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Information Design and Entry in Auctions*

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Abstract

This paper is about a two-bidder auction setting with endogenous and costly entry in which, before the bidders' entry decisions, the seller may release information about the object on sale. This information affects each bidder's belief about the own distribution of value for the object on sale, hence it affects the bidder's incentive to enter. The seller uses a second price auction and we consider the class of unrestricted information structures using the techniques of information design in which the seller sends private messages to the bidders. We characterize the optimal information structure, which optimally trades off providing rents to the bidders in favorable (to the bidders) states of the world against inducing entry of all bidders in other states of the world (in order to generate a positve auction revenue). We compare the optimal information structure with some specific information structures examined in the literature and then show that a restriction to public messages hurts the seller significantly. We also show that using a first price auction allows the seller to earn the same revenue as when a second price auction is used. We then allow the seller to use an entry fee and jointly optimize, under some restrictions, with respect to the entry fee and the information structure. In this case the seller does not need to induce entry of all bidders to earn a positive revenue, and indeed induces entry of a single bidder, who is required to pay a high entry fee, if the entry cost is not small. But if the seller can also use a reserve price, then it is optimal to (almost) fully subsidize the entry cost and use the reserve price to extract all the bidders' rents while inducing the socially optimal entry.

Keywords: Second-Price Auction, First-Price Auction, Endogenous Entry, Information Design, Entry Fee, Reserve Price.

1 Introduction

In the recent decades, the empirical literature on auctions has emphasized the relevance of participation costs and the ensuing endogeneity of the number of bidders that take part in an auction. Therefore, a seller should take into account how a particular auction procedure affects the entry decisions of potential bidders. The theoretical literature on auctions with endogenous entry emphasizes how the optimal auction strongly depends on the information available to bidders when they choose whether to enter or not, but most of these papers try

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to identify the optimal auction for a given information regime. That is, they neglect the possibility for the seller to increase the revenue by influencing the bidders' information.

This paper tries to fill this gap by considering a specific two-bidder setting in which the seller offers the object through a second price auction and in which entry is endogenous and costly. Each bidder's value for the object on sale is either low (0) or high (1), and h_i denotes the probability of a high value for bidder i. No bidder knows h_1, h_2, but the ex ante probability distribution of h_1, h_2 is common knowledge. Upon entry, a bidder privately learns the own valuation. Before the bidders' entry decisions, the seller may transmit information which affects the bidders' incentives to enter, for instance by disclosing some details about the object on sale which allow each bidder to refine his beliefs about h_1 and/or about h_2. In fact, we apply the techniques of Information Design and assume that the seller commits ex ante to an information transmission mechanism, called information structure, which sends a message to each bidder as a function of h_1, h_2 (the state of the world). We rely on Taneva (2019) to restrict attention to direct information structures, in which the seller's possible messages to each bidder consist of a recommendation to enter, or not to enter, and such that each bidder has incentive to follow the recommendation he receives. This enables us to formulate the seller's problem directly in terms of distributions over entry recommendations subject to incentive constraints. We characterize the optimal information structure, which needs to induce entry of both bidders to generate a revenue in the auction. However, except in trivial cases, this often leads to a negative utility for bidders due to incomplete recovery of the entry cost. Thus the optimal information structure also needs to induce entry of a single bidder in some states of the world in order to make the bidder earn a non negative expected utility upon entry. We describe the optimal information structure as a function of the environment parameters, and in particular as a function of the entry cost $c > 0$ by determining the states of the world in which it is optimal to reduce entry as c increases.

We then examine the limits associated to the adoption of a specific information structure as No Disclosure (ND), Partial Disclosure (PD), Full Disclosure (FD), that in the economic literature are frequently investigated as potential options for the information designer.[1] Although these information structures are intuitive and simple to interpret, they are optimal only when c is small enough that entry of each bidder occurs even without any information transmission from the seller to the bidders. If instead entry of each bidder is infeasible, the optimal information structure is superior as it leaves no rent and induces more frequent entry by both bidders. We also show that requiring the seller to use public messages significantly restrict the set of feasible information structures, and prove that by using the first price auction the seller can earn the same revenue as when se uses the second price auction .

We extend our initial setting by allowing the seller to use an entry fee f and derive first the optimal information structure as a function f, then we optimize jointly with respect to the information structure and with respect to f. For the case of the specific information structures, ND is optimal when c is small, jointly with a suitable entry fee which extracts each bidder's expected surplus. Conversely, FD is optimal for large c as it induces just one bidder to participate, but the seller can tailor f suitably and earn more than under PD because under PD the bidders are on a symmetric footing and each bidder knows he may face competition in the auction; this reduces his incentive to enter. Thus, on one side we have entry of a single bidder; on the other side we have two bidders, each of whom is less likely to enter; in our setting the first alternative is superior as the latter induce too little entry. However, PD is optimal for intermediate c when the likelihood that h_i is high for each bidder i as this determines a high expected auction price. As a general result, the amount of information to release decreases as c increases. We also determine the optimal information structure in the set

[1] See for instance Gal-or et al. (2007) in a procurement context and Lu et al. (2018) and Serena (2022) in an all-pay auction setup.

of unrestricted information structures for given f, and then provide a specific result about revenue optimization with respect to f.

Finally, we examine the case in which the seller can use both an a reserve price, denoted with r, and the entry fee, denoted with f. We show it is profitable for the seller to pick r close to the high valuation 1 and f close to $-c$, which means that the entry cost is almost entirely subsidized by the seller. Therefore each bidder faces an almost zero net entry cost $c + f$ but earns a small rent from playing the auction since r is high. As a consequence, the seller extracts (almost) completely the bidders' rents and his revenue (almost) coincides with social welfare. Moreover, it is possible for the seller to induce bidders' entry as in the social optimum, which yields the seller the maximal social welfare. To this purpose, for instance, the simple information structure FD can be used.

In the rest of the introduction we explain how our paper is related to the existing literature. Then, in Section 2 we illustrate the model we analyze and in Section 3 we examine the optimal information structure, and some extensions. In Section 4 we introduce an entry fee, and in Section 14 we allow an entry fee and a reserve price.

1.1 Related Literature

This paper bridges different strands of the auction literature. First, we draw on models with endogenous entry, which typically investigate optimal auction design under an exogenous information structure. This literature considers various informational environments at the entry stage. McAfee and McMillan (1987) and Levin and Smith (1994) analyze settings in which bidders only know the distribution of their own valuation (and of their rivals' valuations), while Samuelson (1985) and Menezes and Monteiro (2000) study the opposite case in which each bidder privately observes his valuation before entry. More recent research examines intermediate environments in which bidders observe a signal statistically linked to their valuation before entry and learn the exact value only after paying the entry cost.[2] Gentry et al. (2017) provide a theoretical analysis of this case under standard auction procedures.[3] Finally, Ye (2004) considers a setting in which, after entry, bidders learn their valuation and receive signals about rivals' valuations.

In these contributions, bidders are symmetric ex ante and revenue equivalence holds despite endogenous participation. As a result, the analysis focuses on how reserve prices and/or entry fees can be used to maximize revenue for a given informational regime. By contrast, we treat the information structure itself as a design instrument and analyze how it can be optimally chosen to shape entry behavior and how this aspect interacts with the optimal entry fee or reserve price.

Second, our paper relates to the literature on information design in auctions with exogenous participation. Bergemann and Pesendorfer (2007) analyze optimal auctions when the seller can jointly design the mechanism and bidders' information structures. Ivanov (2021) characterizes optimal signal structures within fixed incentive-compatible mechanisms, including second-price auctions. Bergemann et al. (2022) study optimal information disclosure in second-price auctions with initially uninformed bidders and show how disclosure trades off efficiency and information rents.[4] In all these papers, information primarily affects bidding behavior within the auction stage (the intensive margin). In contrast, we study how optimal information design influences entry decisions (the extensive margin), and through this channel, competition and revenue.

[2]See, for instance, Ye (2007), Marmer et al. (2013), Roberts and Sweeting (2013), Gentry and Li (2014), Bhattacharya and Sweeting (2015), and Lu and Ye (2021).

[3]Gentry et al. (2017) study simultaneous and unregulated entry. Other contributions analyze alternative selection mechanisms. Bulow and Klemperer (2009) consider sequential entry, while Ye (2007) and Bhattacharya et al. (2014) study regulated entry via bidding mechanisms. Bhattacharya and Sweeting (2015) compare these mechanisms in terms of efficiency and revenue.

[4]Similar themes appear in Ganuza (2004), Ganuza and Penalva (2010), and Esö and Szentes (2007).

Interestingly, almost no paper has investigated the interaction between endogenous information released by the seller and endogenous bidders' participation in a standard auction setup.[5] The only exception is Vagstad (2007), which considers two alternative information settings. The first one is such that at the entry stage no bidder knows the own valuation, as in McAfee and McMillan (1987) and Levin and Smith (1994). The second one is such that each bidder (privately) observes the own valuation, as in Samuelson (1985) and Menezes and Monteiro (2000). Before the entry decisions, the seller can release information which converts the information regime from the first one to the second one. Vagstad (2007) shows that with only two bidders, if the entry cost is low (high) then information disclosure decreases (increases) entry, and gives an example in which information disclosure is revenue decreasing (increasing) when the entry cost is sufficiently low (high). Our paper departs from the assumptions in Vagstad (2007) in several ways. First, although the information revealed by the seller allows each bidder to have a more precise estimate of the own value, we assume that in no case a bidder learns the own value before entry. Second, the seller's transmission of information affects the beliefs of each bidder also about his rival's valuation. Third, we generalize the information disclosure policies available to the seller and we let him adopt further instruments to extract buyers' surplus (i.e. entry fee or reserve price).

Finally, we contribute to the literature on information design.[6] Our setting combines Bayesian persuasion (Kamenica and Gentzkow, 2011) with endogenous entry into a second-price auction. Our main methodological reference is Taneva (2019)[7] who extends persuasion to environments with multiple interacting receivers using Bayes Correlated Equilibrium (BCE) as a characterization tool.[8] We adopt this approach and allow the auctioneer to commit ex ante to private (and potentially correlated) entry recommendations. To our knowledge, ours is the first analysis of optimal information design in a standard second-price auction environment in which the primary strategic channel operates through endogenous entry rather than bidding behavior.

2 The model

2.1 Information and preferences

A (female) seller owns an object which is worthless to her and which she offers for sale through a second price auction (SPA henceforth). There are two (male) bidders who may be interested in buying the object: bidder 1 and bidder 2, who have independently distributed valuations for the object. For $i = 1, 2$, v_i denotes the value of bidder i and the probability distribution for v_i is characterized as follows by a parameter $h_i \in (0, 1)$:

$$\Pr\{v_i = 1\} = h_i \qquad \text{and} \qquad \Pr\{v_i = 0\} = 1 - h_i$$

That is, v_i has Bernoulli distribution in which h_i is the probability that v_i is equal to 1. Bidder i is risk neutral and wants to maximize v_i times his probability to win the object minus his expected outlay.

Bidder i needs to incur an entry cost $c > 0$ in order to participate in the auction, but at the time of his entry decision he does not observe v_i nor h_i. However, it is c.k. that h_1, h_2 are realizations of stochastically independent random variables and that there exist h^w, h^s, α such that

$$\Pr\{h_i = h^w\} = \alpha \qquad \text{and} \qquad \Pr\{h_i = h^s\} = 1 - \alpha$$

[5] Feng (2023) investigates how the information design may influence the participation and the effort of contestants in an all-pay auction context.

[6] See, e.g., Kamenica (2019), Bergemann and Morris (2019).

[7] Taneva (2019) is one of the main references also for Antsygina and Teteryatnikova (2023) who study an information design problem in an all-pay auction in which contestants effort may be influenced by the information structure.

[8] This type of methodology is based on the paper of Bergemann and Morris (2016).

with $0 < h^w < h^s < 1$ and $\alpha \in (0, 1)$. That is, each bidder i is either weak or strong in terms of the distribution of v_i, depending on whether h_i is equal to h^w or to h^s, and there exists four states of the world in terms of (h_1, h_2), that is $(h_1, h_2) \in \{(h^w, h^w), (h^w, h^s), (h^s, h^w), (h^s, h^s)\}$. In the following, sometimes the states of the world are denoted, respectively, as ww, ws, sw, ss.

Before the bidders' entry decisions, the seller observes the state of the world (h_1, h_2) and sends a private message m_1 to bidder 1, m_2 to bidder 2 in order to influence the bidders' entry decisions. The message is based upon the state of the world according to an information transmission rule, called *information structure* in the following, to which the seller has committed before observing (h_1, h_2). Bidder i knows the information structure, and after receiving a message from the seller he decides whether to enter the auction or not. If he enters, then he incurs the cost c but learns the own valuation v_i (thus, c can be interpreted as the bidder's cost of evaluating the object) and then bids the own valuation as that is a weakly dominant strategy in the SPA. We are interested in determining the information structure which maximizes the seller's expected revenue. We later consider cases in which the seller can use an entry fee and/or a reserve price.

2.2 Direct information structures

Proposition 2 in Taneva (2019) establishes that for each Bayes-Nash Equilibrium (BNE in the following) of the entry game augmented with the information structure designed by the seller, there exists an information structure for which the message each bidder receives is in the set $\{E, NE\}$, and in the resulting game there exists a BNE in which each bidder i enters when $m_i = E$, does not enter when $m_i = NE$. This means that without loss of generality we can focus on information structures for which $\{E, NE\}$ is the set of possible messages[9] and such that a BNE exists with the property that each bidder obeys the indication of the message he receives.

Since $\{E, NE\}$ is the set of possible messages, it follows that we can view the information structure as consisting of the following four probability distributions over messages:

$$\begin{aligned} x_0, x_1, x_2, x_{12} \text{ is the message distribution when } (h_1, h_2) &= (h^w, h^w) \\ y_0, y_1, y_2, y_{12} \text{ is the message distribution when } (h_1, h_2) &= (h^w, h^s) \\ z_0, z_1, z_2, z_{12} \text{ is the message distribution when } (h_1, h_2) &= (h^s, h^w) \\ w_0, w_1, w_2, w_{12} \text{ is the message distribution when } (h_1, h_2) &= (h^s, h^s) \end{aligned}$$

For instance, the distribution x_0, x_1, x_2, x_{12} refers to the state of the world $(h_1, h_2) = (h^w, h^w)$ and x_1 is the probability that $m_1 = E$, $m_2 = NE$, that is the probability that the seller sends message E only to bidder 1 when $(h_1, h_2) = (h^w, h^w)$. Likewise, x_2 is the probability that the sellers' messages are $(m_1, m_2) = (NE, E)$, and $x_0 = \Pr\{(m_1, m_2) = (NE, NE)|(h_1, h_2) = (h^w, h^w)\}$, $x_{12} = \Pr\{(m_1, m_2) = (E, E)|(h_1, h_2) = (h^w, h^w)$; hence $x_0 + x_1 + x_2 + x_{12} = 1$. Similar interpretations apply to the 4-tuples y_0, y_1, y_2, y_{12}, z_0, z_1, z_2, z_{12}, w_0, w_1, w_2, w_{12} which refer to the states of the world $(h_1, h_2) = (h^w, h^s)$, $(h_1, h_2) = (h^s, h^w)$, $(h_1, h_2) = (h^s, h^s)$, respectively.

About the condition that there exists a BNE in which each bidder obeys the message he receives, we begin with bidder 1. We normalize to 0 the utility from staying out of the auction, while entering yields expected utility $h_1(1 - h_2) - c$ because of the entry cost and because bidder 1's utility in the SPA is 0 if $v_1 = 0$ or if $v_1 = v_1 = 1$, but is 1 is $v_1 = 1$, $v_2 = 0$, and the latter event has probability $h_1(1 - h_2)$. Then consider bidder 1 who has received message $m_1 = E$, and expects that bidder 2 obeys message m_2. First notice that $p_{1E} = \alpha^2(x_1 + x_{12}) + \alpha(1 - \alpha)(y_1 + y_{12}) + (1 - \alpha)\alpha(z_1 + z_{12}) + (1 - \alpha)^2(w_1 + w_{12})$ denotes the probability that

[9] An information structure with this feature is said to be "direct".

bidder 1 receives $m_1 = E$. Then we let

$$\begin{aligned} u_{1E} &= \alpha^2 \left(x_1(h-c) + x_{12}(h(1-h)-c)\right) + \alpha(1-\alpha)\left(y_1(h-c) + y_{12}(h^2-c)\right) \\ &\quad + \alpha(1-\alpha)\left(z_1(1-h-c) + z_{12}((1-h)^2-c)\right) + (1-\alpha)^2\left(w_1(1-h-c) + w_{12}((1-h)h-c)\right) \end{aligned} \quad (1)$$

In case $p_{1E} > 0$, the quotient $\frac{u_{1E}}{p_{1E}}$ is bidder 1's expected utility from entering given $m_1 = E$. Since $\frac{u_{1E}}{p_{1E}} \geq 0$ is equivalent to $u_{1E} \geq 0$, from now on we label the latter inequality as IC_{1E}:[10]

$$u_{1E} \geq 0 \qquad (\text{IC}_{1E})$$

Likewise, entering is a best reply for bidder 2 when $m_2 = E$ if and only if u_{2E} below is non-negative (provided that the seller sends $m_2 = E$ with positive probability) with

$$\begin{aligned} u_{2E} &= \alpha^2 \left(x_2(h-c) + x_{12}(h(1-h)-c)\right) + \alpha(1-\alpha)\left(y_2(1-h-c) + y_{12}((1-h)^2-c)\right) \\ &\quad + \alpha(1-\alpha)\left(z_2(h-c) + z_{12}(h^2-c)\right) + (1-\alpha)^2\left(w_2(1-h-c) + w_{12}((1-h)h-c)\right) \end{aligned}$$

In the following we label the inequality $u_{2E} \geq 0$ as IC_{2E}.

