y)^3} < 0\end{aligned}$$

$$\frac{\partial D_1}{\partial y} = x\frac{2xs_2 - 4xy - 2x^2 - y^2}{(2x+y)^2}, \qquad \frac{\partial D_1(h_1,m_1)}{\partial y} = h_1\frac{2h_1(s_2-h_1) - 4h_1 m_1 - m_1^2}{(2h_1+m_1)^2}$$

$$\frac{\partial^2 D_1}{\partial y^2} = -4x^2\frac{x+s_2}{(2x+y)^3} < 0, \qquad \frac{\partial^2 D_1}{\partial x \partial y} = \frac{4xys_2 - 6xy^2 - 6x^2 y - 4x^3 - y^3}{(2x+y)^3}$$

The determinant of the Hessian matrix is

$$\begin{aligned}
&\frac{12x^4 - 12x^2y^2 - 8xy^3 - y^4 + (16x^3 + 16x^2y + 8xy^2)s_2}{(2x+y)^4} \\
> \quad &\frac{12x^4 - 12x^2y^2 - 8xy^3 - y^4 + (16x^3 + 16x^2y + 8xy^2)(x+y)}{(2x+y)^4} = \frac{32x^3y + 12x^2y^2 + 28x^4 - y^4}{(2x+y)^4}
\end{aligned}$$

and the latter quotient is positive if $x \geq \frac{9}{40}y$. Hence D_1 is concave in the set of (x, y) which satisfies this condition.

Proof of Lemma 12(i) The expression of the plane tangent to the graph of D_1 at $(x, y) = (h_1, m_1)$ is $P(x, y) = D_1(h_1, m_1) + \frac{\partial D_1(h_1,m_1)}{\partial x}(x - h_1) + \frac{\partial D_1(h_1,m_1)}{\partial y}(y - m_1)$, in which $D_1(h_1, m_1) = 0$. If $0 \leq \frac{\partial D_1(h_1,m_1)}{\partial y} \leq \frac{\partial D_1(h_1,m_1)}{\partial x}$, then $P(x, y) \leq 0$ for each (x, y) in the subset of $R_{2M} \cap \Sigma_1$ in which D_1 is concave, hence $D_1(x, y) \leq 0$ for each (x, y) in such set. The inequalities $0 \leq \frac{\partial D_1(h_1,m_1)}{\partial y} \leq \frac{\partial D_1(h_1,m_1)}{\partial x}$ are equivalent to $0 \leq \alpha_m \leq \alpha_h$.

For $(x, y) \in R_{2M}$ such that $x < \frac{9}{40}y$ we prove that $\frac{\partial D_1(x,y)}{\partial x} > 0$, which implies that $D_1(x, y) < 0$ for each $(x, y) \in R_{2M}$ such that $x < \frac{9}{40}y$. The sign of $\frac{\partial D_1(x,y)}{\partial x}$ coincides with the sign of $-8x^3 + (4h_2 - 10y + 4)x^2 + 4y(1 - y + h_2)x + y^2(1 - y + h_2 + s_2)$, which we denote $\omega(x, y)$. It is immediate that $\omega(0, y) > 0$, $\omega(\frac{9}{40}y, y) = \frac{y^2}{8000}(16\,820 + 16\,820h_2 + 8000s_2 - 19\,979y) > 0$ as $y \leq s_2$. We prove below that $\omega(x, y) > 0$ for each $x \in (0, \frac{9}{40}y)$.

To the purpose we use

$$\frac{\partial \omega}{\partial x} = -24x^2 + 2x(4h_2 - 10y + 4) + 4y(h_2 - y + 1) \text{ and } \frac{\partial^2 \omega}{\partial x^2} = -48x + 8h_2 - 20y + 8$$

Hence $\frac{\partial \omega}{\partial x}(0, y) > 0$. We examine three cases depending on the sign of $\frac{\partial^2 \omega}{\partial x^2}$.

- Case 1: $\frac{\partial^2 \omega}{\partial x^2} > 0$ for each $x < \frac{9}{40}y$. Then ω is convex and $\frac{\partial \omega}{\partial x}(x, y) > 0$ for each x. Since $\omega(0, y) > 0$, it follows that $\omega(x, y) > 0$ for each $(x, y) \in R_{2M}$ such that $x < \frac{9}{40}y$.
- Case 2: $\frac{\partial^2 \omega}{\partial x^2}$ is first positive and then negative. Then ω is convex and then concave, and $\omega(0, y) > 0$, $\frac{\partial \omega}{\partial x}(0, y) > 0$, $\omega(\frac{9}{40}y, y) > 0$ imply $\omega(x, y) > 0$ for each $(x, y) \in R_{2M}$ such that $x < \frac{9}{40}y$.
- Case 3: $\frac{\partial^2 \omega}{\partial x^2} < 0$ for each $x < \frac{9}{40}y$. Then ω is concave, is minimized at $x = 0$ or at $x = \frac{9}{40}y$, and $\omega(0, y) > 0$, $\omega(\frac{9}{40}y, y) > 0$ imply $\omega(x, y) > 0$ for each $(x, y) \in R_{2M}$ such that $x < \frac{9}{40}y$.

Proof of Lemma 12(iia) If $\frac{\partial D_1(h_1,m_1)}{\partial y} < 0$, then $D_1(h_1, m_1) = 0$ implies $D_1(h_1, y) > 0$ if y is slightly smaller than m_1. The inequality $\frac{\partial D_1(h_1,m_1)}{\partial y} < 0$ is equivalent to $\alpha_m < 0$.

Proof of Lemma 12(iib) If $\frac{\partial D_1(h_1,m_1)}{\partial x} < 0$, then $D_1(h_1, m_1) = 0$ implies $D_1(x, m_1) > 0$ if x is slightly smaller than h_1. The inequality $\frac{\partial D_1(h_1,m_1)}{\partial x} < 0$ is equivalent to $\alpha_h < 0$.

Proof of Lemma 12(iic) If $\frac{\partial D_1(h_1,m_1)}{\partial y} > \frac{\partial D_1(h_1,m_1)}{\partial x}$, then $D_1(h_1, m_1) = 0$ implies $D_1(x, y) > 0$ if $(x, y) = (h_1 - \varepsilon, m_1 + \varepsilon)$ with $\varepsilon > 0$ close to 0. The inequality $\frac{\partial D_1(h_1,m_1)}{\partial y} > \frac{\partial D_1(h_1,m_1)}{\partial x}$ is equivalent to $\alpha_m > \alpha_h$.

7.6.2 Proof of Lemma 5

By Lemma 12(iib), it suffices to prove $\frac{\partial D_1(h_1,m_1)}{\partial x} < 0$ if $h_1 > \frac{1}{2} + \frac{1}{2}h_2$. Indeed,

$$\begin{aligned}\frac{\partial D_1(h_1,m_1)}{\partial x} &= \frac{(s_2 - 4h_1 + h_2 + 1)\, m_1^2 - m_1^3 + h_1 (4 + 4h_2 - 10h_1)\, m_1 + 4h_1^2 (h_2 + 1 - 2h_1)}{(2h_1 + m_1)^2} \\ &< \frac{\left(s_2 - 4(\frac{1}{2} + \frac{1}{2}h_2) + h_2 + 1\right) m_1^2 - m_1^3 + h_1 \left(4 + 4h_2 - 10(\frac{1}{2} + \frac{1}{2}h_2)\right) m_1 + 4h_1^2 \left(h_2 + 1 - 2(\frac{1}{2} + \frac{1}{2}h_2)\right)}{(2h_1 + m_1)^2} \\ &= -\frac{(h_2 + 1 - s_2)m_1^2 + m_1^3 + h_1 (1 + h_2)\, m_1}{(2h_1 + m_1)^2} < 0\end{aligned}$$

7.6.3 Proof of Lemma 6

We rely on Lemma 12(i) and notice that the second inequality in Lemma 6 is equivalent to $\alpha_m > 0$, hence in the following we prove that $\alpha_h > \alpha_m$ for each $h_1 \le h_2$.[30]

The inequality $\alpha_h > \alpha_m$ is equivalent to

$$\frac{(s_2 - 4h_1 + h_2 + 1)\, m_1^2 - m_1^3 + \left(4h_1 + 4h_1h_2 - 10h_1^2\right) m_1 + 4h_1^2h_2 + 4h_1^2 - 8h_1^3}{(2h_1 + m_1)^2} > h_1 \frac{2h_1(s_2 - h_1) - 4h_1m_1 - m_1^2}{(2h_1 + m_1)^2}$$

that is to $\tau(h_1, m_1) = m_1^2 (1 + s_2 - m_1 + h_2) + m_1 (4 + 4h_2 - 3m_1)\, h_1 + 2 (2 - s_2 - 3m_1 + 2h_2)\, h_1^2 - 6h_1^3 > 0$. We prove that this inequality holds for each $h_1 \in [0, h_2]$.[31]

- Step 1: $\tau(h_1, m_1) > 0$ if $2 - s_2 - 3m_1 + 2h_2 \le 0$. Proof: From $2 - s_2 - 3m_1 + 2h_2 \le 0$ it follows that τ is concave in h_1, and $\tau(0, m_1) > 0$. At $h_1 = h_2$ we find $\tau(h_2, m_1) = -m_1^3 + m_1^2 (s_2 + 1 - 2h_2) + h_2(4 - 2h_2)m_1 + h_2^2 (4 - 2s_2 - 2h_2)$, which is greater than $(s_2 - 2h_2)\, m_1^2 + h_2(4 - 2h_2)m_1 + 2h_2^2 (2 - s_2 - h_2)$ as $m_1^2 > m_1^3$. This is positive if $s_2 - 2h_2 \ge 0$, whereas if $s_2 - 2h_2 < 0$ then $m_1^2 (s_2 - 2h_2) + h_2 (4 - 2h_2)\, m_1 + 2h_2^2 (2 - s_2 - h_2) > m_1 \left(s_2 + 2h_2 - 2h_2^2\right) + 2h_2^2 (2 - s_2 - h_2) > 0$.
- Step 2: $\tau(h_1, m_1) > 0$ if $2 - s_2 - 3y + 2h_2 > 0$. Proof: From $2 - s_2 - 3m_1 + 2h_2 > 0$ it follows that τ is positive and convex in h_1 for h_1 close to 0 (as the second derivative is positive for h_1 close to 0) and then maybe concave. But the first derivative is positive at $h_1 = 0$, hence $\tau > 0$ in the interval in which τ is convex. If τ is convex in the whole interval, then it increasing and that is enough. If it becomes concave, then τ is positive in the whole interval because $\tau(h_2, m_1) > 0$.

7.7 Proof of Proposition 2

Consider $h_1 = h_2 + m_2$, $m_1 = 0$, $\tilde{m}_1 = 0$ and $\tilde{h}_1 < h_1$. Then $D_1(\tilde{h}_1, \tilde{m}_1) = (h_1 - \tilde{h}_1)(\tilde{h}_1 - (1 + h_2 - h_1))$, hence $1 + h_2 - h_1$ is the smallest $\tilde{h}_1$ such that $D_1(\tilde{h}_1, m_1) \ge 0$ and we prove that $P^S > \tilde{P}^F$ at $\tilde{h}_1 = 1 + h_2 - h_1$ if $h_1 > \frac{2}{3} + \frac{1}{3}h_2$.

At $\tilde{h}_1 = 1 + h_2 - h_1$, we find

$$P^S - \tilde{P}^F = -2 - h_1^2 + 4h_1 - h_2 - (1 + h_2 - h_1)(1 + h_2) \ln \frac{1 + h_2}{2(1 + h_2 - h_1)}$$

We use $\delta(h_1, h_2)$ to denote the right hand side of the above expression and prove that $\delta(h_1, h_2) > 0$ for each $h_2 \in (0, 1)$, $h_1 > \frac{2}{3} + \frac{1}{3}h_2$.

[30] It actually possible to prove $\alpha_h > 0$ for each $h_1 \le h_2$. Since $\frac{\partial^2 D_1}{\partial x^2} < 0$, hence it suffices to prove $\frac{\partial D_1(h_2,m_1)}{\partial x} > 0$. The numerator of $\frac{\partial D_1(h_2,m_1)}{\partial x}$ is equal to $m_1^2 s_2 + 4h_2m_1 + 4h_2^2 - 4h_2^3 - 6h_2^2m_1 - 3h_2m_1^2 + m_1^2 - m_1^3$ and since $m_1 + h_2 < s_2$ it follows that the expression is greater than $m_1^2(m_1 + h_2) + 4h_2m_1 + 4h_2^2 - 4h_2^3 - 6h_2^2m_1 - 3h_2m_1^2 + m_1^2 - m_1^3 = (2h_2 + m_1)\,(m_1 + 2h_2\,(1 - h_2 - m_1))$, which is positive as $m_1 \le m_2 \le 1 - h_2$.

[31] In fact, in some cases the lowest value for h_1 such that $(h_1, m_1) \in R_{2M}$ is greater than 0 and such that $(h_1, m_1) \in C$.

Step 1: δ is strictly increasing with respect to h_1 The partial derivative of δ with respect to h_1 is $\frac{\partial\delta(h_1,h_2)}{\partial h_1} = (h_2+1)\ln\frac{h_2+1}{2h_2-2h_1+2} + 3 - h_2 - 2h_1$ and $\frac{\partial^2\delta(h_1,h_2)}{\partial h_1^2} = 2\frac{h_1-\frac{1}{2}-\frac{1}{2}h_2}{1-h_1+h_2} > 0$ as $h_1 > \frac{2}{3}+\frac{1}{3}h_2$. Since $\frac{\partial\delta(\frac{2}{3}+\frac{1}{3}h_2,h_2)}{\partial h_1} = \frac{5}{3}(1-h_2)+(1+h_2)\ln\frac{3h_2+3}{4h_2+2} > 0$, it follows that $\frac{\partial\delta(h_1,h_2)}{\partial h_1} > 0$ for each $h_1 \geq \frac{2}{3}+\frac{1}{3}h_2$.

Step 2: $\delta(\frac{2}{3}+\frac{1}{3}h_2,h_2) > 0$ for each $h_2 \in (0,1)$ We find $\delta(\frac{2}{3}+\frac{1}{3}h_2,h_2) = \frac{2}{9}-\frac{1}{9}h_2^2-\frac{1}{9}h_2-\frac{1}{3}(h_2+1)(2h_2+1)\ln\frac{3h_2+3}{4h_2+2}$, which is strictly decreasing with respect to h_2 since $\frac{d\delta(\frac{2}{3}+\frac{1}{3}h_2,h_2)}{dh_2} = \frac{2}{9}-\frac{2}{9}h_2-\frac{4h_2+3}{3}\ln\left(1+\frac{1-h_2}{4h_2+2}\right) < \frac{2}{9}-\frac{2}{9}h_2-\frac{4h_2+3}{3}\left(\frac{1-h_2}{4h_2+2}-\frac{1}{2}\left(\frac{1-h_2}{4h_2+2}\right)^2\right) = -\frac{(1-h_2)(11+53h_2+44h_2^2)}{72(2h_2+1)^2} < 0$. Finally, $\delta(1,1)=0$; thus $\delta(\frac{2}{3}+\frac{1}{3}h_2,h_2) > 0$ for each $h_2 \in (0,1)$.

7.8 Proof of Proposition 3

Since the status quo is symmetric, we use (h,m) to denote both (h_1,m_1) and (h_2,m_2), and use s to denote $h+m$. Moreover, we let $\Pi = mh+(1-h-m)(2h+m)$ denote seller 1's profit in either auction under the status quo; hence $D_1(\tilde{h}_1,\tilde{m}_1) = \tilde{\Pi}_1^F - \Pi - (\tilde{\Pi}_1^S - \Pi) = \tilde{\Pi}_1^F - \tilde{\Pi}_1^S$. Through Subsections 7.8.1-7.8.3 we consider the case of $(h,m) \in F_2$, $(\tilde{h}_1,\tilde{m}_1) \in F_1$; the case of $(h,m) \in F_2$, $(\tilde{h}_1,\tilde{m}_1) \notin F_1$, $D_1(\tilde{h}_1,\tilde{m}_1) \leq 0$; the case of $(h,m) \in F_2$, $(\tilde{h}_1,\tilde{m}_1) \notin F_1$, $D_1(\tilde{h}_1,\tilde{m}_1) > 0$.

7.8.1 Proof of Proposition 3(ib): The case of $(h,m) \in F_2$ and $(\tilde{h}_1,\tilde{m}_1) \in F_1$

We show that in the following that $(h,m) \in F_2$ and $(\tilde{h}_1,\tilde{m}_1) \in F_1$ imply $\tilde{\Pi}_1^F > \Pi$ and $D_1(\tilde{h}_1,\tilde{m}_1) \leq 0$.

Step 1: $\tilde{\Pi}_1^F > \Pi$ We begin by showing that $\tilde{\Pi}_1^F$ is minimized along curve C by proving that for each $(\tilde{h}_1,\tilde{m}_1)$ there exists another $(\tilde{h}_1,\tilde{m}_1) \in C$ which lowers $\tilde{\Pi}_1^F$.

- If $(\tilde{h}_1,\tilde{m}_1) \in R_{1M}$, then $\tilde{\Pi}_1^F = \tilde{m}_1 h + (1-\tilde{h}_1-\tilde{m}_1)(\tilde{\rho}_1+h+m)$ and reducing $\tilde{m}_1$ while leaving $\tilde{h}_1+\tilde{m}_1$ constant makes $\tilde{\rho}_1$ stay constant, hence $\tilde{\Pi}_1^F$ decreases.
- If $(\tilde{h}_1,\tilde{m}_1) \in R_{1ML}$ (which requires $h > m$), then $\tilde{\Pi}_1^F = \tilde{m}_1 h + (1-\tilde{h}_1-\tilde{m}_1)2h$, which is minimized at $(\tilde{h}_1,\tilde{m}_1) = (h-m,0)$.
- If $(\tilde{h}_1,\tilde{m}_1) \in R_{2M}$, then $\tilde{\Pi}_1^F = \tilde{m}_1\tilde{h}_1\frac{\tilde{h}_1+s}{2\tilde{h}_1+\tilde{m}_1} + (1-\tilde{h}_1-\tilde{m}_1)(\tilde{h}_1+s)$ is decreasing with respect to $\tilde{m}_1$ as $\frac{\partial\tilde{\Pi}_1^F}{\partial\tilde{m}_1} = -\frac{(2\tilde{h}_1^2+\tilde{m}_1^2+4\tilde{h}_1\tilde{m}_1)(s+\tilde{h}_1)}{(2\tilde{h}_1+\tilde{m}_1)^2} < 0$.

Now we prove that $\tilde{\Pi}_1^F - \Pi > 0$ for each $(\tilde{h}_1,\tilde{m}_1) \in C$. Since $\tilde{m}_1 = \frac{1}{h}\tilde{h}_1^2 + \frac{m-h}{h}\tilde{h}_1$ along curve C, we have that $\tilde{\Pi}_1^F - \Pi = (\frac{1}{h}\tilde{h}_1^2+\frac{m-h}{h}\tilde{h}_1)h + (1-h_1-(\frac{1}{h}\tilde{h}_1^2+\frac{m-h}{h}\tilde{h}_1))(\tilde{h}_1+h+m) - (mh+(1-h-m)(2h+m)) = \frac{h-\tilde{h}_1}{h}(\tilde{h}_1^2+(h+2m)\tilde{h}_1+2h^2+2hm+m^2-h)$. In case that $m \geq h$, the minimum for $\tilde{h}_1^2+(h+2m)\tilde{h}_1+2h^2+2hm+m^2-h$ with respect to $\tilde{h}_1$ is $2h^2+2hm+m^2-h$ (achieved at $\tilde{h}_1=0$), which is positive since $(h,m) \in F_2$ implies $h \geq \frac{1}{4}$, $m \geq \frac{1}{4}$, hence $2h^2+2hm+m^2-h \geq 2h^2+2h\frac{1}{4}+\frac{1}{16}-h = (h-\frac{1}{4})^2+h^2 > 0$. In case that $m < h$, the minimum for $\tilde{h}_1^2+(h+2m)\tilde{h}_1+2h^2+2hm+m^2-h$ is $h(4h+m-1)$ (achieved at $\tilde{h}_1 = h-m$), which is positive since $(h,m) \in F_2$ implies $h > \frac{1}{4}$, $m \geq \frac{1}{4}$, hence $4h+m-1 \geq \frac{1}{4}$.

Step 2: $(\tilde{h}_1,\tilde{m}_1) \in F_1$ implies $D_1(\tilde{h}_1,\tilde{m}_1) \leq 0$ In view of a contradiction, suppose that $D_1(\tilde{h}_1,\tilde{m}_1) > 0$, that is $\tilde{\Pi}_1^F > \tilde{\Pi}_1^S$. Furthermore, we know from Lemma 2 in CDM that $\tilde{\pi}_{2M}^S \leq \tilde{\pi}_{2M}^F$ and $\tilde{\pi}_{2L}^S \leq \tilde{\pi}_{2L}^F$ for each $(\tilde{h}_1,\tilde{m}_1) \in \Sigma_1$, hence $\tilde{\Pi}_2^F \geq \tilde{\Pi}_2^S$. Therefore the total sellers' profit in case seller 1 makes the investment is higher in the FPA than in the SPA. This implies $\tilde{P}^F > \tilde{P}^S$, which cannot hold since $(\tilde{h}_1,\tilde{m}_1) \in F_1$.

Step 3: The FPA is preferable to the SPA if $k < \Pi_1^F - \Pi$ but not if $k > \Pi_1^F - \Pi$ Steps 1 and 2 imply $0 < \tilde{\Pi}_1^F - \Pi \leq \tilde{\Pi}_1^S - \Pi$. When $k < \Pi_1^F - \Pi$, seller 1 makes the investment both in the FPA as in the SPA, and $\tilde{P}^F < \tilde{P}^S$ since $(\tilde{h}_1, \tilde{m}_1) \in F_1$. If instead $\tilde{\Pi}_1^F - \Pi < k$, then the investment does not occur in the FPA, and the SPA is preferable since $P^F = P^S > \tilde{P}^S$.

7.8.2 Proof of Proposition 3(ia): The case of $(h, m) \in S_2$ (or $(h, m) \in F_2$ and $(\tilde{h}_1, \tilde{m}_1) \notin F_1$) and $D_1(\tilde{h}_1, \tilde{m}_1) > 0$

In this case $(\tilde{h}_1, \tilde{m}_1) \in \mathfrak{F}_1$ (see Figure 5), that is $\tilde{\Pi}_1^F - \Pi > \tilde{\Pi}_1^S - \Pi$ as $D_1(\tilde{h}_1, \tilde{m}_1) > 0$.

- When k is such that $k < \tilde{\Pi}_1^S - \Pi < \tilde{\Pi}_1^F - \Pi$, the investment occurs in both auctions and the SPA is weakly preferable because of the assumptions on (h, m) and on $(\tilde{h}_1, \tilde{m}_1)$.
- When k is such that $\tilde{\Pi}_1^F - \Pi < k$, seller 1 does not invest in either auction and the two auctions are equivalent.
- When k is such that $\tilde{\Pi}_1^S - \Pi < k < \tilde{\Pi}_1^F - \Pi$, the investment occurs only in the FPA. But it is not straightforward whether $P^S > \tilde{P}^F$ or not. We now show that there exists $(\tilde{h}_1, \tilde{m}_1) \in R_{2M}$ such that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$ and $\tilde{P}^F < P^S$. We let $B(\tilde{h}_1, \tilde{m}_1) = P^S - \tilde{P}^F$ and notice that $D_1(h, m) = 0$, $B(h, m) = 0$. We prove that $D_1(\tilde{h}_1, \tilde{m}_1) > 0$, $B(\tilde{h}_1, \tilde{m}_1) > 0$ if $(\tilde{h}_1, \tilde{m}_1) = (h - \varepsilon, m - \sigma\varepsilon)$, with σ between $\frac{2h-2h^2+m-2hm}{hm}$ and $\frac{2h-2h^2+m-2hm}{hm} + \frac{(1-h)(2h+m)}{hm}$, and $\varepsilon > 0$ is small; notice that $\frac{3h-3h^2+m-2hm}{hm}$ which appears in the statement of Proposition 3(ia) is between $\frac{2h-2h^2+m-2hm}{hm}$ and $\frac{2h-2h^2+m-2hm}{hm} + \frac{(1-h)(2h+m)}{hm}$.

  We first prove $\left.\frac{dD_1(h-\varepsilon, m-\sigma\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} > 0$ for $\sigma > \frac{2h-2h^2+m-2hm}{hm}$, hence $D_1(h - \varepsilon, m - \sigma\varepsilon) > 0$ if $\varepsilon > 0$ is small. The partial derivatives of D_1 are

  $$\frac{\partial D_1}{\partial \tilde{h}_1} = \tilde{m}_1 \frac{(2\tilde{h}_1 + s)(2\tilde{h}_1 + \tilde{m}_1) - 2\tilde{h}_1^2 - 2s\tilde{h}_1}{(2\tilde{h}_1 + \tilde{m}_1)^2} + 1 - 2\tilde{h}_1 - \tilde{m}_1 + h \quad \text{and} \quad \frac{\partial D_1(h,m)}{\partial \tilde{h}_1} = \frac{2h + m - 2h^2 - 2hm}{h + s}$$

  $$\frac{\partial D_1}{\partial \tilde{m}_1} = \frac{(\tilde{h}_1^2 + s\tilde{h}_1)2\tilde{h}_1}{(2\tilde{h}_1 + \tilde{m}_1)^2} - \tilde{h}_1 \quad \text{and} \quad \frac{\partial D_1(h,m)}{\partial \tilde{m}_1} = -\frac{hm}{h+s}$$

  hence

  $$\left.\frac{dD_1(h-\varepsilon, m-\sigma\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \frac{2h + m - 2h^2 - 2hm}{h+s}(-1) - \frac{hm}{h+s}(-\sigma) = \left(\sigma - \frac{2h - 2h^2 + m - 2hm}{hm}\right)\frac{hm}{h+s} > 0$$

  Now we prove that $\left.\frac{dB(h-\varepsilon, m-\sigma\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} > 0$ for $\sigma < \frac{2h-2h^2+m-2hm}{hm} + \frac{(1-h)(2h+m)}{hm}$, hence $B(h - \varepsilon, m - \sigma\varepsilon) > 0$ if $\varepsilon > 0$ is small. From Subsection 7.1 we obtain $P^S - \tilde{P}^F$ and

  $$\begin{aligned}
  \frac{\partial B}{\partial \tilde{h}_1} &= -(2\tilde{h}_1 + s)\left(\frac{\tilde{m}_1}{2\tilde{h}_1 + \tilde{m}_1} + \ln\frac{\tilde{h}_1 + s}{2\tilde{h}_1 + \tilde{m}_1}\right) - \tilde{h}_1\frac{\tilde{m}_1^2 - 4s\tilde{h}_1 - 4s\tilde{m}_1}{(2\tilde{h}_1 + \tilde{m}_1)^2} - (2 - s) \\
  \frac{\partial B(h,m)}{\partial \tilde{h}_1} &= \frac{-4h - 2m + 4h^2 + 3hm}{h+s}
  \end{aligned}$$

  $$\frac{\partial B}{\partial \tilde{m}_1} = \frac{\tilde{h}_1\tilde{m}_1(\tilde{h}_1 + s)}{(2\tilde{h}_1 + \tilde{m}_1)^2} \quad \text{and} \quad \frac{\partial B(h,m)}{\partial \tilde{m}_1} = \frac{hm}{h+s}$$

  hence

  $$\begin{aligned}
  \left.\frac{dB(h-\varepsilon, m-\sigma\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} &= \frac{-4h - 2m + 4h^2 + 3hm}{h+s}(-1) + \frac{hm}{h+s}(-\sigma) \\
  &= \left(\frac{2h - 2h^2 + m - 2hm}{hm} + \frac{(1-h)(2h+m)}{hm} - \sigma\right)\frac{hm}{h+s} > 0
  \end{aligned}$$

Since $\tilde{P}^F$ is strictly increasing in $\tilde{h}_1$ and strictly decreasing in $\tilde{m}_1$, we conclude that $\mathfrak{F}_1$ always includes $(\tilde{h}_1, \tilde{m}_1)$ such that $\tilde{P}^F < P^S$, and in particular $(\tilde{h}_1, \tilde{m}_1)$ with this property are found in $\mathfrak{F}_1$ near its north west border.

7.8.3 Case of $(h,m) \in S_2$ (or $(h,m) \in F_2$ and $(\tilde{h}_1, \tilde{m}_1) \notin F_1$) and $D_1(\tilde{h}_1, \tilde{m}_1) \leq 0$

Proposition 1 proves that in this case the SPA is weakly preferable to the FPA for each k.

7.9 Proof of Lemma 7

7.9.1 A BNE in the FPA when (1) is violated

In order to examine the case in which $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$, we determine in the following a BNE in the FPA when $(\tilde{h}_1, \tilde{m}_1)$ violates (1), that is when $\tilde{h}_1 + \tilde{m}_1 > h_2 + m_2$.

When (1) does not hold, each BNE still satisfies (20)-(23) and $\underline{b}_{1M} < \underline{b}_{2M} \leq c_H$, $1 - G_2(\underline{b}_{1M}) = h_1 + m_1$ – these properties are analogous to (24). From (20) evaluated at $b = \underline{b}_{2M}$ and at $b = \underline{b}_{1M}$ we obtain $\underline{b}_{2M}, \underline{b}_{1M}$ as a function of ρ_2, and (23) evaluated at $b = \underline{b}_{1M}$ yields $\underline{b}_L$:

$$\underline{b}_{2M} = c_M + \frac{\rho_2}{h_2 + m_2}\Delta, \qquad \underline{b}_{1M} = c_M + \frac{\rho_2}{h_1 + m_1}\Delta, \qquad \underline{b}_L = c_L + (\rho_2 + h_1 + m_1)\Delta \tag{29}$$

In order to determine ρ_1, ρ_2, notice that the bid $\underline{b}_{2M}$ belongs both to the interval $[\underline{b}_{2M}, c_H]$ in (22) and to the interval $[\underline{b}_L, \underline{b}_{2M}]$ in (23). Hence, both (22) and (23) determine a value of $G_1(\underline{b}_{2M})$ and the two values turn out to agree if and only if[32]

$$F(\rho_1, \rho_2) = 0, \qquad \text{with} \qquad F(\rho_1, \rho_2) = \rho_1(1 + \frac{h_2 + m_2}{\rho_2}) - h_1 - m_1 - \rho_2 \tag{30}$$

and F is strictly increasing in ρ_1, strictly decreasing in ρ_2. Depending on the sign of $F(h_1, h_2)$ and on the sign of $F(h_1, h_2 + m_2)$, one of the three following strategy profiles is the unique equilibrium of the FPA, as described by next Lemma[33]

$$E^*_{1M} : \begin{cases} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying (20)-(23), with } \underline{b}_{1M}, \underline{b}_{2M}, \underline{b}_L \\ \text{in (29) and } \rho_2 = h_2,\ \rho_1 = h_2 \frac{h_1 + h_2 + m_1}{2h_2 + m_2} \text{ is the unique solution to } F(\rho_1, h_2) = 0 \end{cases} \tag{31}$$

$$E^*_{2M} : \begin{cases} \text{the distributions of bids are given by } G_1, G_2 \text{ satisfying (20)-(23), with } \underline{b}_{1M}, \underline{b}_{2M}, \underline{b}_L \text{ in (29) and} \\ \rho_1 = h_1,\ \rho_2 = \sqrt{\frac{1}{4}m_1^2 + h_1(h_2 + m_2)} - \frac{1}{2}m_1 \text{ is the unique solution to } F(h_1, \rho_2) = 0 \text{ in } [h_2, h_2 + m_2) \end{cases} \tag{32}$$

$$E^*_{2ML} : \begin{cases} \text{type } 2_M \text{ bids } c_H \text{ (that is, } \underline{b}_{2M} = c_H\text{); the distributions of bids are given by } G_1, G_2 \\ \text{satisfying (20), (21), (23), with } \underline{b}_{1M}, \underline{b}_L \text{ given by (29), and } \rho_1 = h_1,\ \rho_2 = h_1 - m_1 \end{cases} \tag{33}$$

Lemma 13 (BNE in FPA when (1) is violated) *Suppose that (1) is violated. Then the unique equilibrium in the FPA is E^*_{1M} if $F(h_1, h_2) < 0$, that is if*

$$h_1(h_2 + m_2) < h_2(h_2 + m_1) \tag{34}$$

The unique equilibrium is E^*_{2M} if $F(h_1, h_2) \geq 0 > F(h_1, h_2 + m_2)$, with $F(h_1, h_2 + m_2) < 0$ if and only if

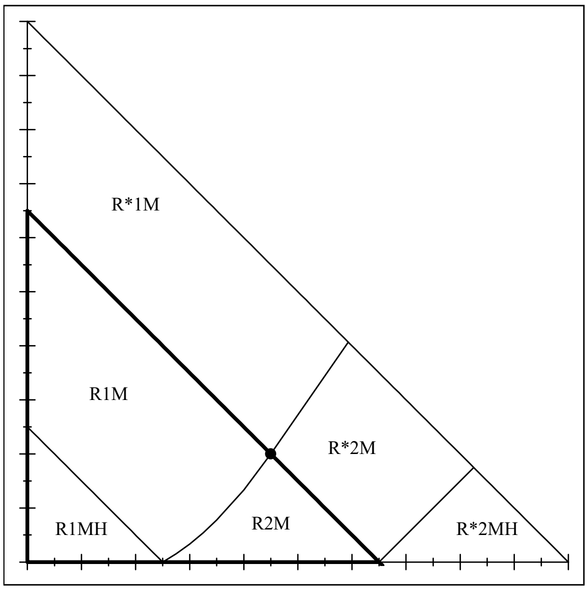
$$h_1 - m_1 < h_2 + m_2 \tag{35}$$

The unique equilibrium is E^*_{2ML} if $F(h_1, h_2 + m_2) \geq 0$.