When bidder i's receives message NE, staying out of the auction needs to be a best reply for bidder i, a condition we label as IC_{iNE}. But in fact, it turns out IC_{1NE} and IC_{2NE} do not affect the optimal information structure, hence we leave them to the proof of Proposition 1 in the appendix

The seller's revenue is equal to 1 if both bidders enter and both turn out to have value 1, but is equal to 0 in each other circumstance. Hence the expected revenue is

$$R = \alpha^2 h^2 x_{12} + \alpha(1-\alpha)h(1-h)y_{12} + (1-\alpha)\alpha(1-h)hz_{12} + (1-\alpha)^2(1-h)^2 w_{12} \quad (2)$$

In the following we maximize R subject to IC_{1E}, IC_{2E}, but since the bidders are ex ante symmetric it turns out that without loss of generality we can restrict to symmetric information structures, that is such that $x_1 = x_2$, $y_1 = z_2$, $y_2 = z_1$, $y_{12} = z_{12}$, $w_1 = w_2$. Under these conditions IC_{2E} is equivalent to IC_{1E} and our problem can be stated as

$$\max R \qquad \text{s.t.} \qquad \text{IC}_{1E} \quad (3)$$

3 The optimal information structure

We begin by introducing an information structure which provides the bidders with no information, but nevertheless is optimal when no information is needed to decide to enter.

3.1 The no-information information structure

At one extreme, it is immediate that the seller induces entry of no bidder if $c > 1 - h$ because in such a case the coefficients of $x_1, x_{12}, y_1, y_{12}, z_1, z_{12}, w_1, w_{12}$ in IC_{1E} are all negative, that is even in the most favorable circumstance – bidder 1 is strong and bidder 2 does not enter – the entry cost c is higher than bidder 1's utility in the SPA, $1 - h$. Hence the only way to satisfy IC_{1E} is to set all these variables equal to 0 and by symmetry it follows $x_0 = 1$, $y_0 = 1$, $z_0 = 1$, $w_0 = 1$. Therefore in the following we assume $c \leq 1 - h$.

At another extreme, since R is increasing in $x_{12}, y_{12}, z_{12}, w_{12}$ it is intuitive that it is optimal to set $x_{12} = 1$, $y_{12} = 1$, $z_{12} = 1$, $w_{12} = 1$ if c is small, as then the coefficients of $x_{12}, y_{12}, z_{12}, w_{12}$ in IC_{1E} are all positive. Hence

[10]If $p_{1E} = 0$, then $m_1 = E$ never occurs and there is no constraint IC_{1E}.

IC_{1E} is satisfied at $x_{12} = y_{12} = z_{12} = w_{12} = 1$. Precisely, IC_{1E} holds when $x_{12} = y_{12} = z_{12} = w_{12} = 1$ if and only if

$$c \leq c_A \equiv h^e(1 - h^e)$$

in which $h^e \equiv \alpha h + (1 - \alpha)(1 - h)$ denotes the expectation of both h_1 and h_2. In the following we denote with S_A the information structure such that $x_{12} = y_{12} = z_{12} = w_{12} = 1$.

It may be worthwhile to notice that for some parameters, S_A induces excessive entry from the social point of view in each state of the world. Indeed, assuming $h_1 \geq h_2$ to fix the ideas (without loss of generality), social welfare in state h_1, h_2 is equal to

$$SW_{12} = 1 - (1 - h_1)(1 - h_2) - 2c$$

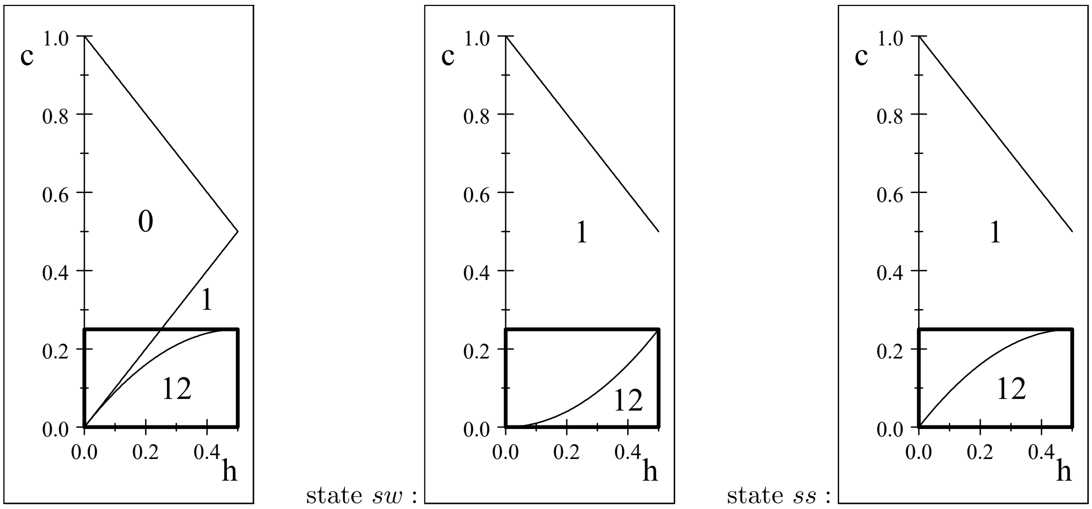
if both bidders enter, and

$$SW_1 = h_1 - c$$

if only bidder 1 enters.[11] Hence $SW_{12} - SW_1 = (1 - h_1)h_2 - c$, which highlight that entry of bidder 2 contributes to social welfare just when $v_1 = 0$ and $v_2 = 1$, an event with probability $h_1(1 - h_2)$.

The following figures describe the socially optimal entry pattern as a function of h, c and the set of (h, c) which satisfy $c \leq c_A$ for the case of $\alpha = \frac{1}{2}$: this is the rectangle with bold edges. Such set strictly includes the set in which entry of both bidders is socially optimal. This reveals that the seller's revenue is maximized by socially excessive entry when $c \leq c_A$. The reason is that entry of both bidders in each state of the world reduces the bidders' expected utility but increases the seller's revenue, even though the former decrease is greater the latter increase.[12]

Figure 1: Socially optimal entry depending on the state of the world



state ww : state sw : state ss :

3.2 The optimal information structure when $c_A < c \leq 1 - h$

We determine below the optimal information structure when $c_A < c \leq 1 - h$. Then S_A is infeasible because it violates IC_{1E}, hence the seller needs to reduce below 1 one or more among the variables $x_{12}, y_{12}, z_{12}, w_{12}$, that is in at least one state of the world at least one bidder does not enter. For instance, $w_{12} < 1$ means that in state

[11] Since $h_1 \geq h_2$, entry of only bidder 2 does not maximize social welfare as it generates a social welfare no higher than SW_1.

[12] Figures 1(a-c) illustrate this result for the case of $\alpha = \frac{1}{2}$, but in fact the result holds for each $\alpha \in (0, 1)$ because for states ww and ss, $h^e(1 - h^e) > h - h^2$, for state sw, $h^e(1 - h^e) > h^2$.

ss with positive probability the seller induces entry of a single bidder if $w_1 = w_2 > 0$, or induces entry of no bidder if $w_0 > 0$. It is important to stress that given $w_{12} < 1$, the values of w_0, w_1, w_2 do not affect the revenue in (2), but only IC_{1E}; hence they are optimally chosen in order to help satisfy IC_{1E}. More in detail, in state ss entry by a single bidder is (at least weakly) preferable for the bidder to no entry as the coefficient of w_1 in IC_{1E}, $1 - h - c$, is non-negative; hence, given $w_{12} < 1$, it is optimal to set $w_1 = w_2 > 0$ and $w_0 = 0$ to help satisfy IC_{1E}. Likewise, in states sw, ws the strong bidder earns a non-negative utility if he enters alone, and higher than the utility of the weak bidder if the latter enters alone.[13] Hence, given $y_{12} = z_{12} < 1$, it is optimal to set $y_2 = 1 - y_{12}$ (and $y_0 = 0$, $y_1 = 0$), $z_1 = 1 - z_{12}$ (and $z_0 = 0$, $z_2 = 1$). Conversely, in state ww matters are less straightforward as the utility of a single entrant is $h - c$. This is negative if $h < c$, and then $x_1 = x_2 = 0$, $x_0 = 1 - x_{12}$ when $x_{12} < 1$. If instead $h > c$, then $x_{12} < 1$ implies $x_1 = x_2 = \frac{1}{2} - \frac{1}{2}x_{12}$, $x_0 = 0$. As these remarks suggest, determining the optimal information structure when $c > c_A$ boils down to reducing $x_{12}, y_{12}, z_{12}, w_{12}$ the least below 1, while choosing $x_0, x_1, x_2, ..., w_0, w_1, w_2$ in the best way to satisfy IC_{1E}. We show below that this leads to entry of both bidders with positive probability at least in state ss for each $c < 1 - h$.

We list below in (4) seven information structures, $S_A, ..., S_G$ such that for each parameter values the optimal information structure is one of them, as described in Proposition 1:[14]

$$\begin{cases} S_A : x_{12} = 1,\ y_{12} = 1,\ z_{12} = 1,\ w_{12} = 1 \\ S_B : x_{12} = 1,\ y_2 \in (0,1), y_{12} = 1 - y_2,\ z_1 = y_2, z_{12} = 1 - z_1,\ w_{12} = 1 \\ S_C : x_0 = 1,\ y_2 \in (0,1), y_{12} = 1 - y_2,\ z_1 = y_2, z_{12} = 1 - z_1,\ w_{12} = 1 \\ S_D : x_0 = 1,\ y_2 = 1,\ z_1 = 1,\ w_1 = w_2 \in (0, \frac{1}{2}), w_{12} = 1 - 2w_1 \\ S_E : x_0 \in (0,1), x_{12} = 1 - x_0,\ y_{12} = 1,\ z_{12} = 1,\ w_{12} = 1, \\ S_F : x_1 = x_2 \in (0, \frac{1}{2}), x_{12} = 1 - 2x_1,\ y_2 = 1,\ z_1 = 1,\ w_{12} = 1 \\ S_G : x_1 = x_2 = \frac{1}{2},\ y_2 = 1,\ z_1 = 1,\ w_1 = w_2 \in (0, \frac{1}{2}), w_{12} = 1 - 2w_1 \end{cases} \tag{4}$$

In order to earn some intuition, consider the case in which c is slightly greater than c_A. Then S_A is infeasible because $u_{1E} = -\gamma$, for some $\gamma > 0$, when $x_{12} = y_{12} = z_{12} = w_{12} = 1$. Starting from S_A, one change in the information structure which satisfies IC_{1E} consists in reducing x_{12} by $\varepsilon > 0$ and increasingly x_0 by ε. The rationale is that $c > c_A$ implies that the coefficient of x_{12} in IC_{1E}, $h(1-h) - c$, is negative,[15] hence $\Delta x_{12} = -\varepsilon$ increases u_{1E} and ε needs to be equal to $\frac{\gamma}{\alpha^2(c-h+h^2)}$ in order to make IC_{1E} satisfied. As a result, the information structure S_E in (4) is identified and $\Delta R^E = -\frac{h^2}{c-h+h^2}\gamma$ with respect to the revenue obtained if both bidders enter in each state of the world. An alternative way to satisfy IC_{1E} consists in reducing y_{12}, z_{12} by $\varepsilon > 0$ each, and increasing y_2, z_1 by ε each. This increases u_{1E} because the coefficient of z_2 in $\text{IC}_{1E}$is positive and is greater than the sum of the coefficients of y_{12} and of z_{12}. Precisely, the resulting change in u_{1E} is $\alpha(1-\alpha)\left(-\varepsilon(h^2 - c) - \varepsilon((1-h)^2 - c) + \varepsilon(1 - h - c)\right)$, or $\alpha(1-\alpha)(c + h - 2h^2)\varepsilon$, hence ε needs to be $\frac{\gamma}{\alpha(1-\alpha)(c+h-2h^2)}$ to satisfy IC_{1E}. Then the information structure S_B in (4) is identified and $\Delta R^B = -\frac{2h(1-h)}{c+h-2h^2}\gamma$.

Comparing $\frac{h^2}{c-h(1-h)}\gamma$ with $\frac{2h(1-h)}{c+h-2h^2}\gamma$ reveals that the first quotient is smaller (is greater) than the second if $h < c$ (if $h > c$), thus S_E is superior to S_B when $h < c$, whereas the seller prefers S_B if $h > c$. It is intuitive that a small h makes S_E superior to S_B, because if h is close to 0 then Δx_{12} in S^E is about equal to $\frac{\gamma}{\alpha^2 c}$, while Δy_{12} and Δz_{12} in S^B are both about equal to $\frac{\gamma}{\alpha(1-\alpha)c}$. Then the difference in the effect on the revenue is determined by the fact that state ww has probability h^2, the states ws, sw have probability $2h(1-h)$, and the former probability is much smaller than the latter for a small h; thus the revenue decrease under S^E is less than under S^B. Conversely if $h > c$ then the decrease in Δx_{12} is relatively large compared with the decrease

[13] Actually, in some cases entry of the weak bidder alone reduces the latter's utility.

[14] We specify only the terms of the information structure which differ from zero, but in some cases full details are left to the proof of Proposition 1.

[15] If $c \le h(1-h)$, then the coefficients of x_{12}, of w_{12}, and of $y_{12} + z_{12}$ in IC_{1E} are all positive, hence S_A is feasible.

in Δy_{12} and Δz_{12} because in state ww the utility of each bidder when both bidders enter is not too negative. Then a large reduction in x_{12} is needed to satisfy IC_{1E}, which hurts the revenue significantly.[16]

Actually, when c is just a bit larger than c_A Proposition 1 establishes that S_E or S_B is optimal overall. But for c significantly greater than c_A, the optimal information structure is not S_B nor S_E, as further entry reduction is needed. Precisely, the optimal information structure, depends on the comparison between c and the following thresholds

$$c_B \equiv \frac{(1-h)\left(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2\right)}{1-\alpha+\alpha^2}, \qquad c_C \equiv \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{1+\alpha}$$

$$c_D \equiv (1-h)\left(h+\alpha-h\alpha\right), \qquad c_E \equiv \frac{2\alpha-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2}{2-2\alpha+\alpha^2}$$

for which in the proof of Proposition 1 we show

$$c_A < \min\{c_B, c_C\} < \max\{c_B, c_C\} < c_D < 1-h \text{ and } c_B < c_E < 1-h$$

Proposition 1 (The optimal information structure with no entry fee nor reserve price) *The optimal information structure is as follows;*
(i) S_A *if* $c \le c_A$
(ii) S_B *if* $c_A \le c \le \min\{c_B, h\}$
(iii) S_C *if* $\max\{c_C, h\} \le c \le c_D$
(iv) S_D *if* $\max\{c_D, h\} \le c \le 1-h$
(v) S_E *if* $\max\{c_A, h\} \le c \le c_C$
(vi) S_F *if* $c_B \le c \le \min\{c_E, h\}$
(vii) S_G *if* $c_E \le c \le h$.

The optimal information structure depends on whether $h < c$ or $h > c$ not only when c is slightly greater than c_A, as the sign of $h-c$ determines whether $x_{12} < 1$ should be accompanied by $x_0 > 0$ and $x_1 = x_2 = 0$, or by $x_0 = 0$ and $x_1 = x_2 > 0$. In detail, when $h < c$ and c is slightly larger than c_A, S_E satisfies IC_{1E} through a reduction in x_{12} below 1, but when c is larger than c_C even a reduction of x_{12} to 0 is not enough. Then S_C is optimal, in which $x_{12} = 0$ ($x_0 = 1$) and also y_{12}, z_{12} are decreased below 1, with $y_2 = z_1 > 0$. As it is intuitive, for greater c, larger than c_D, in order to keep u_{1E} from being negative it is necessary to reduce both y_{12} and z_{12} to 0 (and $y_2 = 1$, $z_1 = 1$), and even w_{12} is reduced below 1, with $w_1 = w_2 > 0$; this is information structure S_D. In this case bidder 1 is invited to enter if and only if he is strong, and he earns a positive (negative) utility in case bidder 2 stays out (enters), in such a way that he breaks even in expectation.

When $h > c$, an increase in c above c_B makes S_B infeasible because reducing just y_{12}, z_{12} and increasing y_2, z_1 is not enough to satisfy IC_{1E}. For $c > c_B$, it is optimal to reduce also x_{12} is reduced while increasing x_1 and x_2; this delivers the information structure S_F. When $c > c_E$, increasing x_1 and x_2 up to $\frac{1}{2}$ is not enough to satisfy $u_{1E} \ge 0$ and it is necessary to reduce w_{12} below 1, while increasing w_1, w_2. This is information structure S_G, which differs from S_D because $x_0 = 1$ in S_D but $x_1 = x_2 = \frac{1}{2}$ in S_G, as now $h > c$ makes it convenient to induce entry of just one bidder in state ww in order to allow more frequently entry of both bidders in state ss.

The figure below describes the optimal information structure as a function of h and c when $\alpha = \frac{1}{2}$.

Please insert here Figure 2, with the following caption
The optimal information structure as a function of h, c when $\alpha = \frac{1}{2}$

[16] In fact, when $h > c$ a better alternative to S_E is such that $\Delta x_{12} = -\varepsilon$, $\Delta x_1 = \frac{1}{2}\varepsilon = \Delta x_2$ because it induces entry by a single bidder with positive probability, which increases u_{1E} more than S_E. In this case the change in u_{1E} given ε is $\alpha^2(\frac{1}{2}\varepsilon(h-c) - \varepsilon(h - h^2 - c))$, hence $\varepsilon = \frac{2\gamma}{\alpha^2\left(c-h+2h^2\right)}$. This identifies the information structure S^F below, with $\Delta R^F = -\frac{2h^2\Delta}{c-h+2h^2}$, which is greater than ΔR^E, but is still smaller than ΔR^B.

3.3 Comparison with standard information structures

In order to understand better the features of the optimal information structure, it may be useful to compare it with some "standard" information structures already analyzed by the literature in different contexts. In particular, we consider here three information structures examined in Gal-Or et al. (2007) in a procurement context (see also Lu et al. (2018), Serena (2022)), which we call no disclosure (ND), full disclosure (FD), partial disclosure (PD). The differences among these information structures lie in the information each bidder i receives about h_i and about h_j.

The information structure FD (full disclosure) Under FD, the message m_1 to bidder 1 belongs to the set of the states of the world, that is $M_1 \equiv \{(h^w, h^w), (h^w, h^s), (h^s, h^w), (h^s, h^s)\}$, and likewise $m_2 \in M_2 = M_1$. In each state of the world (h_1, h_2), the seller's messages are $m_1 = (h_1, h_2)$, $m_2 = (h_1, h_2)$; hence h_1 and h_2 become common knowledge between the bidders. Bidder i's expected utility in case of entry by both bidders is $h_i(1-h_j) - c$; his expected utility if he enters alone is $h_i - c$. The entry game between the bidders in state (h_1, h_2) is therefore

1\2	E	NE
E	$h_1(1-h_2)-c, h_2(1-h_1)-c$	$h_1-c, 0$
NE	$0, h_2-c$	$0, 0$

and it is useful to keep in mind the following inequalities:

$$h^2 < h(1-h) < \min\{h, (1-h)^2\} \leq \max\{h, (1-h)^2\} < 1-h$$

Proposition 2 (Equilibrium under FD) *Under FD*
(i) if $c \leq h^2$, then in each state of the world (E, E) is the unique equilibrium in the entry game;
(ii) if $h^2 < c \leq h(1-h)$, then in states ww, ss the unique equilibrium is (E, E); in states ws,sw the only equilibrium is such that only the strong bidder enters;
(iii) if $h(1-h) < c \leq \min\{h, (1-h)^2\}$, then in states ww, ss the equilibria are (E, NE) and (NE, E); in states ws, sw the only equilibrium is such that only the strong bidder enters;
(iv) if $h < c \leq (1-h)^2$ (this requires $h < 0.382$), then in state ww the only equilibrium is (NE, NE); in states ws, sw the only equilibrium is such that only the strong bidder enters; in state ss the equilibria are (E, NE) and (NE, E);
(v) if $(1-h)^2 < c \leq h$ (this requires $0.382 < h$), then in each state the equilibria are (E, NE) and (NE, E);
(vi) if $\max\{h, (1-h)^2\} < c \leq 1-h$, then in state ww the unique equilibrium is (NE, NE); in states ws, sw the only equilibrium is such that only the strong bidder enters; in state ss the equilibria are (E, NE) and (NE, E).

It is especially relevant for us that if $c > h(1-h)$, then in no state of the world (E, E) is an equilibrium of the entry game, and therefore the revenue is zero. The intuition for this result is that under FD the bidders' strength in terms of h_1, h_2 is common knowledge among bidders, and $c > h(1-h)$ makes it unprofitable that both bidders enter when $h_1 = h_2 = h$ and when $h_1 = h_2 = 1-h$. When instead $h_1 \neq h_2$, the weak bidder's expected utility under entry of both bidders is h^2, smaller than $h(1-h)$, thus $c > h(1-h)$ implies $c > h^2$ and no state of the world exists such that both bidders enter. Therefore R^{FD}, the revenue under FD, is equal to zero for each $c > h(1-h)$.

The information structure ND (no disclosure) Under ND, the bidders receive no information from the seller, that is $M_1 = M_2 = \{m\}$ for an arbitrary m and $m_1 = m$, $m_2 = m$ in each state of the world. Then bidder i's expected utility if both bidders enter is the expectation of $h_i(1-h_j)-c$, which is $h^e(1-h^e)-c$, or c_A-c. If instead only bidder i enters, then his utility is h^e-c. Thus the entry game is

$1\backslash 2$	E	NE
E	c_A-c, c_A-c	$h^e-c, 0$
NE	$0, h^e-c$	$0, 0$

Proposition 3 (Equilibrium under ND) *Under ND*
(i) if $c \leq c_A$, then the unique equilibrium is (E, E);
(ii) if $c_A < c \leq h^e$, then the equilibria are (E, NE) and (NE, E);
(iii) if $h^e \leq c$, then the unique equilibrium is (NE, NE).

Proposition 3(ii) establishes that ND generates the same outcome as S_A when $c \leq c_A$ – because ND provides the same information to bidders as S_A – but generates zero revenue, that is $R^{ND}=0$, if $c > c_A$.

The information structure PD (partial disclosure) Under PD, each bidder i learns h_i but not h_j, that is $M_1 = M_2 = \{h^w, h^s\}$ and in each state of the world (h_1, h_2) the messages are $m_1 = h_1$, $m_2 = h_2$. Entering by a weak bidder is less frequent than under ND because if bidder i learns that h_i is equal to h^w then he is less willing to enter than if he has no information on h_i and relies on h^e. However, PD induces more entry by a bidder i who learns $h_i = h^s$ than when bidder i relies on h^e, as next proposition establishes. In order to describe the equilibrium under PD, we use $\beta = \alpha + (1-\alpha)h$ to denote the probability for an entrant bidder to face no opponent in the SPA or an opponent with value 0, assuming that the other bidder stays out if weak, enters if strong.

Proposition 4 (Equilibrium under PD) *Let $\beta = \alpha + (1-\alpha)h$. The unique symmetric equilibrium under PD is as follows:*

(i) if $c \leq h(1-h^e)$, then each bidder i enters for each h_i;
(ii) if $h(1-h^e) < c \leq h\beta$, then each bidder i enters if $h_i = h^s$, enters with probability $e_w = \frac{h\beta - c}{h^2\alpha}$ if $h_i = h^w$;
(iii) if $h\beta < c \leq (1-h)\beta$, then each bidder i enters if $h_i = h^s$, does not enter if $h_i = h^w$;
(iv) if $(1-h)\beta < c \leq 1-h$, then each bidder i enters with probability $e_s = \frac{1-c-h}{(1-\alpha)(1-h)^2}$ if $h_i = h^s$, does not enter if $h_i = h^w$.

Ranking the revenue from FD, ND, PD The ranking between ND and FD in terms of revenue is unambiguous as $R^{ND} \geq R^{FD}$ for each c. In order to see why, notice that under ND no bidder can condition entry on his signal, therefore either both types w and s of a bidder enter, or both stay out depending on the average utility from entering. Assuming that one bidder enters, Such an average utility for the bidders is positive if and only if $c < c_A$, hence this inequality is the condition under which entry of both bidders occurs under ND. Conversely, under FD each bidder observes h_1, h_2 and can condition his entry on this information. In states sw or ws, if the strong bidder enters then the weak bidder wants to enter if and only if $c < h^2$, and under this condition both bidders enter also in states ss and ww, hence entry of both bidders occurs under FD if and only if $c \leq h^2$. Since $h^2 < c_A$, the inequality $R^{ND} \geq R^{FD}$ holds for each c smaller than c_A.

In fact, $R^{ND} \geq R^{PD}$ holds also for each $c < c_A$ since ND maximizes the revenue as it induces entry of both bidders in each state of the world. If instead $c > c_A$, then the average utility does not allow the entry of two bidders and Proposition 3(ii) reveals $R^{ND} = 0$, but also R^{FD} is zero as Proposition 2 implies that under FD there is no state of the world in which both bidders enter since $h(1-h) < c_A$. However, PD is superior to both FD and PD when $c > c_A$ because by Proposition 4 for each bidder type s enters with positive probability $e_s \in (0,1]$ for each $c < 1-h$, hence entry of both bidders occurs under PD with probability at least $\alpha^2 e_s^2$ which is positive for each $c < 1-h$. This makes R^{PD} positive. Indeed, it is intuitive that some entry occurs for each $c < 1-h$, as if there was an equilibrium in which no type of a bidder i enters, then entering would be a best reply for type s of bidder j as long as $1-h > c$ since he expects to face no competition in the auction.

The following proposition summarizes the revenue comparison above: It is profitable to give no information when the conditions favor entry (c is small), in order to avoid that weak bidders stay out. But it is better to give more information when conditions are unfavorable for entry (c is large) in order to induce entry at least of strong bidders.

Proposition 5 (Ranking the revenue for R^{FD}, R^{ND}, R^{PD}) $R^{ND} \geq \max\{R^{FD}, R^{PD}\}$ *if* $c \leq c_A$*;* $R^{PD} > R^{ND} = R^{FD} = 0$ *for each* c *between* c_A *and* $1-h$*.*

Direct information structures and FD, ND, PD From Proposition 2 in Taneva (2019) we know that for each of the information structures FD, ND, PD there exists a direct information structure which delivers the same outcome. Hence, for each parameter values our Proposition 1 determines on information structure which is (weakly) superior to the best information structure among FD, ND, PD. In particular, for $c \leq c_A$, Proposition 1 identifies S_A as the optimal information structure, which essentially coincides with ND and generates the same revenue.

However, for $c > c_A$ the optimal information structure in Proposition 1 is different from PD determined by Proposition 5and generates a higher revenue. For instance, it is especially simple to see why this result holds in the case covered by Proposition 4(iii), which is such that under PD bidder i enters if and only if $h_i = h^s$, hence entry of both bidders occurs only in state ss. Bidder i with type s earns a positive utility if he enters alone, a negative utility if also the other bidder enters, but in expectation his utility is positive as $(1-h)\beta > c$. The optimal information structure improves upon PD by inducing more often entry of both bidders in order to increase the revenue. This increases the probability of a negative utility for each bidder, but in such a way to satisfy IC_{1E}. For instance, consider parameters such that Proposition 1(iii) and Proposition 4(iii) apply (for instance, $h = \frac{1}{4}$, $\alpha = \frac{1}{2}$, $c = \frac{1}{3}$) so that S_C is the optimal information structure and PD is the best alternative among FD, ND, PD. Then in S_C, in state ww no bidder enters (that is, $x_0 = 1$) – this is the same as in PD; in state ss, both bidders enter (that is $w_{12} = 1$) – this is the same as in PD; in states ws and sw, only the strong bidder enters with a positive probability $y_2 = z_1 \in (0,1)$, but both bidders enter with complementary probability $y_{12} = z_{12} = 1 - y_2$, whereas under PD only the strong bidder enters with probability 1. The latter difference is what makes S_C superior to PD as the revenue is the same in the states ww, ss, but is higher under S_C in the states ws, sw since both bidders enter with positive probability. Precisely, under PD the revenue is zero in states sw, ws as only one bidder enters, who earns a positive rent. Conversely, S_C makes the strong bidder enter with probability less than one, which produces a rent to the bidder, but also makes both bidders enter with positive probability, which produces a positive revenue. Even if the bidders have a negative utility in the latter case, the probabilities of entry are such that the rent compensates the loss and the bidder breaks even overall.[17]

As another example, consider the case of c slightly smaller than $1-h$. Under PD, Proposition 4(iv) applies and a weak bidder does not enter, a strong bidder enters with a probability $e_s \in (0,1)$. The optimal information structure in this case in S_D, which in states ws, sw induces entry of the strong bidder with probability 1 rather

[17] An even stronger example is the one in which Proposition 1(i) and 4(iii) apply (for instance, $h = \frac{1}{4}$, $\alpha = \frac{1}{2}$, $c = \frac{1}{5}$), so that only the strong bidder enters under PD, but the optimal information structure is such that both bidders enter in each state of the world.

than e_s; this contributes to satisfy IC_{1E} and allows to induce entry of both bidders in state ss with probability greater than e_s^2.

3.4 Public messages

Here we consider information structures in which the seller sends public messages, that is there is a set M such that the message m_1 to bidder 1 is in M, the message m_2 to bidder 2 is in M, and the two messages are the same, that is $m_1 = m_2$, in each state of the world. For instance, FD, ND are information structures in which the seller sends public messages. We want to compare information structures with public messages with the general information structures examined in Subsections 3.1, 3.2.

Proposition 2 in Taneva (2019) implies that for each information structure with public messages there exists a direct information structure with private messages which leads to the same outcome as the original information structure. Hence, whatever revenue the seller may earn through an information structure with public messages, she can earn the same revenue through an information structure with private messages like the ones considered in Subsections 3.1, 3.2. Our Proposition 1 determines the best information structure among all the information structures with private messages, which delivers the following result, essentially a corollary of Proposition 2 in Taneva (2019).

Proposition 6 (Public messages 1) *For any given parameter values, no information structure with public messages is superior to the information structure with private messages identified by Proposition 1.*

Notwithstanding Proposition 6, we think it is interesting to characterize the set of direct information structures which can be derived from information structures with public messages, because that clarifies how public messages restrict the space of information structures available to the seller with respect to the unconstrained set of problem (3).

In the proof of Proposition 7 we show that for each public message, the entry game played by the bidders has at least one pure-strategy equilibrium (although also mixed-strategy equilibria may exist). Then we assume that after each public message, a pure-strategy equilibrium of the entry game is played by the bidders. Hence in the direct information structure which is equivalent to the information structure with public messages, x_{12} coincides with the probability that the seller sends, in state ww, one of the messages after which both bidders enter; x_1 (x_2, x_0) is the probability that the seller sends a message, in state ww, after which only bidder 1 enters (only bidder 2 enters, no bidder enters). The variables $y_0, y_1, y_2, ..., w_1, w_2, w_{12}$ are determined likewise.

Therefore $x_{12}, y_{12}, z_{12}, w_{12}$ need to satisfy the following inequalities (5) (about bidder 1) and (6) (about bidder 2) establishing that after receiving a message which is supposed to induce both bidders to enter, (E, E) is indeed an equilibrium in the entry game:

$$\alpha^2 x_{12}(h(1-h)-c) + \alpha(1-\alpha)y_{12}(h^2-c) + \alpha(1-\alpha)z_{12}((1-h)^2-c) + (1-\alpha)^2 w_{12}(h(1-h)-c) \geq 0 \quad (5)$$
$$\alpha^2 x_{12}(h(1-h)-c) + \alpha(1-\alpha)y_{12}((1-h)^2-c) + \alpha(1-\alpha)z_{12}(h^2-c) + (1-\alpha)^2 w_{12}(h(1-h)-c) \geq 0 \quad (6)$$

Likewise,

$$\alpha^2 x_1(h-c) + \alpha(1-\alpha)y_1(h-c) + \alpha(1-\alpha)z_1(1-h-c) + (1-\alpha)^2 w_1(1-h-c) \geq 0 \quad (7)$$
$$\alpha^2 x_1(h(1-h)-c) + \alpha(1-\alpha)y_1((1-h)^2-c) + \alpha(1-\alpha)z_1(h^2-c) + (1-\alpha)^2 w_1(h(1-h)-c) \leq 0 \quad (8)$$

are the incentive constraints for the case in which the seller sends a message after which only bidder 1 is supposed to enter and

$$\alpha^2 x_2(h(1-h)-c)+\alpha(1-\alpha)y_2((1-h)^2-c)+\alpha(1-\alpha)z_2(h^2-c)+(1-\alpha)^2 w_2(h(1-h)-c) \leq 0 \quad (9)$$

$$\alpha^2 x_2(h-c)+\alpha(1-\alpha)y_2(1-h-c)+\alpha(1-\alpha)z_2(h-c)+(1-\alpha)^2 w_2(1-h-c) \geq 0 \quad (10)$$

are the incentive constraints for the case the seller sends a message after which only bidder 2 is supposed to enter (for brevity, we skip the case of messages after which no bidder is supposed to enter).

It is immediate that the sum of the two left hand sides in (5), (7) is equal to u_{1E} in (1). Hence, when (5), (7) are satisfied it follows that $u_{1E} \geq 0$ holds. But if $u_{1E} \geq 0$, then it is not necessarily the case that both (5), (7) hold. A very similar argument applies to the two left hand sides in (6), (10) and u_{2E}.[18]

This reveals that information structures with public messages need to satisfy more restrictive constraints with respect to IC_{1E} and IC_{2E}. In particular (5) and (7) require that when $m_1 = E$, entry is a best reply for bidder 1 both if $m_2 = E$ and if $m_2 = NE$, and (6), (10) have a similar interpretation for bidder 2. In IC_{1E}, conversely, entry of bidder 1 when bidder 2 does not enter is more likely to generate a positive utility for bidder 1 than when also bidder 2 enters and the former utility compensates the latter even though the latter is negative, for suitable $x_1, x_{12}, y_1, y_{12}, z_1, z_{12}, w_1, w_{12}$. This compensation does not apply when (5) and (7) both need to hold

Even with public messages the seller may restrict to symmetric information structures without loss of generality, hence y_{12} can be set equal to z_{12}. It is still optimal to let all bidders enter, that is $x_{12} = y_{12} = z_{12} = w_{12} = 1$, if $c \leq h^e(1-h^e)$. If instead c is slightly greater than $h^e(1-h^e)$ then some of these variables must be reduces below 1 to satisfy (5), (6), starting with x_{12} which delivers the lower utility to bidders and the lower revenue to the seller. For brevity we do not provide the complete solution to the maximization problem, but it is immediate to see that if $c > \frac{1}{2}h^2 + \frac{1}{2}(1-h)^2$, then the coefficients of x_{12}, of $y_{12} = z_{12}$, of w_{12} in (5), (6) are all negative and these constraints can be satisfied only by $x_{12} = y_{12} = z_{12} = w_{12} = 0$. Therefore under public messages that the revenue is zero for each $c > \frac{1}{2}h^2 + \frac{1}{2}(1-h)^2$, whereas with unrestricted messages the revenue is positive for each $c < 1-h$.