[32]Since $\bar{b}_{2M} \in (v_L, \bar{b}_{2M}]$, (22) implies $\frac{\rho_2}{h_2+m_2}G_1(\bar{b}_{2M}) = \rho_1\Delta$. Since $\bar{b}_{2M} \in [\bar{b}_{2M}, \bar{b}_H]$, (23) implies $(1 + \frac{\rho_2}{h_2+m_2})G_1(\bar{b}_{2M}) = \rho_2 + h_1 + m_1$. Hence $F(\rho_1, \rho_2) = 0$ in (30) needs to hold.

[33]The superscript * in E^*_{1M} and elsewhere below is a remainder that we are considering the case in which (1) does not hold.

The figure below, in the Cartesian plane (h_1, m_1) identifies the regions $R^*_{1M}, R^*_{2M}, R^*_{2ML}$ of pairs (h_1, m_1) (each of them is such that $h_1 + m_1 > h_2 + m_2$) in which $E^*_{1M}, E^*_{2M}, E^*_{2ML}$ is the equilibrium, respectively.



The border between R^*_{1M} and R^*_{2M} has equation $m_1 = \frac{s_2}{h_2}h_1 - h_2$ for $h_1 \in [h_2, \frac{h_2+h_2^2}{2h_2+m_2}]$; the border between R^*_{2M} and R^*_{2ML} has equation $m_1 = h_1 - h_2 - m_2$ for $h_1 \in [h_2 + m_2, \frac{1+h_2+m_2}{2}]$. In the three BNE identified by the above lemma, the sellers' profits are as follows, with $\rho_2^* = \sqrt{\frac{1}{4}m_1^2 + h_1(h_2 + m_2)} - \frac{1}{2}m_1$:

equilibrium\type	1_M	1_H	2_M	2_H
E^*_{1M}	h_2	$h_1 + h_2 + m_1$	$h_2\frac{h_2+h_1+m_1}{2h_2+m_2}$	$h_1 + h_2 + m_1$
E^*_{2M}	ρ_2^*	$\rho_2^* + h_1 + m_1$	h_1	$\rho_2^* + h_1 + m_1$
E^*_{2ML}	$h_1 - m_1$	$2h_1$	h_1	$2h_1$

7.9.2 Proof that $D_1(\tilde{h}_1, \tilde{m}) \geq 0$ for each $(\tilde{h}_1, \tilde{m}_1) \notin \Sigma_1$

We use (h, m) to denote the common initial distribution $(h_1, m_1) = (h_2, m_2)$ and use (x, y) instead of $(\tilde{h}_1, \tilde{m}_1)$.

Step 1: Proof when $(x, y) \in R_{2M}\backslash\Sigma_1$ We know from the proof of Lemma 3 that $D_1(x, y) = y(\tilde{\rho}_2 - h) + (1 - x - y)(x - h)$, and for each $(x, y) \in R_{2M}\backslash\Sigma_1$ both terms in $D_1(x, y)$ are non-negative; they are both zero if and only if $x = 1$.

Step 2: Proof when $(x, y) \in R^*_{1M}$ We find that $D_1(x, y) > 0$ for each $(x, y) \in R^*_{1M}$ because $D_1(x, y) = yh + (1 - x - y)(x + y + h) - (yh + (1 - x - y)(2h + m)) = (1 - x - y)(x + y - h - m) > 0$.

Step 3: Proof when $(x, y) \in R^*_{2ML}$ We find that $D_1(x, y) \geq 0$ for each $(x, y) \in R^*_{2ML}$ because $D_1(x, y) = y(x-y)+(1-x-y)2x-yh-(1-x-y)(2h+m) = (2h + m + 2)\,x - 2x^2 + (h + m)\,y - y^2 - yx - 2h - m$, which is concave. We find that $D_1(h+m, 0) = m\,(1 - h - m) \geq 0$, $D_1(1, 0) = 0$, $D_1(\frac{1}{2}(1+h+m), \frac{1}{2}(1-h-m)) = \frac{1}{2}m\,(1 - h - m) \geq 0$, hence $D_1(x, y) \geq 0$ for each $(x, y) \in R^*_{2ML}$.

Step 4: Proof when $(x, y) \in R^*_{2M}$ We find that $D_1(x, y) > 0$ for each $(x, y) \in R^*_{2M}$. In detail, $D_1(x, y) = y(\tilde{\rho}_2 - h) + (1 - x - y)(\tilde{\rho}_2 + x + y - 2h - m)$, which is non-negative for each $(x, y) \in R^*_{2M}$.

7.9.3 Proof that $\tilde{P}^F > P^F$ for each $(x, y) \notin \Sigma_1$

Step 1: Proof when $(x, y) \in R_{2M} \backslash \Sigma_1$ Each $(x, y) \in R_{2M} \backslash \Sigma_1$ is such that $x > h$, $y < m$, which implies $\tilde{\rho}_1 = x > h$, $\tilde{\rho}_2 > h$, $\tilde{\underline{b}}_L = \underline{b}_L$. Then $\tilde{P}^F > P$ by Lemma 1.

Step 2: Proof when $(x, y) \in R^*_{1M}$ For each $(x, y) \in R^*_{1M}$ we have $\tilde{P}^F > P^F$ because $\tilde{\rho}_1 > h$, $\tilde{\rho}_2 = h$, $\tilde{\underline{b}}_L > \underline{b}_L$.

Step 3: Proof when $(x, y) \in R^*_{2ML}$ For each $(x, y) \in R^*_{2ML}$ we have $\tilde{P}^F > P^F$ because $\tilde{\rho}_1 = x > h$, $\tilde{\rho}_2 = x - y > h$, $\tilde{\underline{b}}_L > \underline{b}_L$.

Step 4: Proof when $(x, y) \in R^*_{2H}$ For each $(x, y) \in R^*_{2M}$ we have $\tilde{P}^F > P^F$ because $\tilde{\rho}_1 = x > h$, $\tilde{\rho}_2 > h$, $\tilde{\underline{b}}_L > \underline{b}_L$.

7.10 Proof of Proposition 4

As in the proof of Proposition 3 we set $\Pi = \tilde{\Pi}^F_1 = \tilde{\Pi}^S_1$. Since $(\tilde{h}, \tilde{m}) \notin \Sigma_1$, Lemma 7 implies $\tilde{\Pi}^S_1 - \Pi \leq \tilde{\Pi}^F_1 - \Pi$.

If $\tilde{\Pi}^F_1 - \Pi < k$, then the investment occurs in neither auction and the auctions are equivalent.

If $\tilde{\Pi}^S_1 - \Pi < k \leq \tilde{\Pi}^F_1 - \Pi$ (this requires $0 < \tilde{\Pi}^F_1 - \Pi$), then the investment occurs only in the FPA and the SPA is superior because $\tilde{P}^F > P^F = P^S$ by Lemma 7.

If $k \leq \tilde{\Pi}^S_1 - \Pi$ (this requires $0 < \tilde{\Pi}^S_1 - \Pi$) then the investment occurs in both auctions and we prove $\tilde{P}^F > \tilde{P}^S$. First notice that $\tilde{\Pi}^S_1 - \Pi > 0$ if and only if $\tilde{m}h + (1 - \tilde{h} - \tilde{m})(2h + m) > mh + (1 - h - m)(2h + m)$, which is equivalent to

$$\tilde{m} < \frac{2h^2 + 2hm + m^2}{h + m} - \frac{2h + m}{h + m}\tilde{h} \tag{36}$$

Suppose that $\tilde{h} < h$. The inequality $\tilde{P}^S < P^S$ is equivalent to $(1 - \tilde{h})(1 - h) + (1 - \tilde{h} - \tilde{m})(1 - h - m) > (1 - h)^2 + (1 - h - m)^2$, which is satisfied since (36) holds. Thus $\tilde{P}^S < P^S = P^F < \tilde{P}^F$.

If $\tilde{h} > h$, then (36) implies that $(\tilde{h}, \tilde{m}) \notin \Sigma_1$ and $(\tilde{h}, \tilde{m}) \in R_{2M}$. Hence $\tilde{P}^S < \tilde{P}^F$ because of the remark in footnote 13 in Subsection 2.2.

7.11 Proof of Lemma 8

Here we prove Lemma 8(i) and the following more general version of Lemma 8(ii), which establishes in more detail the set of $(\tilde{h}_2, \tilde{m}_2)$ such that $\tilde{\pi}^F_{2M} > \pi^F_{2M}$.

Lemma 8(ii) *The inequality* $\tilde{\pi}^F_{2M} \leq \pi^F_{2M}$ *holds for each* $(\tilde{h}_2, \tilde{m}_2) \in \Sigma_2 \cap (\Psi_2 \cup R_{2M} \cup R^*_{2M} \cup R^*_{2ML})$, *but not necessarily otherwise. In particular, if* $h_2 > \max\{0, h_1 - m_1\}$ *then there exists* h^*_2 *between* $\max\{0, h_1 - m_1\}$ *and* h_2, *and a strictly increasing function* $\gamma_2 : (h^*_2, h_2] \to (0, m_2]$ *such that* $\lim_{\tilde{h}_2 \downarrow h^*_2} \gamma_2(\tilde{h}_2) = 0$ *and* $\tilde{\pi}^F_{2M} > \pi^F_{2M}$ *if and only if* $\tilde{h}_2 \in (h^*_2, h_2]$ *and* $\tilde{m}_2 < \gamma_2(\tilde{h}_2)$.

7.11.1 Case of $(h_2, m_2) \in \mathcal{R}_{2M}$

$\pi^F_{2M} = h_1$, $\pi^S_{2M} = h_1$, $\pi^F_{2L} = h_1 + h_2 + m_2$, $\pi^S_{2L} = 2h_1 + m_1$. We prove below that $h^*_2 = \max\{0, h_1 - m_1\}$ and the function γ_2 mentioned in the statement is such that $\gamma_2(\tilde{h}_2) = \frac{1}{h_1}\tilde{h}^2_2 + \frac{m_1 - h_1}{h_1}\tilde{h}_2$ if $\tilde{h}_2 \in [h^*_2, h_1]$, $\gamma_2(\tilde{h}_2) = \frac{h_1 + m_1}{h_1}\tilde{h}_2 - h_1$ if $\tilde{h}_2 \in (h_1, h_2]$: this is the border between $\mathcal{R}^*_{1M} \cup \mathcal{R}_{1M}$ and $\mathcal{R}^*_{2M} \cup \mathcal{R}_{2M}$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1ML}$ **(requires** $s_1 < h_2$**, otherwise** $\Sigma_2 \cap \mathcal{R}_{1ML} = \varnothing$**)** Type 2_M: $\tilde\pi^F_{2M} = \tilde h_2 - \tilde m_2$ and $\Delta\pi^F_{2M} = \tilde h_2 - \tilde m_2 - h_1 \geq 0$ as $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1ML}$ implies $\tilde h_2 - \tilde m_2 \geq h_1 + m_1$.[34]

Type 2_L: $\tilde\pi^F_{2L} = 2\tilde h_2$ and $\Delta\pi^F_{2L} = 2\tilde h_2 - h_1 - h_2 - m_2 \leq 2\tilde h_2 - (h_2 - m_2) - h_2 - m_2 = 2(\tilde h_2 - h_2) \leq 0$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1M}$ Type 2_M: $\tilde\pi^F_{2M} = \tilde\rho_1$ and $\Delta\pi^F_{2M} = \tilde\rho_1 - h_1 \geq 0$.[35]

Type 2_L: $\tilde\pi^F_{2L} = \tilde\rho_1 + \tilde h_2 + \tilde m_2$ and $\Delta\pi^F_{2L} = \tilde\rho_1 - h_1 + \tilde h_2 + \tilde m_2 - h_2 - m_2 \leq 0$ is equivalent to $\sqrt{\frac{1}{4}\tilde m_2^2 + \tilde h_2 s_1} \leq h_2 + m_2 - \tilde h_2 - \tilde m_2 + h_1 + \frac{1}{2}\tilde m_2$, or to $\tilde h_2 s_1 \leq (h_2 + m_2 - \tilde h_2 - \tilde m_2)^2 + h_1^2 + h_1 \tilde m_2 + (2h_1 + \tilde m_2)(h_2 + m_2 - \tilde h_2 - \tilde m_2)$ and the right hand side is greater than $h_1(h_1 + \tilde m_2) + (2h_1 + \tilde m_2)(h_2 + m_2 - \tilde h_2 - \tilde m_2) \geq h_1(h_1 + \tilde m_2) + 2h_1(h_2 + m_2 - \tilde h_2 - \tilde m_2) \geq h_1(h_1 + \tilde m_2) + h_1(h_2 + m_2 - \tilde h_2 - \tilde m_2) \geq h_1(h_1 + \tilde m_2) + h_1(m_2 - \tilde m_2) = h_1(h_1 + m_2)$, and this is no less than $h_2 s_1$ as $(h_1, m_1) \in R_{2M}$, which is no less than $\tilde h_2 s_1$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{2M}$ Type 2_M: $\tilde\pi^F_{2M} = h_1$ and $\Delta\pi^F_{2M} = 0$.

Type 2_L: $\tilde\pi^F_{2L} = h_1 + \tilde h_2 + \tilde m_2$ and $\Delta\pi^F_{2L} \leq 0$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}^*_{1M}$ Type 2_M: $\tilde\pi^F_{2M} = \tilde\rho_1 = \tilde h_2 \frac{\tilde h_2 + s_1}{2\tilde h_2 + \tilde m_2}$ and $\pi^F_{2M} \leq \tilde\pi^F_{2M}$.

Type 2_L: $\tilde\pi^F_{2L} = \tilde h_2 + s_1$ and $\tilde h_2 \leq h_2$, $h_2 + s_1 \leq h_1 + s_2$ as $(h_1, m_1) \in R_{2M}$ implies $m_1 \leq m_2$; hence $\tilde\pi^F_{2L} \leq \pi^F_{2L}$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}^*_{2M}$ Type 2_M: $\tilde\pi^F_{2M} = h_1$ and $\tilde\pi^F_{2M} = \pi^F_{2M}$.

Type 2_L: $\tilde\pi^F_{2L} = \tilde\rho_2 + s_1 = \sqrt{\frac{1}{4}m_1^2 + h_1 \tilde s_2} - \frac{1}{2}m_1 + s_1 \leq \sqrt{\frac{1}{4}m_1^2 + h_1 s_1} - \frac{1}{2}m_1 + s_1 \leq \pi^F_{2L}$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}^*_{2ML}$ Type 2_M: $\tilde\pi^F_{2M} = h_1$ and $\tilde\pi^F_{2M} = \pi^F_{2M}$.

Type 2_L: $\tilde\pi^F_{2L} = 2h_1$ and $\tilde\pi^F_{2L} \leq \pi^F_{2L}$.

7.11.2 Case of $(h_2, m_2) \in \mathcal{R}_{1M}$

Then $\pi^F_{2M} = \rho_1$, $\pi^S_{2M} = h_1$, $\pi^F_{2L} = \rho_1 + h_2 + m_2$, $\pi^S_{2L} = 2h_1 + m_1$. We prove below that $h_2^* = 2\rho_1 - s_1$ and the function γ_2 mentioned in the statement is such that $\gamma_2(\tilde h_2) = \frac{1}{\rho_1}\left(\tilde h_2^2 + (s_1 - 2\rho_1)\tilde h_2\right)$ if $\tilde h_2 \in [h_2^*, \rho_1]$, $\gamma_2(\tilde h_2) = m_2 + \frac{h_1 + m_1}{\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2}(\tilde h_2 - h_2)$ if $\tilde h_2 \in (\rho_1, h_2]$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1ML}$ **(requires** $s_1 \leq h_2$**, otherwise** $\Sigma_2 \cap \mathcal{R}_{1ML} = \varnothing$**)** Type 2_M: $\tilde\pi^F_{2M} = \tilde h_2 - \tilde m_2$ and $\Delta\pi^F_{2M} = \tilde h_2 - \tilde m_2 - \rho_1 > 0$ since $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1ML}$, $(h_2, m_2) \in \mathcal{R}_{1M}$ imply $\tilde h_2 - \tilde m_2 \geq h_1 + m_1 \geq \rho_1$.[36]

Type 2_L: $\tilde\pi^F_{2L} = 2\tilde h_2$ and $\Delta\pi^F_{2L} = 2\tilde h_2 - \rho_1 - h_2 - m_2 \leq h_2 - \rho_1 - m_2 \leq 0$ because $\rho_1 \geq h_2 - m_2$ as equality holds when $s_1 = h_2 - m_2$, but $s_1 \geq h_2 - m_2$ in $\mathcal{R}_{1M}$.

Suppose $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{1M}$ Type 2_M: $\tilde\pi^F_{2M} = \tilde\rho_1$ and $\Delta\pi^F_{2M} = \tilde\rho_1 - \rho_1$. The inequality $\tilde\rho_1 > \rho_1$ is equivalent to $\sqrt{\frac{1}{4}\tilde m_2^2 + \tilde h_2 s_1} - \frac{1}{2}\tilde m_2 > \sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2$, or to $\tilde m_2 < m_2 + \frac{s_1}{\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2}(\tilde h_2 - h_2)$. [37]

---

[34] If $(\tilde h_2, \tilde m_2) \in \Psi_2$, that is if $\tilde m_2 > \frac{m_2}{h_2}\tilde h_2$, then $(\tilde h_2, \tilde m_2) \notin \mathcal{R}_{1ML}$ because $(h_2, m_2) \in \mathcal{R}_{2M}$ implies $\frac{m_2}{h_2}\tilde h_2 \geq \frac{s_1}{h_1}\tilde h_2 - \frac{\tilde h_2}{h_2}h_1 > \frac{s_1}{h_1}\tilde h_2 - h_1$ (as $\tilde h_2 < h_2$). Hence $\tilde m_2 \geq \frac{s_1}{h_1}\tilde h_2 - h_1$ and $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{2M}$.

[35] The previous footnote proves that if $(\tilde h_2, \tilde m_2) \in \Psi_2$, $\tilde m_2 \geq \frac{m_2}{h_2}\tilde h_2$, then $(\tilde h_2, \tilde m_2) \in \mathcal{R}_{2M}$ hence $(\tilde h_2, \tilde m_2) \notin \mathcal{R}_{1M}$.

[36] If $(\tilde h_2, \tilde m_2) \in \Psi_2$, that is if $\tilde m_2 > \frac{m_2}{h_2}\tilde h_2$, then $(\tilde h_2, \tilde m_2) \notin \mathcal{R}_{1ML}$ as $\tilde h_2 - \tilde m_2 \leq \frac{h_2 - m_2}{h_2}\tilde h_2 \leq h_2 - m_2 \leq h_1 + m_1$ as $(h_2, m_2) \in \mathcal{R}_{1M}$.

[37] If $(\tilde h_2, \tilde m_2) \in \Psi_2$, that is if $\tilde m_2 > \frac{m_2}{h_2}\tilde h_2$, then $\tilde m_2 < m_2 + \frac{s_1}{\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2}(\tilde h_2 - h_2)$ cannot hold because $\frac{m_2}{h_2}\tilde h_2 > m_2 + \frac{s_1}{\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2}(\tilde h_2 - h_2)$ is equivalent to $\frac{m_2}{h_2} < \frac{s_1}{\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2}$ if and only if $m_2\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} - \frac{1}{2}m_2^2 < s_1 h_2$ if

Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{\rho}_1 + \tilde{h}_2 + \tilde{m}_2$. Thus $\Delta\pi^F_{2L} = \tilde{\rho}_1 + \tilde{h}_2 + \tilde{m}_2 - \rho_1 - h_2 - m_2$ is increasing in $\tilde{h}_2$ and in $\tilde{m}_2$, hence the max point is such that $\tilde{h}_2 + \tilde{m}_2 = h_2 + m_2$ with $\Delta\pi^F_{2L} = \tilde{\rho}_1 - \rho_1$, which is increasing in $\tilde{h}_2$, decreasing in $\tilde{m}_2$. Hence $\Delta\pi^F_{2L}$ is maximized at $\tilde{h}_2 = h_2$, $\tilde{m}_2 = m_2$, where $\Delta\pi^F_{2L} = 0$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$ and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = h_1 + \tilde{h}_2 + \tilde{m}_2$ and $\Delta\pi^F_{2L} = (h_1 - \rho_1) + (\tilde{h}_2 + \tilde{m}_2 - h_2 - m_2) \leq 0$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$ Type 2_M: $\tilde{\pi}^F_{2M} = \tilde{h}_2 \frac{\tilde{h}_2 + s_1}{2\tilde{h}_2 + \tilde{m}_2}$. The inequality $\tilde{h}_2 \frac{\tilde{h}_2 + s_1}{2\tilde{h}_2 + \tilde{m}_2} > \rho_1$ is equivalent to $\tilde{m}_2 < \frac{\tilde{h}_2^2 + (s_1 - 2\rho_1)\tilde{h}_2}{\rho_1}$ and it is satisfied for instance at $(\tilde{h}_2, \tilde{m}_2) = (h_1 + m_1, 0)$.
Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{h}_2 + s_1$ and $\tilde{h}_2 \leq h_2$. Then prove $h_2 + s_1 \leq \rho_1 + s_2$, which is equivalent to $s_1 \leq \rho_1 + m_2$, that is $s_1 - \frac{1}{2}m_2 \leq \sqrt{\frac{1}{4}m_2^2 + h_2 s_1}$, which is true; hence $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$ and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{\rho}_2 + s_1$ and $\tilde{\rho}_2 + s_1 \leq h_1 + s_1$ because $\tilde{s}_2 \leq s_1$, and $h_1 + s_1 \leq \rho_1 + s_2$; hence $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2ML}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$ and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = 2h_1 \leq (\rho_1 + h_2) + m_2$ as $(h_2, m_2) \in \mathcal{R}_{1M}$; hence $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$.

7.11.3 Case of $(h_2, m_2) \in \mathcal{R}_{1ML}$

$\pi^F_{2M} = h_2 - m_2$, $\pi^S_{2M} = h_1$, $\pi^F_{2L} = 2h_2$, $\pi^S_{2L} = 2h_1 + m_1$. We prove below that $h_2^* = h_2 - m_2$ and the function γ_2 mentioned in the statement is such that $\gamma_2(\tilde{h}_2) = \tilde{h}_2 - (h_2 - m_2)$ for each $\tilde{h}_2 \in [h_2^*, h_2]$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1ML}$ Type 2_M: $\tilde{\pi}^F_{2M} = \tilde{h}_2 - \tilde{m}_2$ and $\Delta\tilde{\pi}^F_{2M} > 0$ if $\tilde{h}_2 - \tilde{m}_2$ is greater than $h_2 - m_2$.[38]
Type 2_L: $\tilde{\pi}^F_{2L} = 2\tilde{h}_2$ and $\Delta\pi^F_{2L} = 2(\tilde{h}_2 - h_2) \leq 0$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1M}$ Type 2_M: $\tilde{\pi}^F_{2M} = \tilde{\rho}_1$ and $\Delta\tilde{\pi}^F_{2M} = \tilde{\rho}_1 - (h_2 - m_2) \leq 0$ holds since (i) it is equivalent to $\tilde{h}_2 s_1 \leq (h_2 - m_2)(h_2 - m_2 + \tilde{m}_2)$; (ii) $\tilde{h}_2 s_1 \leq \tilde{h}_2(h_2 - m_2)$ as $(h_2, m_2) \in \mathcal{R}_{1ML}$; (iii) $\tilde{h}_2(h_2 - m_2) \leq (h_2 - m_2)(h_2 - m_2 + \tilde{m}_2)$ is equivalent to $\tilde{h}_2 - \tilde{m}_2 \leq h_2 - m_2$, which holds since $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{1M}$.
Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{\rho}_1 + \tilde{h}_2 + \tilde{m}_2$, $\Delta\tilde{\pi}^F_{2L} = \tilde{\rho}_1 + \tilde{h}_2 + \tilde{m}_2 - 2h_2 \leq \tilde{\rho}_1 - (h_2 - m_2) \leq 0$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$, and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = h_1 + \tilde{h}_2 + \tilde{m}_2$ and $\Delta\tilde{\pi}^F_{2L} = h_1 + \tilde{h}_2 + \tilde{m}_2 - 2h_2 \leq h_1 + h_2 + m_2 - 2h_2 = h_1 + m_2 - h_2 \leq 0$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{1M}$ Type 2_M: $\tilde{\pi}^F_{2M} = \tilde{h}_2 \frac{\tilde{h}_2 + s_1}{2\tilde{h}_2 + \tilde{m}_2}$ is maximized in $\mathcal{R}^*_{1M}$ at $(\tilde{h}_2, \tilde{m}_2) = (s_1, 0)$ ($s_1 \leq h_2$ holds as $(h_2, m_2) \in \mathcal{R}_{1ML}$) with value s_1 and $\Delta\pi^F_{2M} \leq s_1 - (h_2 - m_2) \leq 0$ as $(h_2, m_2) \in \mathcal{R}_{1ML}$.
Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{h}_2 + s_1$ and $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$ as $\tilde{h}_2 \leq h_2$ and $(h_2, m_2) \in \mathcal{R}_{1ML}$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2M}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$, and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = \tilde{\rho}_2 + s_1$ and $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$ as $\tilde{\rho}_2 \leq h_1$ and $h_1 + s_1 \leq 2h_2$ as $(h_2, m_2) \in \mathcal{R}_{1ML}$.

Suppose $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}^*_{2ML}$ Type 2_M: $\tilde{\pi}^F_{2M} = h_1$, and $\Delta\pi^F_{2M} \leq 0$.
Type 2_L: $\tilde{\pi}^F_{2L} = 2h_1$ and $\tilde{\pi}^F_{2L} \leq \pi^F_{2L}$ as $(h_2, m_2) \in \mathcal{R}_{1ML}$.

---

and only if $m_2\sqrt{\frac{1}{4}m_2^2 + h_2 s_1} < \frac{1}{2}m_2^2 + s_1 h_2$, which is satisfied.

[38]If $\tilde{m}_2 \geq \frac{m_2}{h_2}\tilde{h}_2$ then $\tilde{\pi}^F_{2M} \leq \tilde{h}_2 - \frac{m_2}{h_2}\tilde{h}_2 = \frac{h_2 - m_2}{h_2}\tilde{h}_2 \leq \pi^F_{2M}$.

7.12 Proof of Corollary 2

We employ $\hat{\Pi}_2^F$ ($\hat{\Pi}_2^S$) to denote the expected profit of seller 2 in the FPA (in the SPA) under the distribution $(\hat{h}_2, \hat{m}_2) = (h_1, m_1)$ for c_2, that is when the two sellers are symmetric. Then notice that $\tilde{\Pi}_2^F - \Pi_2^F$ can be written as $\tilde{\Pi}_2^F - \hat{\Pi}_2^F + \hat{\Pi}_2^F - \Pi_2^F$, decomposing seller 2's profit change in the change, $\hat{\Pi}_2^F - \Pi_2^F$, when the distribution of c_2 moves from the initial distribution to $(\hat{h}_2, \hat{m}_2)$, plus the change, $\tilde{\Pi}_2^F - \hat{\Pi}_2^F$, when it moves from $(\hat{h}_2, \hat{m}_2)$ to the final distribution. Likewise, $\tilde{\Pi}_2^S - \Pi_2^S$ is equal to $\tilde{\Pi}_2^S - \hat{\Pi}_2^S + \hat{\Pi}_2^S - \Pi_2^S$. As a result,

$$D_2(\tilde{h}_2, \tilde{m}_2) = \tilde{\Pi}_2^F - \hat{\Pi}_2^F - (\tilde{\Pi}_2^S - \hat{\Pi}_2^S) + \hat{\Pi}_2^F - \Pi_2^F - (\hat{\Pi}_2^S - \Pi_2^S)$$

We know from Corollary 1 that $D_2(\hat{h}_2, \hat{m}_2) = \hat{\Pi}_2^F - \Pi_2^F - (\hat{\Pi}_2^S - \Pi_2^S)$ is negative or zero, hence we can conclude that $D_2(\tilde{h}_2, \tilde{m}_2) \leq 0$ if $\tilde{\Pi}_2^F - \hat{\Pi}_2^F - (\tilde{\Pi}_2^S - \hat{\Pi}_2^S) \leq 0$. Since $\hat{\Pi}_2^F, \hat{\Pi}_2^S$ refer to a symmetric status quo, Lemma 3 in Subsection 3.1 applies to reveal that $\tilde{\Pi}_2^F - \hat{\Pi}_2^F - (\tilde{\Pi}_2^S - \hat{\Pi}_2^S) \leq 0$ for each $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}^* \cup \mathcal{R}_{2ML}^*$.