Proposition 7 (Public messages 2) *Under public messages, the seller's revenue coincides with the revenue under private messages if $c \leq h^e(1-h^e)$, but is strictly lower if $c > h^e(1-h^e)$, and in particular is zero if $c > \frac{1}{2}h^2 + \frac{1}{2}(1-h)^2$.*

3.5 First price auction

We have assumed up to now that the seller offers her object through a SPA, but in this subsection we suppose the seller uses a first price auction (FPA in the following) and that if a bidder enters the FPA, then before he bids he observes whether the other bidder has entered or not. In the FPA, bidding is not as straightforward as in the SPA, but the following lemma identifies a Bayes-Nash Equilibrium for the FPA given the following arbitrary probability distribution over (v_1, v_2), in which $\gamma, \theta, \tau, \delta$ are arbitrary non negative numbers which add up to 1:[19]

$$\begin{array}{ccc} v_1 \backslash v_2 & 0 & 1 \\ 0 & \gamma & \theta \\ 1 & \tau & \delta \end{array} \quad (11)$$

[18] An analogous link exists for the constraints about non-entry.

[19] We consider a FPA with the "Vickrey tie-breaking rule" introduced by Maskin and Riley (2000) which has the consequence that if the two sellers submit the same bid, then the seller with the higher value wins and pays the other bidder's value. See Maskin and Riley for additional details on this tie-breaking rule.

Lemma 1 (Equilibrium in the FPA) *Given the probability distribution over* (v_1, v_2) *in (11), the following strategy profile is a Bayes-Nash Equilibrium in the FPA.*

(i) Each bidder with type 0 *bids 0.*

(ii) Type 1 of bidder i*, for* $i = 1, 2$*, bids according to a mixed strategy with support* $[0, \frac{\delta}{m+\delta}]$ *– in which* $m = \max\{\tau, \theta\}$ *– and c.d.f.* G_i *such that* $G_1(0) > 0 = G_2(0)$ *if* $\tau > \theta$*,* $G_1(0) = G_2(0) = 0$ *if* $\tau = \theta$*,* $G_1(0) = 0 < G_2(0)$ *if* $\tau < \theta$*. For each type* 1*, the equilibrium expected utility is* $\frac{m}{m+\delta}$*.*

We derive now the incentive constraint for bidder 1 after he has received message $m_1 = E$. First notice that $m_1 = E$ is consistent with $m_2 = E$ and with $m_2 = NE$, and bidder 1 does not observe m_2. However, upon entering the auction bidder 1 learns whether bidder 2 has entered or not, thus concludes that $m_2 = E$ (that $m_2 = NE$) if bidder 2 has entered (if bidder 2 has not entered). In the first case, bidder 1 can win the good by bidding zero (or an arbitrarily small positive bid). Therefore the expected utility from entering is as follows, in which $p_{1E} = \alpha^2(x_1 + x_{12}) + \alpha(1-\alpha)(y_1 + y_{12}) + (1-\alpha)\alpha(z_1 + z_{12}) + (1-\alpha)^2(w_1 + w_{12})$ is the probability that 1 receives message E, $p_{E,NE} = \alpha^2 x_1 + \alpha(1-\alpha)y_1 + (1-\alpha)\alpha z_1 + (1-\alpha)^2 w_1$ is the probability of $(m_1, m_2) = (E, NE)$, $p_{E,E} = \alpha^2 x_{12} + \alpha(1-\alpha)y_{12} + (1-\alpha)\alpha z_{12} + (1-\alpha)^2 w_{12}$ is the probability of $(m_1, m_2) = (E, E)$, and u_{1EE} is bidder 1's utility in the FPA given $(m_1, m_2) = (E, E)$:

$$\frac{1}{p_{1E}}\left[p_{E,NE}\left(h^w \Pr\{h_1 = h^w|(m_1, m_2) = (E, NE)\} + h^s \Pr\{h_1 = h^s|(m_1, m_2) = (E, NE)\} - c\right) + p_{EE}\left(u_{1EE} - c\right)\right] \tag{12}$$

In order to determine u_{1EE}, we need to determine the probabilities in (11) given $m_1 = m_2 = E$, which yields

$$\gamma = \frac{1}{p_{E,E}}\hat{\gamma} \text{ with } \hat{\gamma} = \alpha^2(1-h^w)^2 x_{12} + \alpha(1-\alpha)(1-h^w)(1-h^s)y_{12} + \alpha(1-\alpha)(1-h^s)(1-h^w)z_{12} + (1-\alpha)^2(1-h^s)^2(w$$

$$\theta = \frac{1}{p_{E,E}}\hat{\theta} \text{ with } \hat{\theta} = \alpha^2(1-h^w)h^w x_{12} + \alpha(1-\alpha)(1-h^w)h^s y_{12} + \alpha(1-\alpha)(1-h^s)h^w z_{12} + (1-\alpha)^2(1-h^s)h^s w_{12}$$

$$\tau = \frac{1}{p_{E,E}}\hat{\tau} \text{ with } \hat{\theta} = \alpha^2 h^w(1-h^w)x_{12} + \alpha(1-\alpha)h^w(1-h^s)y_{12} + \alpha(1-\alpha)h^s(1-h^w)z_{12} + (1-\alpha)^2 h^s(1-h^s)w_{12}$$

$$\delta = \frac{1}{p_{E,E}}\hat{\delta} \text{ with } \hat{\delta} = \alpha^2(h^w)^2 x_{12} + \alpha(1-\alpha)h^w h^s y_{12} + \alpha(1-\alpha)h^s h^w z_{12} + (1-\alpha)^2(h^s)^2 w_{12}$$

Using $\Pr\{h_1 = h^w|(m_1, m_2) = (E, NE)\} = \alpha^2 x_1 + \alpha(1-\alpha)y_1$ and $\Pr\{h_1 = h^s|(m_1, m_2) = (E, NE)\} = \alpha(1-\alpha)z_1 + (1-\alpha)^2 w_1$ we find that (12) reduces to

$$\frac{1}{p_{1E}}\left[(h^w - c)(\alpha^2 x_1 + \alpha(1-\alpha)y_1) + (h^s - c)(\alpha(1-\alpha)z_1 + (1-\alpha)^2 w_1) + p_{E,E}((\tau+\delta)\frac{m}{m+\delta} - c)\right] \geq 0 \tag{17}$$

The incentive constraint for bidder 2 in case of $m_2 = E$ is obtained likewise.

The revenue is zero unless both bidders enter, hence it coincides with the probability that both bidders enter, times the revenue when both bidders participate in the FPA. The latter is equal to the expected social surplus (that is, the probability that $\max\{v_1, v_2\}$) minus the bidders' utilities. Hence it is equal to

$$p_{E,E}(\theta + \tau + \delta - (\tau+\delta)\frac{m}{m+\delta} - (\theta+\delta)\frac{m}{m+\delta})$$

Next proposition establishes that by using the FPA, the seller can earn the same revenue as when using the SPA.

Proposition 8 (Revenue with the FPA) *Given any parameters* h, α, c*, let* S^* *denote the optimal information structure identified by Proposition 1 for the case in which the seller uses the SPA. Then* S^* *yields the same revenue if the seller uses the FPA.*

The argument for this proposition is very simple, as each information structure identified by Proposition 1 is such that $y_{12} = z_{12}$. From (14), (15) we see that this implies $\tau = \theta = m$, hence $p_{E,E}((\tau + \delta)\frac{m}{m+\delta} - c)$ in (17) is equal to $p_{E,E}(\tau - c) = \alpha^2 x_{12}(h^w(1-h^w) - c) + \alpha(1-\alpha)y_{12}(h^w(1-h^s) - c) + \alpha(1-\alpha)(h^s(1-h^w) - c)z_{12} + (1-\alpha)^2 w_{12}(h^s(1-h^s) - c)$. As a result, the left hand side in (17) has the same sign as u_{1E} in (1), and since $u_{1E} \geq 0$ at S^* it follows that (17) is satisfied. Moreover, equalities $\tau = \theta = m$ imply that the revenue reduces to

$$\begin{aligned} & \hat{\theta} + \hat{\tau} + \hat{\delta} - p_{E,E}\tau - p_{E,E}\theta \\ = \quad & \hat{\delta} = \alpha^2(h^w)^2 x_{12} + \alpha(1-\alpha)h^w h^s y_{12} + \alpha(1-\alpha)h^s h^w z_{12} + (1-\alpha)^2(h^s)^2 w_{12} \end{aligned}$$

which is just the revenue in the SPA in (2). Therefore, when the seller uses the FPA, S^* satisfies the constraints and yields the same revenue as in the SPA.

4 Entry fee

In this section we suppose that the seller can use an entry fee $f \geq 0$. From a bidder's perspective, $f > 0$ is equivalent to an increase in the entry cost and makes entry less profitable. In particular, f needs to satisfy $c + f \leq 1 - h$, otherwise no bidder will enter. From the point of view of the seller, $f > 0$ increases the revenue when both bidders enter from $h_1 h_2$ to $2f + h_1 h_2$, and from 0 to f when a single bidder enters; in particular, the revenue is positive even if just one bidder enters. We first examine the effect of entry fee on the information structures FD, ND, PD, and then consider the set of unrestricted information structures.

4.1 Entry fee for FD, ND, PD

It is immediate to derive results about the equilibrium when $f > 0$ under FD, ND, PD as it suffices to replace c with $c + f$ in Propositions 2, 3, 4. Moreover, the new revenue is obtained by adding $2f$ to the old revenue when both bidders enter, by adding f when only one bidder enters.

Proposition 9 *(i) Under FD*
(ia) If $c + f \leq h^2$, *then in each state of the world the unique equilibrium in the auction is* (E, E); $R^{FD} = 2f + (h^e)^2$.
(ib) If $h^2 < c + f \leq h(1-h)$, *then in states* ww, ss *the unique equilibrium is* (E, E); *in states* ws,sw *the only equilibrium is such that only the strong bidder enters;* $R^{FD} = \alpha^2(2f + h^2) + (1-\alpha)^2(2f + (1-h)^2) + 2\alpha(1-\alpha)f$.
(ic) If $h(1-h) < c + f \leq \min\{h, (1-h)^2\}$, *then in states* ww, ss *the equilibria are* (E, NE) *and* (NE, E); *in states* ws, sw *the only equilibrium is such that only the strong bidder enters;* $R^{FD} = f$.
(id) If $h < c + f \leq (1-h)^2$ *(this requires* $h < 0.382$*), then in state* ww *the only equilibrium is* (NE, NE); *in states* ws, sw *the only equilibrium is such that only the strong bidder enters; in state* ss *the equilibria are* (E, NE) *and* (NE, E); $R^{FD} = (1-\alpha^2)f$.
(ie) If $(1-h)^2 < c + f \leq h$ *(this requires* $0.382 < h$*), then in each state the equilibria are* (E, NE) *and* (NE, E); $R^{FD} = f$.
(if) If $\max\{h, (1-h)^2\} < c + f \leq 1 - h$, *then in state* ww *the unique equilibrium is* (NE, NE); *in states* ws, sw, *the only equilibrium is such that only the strong bidder enters; in state* ss *the equilibria are* (E, NE) *and* (NE, E); $R^{FD} = (1-\alpha^2)f$.
(ii) Under ND
(iia) if $c + f \leq h^e(1-h^e)$, *then the unique equilibrium is* (E, E); $R^{ND} = 2f + (h^e)^2$.

(iib) if $h^e(1-h^e) < c+f \leq h^e$, then the equilibria are (E, NE) and (NE, E); $R^{ND} = f$.
(iic) if $h^e < c+f$, then the unique equilibrium is (NE, NE); $R^{ND} = 0$.

(iii) Under PD, let $\beta = \alpha + (1-\alpha)h$. Then
(iiia) if $c+f \leq h(1-h^e)$, then the unique equilibrium is such that each type of each bidder enters; $R^{PD} = 2f + (h^e)^2$.
(iiib) if $h(1-h^e) < c+f \leq h\beta$, then the unique equilibrium is that each type s enters, each type w enters with probability $e_w = \frac{h\beta - c - f}{h^2\alpha}$; $R^{PD} = 2(1-\alpha+\alpha e_w) f + (1-h-\alpha+h\alpha+h\alpha e_w)^2$.
(iiic) if $h\beta < c+f \leq (1-h)\beta$, then the unique equilibrium is such that each type s enters, each type w does not; $R^{PD} = 2(1-\alpha) f + (1-\alpha)^2(1-h)^2$.
(iiid) if $(1-h)\beta < c+f \leq 1-h$, then the unique equilibrium is such that each type s enters with probability $e_s = \frac{1-c-f-h}{(1-\alpha)(1-h)^2}$, each type w does not enter; $R^{PD} = 2(1-\alpha)e_s f + (1-\alpha)^2 e_s^2 (1-h)^2$.

The comparison between R^{FD} and R^{ND} is straightforward. In particular, $R^{ND} \geq R^{FD}$ when ND induces entry of both bidders, an event which occurs more frequently under ND than under FD (see Proposition 9(ia,iia)) for the same reason which applies when $f = 0$: see Subsection 3.3 – precisely, both bidders enter under ND if and only if $c+f \leq h^e(1-h^e)$. When instead $h^e(1-h^e) < c+f \leq h^e$, a single bidder enters under ND but still $R^{ND} \geq R^{FD}$ as under FD entry is not larger, in each state of the world: see Proposition 9(ic-f). Conversely, when $h^e < c+f \leq 1-h$ no bidder enters under ND but FD induces entry of one (strong) bidder in states ws, sw, ss, thus $R^{FD} = (1-\alpha^2)f > R^{ND} = 0$. As a result, ND is (at least weakly) superior to FD if $c+f \leq h^e$ but the opposite result holds if $h^e < c+f$.

This is analogous to the principle which applies when $f = 0$. Under favorable entry conditions, that is when $c+f$ is not large, ND is superior as it induces entry of all bidder types on bidders' expected utility. When instead $c+f$ is large, entry is unprofitable in expectation – hence $R^{ND} = 0$ – but is profitable for a single bidder in some states of the world and FD produces a positive revenue in such states because $f > 0$.

When we consider PD, in order to limit the number of possible cases we suppose $\alpha = \frac{1}{2}$, which implies

$$h(1-h^e) < h^e(1-h^e) < h\beta < (1-h)\beta < h^e$$

Of course, $R^{ND} \geq R^{PD}$ when $c+f \leq h^e(1-h^e)$, since then both bidders enter under ND. But if $c+f$ is between $h^e(1-h^e)$ and $h\beta$ then $R^{PD} > R^{ND} = f$ because under ND a single bidder enters, whereas under PD on average more than one bidder enters see Proposition 9(iiib).[20] In fact, $R^{PD} > R^{ND}$ also if $h\beta < c+f \leq (1-h)\beta$ because in this case Proposition 9(iiic) applies and only type s of bidder enters, hence exactly one bidder enters in average, but in state ss the expected sale price in the SPA is positive because both bidders enter in such state. When $(1-h)\beta < c+f$, R^{PD} remains higher than R^{FD} and R^{ND} as long as $c+f$ is not too larger than $(1-h)\beta$ and h is small, as then type s enters with probability less than 1, but a small h implies a high expected sale price in the SPA in the state ss. However, if $c+f$ is large then the entry probability of each type s is small and R^{PD} is close to zero, whereas under FD one bidder enters in states ws, sw, ss and $R^{FD} > 0$.

Proposition 10 (Comparison among FD, ND, PD for a given $f > 0$) *Suppose that $\alpha = \frac{1}{2}$. Then*
(i) $R^{ND} \geq \max\{R^{FD}, R^{PD}\}$ for each $c+f \leq c_A$;
(ii) $R^{PD} > \max\{R^{ND}, R^{FD}\}$ for each $c+f$ between c_A and $(1-h)\beta$;
(iii) $R^{PD} > \max\{R^{ND}, R^{FD}\}$ for each $c+f$ between $(1-h)\beta$ and h^e if $h \leq \frac{1}{4}$, but if $h > \frac{1}{4}$ then $R^{ND} > R^{PD}$ may hold;

[20] This conclusion does not necessarily hold if α is greater than $\frac{1}{2}$, as then less than one bidder enters in expectation.

(iv) When $h^e < c+f < 1-h$, the inequality $R^{PD} > \max\{R^{ND}, R^{FD}\}$ holds if $c+f$ is close to h^e and $h \leq \frac{9}{25}$, but $R^{FD} > R^{PD}$ if $c+f$ is close to $1-h$.

Figure 3 below shows the plots of R^{FD}, R^{ND}, R^{PD} as a function of f for a special case in which $h > \frac{9}{25}$.

Please insert here Figure 3, with the following caption
R^{FD} (green), R^{ND} (black), R^{PD} (blue) as a function of f, when $\alpha = \frac{1}{2}$, $h = \frac{2}{5}$, $c = \frac{15}{100}$.

Proposition 10 compares FD, ND, PD *for given* f, but in fact the seller *can choose* f, hence it is interesting to compare the three information structures when the seller selects f to maximize her revenue. We use R^{ND*} to denote the value of R^{ND} at the f which is optimal under ND and define R^{FD*}, R^{PD*} likewise.

Before stating the main result, we introduce on a few simple remarks which reduce the cases to consider to a small set.

- About R^{ND*}: Proposition 9(ii) shows that if $c \leq h^e(1-h^e)$, then

$$f_1 = h^e(1-h^e) - c, \qquad f_2 = h^e - c$$

  are both local max points for R^{ND}. But it is readily seen that f_1 is the global maximum point and $R^{ND*} = 2h^e - (h^e)^2 - 2c$; (ii) if $h^e(1-h^e) < c \leq h^e$, then f_2 is the global maximum point for R^{ND} and $R^{ND*} = h^e - c$; (iii) if $h^e < c$, then $R^{ND} = 0$ for each f and $R^{ND*} = 0$.

- About R^{PD*}: Proposition 9(iii) reveals that if $c \leq h(1-h^e)$, then R^{PD} is increasing for $f < h(1-h^e) - c$, is decreasing for f between $h(1-h^e) - c$ and $h\beta - c$, is increasing for f between $h\beta - c$ and $(1-h)\beta - c$, is decreasing for f between $(1-h)\beta - c$ and $1-h$. Hence

$$f_3 = h(1-h^e) - c, \qquad f_4 = (1-h)\beta - c$$

  are both local max points for R^{PD}. However, since $f_3 < f_1$ it follows that R^{PD} when $f = f_3$, equal to $2f_3 + (h^e)^2$, is smaller than R^{ND} when $f = f_1$, equal to $2f_1 + (h^e)^2$. Hence PD with $f = f_3$ is never optimal as ND with $f = f_1$ is superior. Therefore $f = f_4$ if PD is optimal and then $R^{PD*} = 2(1-\alpha)f_4 + (1-\alpha)^2(1-h)^2$.

- About R^{FD*}: Proposition 9(i) shows that determining R^{FD*} requires to consider various cases, but it is useful to notice that if $R^{FD*} > \max\{R^{ND*}, R^{PD*}\}$, then $f = 1-h-c$ and $R^{FD*} = (1-\alpha^2)(1-h-c)$. Precisely, Proposition 9(i) shows that (i) $R^{FD} > (1-\alpha^2)f$ requires $c+f \leq h$, and then we prove as follows that FD is suboptimal: (i) in case that $h(1-h) < c+f \leq h$, Proposition 9(ic,e) shows that $f = R^{FD}$ but $f \leq R^{ND*}$ by Proposition 9(iia,b); (ii) in case of $c+f \leq h(1-h)$ we have $c \leq h(1-h) < h^e(1-h^e)$, thus $R^{FD} < 2h^e - (h^e)^2 - 2c = R^{ND*}$.

As a result, in order to determine jointly the optimal information structure among FD, ND, PD and the optimal f it suffices to compare the following four alternatives: ND with f_1, ND with f_2, PD with f_4, FD with $1-h-c$.

Proposition 11 (Comparison among FD, ND, PD with optimal f) *(i) Suppose that $2+3h\alpha > 4h+2\alpha$, that is neither h nor α is large. Then the optimal information structure and f are*
ND with $f = f_1$ if $c \leq h(h+\alpha-\frac{3}{2}h\alpha)$,
PD with $f = f_4$ if c is between $h(h+\alpha-\frac{3}{2}h\alpha)$ and $h(1-h)$,
FD with $f = 1-h-c$ if c is between $h(1-h)$ and $1-h$.

(ii) Suppose that $2+3h\alpha \leq 4h+2\alpha$. Then there exists $\bar{c} \in (0, h^e)$ such that the optimal information structure and f is
ND with $f = f_1$ if $c < \min\{c_A, \bar{c}\}$, ND with $f = f_2$ if $c_A < c \leq \bar{c}$,,
FD with $f = 1 - h - c$ if $c \in (\bar{c}, 1 - h)$.

In both cases covered by Proposition 11, ND is optimal for c small and $f = f_1$, FD is optimal if c is large, with $f = 1 - h - c$. The first result holds because for small c, all types have incentive to enter and ND allows the seller to extract all the bidders surplus based on the expected utility by charging the entry fee f_1. This is not doable if each bidder i knows h_i, as then in order to induce full entry f is constrained by the utility of type w. The second result follows since entry is unlikely when c is large and in particular no bidder enters under ND, only strong types enter under PD but with probability close to zero; however, under FD a bidder has incentive to enter in state ws, sw, ss, thus the revenue coincides with the entry fee paid by the bidder and is higher than under the alternatives. However, when $2 + 3h\alpha > 4h + 2\alpha$, that is when α and/or h is small, are small and c is intermediate, PD with $f = f_4$ is optimal. This extracts the surplus from each type s (which leaving out type w). That allows to charge a higher fee than f_1 under ND, and even though no type w pays the fee, the probability of type w is close to 0 when α is small. The upper bound $h(1 - h)$ on c about the comparison between R^{FD*} and R^{PD*} is due to the fact that f_4 and $1 - h - c$ both decrease with c, but a decrease in f_4 reduces the revenue in PD more than a decrease in $1-h-c$ in ND. The reason is that under ND, f_5 is earned by the seller in state ws, sw,ss, whereas under PD, f_4 is earned by the seller in the same states but is earned twice in in state ss.

One significant difference with respect to the setting with $f = 0$ is that now FD is sometimes optimal, in particular when c is not too small. The reason is that FD induces the entry of a single bidder, who pays the entry fee, more often than PD and than ND, that is even for large c. This occurs because the information FD provides is such that the entering bidder knows he is not facing competition, thus he is willing to enter as long as $c + f < 1 - h$.

4.2 Entry fee and unrestricted information structures

Now we suppose the seller can design general information structures, as in Subsections 3.1 and 3.2. The bidders' incentive constraints are obtained by replacing c with $c + f$, and for instance IC_{1E} is $u^f_{!E} \geq 0$ with

$$\begin{aligned} u^f_{!E} &= \alpha^2 \left(x_1(h-f-c) + x_{12}(h(1-h)-f-c)\right) + \alpha(1-\alpha)\left(y_1(h-f-c) + y_{12}(h^2-f-c)\right) \\ &\quad +\alpha(1-\alpha)\left(z_1(1-h-f-c) + z_{12}((1-h)^2-f-c)\right) + (1-\alpha)^2\left(w_1(1-h-f-c) + w_{12}(h(1-h)-f-c)\right) \end{aligned}$$

and the revenue is equal to

$$\begin{aligned} R^f &= \alpha^2\left(x_{12}(2f+h^2) + x_1 f + x_2 f\right) + \alpha(1-\alpha)\left(y_{12}(2f + h(1-h)) + y_1 f + y_2 f\right) \\ &\quad +(1-\alpha)\alpha\left(z_{12}(2f + (1-h)h) + z_1 f + z_2 f\right) + (1-\alpha)^2\left(w_{12}(2f + (1-h)^2) + w_1 f + w_2 f\right) \end{aligned}$$

In this context we denote information structures using S^f and it turns out that eleven different information structures may be optimal depending on the parameters:

$$\begin{cases} S^f_A : x_{12} = 1,\ y_{12} = 1,\ z_{12} = 1,\ w_{12} = 1 \\ S^f_B : x_{12} = 1,\ y_2 = \frac{c+f-c_A}{\alpha(1-\alpha)(c+f+h-2h^2)}, y_{12} = 1 - y_2,\ z_1 = y_2, z_{12} = 1 - z_1,\ w_{12} = 1 \\ S^f_C : x_0 = 1,\ y_2 = \frac{(1+\alpha)(c+f-c_C)}{\alpha(c+f+h-2h^2)}, y_{12} = 1 - y_2,\ z_1 = y_2, z_{12} = 1 - z_1,\ w_{12} = 1 \\ S^f_D : x_0 = 1,\ y_2 = 1,\ z_1 = 1,\ w_1 = w_2 = \frac{c+f-c_D}{(1-\alpha)(c+f+(1-2h)(1-h))}, w_{12} = 1 - 2w_1 \\ S^f_E : x_0 = \frac{c+f-c_A}{\alpha^2(c+f-h+h^2)}, x_{12} = 1 - x_0,\ y_{12} = 1,\ z_{12} = 1,\ w_{12} = 1 \\ S^f_F : x_1 = x_2 = \frac{(1-\alpha+\alpha^2)(c+f-c_B)}{\alpha^2(c+f+2h^2-h)}, x_{12} = 1 - 2x_1,\ y_2 = 1,\ z_1 = 1,\ w_{12} = 1 \\ S^f_G : x_1 = x_2 = \frac{1}{2},\ y_2 = 1,\ z_1 = 1,\ w_1 = w_2 = \frac{(\alpha^2-2\alpha+2)(c+f-c_E)}{2(1-\alpha)^2(c+f+(1-2h)(1-h))}, w_{12} = 1 - 2w_1 \\ S^f_H : x_1 = x_2 = \frac{(1-\alpha^2)(1-h-c-f)}{2\alpha^2(c+f-h)}, x_0 = 1 - 2x_1, y_2 = 1, z_1 = 1, w_1 = \frac{1}{2}, w_2 = \frac{1}{2} \\ S^f_I : x_{12} = 1, y_2 = 1, z_1 = 1, w_1 = w_2 = \frac{(1-\alpha+\alpha^2)(c+f-c_B)}{(1-\alpha)^2(c+f+(1-2h)(1-h))}, w_{12} = 1 - 2w_1 \\ S^f_J : x_1 = x_2 = \frac{(\alpha^2+1)(f+c-c_F)}{2\alpha^2(c+f+h(2h-1))}, x_{12} = 1 - 2x_1, y_2 = 1, z_1 = 1, w_1 = w_2 = \frac{1}{2} \\ S^f_K : x_1 = x_2 = \frac{(1-\alpha)(c_D-f-c)}{\alpha^2(c+f-h)},\ x_0 = 1 - 2x_1, y_2 = 1, z_1 = 1, w_{12} = 1 \end{cases} \tag{18}$$

The information structure S^f_A is analogous to S_A when $f = 0$ as it induces entry by both bidders in all states of the world if and only if $c + f \leq c_A$, and is optimal when such inequality holds. Likewise S^f_B, analogous to S_B but requires c between $c_A - f$ and $c_B - f$ rather than between c_A and c_B as when $f = 0$. A similar remark applies to $S^f_C, S^f_D, S^f_E, S^f_F, S^f_G$. It is intuitive that the condition under which, given $x_{12} < 1$, it is optimal to set $x_1 = x_2 > 0$ is not $h > c$ as when $f = 0$, but is a weaker condition as an increase x_1, x_2 affects not just IC_{1E}, but also the revenue and in a positive way. Therefore, given $x_{12} < 1$, when $f > 0$ it is more likely that $x_1 = x_2 > 0$ is optimal rather than when $f = 0$ (equivalently, information structures with $x_0 > 0$ are now less likely to be optimal).

In addition to c_A, c_B, c_C, c_D, c_E identified by Subsection 3.2, we need two further bounds on c, which we denote c_F, c_G:

$$c_F = \frac{(1-h)(2h\alpha^2 - \alpha^2 + 1)}{\alpha^2 + 1}, \quad c_G = 1 + 2h\alpha^2 - h - \alpha^2$$

and then it is possible to state the optimal information structure as a function of h, α, c, f.

Proposition 12 *The optimal information structure when the seller uses an entry fee f is determined as follows;*

(a) S^f_A *if* $c \leq c_A - f$

(b) S^f_B *if* $c_A - f \leq c \leq \min\{c_B - f, h + f\}$

(c) S^f_C *if* $\max\{c_C - f, h + f\} \leq c \leq c_D - f$

(d) S^f_D *if* $\max\{c_D - f, h + \frac{h^2}{(1-h)^2} f\} \leq c \leq 1 - h - f$.

(e) S^f_E *if* $\max\{c_A - f, h + f\} \leq c \leq c_C - f$.

(f) S^f_F *if* $\max\{h - h^2, c_B - f\} \leq c \leq \min\{c_E - f, h + f\}$

(g) S^f_G *if* $\max\{h - h^2, c_E - f\} \leq c \leq \min\{h + \frac{h^2}{(1-h)^2} f, c_G - f\}$.

(h) S^f_H *if* $c_G - f < c \leq h + \frac{h^2}{(1-h)^2} f$.

(i) S^f_I *if* $c_B - f \leq c < \min\{h - h^2, c_F - f\}$ *when the latter is violated, we to go to* S^f_F.

(j) S^f_J *if* $c_F - f \leq c \leq \min\{h - h^2, c_G - f\}$.

(k) S^f_K *if* $\max\{c_E - f, h + \frac{h^2}{(1-h)^2} f\} \leq c \leq \min\{c_D - f, h + f\}$.

Some argument similar to those for Proposition 1 apply in this case. In particular, if $c + f$ is slightly greater than c_A then it is optimal to keep the weak bidder from entering in states ws and sw (information structure S^f_B), but as we mentioned above the upper bound on c is not h but a weaker one to take into account that entry of a single bidder affects not only IC_{1E} but also increases the revenue (if such a condition is violated, then it is best to reduce x_{12} and increase x_0 rather than reducing y_{12}, z_{12}: information structure S_E^f). A further increase in c leads to a reduction y_{12}, z_{12} to 0 (and to set $y_2 = z_1$ to 1), and to reduce also x_{12} or w_{12} and increase x_1, x_2 or w_1, w_2 depending on whether c is larger or smaller than $h - h^2$. When $f = 0$, the condition $c > c_A$ implies $c > h - h^2$, therefore it is more convenient to reduce x_{12} and increase x_1, x_2 (information structure S_F^f) if $f = 0$. But when $f > 0$, it is possible that $c > c_A$ and $c < h - h^2$; in such case w_{12} is optimally set below 1 (information structure S_I^f). As c becomes even greater, the optimal information structure has $x_{12} = 0$ especially if h is small, because then the revenue generated by the entry of both bidders is small. The optimal information structure ends up with only the strong bidder entering in states sw, ws, and either no bidder entering in state ww but at least one bidder entering in state ss (if h is small, information structure S_D^f), or just one bidder entering in state ss and one bidder entering in state ww with positive probability (if h is large, information structure S_H^f).

The figure below describes the optimal information structure as a function of h and c when $\alpha = \frac{1}{2}$ and $f = 0.14$

Please insert here Figure 3, with the following caption
The optimal information structure as a function of h, c when $\alpha = \frac{1}{2}, f = 0.14$

Of course it is interesting to determine the optimal f given h, α, c, but as figure 3 suggests, the expression of the revenue is a quite complicated function of the parameters and of f. Therefore a solution to the maximization problem is not available, except for the following result which is consistent with Proposition 11 in suggesting that the optimal information structure allows entry of a single bidder when c is large.

Proposition 13 *Suppose that $\alpha = \frac{1}{2}$ and $c \geq \frac{1}{2}$. Then the optimal information information structure is S_D^f with $f = 1 - h - c$, which is equivalent to S_H^f with $f = 1 - h - c$ in the sense that in both cases $x_0 = 1$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 = \frac{1}{2}$.*

5 Entry fee and reserve price

Here we suppose that the seller can use an entry fee f and a reserve price r and show that the optimal information structure and the optimal f, r allow the seller to implement the socially efficient entry while extracting all the surplus from the bidders. Therefore the seller essentially achieves the same outcome as if he had complete information. The basic idea behind this result is that by setting r close to 1 and f close to $-c$, the seller subsidizes (about) entirely the bidders' entry costs and leaves (about) no rent to active bidders with value 1. This makes the revenue almost coincide with social welfare, hence the seller's goal in the choosing the information structure consists in maximizing social welfare.

More in detail, we know from Subsection 3.1 that the optimal entry depends on (h, c) as described by Figure 1. Then setting $f = -c + \varepsilon_f$, $r = 1 - \varepsilon_r$, we can find $\varepsilon_f > 0$, $\varepsilon_r > 0$ close to zero such that the incentive constraints are (strictly) satisfied when the information structure maximizes social welfare. As a result, the revenue approximately coincides with social welfare as each bidder i's utility is close to zero: it is $-\varepsilon_f$ if he enters and does not win good 1, or wins the good by paying 1, is $\varepsilon_r - \varepsilon_f$ if he enters and wins the good by paying r. Thus, for each h, α, c it is possible for the seller to earn the full social surplus.

Proposition 14 *If the seller can use both an entry fee f and a reserve price r, then she can choose f, r and the information structure in such a way that the revenue is arbitrarily close to the maximal social welfare.*

The result of Proposition 14 relies on the use of a negative entry fee. This might be in some cases problematic, hence an interesting question is determining the optimal (f, r) in case f needs to be non negative. Moreover, Proposition 14 is a consequence of the particular structure of our setting, such that each bidder's value is either 0 or 1. If each bidder's possible values were each number in the interval $[0, 1]$, then it would be necessary to set $r = 0$ to achieve social efficiency from the point of view of the allocation of the good, but then it would not be possible to choose the entry fee to achieve socially efficient entry and full surplus extraction.

6 Conclusions

In this paper we have examined a setting in which the seller affect the bidders' information before they decide whether to enter an auction or not. In order to extend our results it would be useful to examine the context in which the seller can use a reserve price and not an entry fee. It would also be interesting to allow for more general value distributions than the binary one we have considered, and to allow for more general auctions, beyond FPA and SPA.