7.13 Proof of Lemma 10

Proof of part (i) (the proof of part (ii) is in the text) For arbitrary (h_2, m_2), $(\tilde{h}_2, \tilde{m}_2)$, from (14) we obtain

$$\begin{aligned} D_2(\tilde{h}_2, \tilde{m}_2) &= \Delta m_2(\tilde{\rho}_1 - h_1) + m_2(\tilde{\rho}_1 - \rho_1) + \Delta l_2(\tilde{\rho}_1 + \tilde{s}_2 - h_1 - s_1) + l_2(\tilde{\rho}_1 + \tilde{s}_2 - \rho_1 - s_2) \\ &= (1 - \tilde{h}_2)\tilde{\rho}_1 + h_1(\tilde{h}_2 - h_2) - (1 - h_2)\rho_1 + (s_2 - \tilde{s}_2)(\tilde{s}_2 - s_1) + (1 - s_2)(\tilde{s}_2 - s_2) \end{aligned}$$

When $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ we have $\tilde{\rho}_1 = h_1$, and when we consider $(\tilde{h}_2', \tilde{m}_2') = (\tilde{h}_2 + \varepsilon, \tilde{m}_2 - \varepsilon)$ with $\varepsilon > 0$ such that $(\tilde{h}_2', \tilde{m}_2') \in \mathcal{R}_{1M} \cup \mathcal{R}_{1ML}$ we have $\tilde{\rho}_1' \geq h_1$. Then $D_2(\tilde{h}_2', \tilde{m}_2') - D_2(\tilde{h}_2, \tilde{m}_2) = (1 - \tilde{h}_2 - \varepsilon)\tilde{\rho}_1' + h_1(\tilde{h}_2 + \varepsilon - h_2) - (1 - \tilde{h}_2)h_1 - h_1(\tilde{h}_2 - h_2) \geq (1 - \tilde{h}_2 - \varepsilon)h_1 + h_1(\tilde{h}_2 + \varepsilon - h_2) - (1 - \tilde{h}_2)h_1 - h_1(\tilde{h}_2 - h_2) = 0$.

7.14 Proof of Proposition 5

Since $(h_2, m_2) \in \mathcal{R}_{2M}$, it follows that $\rho_1 = h_1$ and $(\tilde{h}_2, \tilde{m}_2) \in \mathcal{R}_{2M}$ implies $D_2(\tilde{h}_2, \tilde{m}_2) > 0$ if and only if $\tilde{s}_2 > 1 + s_1 - s_2$ (by Lemma 9) which is satisfied since s_2 is close to 1. Moreover, $\tilde{P}^F < P^S$ if $\tilde{s}_2$ is close to s_1 as then $\tilde{P}^F$ is close to $c_H - (2 - h_1 m_1 - (2 - s_1)(h_1 + s_1))\,\Delta$, P^S is about $c_H - (1 - h_1)(1 - h_2)$ (see Subsection 7.1) and the former is less than the latter since $h_2 > h_1 - \frac{(1-s_1)^2}{1-h_1}$.

7.15 Proof of Proposition 6

From $D_2(\tilde{h}_2, 0) = 2(h_2 - \tilde{h}_2)(\tilde{h}_2 - (1 + h_1 - h_2))$ it is immediate that $D_2(\tilde{h}_2, 0) > 0$ if and only if $1 + h_1 - h_2 < \tilde{h}_2 < h_2$. Since $\tilde{P}^F = c_H - 2(1 - \tilde{h}_2)^2$ is increasing in $\tilde{h}_2$, we consider $\tilde{h}_2 = 1 + h_1 - h_2$, the smallest $\tilde{h}_2$ consistent with $D_2(\tilde{h}_2, 0) \geq 0$. Then $P^S - \tilde{P}^F = 2(h_2 - h_1)^2 - 2(1 - h_2)(1 - h_1) = 2\left(h_2 - \left(\frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{2}h_1\right)\right)\left(h_2 - \left(\frac{\sqrt{5}+3}{2}h_1 - \frac{\sqrt{5}+1}{2}\right)\right)$ and $h_2 - \left(\frac{\sqrt{5}+3}{2}h_1 - \frac{\sqrt{5}+1}{2}\right) > 0$. Hence $\tilde{P}^F < P^S$ if and only if $h_2 > \frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{2}h_1$.

7.16 Proof of Lemma 11

7.16.1 Proof of the first inequality in (18)

We prove that $\delta^F = \Pi_{IN}^F - \Pi_{NN}^F - (\Pi_{II}^F - \Pi_{NI}^F) \geq 0$ for each (h, m) and each $(\tilde{h}, \tilde{m}) \in \Sigma_1$. In this proof we use (x, y) instead of $(\tilde{h}, \tilde{m})$ and we minimize δ^F with respect to $(x, y) \in \Sigma_1$. Precisely, first we prove that the minimum point of δ^F lies in C, then we prove that $\delta^F(x, y) \geq 0$ for each $(x, y) \in C$.

Case of $(x,y) \in R_{1ML} \cap \Sigma_1$ When $(x,y) \in R_{1ML}$, (4) yields

$$\begin{aligned}\delta^F(x,y) &= (yh + (1-x-y)2h - mh - (1-h-m)(2h+m)) \\ &\quad - (yx + (1-x-y)(2x+y) - m(h-m) - (1-h-m)2h)\end{aligned}$$

The Hessian matrix of δ^F is $\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, hence δ^F is a convex function. Moreover, the gradient of δ^F is $(4x - 2h + 2y - 2, 2x - h + 2y - 1)$ and there are no critical points in the interior or on the edges of R_{1ML}. Since $\delta^F(0,0) = 2h - m + hm$, $\delta^F(0, h-m) = h + m^2 < 2h - m + hm$, $\delta^F(h-m, 0) = m(1+2m-h) < h + m^2$, it follows that the minimum point for δ^F is $(x,y) = (h-m, 0)$, which is on the curve C.

Case of $(x,y) \in R_{1M} \cap \Sigma_1$ When $(x,y) \in R_{1M}$, (4) yields

$$\begin{aligned}\delta^F(x,y) &= yh + (1-s)(\rho_1 + h + m) - mh - (1-h-m)(2h+m) \\ &\quad - (yx + (1-s)(x+s) - m\rho_1 - (1-h-m)(\rho_1 + h + m)) \\ &= x^2 - x + yh + m - s + 2\rho_1 - h\rho_1 - s\rho_1 - hs - ms + h^2 + s^2\end{aligned}$$

in which $s = x + y$ and $\rho_1 = \sqrt{\frac{1}{4}m^2 + hs} - \frac{1}{2}m$. Given a value for s, δ^F depends on $x^2 - x + yh$, which is equal to $x^2 - (h+1)\,x + sh$ and this is decreasing in $x \in [0,h]$. Hence δ^F is minimized in R_{1M} at a point in C.

Case of $(x,y) \in R_{2M} \cap \Sigma_1$ When $(x,y) \in R_{2M}$, (4) yields

$$\begin{aligned}\delta^F(x,y) &= yx\frac{x+h+m}{2x+y} + (1-x-y)(x+h+m) - mh - (1-h-m)(2h+m) \\ &\quad - (yx + (1-x-y)(2x+y) - mx - (1-h-m)(x+h+m)) \\ &= yx\frac{x+h+m}{2x+y} - y - 2xh - xm - yh - ym + xy + x^2 + y^2 + h^2 + m\end{aligned}$$

We now prove that $\frac{\partial \delta^F(x,y)}{\partial y}$ is negative, which implies that δ^F is minimized at a point in C. Since $\frac{\partial \delta^F}{\partial y} = 2x^2\frac{x+h+m}{(2x+y)^2} + x - h + 2y - m - 1$, it follows that $\frac{\partial \delta^F}{\partial y}$ is increasing with respect to x and $\frac{\partial \delta^F(h,y)}{\partial y} = \frac{4h^3 + 2h^2 m}{(y+2h)^2} + 2y - m - 1$, which is convex in y. Hence $\frac{\partial \delta^F(h,y)}{\partial y}$ is maximized at $y = 0$ or $y = m$, with $\frac{\partial \delta^F(h,0)}{\partial y} = h - \frac{1}{2}m - 1 < 0$, $\frac{\partial \delta^F(h,m)}{\partial y} = -\frac{(1-h-m)(h+m) + h - h^2}{2h+m} < 0$.

$\delta^F(x,y) > 0$ **for each** $(x,y) \in C$ Given $(x,y) \in C$, we have

$$\begin{aligned}\delta^F(x,y) &= \left[yx\frac{x+h+m}{2x+y} - y - 2xh - xm - yh - ym + xy + x^2 + y^2 + h^2 + m\right]_{y = \frac{1}{h}x^2 + \frac{m-h}{h}x} \\ &= \frac{h-x}{h^2}\left(-x^3 - 2mx^2 - \left(h^2 + m^2 - h\right)x + h^3 + mh\right) \text{ with } x \in [\max\{0, h-m\}, h]\end{aligned}$$

and $-x^3 - 2mx^2 - \left(h^2 + m^2 - h\right)x + h^3 + mh$ is a concave function of x, with value $h^3 + mh > 0$ at $x = 0$, value $h^2(1 - h + 2m) > 0$ at $x = h - m$, value $h(1-h-m)(h+m) > 0$ at $x = h$. Hence $\delta^F(x,y) \geq 0$ for each $(x,y) \in C$.

7.16.2 Proof of the second inequality in (18)

We prove that $\delta^S = \Pi^S_{IN} - \Pi^S_{NN} - (\Pi^S_{II} - \Pi^S_{NI}) \geq 0$ for each (h,m) and each $(\tilde{h}, \tilde{m}) \in \Sigma_1$ and we minimize δ^S with respect to $(\tilde{h}, \tilde{m}) \in \Sigma_1$. To simplify notation, we replace with $(\tilde{h}, \tilde{m})$ with (x,y). From (4) it follows that

$$\delta^S(x,y) = yh + (1-x-y)(2h+m) - mh - (1-h-m)(2h+m)$$
$$- (yx + (1-x-y)(2x+y) - mx - (1-h-m)(2x+y))$$

The Hessian matrix of δ^S is $\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$, hence δ^S is a convex function. Moreover, the gradient of δ^S is $(4x+2y-4h-2m, 2x+2y-2h-2m)$ and the unique critical point is $(x,y)=(h,m)$. Hence the minimum point for δ^S in Σ_1 is $(x,y)=(h,m)$, with $\delta^S(h,m)=0$.

7.16.3 Proof of (19)

Inequality (19) follows immediately from Corollary 1, as given that seller 1 has made the investment, if seller 2 decides to invest then the sellers become symmetric and Corollary 1 applies.

7.17 Proof of Proposition 7

Lemma 11 implies $\Pi^F_{II} - \Pi^F_{NI} \le \Pi^S_{II} - \Pi^S_{NI} < \Pi^S_{IN} - \Pi^S_{NN}$ and $\Pi^F_{II} - \Pi^F_{NI} < \Pi^F_{IN} - \Pi^F_{NN}$ but does not determine the position of $\Pi^F_{IN} - \Pi^F_{NN}$ relative to $\Pi^S_{II} - \Pi^S_{NI}$ and relative to $\Pi^S_{IN} - \Pi^S_{NN}$. hence we consider the three possible cases.

Case 1: $\Pi^F_{II} - \Pi^F_{NI} \le \Pi^F_{IN} - \Pi^F_{NN} < \Pi^S_{II} - \Pi^S_{NI} < \Pi^S_{IN} - \Pi^S_{NN}$

- Case 1.1: If $k < \Pi^F_{II} - \Pi^F_{NI}$, then (I,I) is the unique NE in G^F and in G^S. Then the buyer's expected payment is the same in the FPA as in the SPA.
- Case 1.2: If $\Pi^F_{II} - \Pi^F_{NI} < k < \Pi^F_{IN} - \Pi^F_{NN}$, then the NE in G^F are (I,N) and (N,I). In G^S, (I,I) is the unique NE. The buyer's expected payment is lower in the SPA.[39]
- Case 1.3: If $\Pi^F_{IN} - \Pi^F_{NN} < k < \Pi^S_{II} - \Pi^S_{NI}$, then (N,N) is the unique NE in G^F. In G^S, (I,I) is the unique NE G^S . The buyer's expected payment is lower in the SPA.
- Case 1.4: If $\Pi^S_{II} - \Pi^S_{NI} < k < \Pi^S_{IN} - \Pi^S_{NN}$, then (N,N) is the unique NE in G^F. In G^S, (I,N) and (N,I) are the NE. The buyer's expected payment is lower in the SPA.

Case 2: $\Pi^F_{II} - \Pi^F_{NI} \le \Pi^S_{II} - \Pi^S_{NI} < \Pi^F_{IN} - \Pi^F_{NN} < \Pi^S_{IN} - \Pi^S_{NN}$

- Case 2.1: If $k < \Pi^F_{II} - \Pi^F_{NI}$, then the conclusions of Case 1.1 apply.
- Case 2.2: If $\Pi^F_{II} - \Pi^F_{NI} < k < \Pi^S_{II} - \Pi^S_{NI}$, then the conclusions of Case 1.2 apply.
- Case 2.3: If $\Pi^S_{II} - \Pi^S_{NI} < k < \Pi^F_{IN} - \Pi^F_{NN}$, then the NE are (I,N) and (N,I) both in G^F and in G^S. The payment comparison is determined by whether (h,m) is in S_2 or in F_2, and in the latter case by whether $(\tilde{h},\tilde{m}) \in F_1$ or $(\tilde{h},\tilde{m}) \in S_1$. In particular, $\tilde{P}^F < \tilde{P}^S$ if and only if $(h,m) \in F_2$ and $(\tilde{h},\tilde{m}) \in F_1$.
- Case 2.4: If $\Pi^F_{IN} - \Pi^F_{NN} < k < \Pi^S_{IN} - \Pi^S_{NN}$, then the conclusions of Case 1.4 apply.

[39]Starting from the symmetric setting with $(h_1,m_1) = (h_2,m_2) = (\tilde{h},\tilde{m})$ and $(\tilde{h}_1,\tilde{m}_1) = (h,m)$, we apply Lemma 7 to conclude that the expected payment is higher in the FPA.

Case 3: $\Pi^F_{II} - \Pi^F_{NI} \leq \Pi^S_{II} - \Pi^S_{NI} < \Pi^S_{IN} - \Pi^S_{NN} < \Pi^F_{IN} - \Pi^F_{NN}$ **(that is,** $(\tilde{h}, \tilde{m}) \in \mathfrak{F}_1$**)**

- Case 3.1: If $k < \Pi^F_{II} - \Pi^F_{NI}$, then the conclusions of Case 1.1 apply.
- Case 3.2: If $\Pi^F_{II} - \Pi^F_{NI} < k < \Pi^S_{II} - \Pi^S_{NI}$, then the conclusions of Case 1.2 apply.
- Case 3.3: If $\Pi^S_{II} - \Pi^S_{NI} < k < \Pi^S_{IN} - \Pi^S_{NN}$, then the conclusions of Case 2.3 apply.
- Case 3.4: If $\Pi^S_{IN} - \Pi^S_{NN} < k < \Pi^F_{IN} - \Pi^F_{NN}$, then (I, N) and (N, I) are the NE of G^F. In G^S, (N, N) is the unique NE. The expected payment in the FPA is lower than in the SPA if and only if $(\tilde{h}, \tilde{m})$ is in the subset of $\mathfrak{F}_1$ identified by Proposition 3(ia).

7.18 Proof of Proposition 8

7.18.1 Proof of Proposition 8(i)

For seller 1, $\tilde{\Pi}^S_1(\varepsilon_1, 0) = (m_1 - \alpha\varepsilon_1)h_2 + (1 - (h_1 - \varepsilon_1) - (m_1 - \alpha\varepsilon_1))(2h_2 + m_2)$ and $\frac{\partial \tilde{\Pi}^S_1(\varepsilon_1,0)}{\partial \varepsilon_1} = \frac{\partial \tilde{\Pi}^S_1(0,0)}{\partial \varepsilon_1} = (h_2 + m_2)\alpha + 2h_2 + m_2$. Likewise, $\frac{\partial \tilde{\Pi}^S_2(0,\varepsilon_2)}{\partial \varepsilon_2} = \frac{\partial \tilde{\Pi}^s_2(0,0)}{\partial \varepsilon_2} = (h_1 + m_1)\alpha + 2h_1 + m_1$. Hence the inequality $\frac{\partial \tilde{\Pi}^S_1(0,0)}{\partial \varepsilon_1} - \frac{\partial \tilde{\Pi}^S_2(0,0)}{\partial \varepsilon_2}$ is equivalent to $(h_2 + m_2 - h_1 - m_1)\alpha + 2h_2 + m_2 - 2h_1 - m_1 > 0$. The term $h_2 + m_2 - h_1 - m_1$ is non-negative since $(h_1, m_1) \in \Sigma_2$, hence

$$(h_2 + m_2 - h_1 - m_1)\alpha + 2h_2 + m_2 - 2h_1 - m_1 \geq (h_2 + m_2 - h_1 - m_1)(-1) + 2h_2 + m_2 - 2h_1 - m_1 = h_2 - h_1 \tag{37}$$

The latter difference is non-negative since $(h_1, m_1) \in \Sigma_2$, but in fact $(h_2 + m_2 - h_1 - m_1)\alpha + 2h_2 + m_2 - 2h_1 - m_1$ is positive because (i) if $h_2 + m_2 - h_1 - m_1 > 0$, then the inequality in (37) is strict; (ii) if $h_2 + m_2 - h_1 - m_1 = 0$, then $(h_1, m_1) \in \Sigma_2$ implies $h_1 - h_2$.