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7 Appendix

7.1 Proof of Proposition 1

In case bidder 1 receives message NE, then his expected utility from entering is u_{1NE}/p_{1NE}, with

$$\begin{aligned} u_{1NE} &= \alpha^2 \left(x_2(h^w(1-h^w)-c)+x_0(h^w-c)\right)+\alpha(1-\alpha)\left(y_2(h^w(1-h^s)-c)+y_0(h^w-c)\right) \\ &\quad +\alpha(1-\alpha)\left(z_2(h^s(1-h^w)-c)+z_0(h^s-c)\right)+(1-\alpha)^2\left(w_2(h^s(1-h^s)-c)+w_0(h^s-c)\right) \end{aligned}$$

and $p_{1NE} = \alpha^2(x_0+x_2)+\alpha(1-\alpha)(y_0+y_2)+\alpha(1-\alpha)(z_0+z_2)+(1-\alpha)^2(w_0+w_2)$. The constraint IC_{1NE} requires $\frac{u_{1NE}}{p_{2NE}} \leq 0$, which is equivalent to $u_{1NE} \leq 0$. Likewise, for bidder 2 we define

$$\begin{aligned} u_{2NE} &= \alpha^2 \left(x_1(h^w(1-h^w)-c)+x_0(h^w-c)\right)+\alpha(1-\alpha)\left(y_1(h^s(1-h^w)-c)+y_0(h^s-c)\right) \\ &\quad +\alpha(1-\alpha)\left(z_1(h^w(1-h^s)-c)+z_0(h^w-c)\right)+(1-\alpha)^2\left(w_1(h^s(1-h^s)-c)+w_0(h^s-c)\right) \end{aligned}$$

and IC_{2NE} reduces to $u_{2NE} \leq 0$.

The Lagrangian function is

$$\mathcal{L} = \alpha^2 h^2 x_{12} + \alpha(1-\alpha)h(1-h)y_{12} + (1-\alpha)\alpha(1-h)hz_{12} + (1-\alpha)^2(1-h)^2 w_{12} + \lambda_1 u_{1E} + \lambda_2 u_{2E}$$

Then we consider $\lambda_1 = \lambda_2 = \lambda$ and obtain

$$\frac{\partial \mathcal{L}}{\partial x_{12}} = \alpha^2\left(h^2 + 2\lambda(h(1-h)-c)\right), \qquad \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \lambda\alpha^2(h-c)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y_{12}} &= \frac{\partial \mathcal{L}}{\partial z_{12}} = \alpha(1-\alpha)\left(h(1-h)+\lambda\left(h^2+(1-h)^2-2c\right)\right) \\ \max\{0, \frac{\partial \mathcal{L}}{\partial y_1} &= \frac{\partial \mathcal{L}}{\partial z_2} = \lambda\alpha(1-\alpha)(h-c)\} \leq \frac{\partial \mathcal{L}}{\partial y_2} = \frac{\partial \mathcal{L}}{\partial z_1} = \lambda\alpha(1-\alpha)(1-h-c) \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial w_{12}} = (1-\alpha)^2\left((1-h)^2 + 2\lambda(h(1-h)-c)\right), \qquad \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = \lambda(1-\alpha)^2(1-h-c) \geq 0$$

Just before Proposition 1 we have described the relation between c_A, c_B, c_C, c_D, c_E, due to the results below. Notice that the relation between h and c_A, c_B, c_C, c_D, c_E is not a constant one.

- $c_A - h = (\alpha+h-2\alpha h)(1-\alpha-h+2\alpha h) - h = \alpha - 4h\alpha - h^2 - 4h^2\alpha^2 - \alpha^2 + 4h\alpha^2 + 4h^2\alpha$ is positive to the left of the thin curve
- $c_A - h(1-h) = (\alpha+h-2\alpha h)(1-\alpha-h+2\alpha h) - h(1-h) = \alpha(1-\alpha)(1-2h)^2 > 0.$
- $c_B - c_A = \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} - (\alpha+h-2\alpha h)(1-\alpha-h+2\alpha h) = \frac{\alpha(1-\alpha)\left(-(1-2h)^2\alpha^2+(1-2h)^2\alpha+h(2-3h)\right)}{1-\alpha+\alpha^2} > 0$ as (...) is concave, and > 0 at $\alpha = 0$, at $\alpha = 1$.
- $c_B - h = \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} - h = \frac{\alpha-2h^2\alpha^2-2h\alpha-\alpha^2+2h\alpha^2+2h^2\alpha-h^2}{-\alpha+\alpha^2+1}$ is positive to the left of the dashed curve. $\left[\alpha - 2h^2\alpha^2 - 2h\alpha - \alpha^2 + 2h\alpha^2 + 2h^2\alpha - h^2\right]_{\alpha=\frac{1}{2}} \left[\alpha - 2h\alpha + h^2\alpha - h^2\right]_{\alpha=\frac{1}{2}} = \frac{1}{2}h^2 - 2h + \frac{1}{2}$
- $c_B - h(1-2h) = \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} - h(1-2h) = \frac{h^2-(1-2h)\alpha^2+(1-2h)\alpha}{1-\alpha+\alpha^2} > 0.$
- $c_B - h(1-h) = \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} - h(1-h) = \frac{\alpha(1-h)^2(1-\alpha)}{1-\alpha+\alpha^2} > 0$
- $c_C - h = \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{\alpha+1} - h = \frac{\alpha-4h\alpha+3h^2\alpha-h^2}{\alpha+1}$ is positive to the left of the red curve

- $c_C - c_A = \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{1+\alpha} - (\alpha + h - 2\alpha h)(1 - \alpha - h + 2\alpha h) = \frac{\alpha^3(1-h)^2}{1-\alpha+\alpha^2} > 0.$
- $c_C - c_B = \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} - \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{1+\alpha} = \alpha\frac{\left(6h-5h^2-2\right)\alpha^2+\left(4h^2-4h+1\right)\alpha+2h-3h^2}{(\alpha+1)(-\alpha+\alpha^2+1)}$ is positive if α is small (in particular if $\alpha \leq \frac{1}{2}$), is negative if $\alpha = 1$.
- $c_D - h = (1-h)(h + \alpha - h\alpha) - h = \alpha - 2h\alpha + h^2\alpha - h^2$ is positive to the left of the thick curve
- $c_D - c_B = (1-h)(h + \alpha - h\alpha) - \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} = \frac{\alpha^3(1-h)^2}{1-\alpha+\alpha^2} > 0$
- $c_D - c_C = (1-h)(h + \alpha - h\alpha) - \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{1+\alpha} = \alpha\frac{\alpha(1-h)^2+h(2-3h)}{\alpha+1} > 0$
- $1 - h - c_D = 1 - h - (1-h)(h + \alpha - h\alpha) = (1-h)^2(1-\alpha) > 0$
- $c_E - c_B = \frac{-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2+2\alpha}{2-2\alpha+\alpha^2} - \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} = \alpha^2\frac{\alpha(1-\alpha)(1-2h)+h^2}{(1-\alpha+\alpha^2)(2-2\alpha+\alpha^2)} > 0$
- $c_E - c_C = \frac{-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2+2\alpha}{2-2\alpha+\alpha^2} - \frac{h+\alpha-3h\alpha+3h^2\alpha-h^2}{1+\alpha} = \alpha\frac{\left(8h-5h^2-3\right)\alpha^2+\left(9h^2-8h+2\right)\alpha+4h-6h^2}{(\alpha+1)(-2\alpha+\alpha^2+2)}$ is positive if α is small (in particular $\alpha \leq \frac{2}{3}$), is negative if α is large and $h \leq 1 - \frac{1}{2}\sqrt{2}$.
- $c_D - c_E = \alpha^2\frac{\alpha-2h\alpha+h^2\alpha-h^2}{2-2\alpha+\alpha^2}$ is positive to the left of the thick curve.
- $c_E - h = \frac{-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2+2\alpha}{2-2\alpha+\alpha^2} - h = 2(1-\alpha)\frac{\alpha-2h\alpha+h^2\alpha-h^2}{2-2\alpha+\alpha^2}$ is positive to the left of the thick curve.
- $c_E - h + h^2 = \frac{-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2+2\alpha}{2-2\alpha+\alpha^2} - h + h^2 = \alpha\frac{4h\alpha-2\alpha-4h-h^2\alpha+2h^2+2}{-2\alpha+\alpha^2+2} > 0$
- $1 - h - c_E = 1 - h - \frac{-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2+2\alpha}{2-2\alpha+\alpha^2} = \frac{\left(2h^2-6h+3\right)\alpha^2-4(1-h)^2\alpha+2(1-h)^2}{-2\alpha+\alpha^2+2} > 0.$
- $c_F - h = \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} - h = \frac{-2h^2\alpha^2+2h\alpha^2-2h-\alpha^2+1}{\alpha^2+1}$
- $c_F - c_B = \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} - \frac{(1-h)(h+\alpha-2h\alpha-\alpha^2+2h\alpha^2)}{1-\alpha+\alpha^2} = \frac{(1-h)^2(1-\alpha)^2}{(\alpha^2+1)(-\alpha+\alpha^2+1)} > 0$
- $c_F - h(1-h) = \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} - h(1-h) = \frac{(1-h)^2(1-\alpha)(\alpha+1)}{\alpha^2+1}$
- $c_F - h(1-2h) = \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} - h(1-2h) = \frac{(1-h)^2+h^2-\alpha^2(1-2h)}{1+\alpha^2} > 0$
- $c_F - c_D = \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} - (1-h)(h + \alpha - h\alpha) = \left(1 - \alpha - \alpha^2 - \alpha^3\right)\frac{(h-1)^2}{\alpha^2+1}$
- $c_G - c_F = 1 + 2h\alpha^2 - h - \alpha^2 - \frac{(1-h)(2h\alpha^2-\alpha^2+1)}{\alpha^2+1} = \alpha^2\frac{(1-\alpha^2)(1-2h)+2h^2}{\alpha^2+1} > 0.$
- $c_G - h = \left(1 + 2h\alpha^2 - h - \alpha^2\right) - h = (1-\alpha)(\alpha+1)(1-2h) > 0$
- $1 - h - c_G = 1 - h - \left(1 + 2h\alpha^2 - h - \alpha^2\right) = \alpha^2(1-2h) > 0$

It is immediate that $\frac{\partial\mathcal{L}}{\partial y_2} = \frac{\partial\mathcal{L}}{\partial z_1} \geq 0 = \frac{\partial\mathcal{L}}{\partial y_0} = \frac{\partial\mathcal{L}}{\partial z_0}$ and $\frac{\partial\mathcal{L}}{\partial w_1} = \frac{\partial\mathcal{L}}{\partial w_2} \geq \frac{\partial\mathcal{L}}{\partial w_0} = 0$. Hence in the following, when considering states ws, sw, we neglect $\frac{\partial\mathcal{L}}{\partial y_0}, \frac{\partial\mathcal{L}}{\partial y_1}$ and $\frac{\partial\mathcal{L}}{\partial z_0}, \frac{\partial\mathcal{L}}{\partial z_2}$. In the state ss we neglect $\frac{\partial\mathcal{L}}{\partial w_0}$.

7.1.1 Proof of Proposition 1(b)

Set $\lambda = \frac{h-h^2}{c+h-2h^2}$ ($\lambda > 0$ as $h < \frac{1}{2}$) and verify that

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = h^2 + 2\frac{h-h^2}{c+h-2h^2}(h(1-h) - c) = \frac{h(2-3h)(h-c)}{c+h-2h^2} > \max\{\frac{\partial \mathcal{L}}{\partial x_0}, \frac{\partial \mathcal{L}}{\partial x_1}, \frac{\partial \mathcal{L}}{\partial x_2}\}$ as $\frac{\partial \mathcal{L}}{\partial x_0} = 0$, $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{(h-h^2)(h-c)}{c+h-2h^2}$ and $c \leq h$, $\frac{\partial \mathcal{L}}{\partial x_{12}} - \frac{\partial \mathcal{L}}{\partial x_1} = \frac{h(1-2h)(h-c)}{c+h-2h^2} \geq 0$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{\partial \mathcal{L}}{\partial y_{12}}$ as $h(1-h) + \frac{h-h^2}{c+h-2h^2}\left(h^2 + (1-h)^2 - 2c\right) = \frac{h-h^2}{c+h-2h^2}(1-h-c)$. In a similar way we prove $\frac{\partial \mathcal{L}}{\partial z_1} = \frac{\partial \mathcal{L}}{\partial z_{12}}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = (1-h)^2 + 2\frac{h-h^2}{c+h-2h^2}((1-h)h - c)$, $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = \frac{(h-h^2)(1-h-c)}{c+h-2h^2} \geq 0$ and $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{c(1-2h)(1-h)}{c+h-2h^2} \geq 0$.

From (i)-(iii) it follows it is optimal to set $x_{12} = 1$, and $y_2 = z_1 > 0$, $y_{12} = z_{12} = 1 - z_1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is such that $\alpha^2 (h(1-h) - c) + \alpha(1-\alpha)(1-z_1)(h^2 - c) + \alpha(1-\alpha)\left(z_1(1-h-c) + (1-z_1)((1-h)^2 - c)\right) + (1-\alpha)^2 (h(1-h) - c) = 0$, which holds if and only if $y_2 = z_1 = \frac{c-c_A}{\alpha(1-\alpha)(c+h-2h^2)}$. Since $c_A < c \leq c_B$, it follows $z_1 \in (0, 1]$.

Finally, $u_{1NE} = \alpha(1-\alpha)\frac{c-c_A}{\alpha(1-\alpha)(c+h-2h^2)}(h^2 - c) = -\frac{(c-c_A)(c-h^2)}{c+h-2h^2} < 0$

7.1.2 Proof of Proposition 1(c)

Set $\lambda = \frac{h-h^2}{c+h-2h^2}$ and verify that

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{h(2-3h)(h-c)}{c+h-2h^2} \leq 0$, $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{(h-h^2)(h-c)}{c+h-2h^2} \leq 0$.
- (ii) Like in the proof of Proposition 1(b), $\frac{\partial \mathcal{L}}{\partial w_{12}} > \max\{\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ and $\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial y_2}$, $\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_1}$.

From (i)-(ii) it follows it is optimal to set $x_0 = 1$, $w_{12} = 1$ and $y_2 = z_1 > 0$, $y_{12} = z_{12} = 1 - z_1$ such that $u_{1E} = 0$, that is such that $\alpha(1-\alpha)(1-z_1)(h^2 - c) + \alpha(1-\alpha)\left(z_1(1-h-c) + (1-z_1)((1-h)^2 - c)\right) + (1-\alpha)^2 (h(1-h) - c) = 0$, which holds if and only if $z_1 = \frac{(1+\alpha)(c-c_C)}{\alpha(c+h-2h^2)}$. Since $c_C \leq c \leq c_D$, it follows $z_1 \in [0, 1]$.

Finally, $u_{1NE} = \alpha^2(h-c) + \alpha(1-\alpha)y_2(h^2 - c) = -\alpha(1-\alpha)\left(c - h^2\right)y_2 - \alpha^2(c-h) < 0$.

7.1.3 Proof of Proposition 1(d)

Set $\lambda = \frac{(1-h)^2}{c+(1-2h)(1-h)}$. Then

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} \leq 0$ since $c \geq h$ (=) and $\frac{\partial \mathcal{L}}{\partial x_{12}} = h^2 + 2\frac{(1-h)^2}{c+(1-2h)(1-h)}(h(1-h) - c) = \frac{h(2-3h)(1-h)-(2-4h+h^2)c}{c-3h+2h^2+1} \leq \frac{\partial \mathcal{L}}{\partial x_0} = 0$ since $c \geq h$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \frac{\partial \mathcal{L}}{\partial y_{12}}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2(1-h-c)}{c+(1-2h)(1-h)}$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = h(1-h) + \frac{(1-h)^2}{c+(1-2h)(1-h)}\left(h^2 + (1-h)^2 - 2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{c(1-2h)(1-h)}{c+(1-2h)(1-h)} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_1} \geq \frac{\partial \mathcal{L}}{\partial z_{12}}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_1}$ as $(1-h)^2 + 2\frac{(1-h)^2}{c+(1-2h)(1-h)}(h(1-h) - c) = \frac{(1-h)^2}{c+(1-2h)(1-h)}(1-h-c)$.

From (i)-(iii) it follows it is optimal to set $x_0 = 1$, $y_2 = 1$, $z_1 = 1$, and $w_1 = w_2 > 0$, $w_{12} = 1 - 2w_1$ such that $u_{1E} = 0$, that is such that $\alpha(1-\alpha)(1-h-c) + (1-\alpha)^2 (w_1(1-h-c) + (1-2w_1)(h(1-h) - c)) = 0$ which holds if and only if $w_1 = \frac{c-c_D}{(1-\alpha)(c+(1-2h)(1-h))}$ (=). Since $c_D \leq c \leq 1-h$, it follows $w_1 \in [0, \frac{1}{2}]$.

Finally, $u_{1NE} = \alpha^2(h-c) + \alpha(1-\alpha)(h^2 - c) + (1-\alpha)^2 w_2(h(1-h) - c) = -(1-\alpha)^2\left(c - h + h^2\right)w_2 - \alpha\left(c - h^2 - h\alpha + h^2\alpha\right) < 0$ as $c \geq h$.

7.1.4 Proof of Proposition 1(e)

Set $\lambda = \frac{h^2}{2c-2h(1-h)}$. Then

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial \mathcal{L}}{\partial x_0} = 0 \geq \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2}$ as $c \geq h$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_{12}} \geq \frac{\partial \mathcal{L}}{\partial y_2}$ since $\frac{\partial \mathcal{L}}{\partial y_{12}} = h(1-h) + \frac{h^2}{2c-2h(1-h)}\left(h^2 + (1-h)^2 - 2c\right)$, $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{h^2(1-h-c)}{2c-2h(1-h)}$ and $\frac{\partial \mathcal{L}}{\partial y_{12}} - \frac{\partial \mathcal{L}}{\partial y_2} = \frac{h(2-3h)(c-h)}{2(c-h+h^2)} \geq 0$. A similar argument applies to prove $\frac{\partial \mathcal{L}}{\partial z_{12}} \geq \frac{\partial \mathcal{L}}{\partial z_1}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = (1-h)^2 + 2\frac{h^2}{2c-2h(1-h)}((1-h)h - c)$ and $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = \frac{h^2(1-h-c)}{2c-2h(1-h)}$, $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(2+h^2-4h)c-h(2-3h)(1-h)}{2(c-h+h^2)} \geq 0$ since $c \geq h$.