7.18.2 Proof of Proposition 8(ii)

For the FPA we have three cases to consider.

Proof of Proposition 8(iia) For seller 1, $\tilde{\Pi}^F_1(\varepsilon_1, 0) = (m_1 - \alpha\varepsilon_1)(h_1 - \varepsilon_1)\frac{h_1 - \varepsilon_1 + s_2}{2(h_1 - \varepsilon_1) + m_1 - \alpha\varepsilon_1} + (1 - (h_1 - \varepsilon_1) - (m_1 - \alpha\varepsilon_1))(h_1 - \varepsilon_1 + s_2)$ and

$$\frac{\partial \tilde{\Pi}^F_1(\varepsilon_1, 0)}{\partial \varepsilon_1} = \frac{\begin{pmatrix} \left(2\alpha^2 + 4\alpha\right)\varepsilon_1^3 + \left(-2m_1 - 10\alpha h_1 - 4\alpha m_1 - 2\alpha s_2 - 2\alpha^2 h_1 - \alpha^2 s_2\right)\varepsilon_1^2 \\ +2\left(2h_1 + m_1\right)\left(m_1 + 2\alpha h_1 + \alpha s_2\right)\varepsilon_1 - 2\alpha h_1^3 - 2h_1^2 m_1 - 2\alpha s_2 h_1^2 - 2h_1 m_1^2 - s_2 m_1^2 \end{pmatrix}}{\left(2h_1 + m_1 - 2\varepsilon_1 - \alpha\varepsilon_1\right)^2} + (\alpha + 1)(h_1 + s_2 - 2\varepsilon_1) - (1 - m_1 - h_1)$$

hence $\frac{\partial \tilde{\Pi}^F_1(0,0)}{\partial \varepsilon_1} = -\frac{2h_1^2(h_1+s_2)\alpha + (2h_1^2 + 2h_1 m_1 + m_1 s_2)m_1}{(2h_1+m_1)^2} + (\alpha + 1)(h_1 + s_2) - (1 - s_1)$.

For seller 2, $\tilde{\Pi}^F_2(0, \varepsilon_2) = (m_2 - \alpha\varepsilon_2)h_1 + (1 - (h_2 - \varepsilon_2) - (m_2 - \alpha\varepsilon_2))(h_1 + h_2 - \varepsilon_2 + m_2 - \alpha\varepsilon_2)$ and $\frac{\partial \tilde{\Pi}^F_2(0,\varepsilon_2)}{\partial \varepsilon_2} = -2\varepsilon_2\alpha^2 + (2s_2 - 1)(\alpha + 1) - 4\varepsilon_2\alpha + h_1 - 2\varepsilon_2$, $\frac{\partial \tilde{\Pi}^F_2(0,0)}{\partial \varepsilon_2} = (2s_2 - 1)(\alpha + 1) + h_1$. Suppose $\alpha \geq 0$, which implies that $-\frac{2h_1^2(h_1+s_2)\alpha + (2h_1^2 + 2h_1 m_1 + m_1 s_2)m_1}{(2h_1+m_1)^2}$ in $\frac{\partial \tilde{\Pi}^F_1(0,0)}{\partial \varepsilon_1}$ is negative. Then a sufficient condition for $\frac{\partial \tilde{\Pi}^F_2(0,0)}{\partial \varepsilon_2} > \frac{\partial \tilde{\Pi}^F_1(0,0)}{\partial \varepsilon_1}$ is $(2s_2 - 1)(\alpha + 1) + h_1 > (\alpha + 1)(h_1 + s_2) - (1 - s_1)$, or $\alpha < \frac{s_2 - s_1}{1 + h_1 - s_2}$. The latter inequality is satisfied at $\alpha = 0$ since $s_2 > s_1$. Finally, $s_2 > \frac{1}{2}$ implies $\frac{\partial \tilde{\Pi}^F_2(0,0)}{\partial \varepsilon_2} > 0$.

Proof of Proposition 8(iib) For seller 1, $\tilde{\Pi}_1^F(\varepsilon_1, 0) = (m_1 - \alpha\varepsilon_1)h_2 + (1 - (h_1 - \varepsilon_1) - (m_1 - \alpha\varepsilon_1))(\hat{\rho}_1(\varepsilon_1) + s_2)$ with $\hat{\rho}_1(\varepsilon_1) = \sqrt{\frac{1}{4}m_2^2 + h_2(h_1 - \varepsilon_1 + m_1 - \alpha\varepsilon_1)} - \frac{1}{2}m_2$, and

$$\begin{aligned}
\frac{\partial \tilde{\Pi}_1^F(\varepsilon_1, 0)}{\partial \varepsilon_1} &= -\alpha h_2 + (\alpha + 1)(\hat{\rho}_1(\varepsilon_1) + s_2) + (1 - \tilde{s}_1)\frac{d\hat{\rho}_1}{d\varepsilon_1} \\
&= (\alpha + 1)\left(s_2 - \frac{1}{2}m_2 + \sqrt{\frac{1}{4}m_2^2 + h_2\tilde{s}_1}\right) - \alpha h_2 - (1 - \tilde{s}_1)\frac{h_2(\alpha + 1)}{2\sqrt{\frac{1}{4}m_2^2 + h_2\tilde{s}_1}} \\
\frac{\partial \tilde{\Pi}_1^F(0, 0)}{\partial \varepsilon_1} &= -\alpha h_2 + (\alpha + 1)(\rho_1 + s_2) - (1 - s_1)\frac{h_2(\alpha + 1)}{2\hat{\rho}_1(0) + m_2}
\end{aligned}$$

For seller 2, $\tilde{\Pi}_2^F(0, \varepsilon_2) = (m_2 - \alpha\varepsilon_2)\check{\rho}_1(\varepsilon_2) + (1 - (h_2 - \varepsilon_2) - (m_2 - \alpha\varepsilon_2))(\check{\rho}_1(\varepsilon_2) + h_2 - \varepsilon_2 + m_2 - \alpha\varepsilon_2)$ with $\check{\rho}_1(\varepsilon_2) = \sqrt{\frac{1}{4}(m_2 - \alpha\varepsilon_2)^2 + (h_2 - \varepsilon_2)(h_1 + m_1)} - \frac{1}{2}(m_2 - \alpha\varepsilon_2)$. Then

$$\begin{aligned}
\frac{\partial \tilde{\Pi}_2^F(0, \varepsilon_2)}{\partial \varepsilon_2} &= -\alpha\check{\rho}_1(\varepsilon_2) + \tilde{m}_2\frac{d\check{\rho}_1}{d\varepsilon_2} + (\alpha + 1)(\check{\rho}_1(\varepsilon_2) + \tilde{s}_2) + (1 - \tilde{s}_2)(\frac{d\check{\rho}_1}{d\varepsilon_2} - 1 - \alpha) \\
&= -\alpha\left(\sqrt{\frac{1}{4}\tilde{m}_2^2 + \tilde{h}_2(h_1 + m_1)} - \frac{1}{2}\tilde{m}_2\right) + \tilde{m}_2\left(\frac{1}{2}\alpha - \frac{h_1 + m_1 + \frac{1}{2}\alpha\tilde{m}_2}{2\sqrt{\frac{1}{4}\tilde{m}_2^2 + \tilde{h}_2(h_1 + m_1)}}\right) \\
&\quad + (\alpha + 1)\left(\tilde{h}_2 + \frac{1}{2}\tilde{m}_2 + \sqrt{\frac{1}{4}\tilde{m}_2^2 + \tilde{h}_2(h_1 + m_1)}\right) \\
&\quad - \left(1 - \tilde{h}_2 - \tilde{m}_2\right)\left(\frac{1}{2}\alpha + \frac{h_1 + m_1 + \frac{1}{2}\alpha\tilde{m}_2}{2\sqrt{\frac{1}{4}\tilde{m}_2^2 + \tilde{h}_2(h_1 + m_1)}} + 1\right) \\
\frac{\partial \tilde{\Pi}_2^F(0, 0)}{\partial \varepsilon_2} &= -\alpha\check{\rho}_1(0) + m_2\left(\frac{-\frac{1}{2}m_2\alpha - s_1}{2\check{\rho}_1(0) + m_2} + \frac{1}{2}\alpha\right) + (\alpha + 1)(\check{\rho}_1(0) + s_2) + (1 - s_2)\left(\frac{-\frac{1}{2}m_2\alpha - s_1}{2\check{\rho}_1(0) + m_2} - \frac{1}{2}\alpha - 1\right) \\
&= -\alpha\check{\rho}_1(0) + m_2\frac{-s_1 + \alpha\check{\rho}_1(0)}{2\check{\rho}_1(0) + m_2} + (\alpha + 1)(\check{\rho}_1(0) + s_2) - (1 - s_2)\frac{2\check{\rho}_1(0) + m_2 + s_1 + \alpha\check{\rho}_1(0) + \alpha m_2}{2\check{\rho}_1(0) + m_2}
\end{aligned}$$

The inequality $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ is equivalent to $\left(h_2 - \rho_1 + \rho_1\frac{m_2}{2\rho_1 + m_2} - (1 - s_2)\frac{\rho_1 + m_2}{2\rho_1 + m_2} + h_2\frac{1 - s_1}{2\rho_1 + m_2}\right)$ $\alpha + \frac{s_2 - 1}{2\rho_1 + m_2}(2\rho_1 + m_2 + s_1) - m_2\frac{s_1}{2\rho_1 + m_2} - h_2\frac{s_1 - 1}{2\rho_1 + m_2} > 0$, with $\rho_1 = \hat{\rho}_1(0) = \check{\rho}_1(0)$. The term $h_2 - \rho_1 + \rho_1\frac{m_2}{2\rho_1 + m_2} - (1 - s_2)\frac{\rho_1 + m_2}{2\rho_1 + m_2} + h_2\frac{1 - s_1}{2\rho_1 + m_2}$ is positive if s_2 is sufficiently close to 1 because $h_2 > \rho_1$, hence $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ when α is large.

Proof of Proposition 8(iic) For seller 1, $\tilde{\Pi}_1^F(\varepsilon_1, 0) = (m_1 - \alpha\varepsilon_1)h_2 + (1 - (h_1 - \varepsilon_1) - (m_1 - \alpha\varepsilon_1))2h_2$, and $\frac{\partial \tilde{\Pi}_1^F(\varepsilon_1,0)}{\partial \varepsilon_1} = \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1} = (2 + \alpha)h_2$.

For seller 2, $\tilde{\Pi}_2^F(0, \varepsilon_2) = (m_2 - \alpha\varepsilon_2)(h_2 - \varepsilon_2 - m_2 + \alpha\varepsilon_2) + (1 - (h_2 - \varepsilon_2) - (m_2 - \alpha\varepsilon_2))2(h_2 - \varepsilon_2)$ and $\frac{\partial \tilde{\Pi}_2^F(0,\varepsilon_2)}{\partial \varepsilon_2} = -2(\alpha + \alpha^2 + 2)\varepsilon_2 + 4h_2 + m_2 + \alpha h_2 + 2\alpha m_2 - 2$, $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} = 4h_2 + m_2 + \alpha h_2 + 2\alpha m_2 - 2$. The inequality $\frac{\partial \tilde{\Pi}_2^F(0,0)}{\partial \varepsilon_2} > \frac{\partial \tilde{\Pi}_1^F(0,0)}{\partial \varepsilon_1}$ is equivalent to $\alpha > \frac{2 - 2h_2 - m_2}{2m_2}$, which is satisfied if α is large (is violated if $\alpha \leq \frac{1}{2}$).