From (i)-(iii) it is optimal to set $x_0 > 0$, $x_{12} = 1 - x_0$, $y_{12} = 1$, $z_{12} = 1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is such that $\alpha^2(1-x_0)(h(1-h)-c) + \alpha(1-\alpha)\left(h^2 + (1-h)^2 - 2c\right) + (1-\alpha)^2(h(1-h)-c) = 0$ which holds if and only if $x_0 = \frac{c-c_A}{\alpha^2(c-h+h^2)}$ (=). Since $c_A < c < c_C$, it follows $0 < x_0 \leq 1$.

Finally, $u_{1NE} = \alpha^2 x_0(h-c) = -\alpha^2 x_0 (c-h) < 0$.

7.1.5 Proof of Proposition 1(f)

Set $\lambda = \frac{h^2}{c-h+2h^2}$. Then

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} \geq \frac{\partial \mathcal{L}}{\partial x_1} = 0$ as $h^2 + 2\frac{h^2}{c-h+2h^2}(h(1-h)-c) = \lambda(h-c)$ and $c \leq h$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \frac{\partial \mathcal{L}}{\partial y_{12}}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{h^2(1-h-c)}{c+h-2h(1-h)}$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = h(1-h) + \frac{h^2}{c-h+2h^2}\left(h^2 + (1-h)^2 - 2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{h(1-2h)(h-c)}{c-h+2h^2} \geq 0$ as $c \leq h$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_1} \geq \frac{\partial \mathcal{L}}{\partial z_{12}}$
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = (1-h)^2 + 2\frac{h^2}{c-h+2h^2}((1-h)h - c)$ and $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = \frac{h^2(1-h-c)}{c+h-2h(1-h)}$, $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(1-2h)(c-h+h^2)}{c-h+2h^2} \geq 0$ as $c > c_A$ implies $c \geq h - h^2$.

From (i)-(iii) it is optimal to set $x_1 = x_2 > 0$, $x_{12} = 1 - 2x_1$, $y_2 = 1$, $z_1 = 1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is such that $\alpha^2(x_1(h-c) + (1-2x_1)(h(1-h)-c)) + \alpha(1-\alpha)(1-h-c) + (1-\alpha)^2(h(1-h)-c) = 0$ which holds if and only if $x_1 = \frac{(1-\alpha+\alpha^2)(c-c_B)}{\alpha^2(c+2h^2-h)}$. Since $c_B \leq c \leq c_E$, it follows $x_1 \in [0, \frac{1}{2}]$.

Finally, $u_{1NE} = \alpha^2 x_2(h(1-h)-c) + \alpha(1-\alpha)(h^2-c) = -\alpha^2(c - h(1-h))x_2 - \alpha(1-\alpha)\left(c-h^2\right) < 0$

7.1.6 Proof of Proposition 1(g)

Set $\lambda = \frac{(1-h)^2}{c+(1-2h)(1-h)}$. Then

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{(1-h)^2(h-c)}{c+(1-2h)(1-h)} \geq 0$ since $c \leq h$, $\frac{\partial \mathcal{L}}{\partial x_{12}} = h^2 + 2\frac{(1-h)^2}{c+(1-2h)(1-h)}(h(1-h)-c)$ and $\frac{\partial \mathcal{L}}{\partial x_2} - \frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{(1-2h)(c-h+h^2)}{c+(1-2h)(1-h)} \geq 0$ as $c > \hat{c}_A$ implies $c \geq h(1-h)$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \frac{\partial \mathcal{L}}{\partial y_{12}}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2(1-h-c)}{c+(1-2h)(1-h)}$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = h(1-h) + \frac{(1-h)^2}{c+(1-2h)(1-h)}\left(h^2 + (1-h)^2 - 2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{c(1-2h)(1-h)}{c+(1-2h)(1-h)} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_2} \geq \frac{\partial \mathcal{L}}{\partial z_1}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_1}$ as $(1-h)^2 + 2\frac{(1-h)^2}{c+(1-2h)(1-h)}(h(1-h)-c) = \frac{(1-h)^2}{c+(1-2h)(1-h)}(1-h-c)$.

From (i)-(iii) it is optimal to set $x_1 = x_2 = \frac{1}{2}$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 > 0$, $w_{12} = 1 - 2w_1$ such that $u_{1E} = 0$, that is such that $\alpha^2\frac{1}{2}(h-c) + \alpha(1-\alpha)(1-h-c) + (1-\alpha)^2(w_1(1-h-c) + (1-2w_1)(h(1-h)-c)) = 0$, which holds if and only if $w_1 = \frac{(\alpha^2-2\alpha+2)(c-c_E)}{2(1-\alpha)^2(c+(1-2h)(1-h))}$. Since $c_E \leq c \leq h$, it follows $0 \leq w_1 \leq \frac{1}{2}$.

Finally, $u_{1NE} = \alpha^2\left(\frac{1}{2}(h(1-h)-c)\right) + \alpha(1-\alpha)(h^2-c) + (1-\alpha)^2 w_2(h(1-h)-c) = -(1-\alpha)^2(c-h(1-h))w_2 - \alpha(1-\alpha)\left(c-h^2\right) - \frac{1}{2}\alpha^2(c-h(1-h)) < 0$.

7.2 Proof of Proposition 7

In order to investigate the link between the two sets of information structures, consider an information structure with public messages such that $M = \{m^1, m^2, ..., m^n\}$, in which n is a arbitrary natural number. Let $x^1, x^2, ..., x^n$ denote the probability distribution over M the seller chooses in state ww, that is $x^j = \Pr\{m_1 = m_2 = m^j | (h_1, h_2) = (h^w, h^w)\}$ for $j = 1, ..., n$. Likewise, $y^1, y^2, ..., y^n$ is the probability distribution over M in state ws; $z^1, z^2, ..., z^n$ is the probability distribution over M in state sw; $w^1, w^2, ..., w^n$ is the probability distribution over M in state ss.

Then the probability for either bidder to receive message m^j is

$$p^j = \alpha^2 x^j + \alpha(1-\alpha)y^j + \alpha(1-\alpha)z^j + (1-\alpha)^2 w^j$$

and suppose $p^j > 0$, otherwise m^j is irrelevant and could be deleted from M. Let e_1^j (e_2^j) denote the probability that bidder 1 (bidder 2) enters in the equilibrium of the entry game the bidder play after receiving message m^j. Then the expected utility of bidder 1 from entering after receiving message m^j is

$$\frac{1}{p^i}\left\{\begin{array}{c} \alpha^2 x^j[h(1-h)e_2^j + h(1-e_2^j) - c] + \alpha(1-\alpha)y^j[h^2 e_2^j + h(1-e_2^j) - c] \\ +\alpha(1-\alpha)z^j[(1-h)^2 e_2^j + (1-h)(1-e_2^j) - c] + (1-\alpha)^2 w^j[(1-h)he_2^j + (1-h)(1-e_2^j) - c] \end{array}\right\} \quad (19)$$

This needs to be non-negative if $e_1^j > 0$, need to be non-positive if $e_1^i < 1$, needs to be zero if $e_1^i \in (0,1)$. A similar argument applies to bidder 2, and this characterizes an equilibrium for an information structure with public messages: Such an equilibrium is characterized by the n pairs (e_1^1, e_2^1), (e_1^2, e_2^2), (e_1^3, e_2^3), ..., (e_1^n, e_2^n), each of whom specifies an equilibrium after the bidders have seen a specific public message.

Notice that in state ww the probability that both bidders enter is

$$p_{EE}^{ww} = x^1 e_1^1 e_2^1 + x^2 e_1^2 e_2^2 + ... + x^n e_1^n e_2^n$$

the probability that bidder 1 enters, bidder 2 does not is

$$p_{ENE}^{ww} = x^1 e_1^1(1-e_2^1) + x^2 e_1^2(1-e_2^2) + ... + x^n e_1^n(1-e_2^n)$$

the probability that bidder 1 does not enter, bidder 2 does is

$$p_{NEE}^{ww} = x^1(1-e_1^1)e_2^1 + x^2(1-e_1^2)e_2^2 + ... + x^n(1-e_1^n)e_2^n$$

the probability that no bidder enters is

$$p_{NENE}^{ww} = x^1(1-e_1^1)(1-e_2^1) + x^2(1-e_1^2)(1-e_2^2) + ... + x^n(1-e_1^n)(1-e_2^n)$$

In a similar manner $p_{hk}^{ws}, p_{hk}^{sw}, p_{hk}^{ss}$ are derived for $h, k = E, NE$.

An equivalent information structure with private messages In order to identify an information structure with private messages which leads to the same outcome it suffices to set $M_1 = \{E, NE\} = M_2$ and $x_0, x_1, ..., w_2, w_{12}$ as follows:

$$x_0 = p_{NENE}^{ww}, x_1 = p_{ENE}^{ww}, x_2 = p_{NEE}^{ww}, x_{12} = p_{EE}^{ww} \quad (20)$$

$$y_0 = p_{NENE}^{ws}, y_1 = p_{ENE}^{ws}, y_2 = p_{NEE}^{ws}, y_{12} = p_{EE}^{ws} \quad (21)$$

$$z_0 = p_{NENE}^{ws}, z_1 = p_{ENE}^{ws}, z_2 = p_{NEE}^{ws}, z_{12} = p_{EE}^{ws} \quad (22)$$

$$w_0 = p_{NENE}^{ss}, w_1 = p_{ENE}^{ssw}, w_2 = p_{NEE}^{ss}, w_{12} = p_{EE}^{ss} \quad (23)$$

This is enough to conclude that obeying the message in the direct information structure is an equilibrium.

The link between the two sets of information structures We begin with an intermediate lemma

Lemma 2 (Existence of a pure-strategy equilbrium) *After each public message m^j, there exists a pure-strategy equilibrium in the entry game.*

After bidders have both received a message m^j, they play the entry game with the following utilities

$1\backslash 2$	E	NE
E	$\rho - c, \tau - c$	$\omega - c, 0$
NE	$0, \lambda - c$	$0, 0$

with

$$\begin{aligned}\rho &= E_{m^j}[h_1(1-h_2)] < \omega = E_{m^j}(h_1) \\ \tau &= E_{m^j}[h_2(1-h_1)] < \lambda = E_{m^j}(h_2)\end{aligned}$$

in which E_{m^j} denotes expectation conditional on observing message m^j.

It is immediate that in this game there exists at least a pure-strategy equilibrium.

- (E, E) is an equilibrium if and only if $\rho - c \geq 0, \tau - c \geq 0$. Hence, suppose in the following $\rho - c < 0$ and/or $\tau - c < 0$, so that (E, E) is not an equilibrium – but we prove that some other pure-strategy equilibrium exists.

- Suppose $\rho - c < 0$. Then (NE, E) is an equilibrium if $\lambda - c \geq 0$. Hence suppose $\lambda - c < 0$, so that (NE, E) is not an equilibrium but now we prove that some other pure-strategy equilibrium exists.

- Given $\rho - c < 0$, $\lambda - c < 0$, notice that (NE, NE) is an equilibrium if $\omega - c \leq 0$. Hence suppose $\omega - c > 0$, so that (NE, NE) is not an equilibrium, but now we show that (E, NE) is an equilibrium.

- Given $\rho - c < 0$, $\lambda - c < 0$, $\omega - c > 0$, we have that (E, NE) is an equilibrium if and only if $\tau - c \leq 0$. Since we know that $\tau < \lambda$ from above and we have $\lambda - c < 0$, it follows that $\tau - c < 0$.

Therefore either (E, E) is an equilibrium, or if it is not an equilibrium because $\rho - c < 0$, then there exists at least one other pure-strategy equilibrium. In a similar way we can prove that if (E, E) is not an equilibrium because $\tau - c < 0$, then there exists at least one other pure-strategy equilibrium (to this purpose we need to use the inequality $\rho < \omega$).

In view of Lemma 2, recall that $M = \{m^1, m^2, ..., m^n\}$ and gather in $M_{E,E}$ the messages in M after which the bidders play the equilibrium (E, E); gather in $M_{NE,E}$ the messages after which the equilibrium (NE, E) is played; gather in $M_{E,NE}$ the messages after which equilibrium (NE, E) is played, and finally place in $M_{NE,NE}$ the messages after which (NE, NE) is played. Thus $M_{E,E}$, $M_{NE,E}$, $M_{E,NE}$, $M_{NE,NE}$ is a partition of M.

In the state ww, let x_{EE} denote the probability the seller's public message is in $M_{E,E}$, that is $x_{E,E} = \sum_{j:m^j \in M_{E,E}} x^j$, and notice that this coincides with p^{ww}_{NENE} defined above. Likewise, let $x_{E,NE}, x_{NE,E}, x_{NE,NE}$ be defined with reference to $M_{E,NE}$, $M_{NE,E}$, $M_{NE,NE}$ and let $y_{h,k}$, $z_{h,k}$, $w_{h,k}$ denote the probability that the seller's public message is in $M_{h,k}$ in the state of the world ws, sw, ss, respectively, for $h, k = E, NE$.

If bidder 1 receives a message in $M_{E,E}$, then bidder 1 knows that bidder 2 has received the same message and expects that bidder 2 enters. Thus a minor extension of (19) (we need to allow for all messages in $M_{E,E}$

rather than just one message m^j) with $e^j = 1$ allows to conclude that entering is a best reply for bidder 1 if and only if[21]

$$\alpha^2 x_{E,E}(h(1-h)-c)+\alpha(1-\alpha)y_{E,E}(h^2-c)+\alpha(1-\alpha)z_{E,E}((1-h)^2-c)+(1-\alpha)^2 w_{E,E}(h(1-h)-c)\geq 0 \quad (24)$$

Likewise, if bidder 1 receives a message in $M_{E,NE}$ then he expects that bidder 2 does not enter ($e^j = 0$ in (19)). Hence entering is best reply for bidder 1 if and only if

$$\alpha^2 x_{E,NE}(h-c)+\alpha(1-\alpha)y_{E,NE}(h-c)+\alpha(1-\alpha)z_{E,NE}(1-h-c)+(1-\alpha)^2 w_{E,NE}(1-h-c)\geq 0 \quad (25)$$

The inequalities (24), (25) are the conditions such that bidder 1 wants to enter if the message is in $M_{E,E}$ or in $M_{E,NE}$. In order to find an equivalent information structure with private messages, we apply (20)-(23) to obtain $x_0 = x_{NE,NE}, x_1 = x_{E,NE}, x_2 = x_{NE,E}, x_{12} = x_{E,E}$; $y_0, y_1, ..., w_2, w_{12}$ are obtained likewise. Hence (24), (25) can be written as

$$\alpha^2 x_{12}(h(1-h)-c)+\alpha(1-\alpha)y_{12}(h^2-c)+\alpha(1-\alpha)z_{12}((1-h)^2-c)+(1-\alpha)^2 w_{12}(h(1-h)-c)\geq 0 \quad (26)$$

and

$$\alpha^2 x_1(h-c)+\alpha(1-\alpha)y_1(h-c)+\alpha(1-\alpha)z_1(1-h-c)+(1-\alpha)^2 w_1(1-h-c)\geq 0 \quad (27)$$

In order to maximize the revenue, the lagrangian function is $\mathcal{L} = \alpha^2 h^2 x_{12} + \alpha(1-\alpha)h(1-h)y_{12} + (1-\alpha)\alpha(1-h)hz_{12} + (1-\alpha)^2(1-h)^2 w_{12} + \lambda_1(5) + \lambda_2(6)$, and with $\lambda_1 = \lambda_2 = \lambda$ we obtain

$$\frac{\partial \mathcal{L}}{\partial x_{12}} = \alpha^2\left(h^2+2\lambda(h(1-h)-c)\right), \qquad \frac{\partial \mathcal{L}}{\partial w_{12}} = (1-\alpha)^2\left((1-h)^2+2\lambda(h(1-h)-c)\right)$$
$$\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial z_{12}} = \alpha(1-\alpha)\left(h(1-h)+\lambda\left(h^2+(1-h)^2-2c\right)\right)$$

If c is a bit greater than c_A, then solution is such that $\lambda = \frac{h^2}{2c+2h(h-1)}$ and x_{12} such that (5), (6) hold with equality if $c < c_C$, that is if reducing x_{12} alone is enough to satisfy (5), (6) as

$$\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial z_{12}} = h(1-h)+\frac{h^2}{2c+2h(h-1)}\left(h^2+(1-h)^2-2c\right) = h(1-2h)\frac{c-\frac{1}{2}h}{c-h(1-h)} > 0$$
$$\frac{\partial \mathcal{L}}{\partial w_{12}} = (1-h)^2+2\frac{h^2}{2c+2h(h-1)}(h(1-h)-c) = 1-2h > 0$$

If instead $c > c_C$, then it is necessary to reduce also $y_{12} = z_{12}$ and or w_{12} below 1. Suppose we reduce just w_{12}, then set $\lambda = \frac{(1-h)^2}{2(c-h+h^2)}$ from $\frac{\partial \mathcal{L}}{\partial w_{12}} = 0$, so that

$$\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial z_{12}} = h(1-h)+\frac{(1-h)^2}{2(c-h+h^2)}\left(h^2+(1-h)^2-2c\right) = (1-2h)(1-h)\frac{\frac{1}{2}-\frac{1}{2}h-c}{c-h+h^2}$$

and we need $c < \frac{1}{2} - \frac{1}{2}h$, with $w_{12} = \frac{\alpha-2c\alpha-2h\alpha+2h^2\alpha}{c-h-c\alpha+h\alpha-h^2\alpha+h^2}$, which is positive if $c < \frac{1}{2} - h + h^2$, is less than 1 if $c > c_C$. Notice that $\frac{1}{2} - h + h^2 < \frac{1}{2} - \frac{1}{2}h$ and $\frac{\partial \mathcal{L}}{\partial x_{12}} = h^2 + 2\frac{1}{2}\frac{(1-h)^2}{c-h+h^2}(h(1-h)-c) = -\frac{3}{2}h^2 + 5h - \frac{5}{2} < 0$.