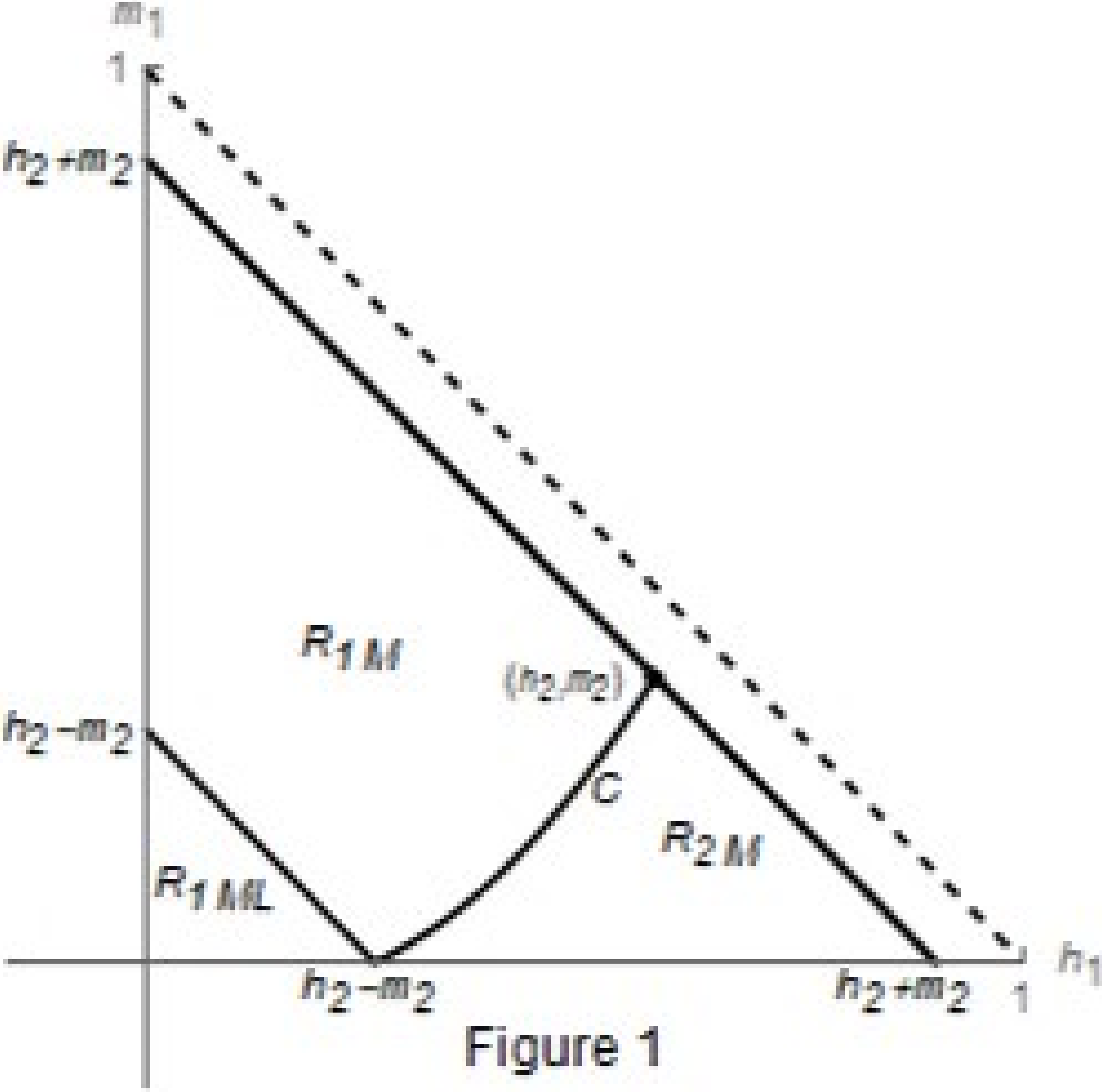


Figure 1

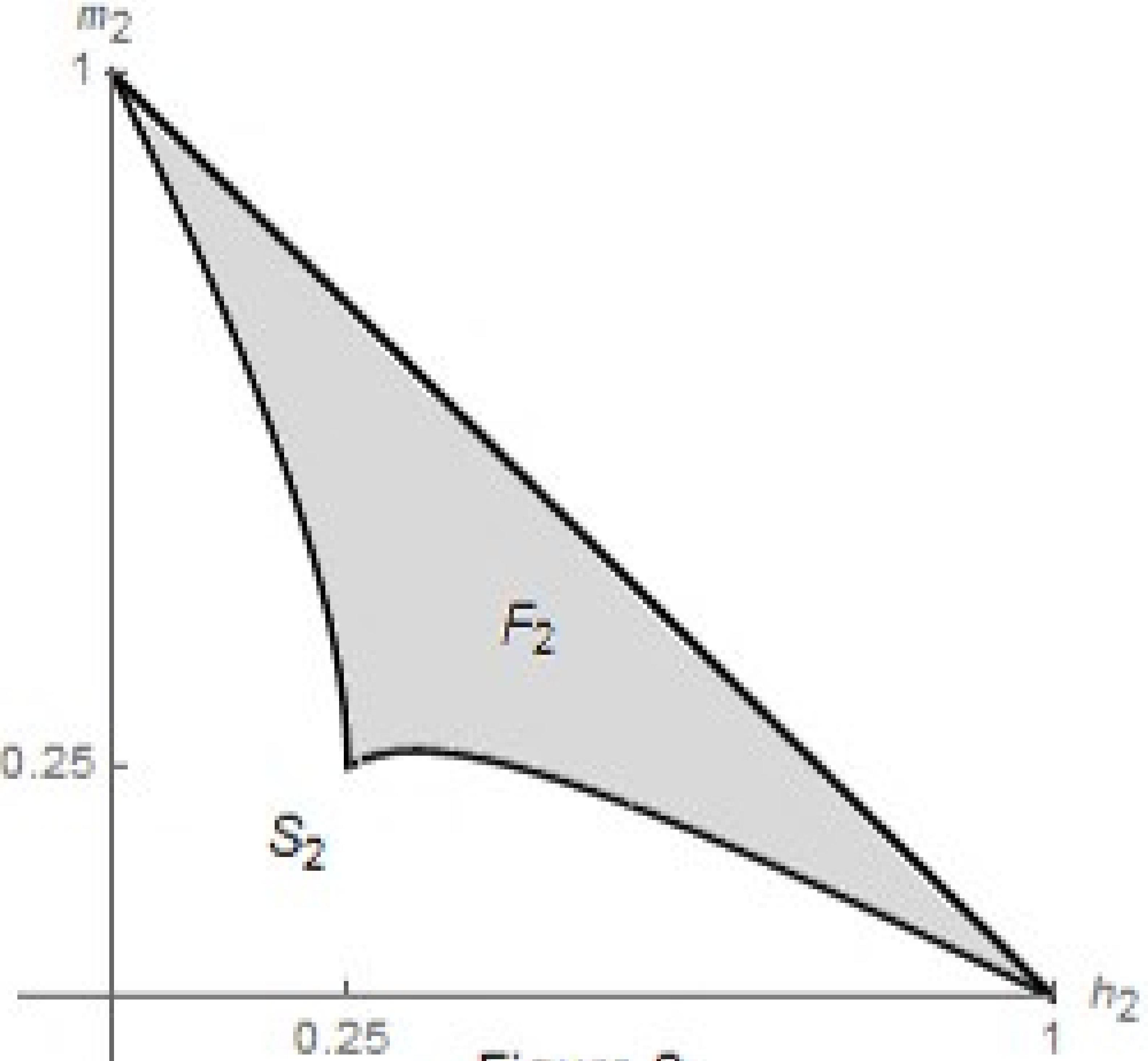


Figure 2

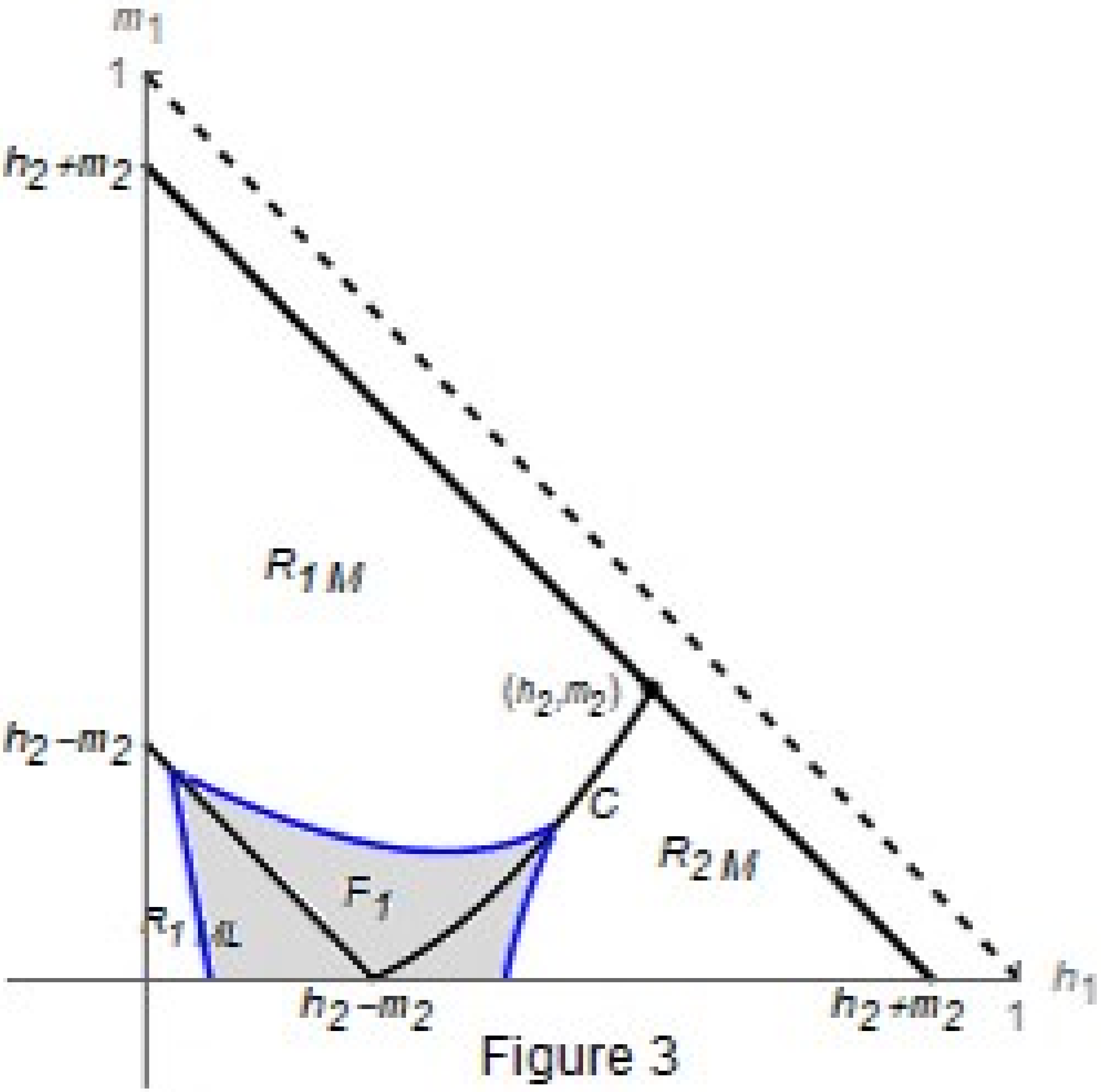


Figure 3

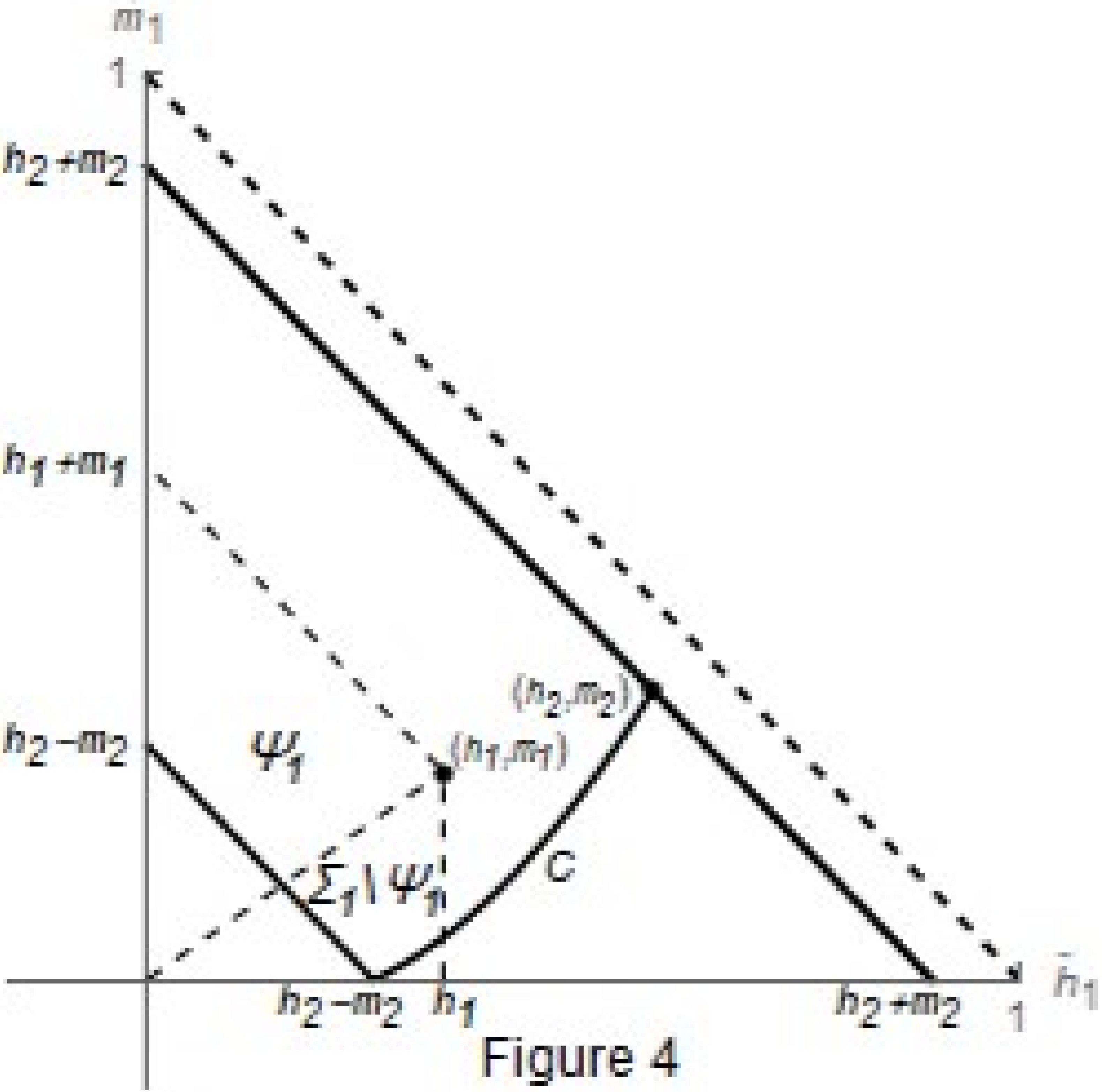

$\bar{m}_1$
1
h_2+m_2
h_1+m_1
h_2-m_2
ψ_1
(h_1,m_1)
(h_2,m_2)
$\Sigma_1 | \psi_1$
C
h_2-m_2
h_1
h_2+m_2
1
$\bar{h}_1$

Figure 4

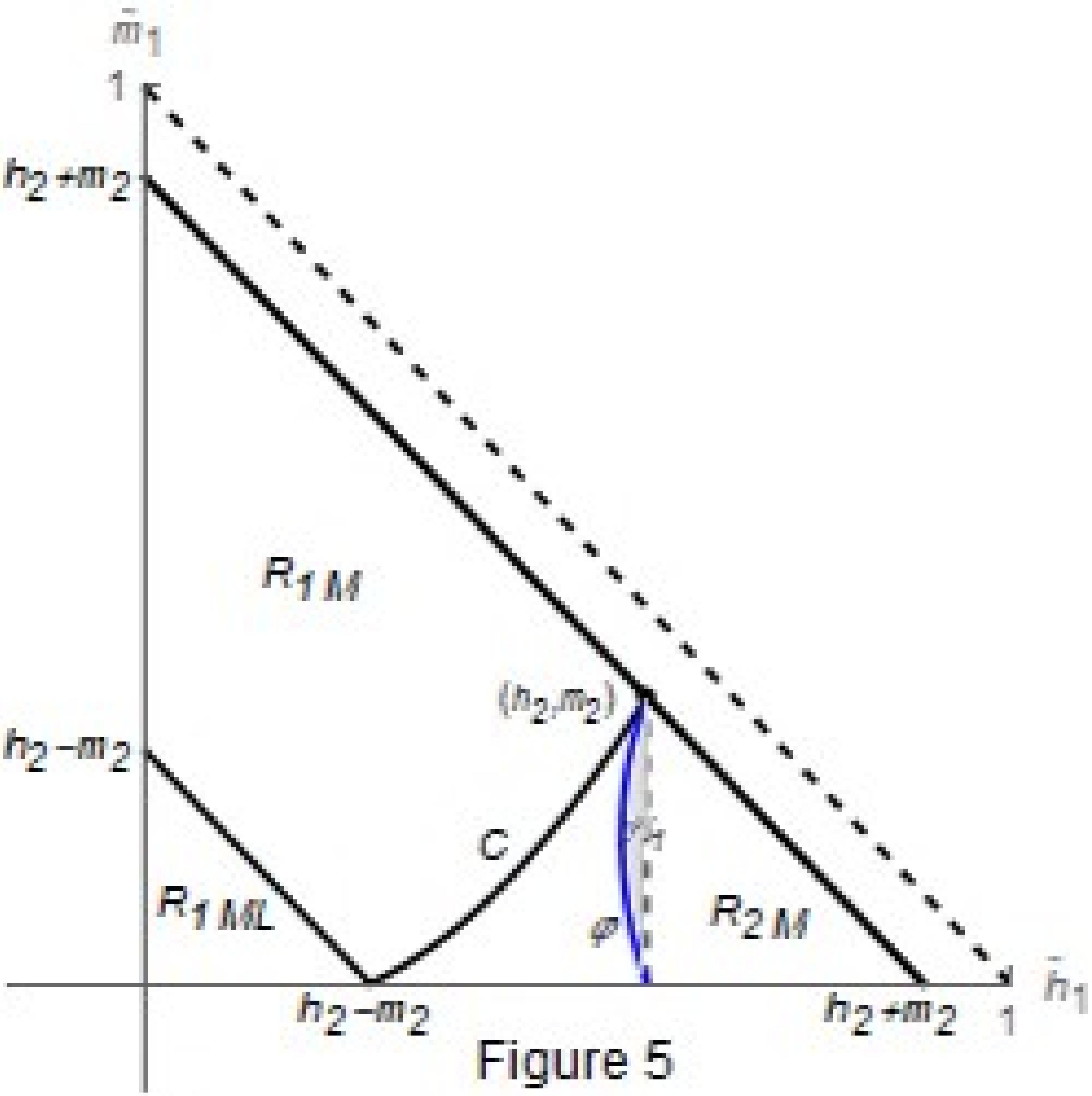
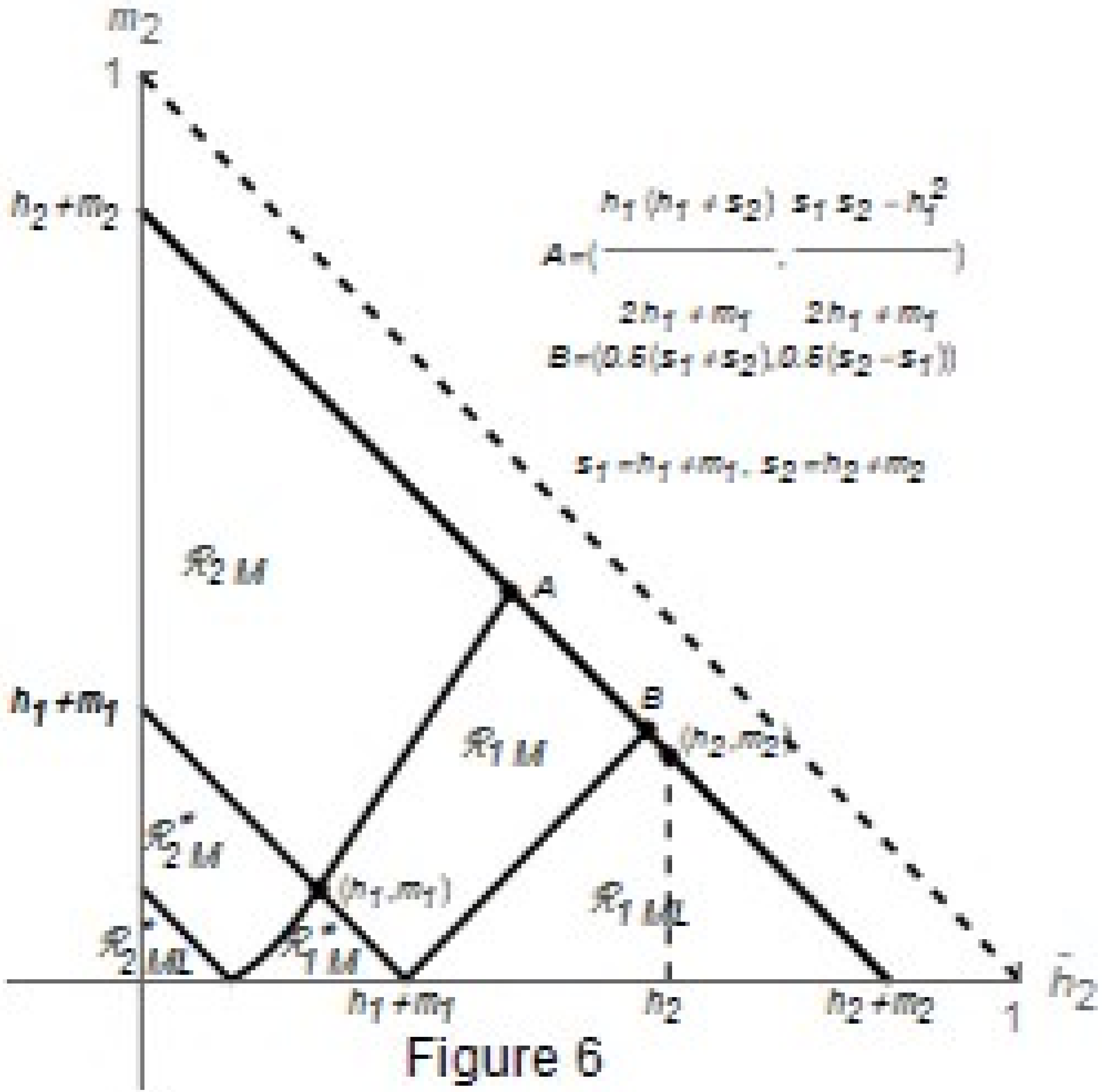
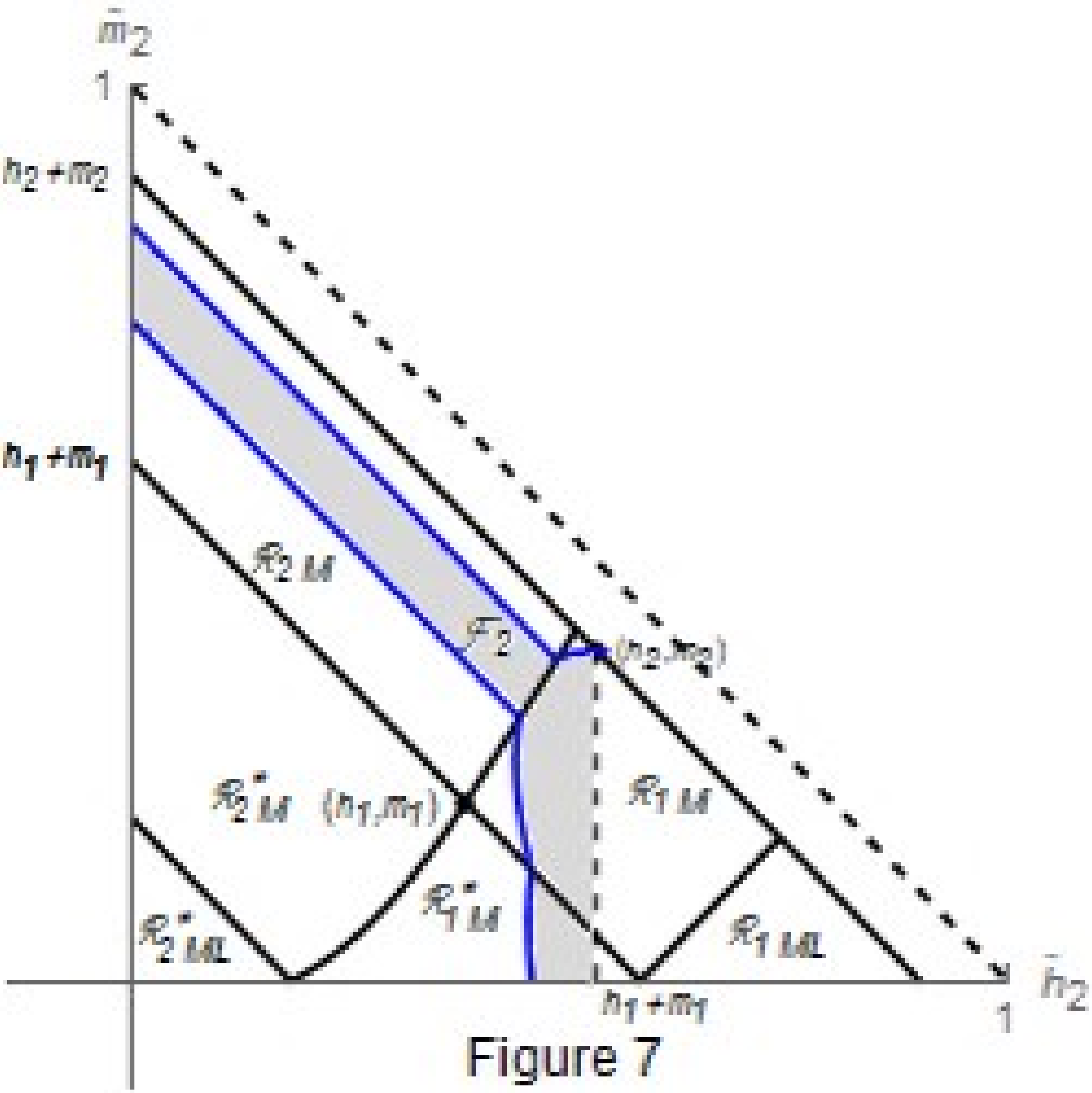


Figure 5



$\bar{m}_2$
1
h_2+m_2
$A = \left(\frac{h_1(h_1+s_2)}{2h_1+m_1}, \frac{s_1 s_2 - h_1^2}{2h_1+m_1}\right)$
$B = (0.5(s_1+s_2), 0.5(s_2-s_1))$
$s_1 = h_1+m_1,\ s_2 = h_2+m_2$
$\Re_{2M}$
A
h_1+m_1
B
$\Re_{1M}$
(h_2, m_2)
$\Re^*_{2M}$
(h_1, m_1)
$\Re_{1ML}$
$\Re^*_{2ML}$
$\Re^*_{1M}$
h_1+m_1
h_2
h_2+m_2
1
$\bar{h}_2$

Figure 6



$\bar{m}_2$
1
h_2+m_2
h_1+m_1
$\mathcal{R}_{2M}$
$\mathcal{R}_2$
(h_2,m_2)
$\mathcal{R}^*_{2M}$
(h_1,m_1)
$\mathcal{R}_{1M}$
$\mathcal{R}^*_{1M}$
$\mathcal{R}^*_{2ML}$
$\mathcal{R}_{1ML}$
$\bar{h}_2$
h_1+m_1
1

Figure 7

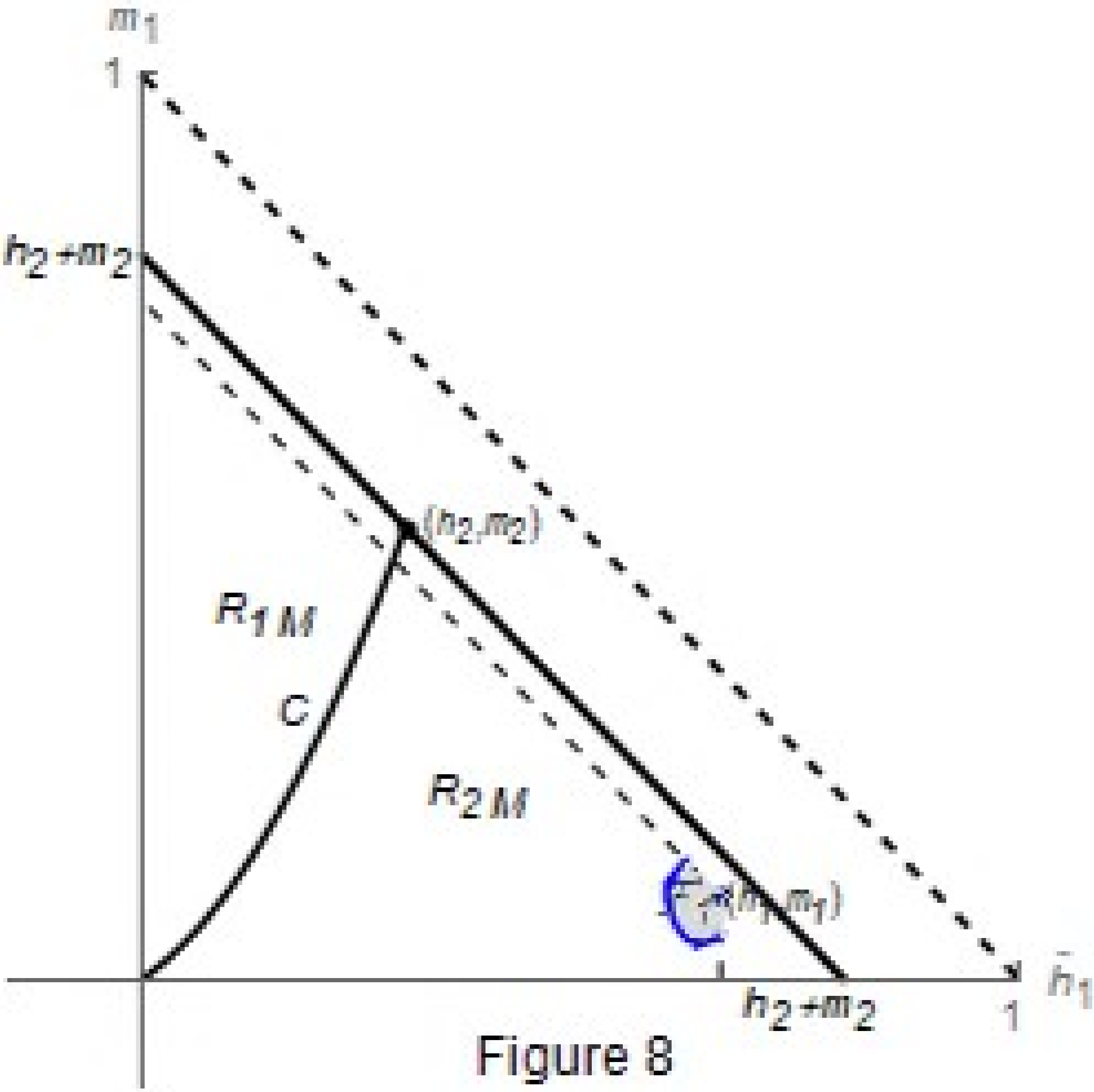


Figure 8

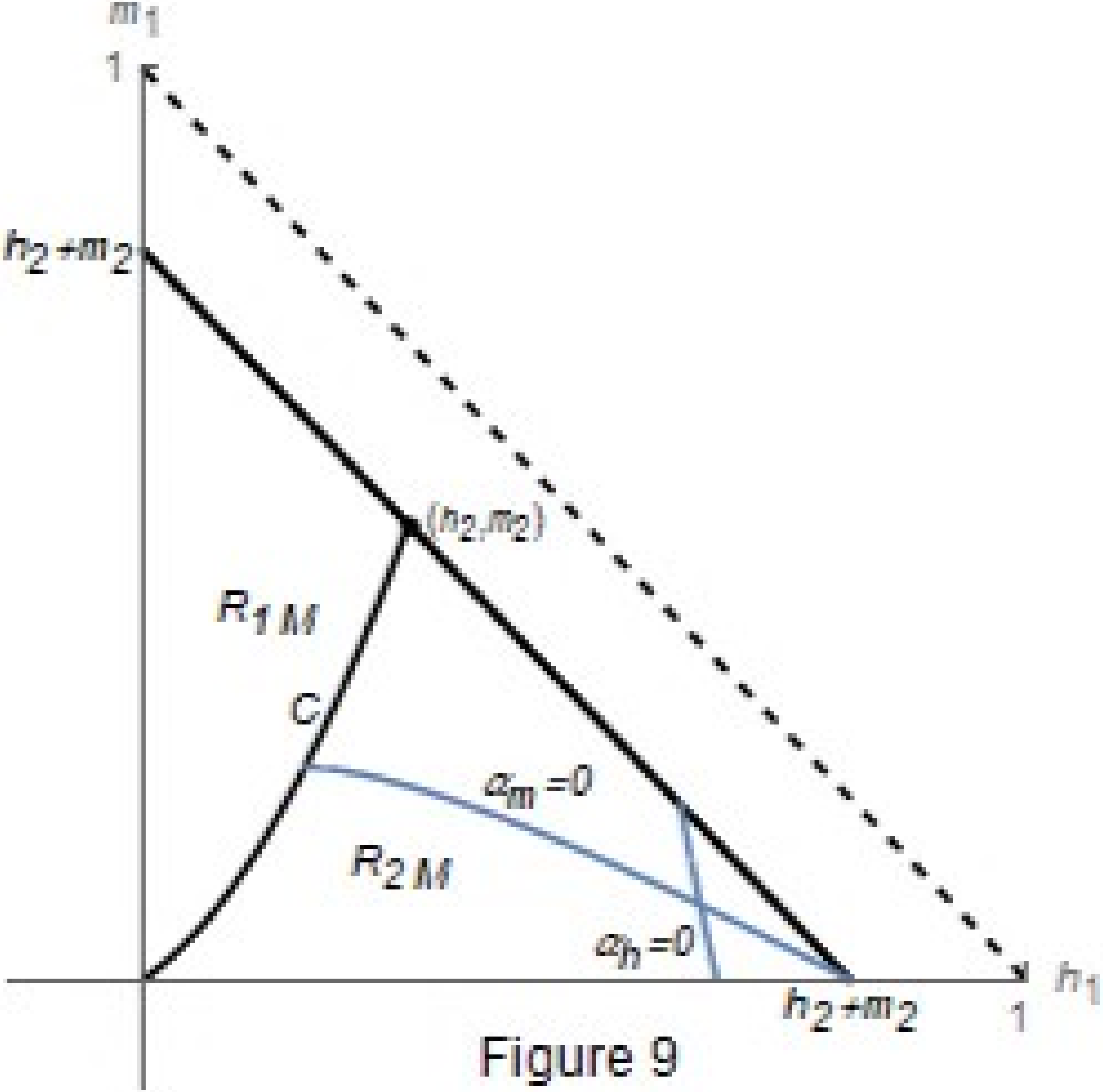


Figure 9