If $c > \frac{1}{2} - h + h^2$, then all coefficients are negative and the contraint is not satisfied.

[21] The utility of bidder 1 from entering is given by the left hand side below divided by $p_{E,E} = \alpha^2 x_{E,E} + \alpha(1-\alpha)y_{E,E} + \alpha(1-\alpha)z_{E,E} + (1-\alpha)^2 w_{E,E}$, which is the probability that the message each bidder receives is E, E.

7.3 Proof of Lemma 1

Suppose that $\tau \geq \theta$. Then $G_2(b)$ satisfies $\frac{\tau}{\tau+\delta} = (1-b)\left(\frac{\tau}{\tau+\delta} + \frac{\delta}{\tau+\delta}G_2(b)\right)$, which means that type 1 of bidder 1 is indifferent among all bids in $[0, \frac{\delta}{m+\delta}]$; hence $G_2(b) = \frac{\mu b}{\delta(1-b)}$ with $G_2(0) = 0$ and $G_2(\bar{b}) = 1$. Likewise, G_1 is such that type 1 of bidder 1 is indifferent among all bids in $[0, \frac{\delta}{m+\delta}]$, that is $\frac{\tau}{\tau+\delta} = (1-b)\left(\frac{\theta}{\theta+\delta} + \frac{\delta}{\theta+\delta}G_1(b)\right)$, hence $G_1(b) = \frac{\delta(\tau-\theta)+\theta(\tau+\delta)b}{\delta(\tau+\delta)(1-b)}$ with $G_1(0) = \frac{\tau-\theta}{\tau+\delta} \geq 0$ and $G_1(\bar{b}) = 1$. The utility of both type 1 of bidder 1 and type 1 of bidder 2 is $\frac{\tau}{\tau+\delta}$. The proof for the case of $\tau < \theta$ is very similar.

7.4 Proof of Proposition 10

The proof of Proposition 10(i-ii) is in the text. In order to prove Proposition 10(iii), consider $(1-h)\beta \leq c+f \leq h^e$ and notice that $R^{PD} = \frac{(1-c-h)^2-f^2}{(1-h)^2}$, $R^{ND} = f$. The inequality $c \leq h^e - f$ implies $\frac{(1-c-h)^2-f^2}{(1-h)^2} - f \geq \frac{(1-(\frac{1}{2}h+\frac{1}{2}(1-h)-f)-h)^2-f^2}{(1-h)^2} - f = \frac{(1-2h)^2-4fh^2}{4(1-h)^2} > \frac{(1-2h)^2-4(1-h)h^2}{4(1-h)^2} = \frac{1-4h+4h^3}{4(1-h)^2}$ (the last inequality follows from $f < 1-h$), which is positive if $h \leq \frac{1}{4}$. Now consider $h^e \leq c+f \leq 1-h$ and notice that $R^{PD} = \frac{(1-c-h)^2-f^2}{(1-h)^2}$, $R^{ND} = \frac{3}{4}f$. When $c+f$ is close to h^e, that is close to $\frac{1}{2}$, we find that $R^{PD} = \frac{(1-(\frac{1}{2}-f)-h)^2-f^2}{(1-h)^2} = \frac{(1-2h)(4f-2h+1)}{4(1-h)^2}$ and $\frac{(1-2h)(4f-2h+1)}{4(1-h)^2} - \frac{3}{4}f = \frac{(1-2h)^2+(h+1)(1-3h)f}{4(1-h)^2}$, which is positive if $h \leq \frac{1}{3}$, but for $h > \frac{1}{3}$ we have that $\frac{(1-2h)^2+(h+1)(1-3h)f}{4(1-h)^2} \geq \frac{(1-2h)^2+(h+1)(1-3h)(1-h)}{4(1-h)^2}$, which is positive for each $h \leq \frac{9}{25}$. As $c+f$ increases, R^{PD} decreases and tends to 0 when $c+f$ is close to $1-h$, whereas $R^{FD} = \frac{3}{4}f > 0$.

7.5 Proof of Proposition 11

We first compare R^{ND*} with $R^{FD*} = (1-\alpha^2)(1-h-c)$ and find that

$$R^{ND*} - R^{FD*} = \begin{cases} h\left((1-\alpha)^2 + 2\alpha^2 - h(2\alpha-1)^2\right) - \left(1+\alpha^2\right)c & \text{if } c \leq h^e(1-h^e) \\ 2h\alpha - \alpha + \alpha^2 - h\alpha^2 - \alpha^2 c & \text{if } h^e(1-h^e) < c \leq h^e \\ -(1-\alpha^2)(1-h-c) & \text{if } h^e < c \leq 1-h \end{cases}$$

This function is continuous (in particular, it is continuous at $c = h^e(1-h^e)$), and is strictly decreasing in c, has a positive value at $c = 0$ because $(1-\alpha)^2 + 2\alpha^2 - h(2\alpha-1)^2 > (1-\alpha)^2 + 2\alpha^2 - \frac{1}{2}(2\alpha-1)^2 = \frac{1}{2} + \alpha^2 > 0$, has a negative value at $c = h^e$. Hence there exists a unique $\bar{c} \in (0, h^e)$ such that $R^{ND*} > R^{FD*}$ for $c \in [0, \bar{c})$ and $R^{ND*} < R^{FD*}$ for $c \in (\bar{c}, h^e]$. In particular, $h\left((1-\alpha)^2 + 2\alpha^2 - h(2\alpha-1)^2\right) - \left(1+\alpha^2\right)h^e(1-h^e) = \alpha^2(2\alpha-1)^2 h^2 + 2\alpha(1-\alpha)\left(2\alpha^2+1\right)h - \left(\alpha^2+1\right)\alpha(1-\alpha)$; if this is negative (which occurs if if α and/or h are small), then $\bar{c} < c_A$; if it is positive (which occurs if $h = \frac{1}{2}$ and/or $\alpha = 1$), then $\bar{c} > c_A$.

In order to consider PD, suppose that $c \leq (1-h)\beta$, so that $R^{PD*} = (1-\alpha)((1-h)(1+h+\alpha-h\alpha) - 2c)$ and $R^{PD*} - R^{FD*} = (1-\alpha)^2(h(1-h)-c)$.[22] This reveals that for each $c > h(1-h)$, the optimal structure is FD or ND as PD is suboptimal. When instead $c \leq h(1-h)$, the difference $R^{ND*} - R^{PD*}$ is equal to $h\alpha(2h+2\alpha-3h\alpha) - 2\alpha c$ and is positive for each $c \leq h(1-h)$ if $4h+2\alpha-3h\alpha-2 \geq 0$. In this case ND or FD is optimal as described by Proposition 11(ii). If instead $4h+2\alpha-3h\alpha-2 < 0$, then $R^{ND*} > R^{PD*}$ for each c between 0 and $h(h+\alpha-\frac{3}{2}h\alpha)$ but $R^{ND*} < R^{PD*}$ for each c between $h(h+\alpha-\frac{3}{2}h\alpha)$ and $h(1-h)$. For $c > h(1-h)$, FD is optimal as $R^{ND*} - R^{FD*} < 0$ at $c = h(1-h)$ and is decreasing in c.

[22] Notice that $h(1-h) - (1-h)(\alpha+(1-\alpha)h) = -\alpha(1-h)^2 < 0$, hence $R^{PD*} < R^{FD}$ for each $c \geq (1-h)\beta$.

7.6 Proof of Proposition 12

The utility of 2 from entering after receiving signal E is $\frac{1}{p_{2E}}$ times

$$\alpha^2\left(x_2(h_A-f-c)+x_{12}(h_A(1-h_A)-f-c)\right)+\alpha(1-\alpha)\left(y_2(h_B-f-c)+y_{12}(h_B(1-h_A)-f-c)\right)$$
$$+\alpha(1-\alpha)\left(z_2(h_A-f-c)+z_{12}(h_A(1-h_B)-f-c)\right)+(1-\alpha)^2\left(w_2(h_B-f-c)+w_{12}(h_B(1-h_B)-f-c)\right)$$

with $p_{2E}=\alpha(x_2+x_{12})+\alpha(1-\alpha)(y_2+y_{12})+(1-\alpha)\alpha(z_2+z_{12})+(1-\alpha)^2(w_2+w_{12})$, and the constraint IC_{2E} imposes that this expression is non-negative.

The utility of bidder 1 from entering after receiving signal NE is $\frac{1}{p_{1NE}}$ times

$$\alpha^2\left(x_2(h_A(1-h_A)-f-c)+x_0(h_A-f-c)\right)+\alpha(1-\alpha)\left(y_2(h_A(1-h_B)-f-c)+y_0(h_A-f-c)\right)$$
$$+\alpha(1-\alpha)\left(z_2(h_B(1-h_A)-f-c)+z_0(h_B-f-c)\right)+(1-\alpha)^2\left(w_2(h_B(1-h_B)-f-c)+w_0(h_B-f-c)\right)$$

with $p_{1NE}=\alpha^2(x_0+x_2)+\alpha(1-\alpha)\left(y_0+y_2\right)+\alpha(1-\alpha)\left(z_0+z_2\right)+(1-\alpha)^2\left(w_0+w_2\right)$. The constraint IC_{1NE} imposes that this expression is non-positive. The utility of bidder 2 from entering after receiving signal NE is $\frac{1}{p_{2NE}}$ times

$$\alpha^2\left(x_1(h_A(1-h_A)-f-c)+x_0(h_A-f-c)\right)+\alpha(1-\alpha)\left(y_1(h_B(1-h_A)-f-c)+y_0(h_B-f-c)\right)$$
$$+\alpha(1-\alpha)\left(z_1(h_A(1-h_B)-f-c)+z_0(h_A-f-c)\right)+(1-\alpha)^2\left(w_1(h_B(1-h_B)-f-c)+w_0(h_B-f-c)\right)$$

with $p_{2NE}=\alpha^2(x_0+x_1)+\alpha(1-\alpha)\left(y_0+y_1\right)+\alpha(1-\alpha)\left(z_0+z_1\right)+(1-\alpha)^2\left(w_0+w_1\right)$.

We neglect the incentive constraint related to $m_1=NE$ and $m_2=NE$ and find that the derivatives of the lagrangian function are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x_{12}} &= \alpha^2\left(2f+h^2+2\lambda\left(h(1-h)-f-c\right)\right), \qquad \frac{\partial\mathcal{L}}{\partial x_1}=\frac{\partial\mathcal{L}}{\partial x_2}=\alpha^2\left(f+\lambda(h-f-c)\right)\\
\frac{\partial\mathcal{L}}{\partial y_{12}} &= \alpha(1-\alpha)\left(2f+h(1-h)+\lambda\left(h^2+(1-h)^2-2f-2c\right)\right),\\
\max\{0,\frac{\partial\mathcal{L}}{\partial y_1} &= \alpha(1-\alpha)\left(f+\lambda(h-f-c)\right)\}\leq\frac{\partial\mathcal{L}}{\partial y_2}=\alpha(1-\alpha)\left(f+\lambda(1-h-f-c)\right)\\
\frac{\partial\mathcal{L}}{\partial z_{12}} &= \alpha(1-\alpha)\left(2f+h(1-h)+\lambda\left((1-h)^2+h^2-2f-2c\right)\right),\\
\frac{\partial\mathcal{L}}{\partial z_1} &= \alpha(1-\alpha)\left(f+\lambda(1-h-f-c)\right)\geq\max\{0,\frac{\partial\mathcal{L}}{\partial z_2}=\alpha(1-\alpha)\left(f+\lambda(h-f-c)\right)\}\\
\frac{\partial\mathcal{L}}{\partial w_{12}} &= (1-\alpha)^2\left(2f+(1-h)^2+2\lambda\left(h(1-h)-f-c\right)\right), \qquad \frac{\partial\mathcal{L}}{\partial w_1}=\frac{\partial\mathcal{L}}{\partial w_2}=(1-\alpha)^2\left(f+\lambda(1-h-f-c)\right)\geq 0
\end{aligned}$$

7.6.1 Proof of Proposition 12(a)

Set $\lambda=0$. Then $\frac{\partial\mathcal{L}}{\partial x_{12}}=2f+h^2>\frac{\partial\mathcal{L}}{\partial x_1}=f$, $\frac{\partial\mathcal{L}}{\partial y_{12}}=\frac{\partial\mathcal{L}}{\partial z_{12}}=2f+h(1-h)>\frac{\partial\mathcal{L}}{\partial y_2}=\frac{\partial\mathcal{L}}{\partial z_1}=f$, $\frac{\partial\mathcal{L}}{\partial w_{12}}=2f+(1-h)^2>\frac{\partial\mathcal{L}}{\partial w_1}=f$.

7.6.2 Proof of Proposition 12(b)

Set $\lambda=\frac{f+h-h^2}{c+f+h-2h^2}$; the denominator is positive as $c+f\geq 0$ and $h<\frac{1}{2}$. The following holds

- (i) $\frac{\partial\mathcal{L}}{\partial x_{12}}=\frac{h(2-3h)(f+h-c)}{c+f+h-2h^2}\geq\max\{\frac{\partial\mathcal{L}}{\partial x_0},\frac{\partial\mathcal{L}}{\partial x_1},\frac{\partial\mathcal{L}}{\partial x_2}\}$ as $\frac{\partial\mathcal{L}}{\partial x_0}=0$, $\frac{\partial\mathcal{L}}{\partial x_1}=\frac{\partial\mathcal{L}}{\partial x_2}=f+\frac{f+h-h^2}{c+f+h-2h^2}(h-f-c)$ and $\mathbf{c}\leq\mathbf{h}+\mathbf{f}$, $\frac{\partial\mathcal{L}}{\partial x_{12}}-\frac{\partial\mathcal{L}}{\partial x_1}=\frac{h(1-2h)(f+h-c)}{c+f+h-2h^2}\geq 0$.

- (ii) $\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}\}$ as $2f+h(1-h)+\frac{f+h-h^2}{c+f+h-2h^2}\left(h^2+(1-h)^2-2f-2c\right) = f+\frac{f+h-h^2}{c+f+h-2h^2}(1-h-f-c) = \frac{-(h-h^2)c+(1-h-h^2)f+h(1-h)^2}{c+f+h-2h^2}$ and the latter expression is non-negative as $\frac{\partial \mathcal{L}}{\partial y_2} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_2}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_0}, \frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = 2f+(1-h)^2+2\frac{f+h-h^2}{c+f+h-2h^2}((1-h)h-c-f)$, $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = f+\frac{f+h-h^2}{c+f+h-2h^2}(1-h-f-c)$ and $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(1-2h)(c-ch+fh)}{c+f+h-2h^2} > 0$, with $\frac{\partial \mathcal{L}}{\partial w_1} \geq 0$.[23]

From (i)-(iii) it follows it is optimal to set $x_{12}=1$, $w_{12}=1$, and $y_2 = z_1 > 0$, $y_{12} = z_{12} = 1-z_1$ such that $u_{1E}=0$, that is $\alpha^2\left(h(1-h)-c-f\right)+\alpha(1-\alpha)(1-z_1)(h^2-c-f)+\alpha(1-\alpha)\left(z_1(1-h-c-f)+(1-z_1)((1-h)^2-c-f)\right)+(1-\alpha)^2\left(h(1-h)-c-f\right)=0$. This equality holds if and only if $y_2 = z_1$ is as in S_B^f (=). The condition $z_1 \in [0,1]$ is equivalent to $\mathbf{c}_A-\mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_B-\mathbf{f}$ (=). Finally, $u_{1NE} = -\alpha\,(1-\alpha)\,y_2\left(c+f-h^2\right) < 0$ as $c+f > c_A > h^2$ (=).

7.6.3 Proof of Proposition 12(c)

Set $\lambda = \frac{f+h-h^2}{c+f+h-2h^2}$; the denominator is positive as $c+f \geq 0$ and $h < \frac{1}{2}$. The following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{h(2-3h)(f+h-c)}{c+f+h-2h^2} \leq 0$, $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{h(1-h)(f+h-c)}{c+f+h-2h^2} \leq 0$ as $\mathbf{c} \geq \mathbf{h}+\mathbf{f}$.
- (ii) Like in the proof of Proposition 1(b), $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_0}, \frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ and $\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}\}$, $\frac{\partial \mathcal{L}}{\partial z_{12}} = \frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_2}\}$.

From (i)-(ii) it follows it is optimal to set $x_0 = 1$, $w_{12}=1$ and $y_2 = z_1 > 0$, $y_{12} = z_{12} = 1-z_1$ such that $u_{1E}=0$, that is $\alpha(1-\alpha)(1-z_1)(h^2-c-f)+\alpha(1-\alpha)\left(z_1(1-h-c-f)+(1-z_1)((1-h)^2-c-f)\right)+(1-\alpha)^2\left(h(1-h)-c-f\right)=0$ which holds if and only if z_1 is as in S_C (=). The condition $z_1 \in [0,1]$ is equivalent to $\mathbf{c}_C-\mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_D-\mathbf{f}$ (=). Finally, $u_{1NE} = -\alpha\,(1-\alpha)\,y_2\left(c+f-h^2\right)-\alpha^2\,(c+f-h) < 0$ as $c > h+f$.[24]

7.6.4 Proof of Proposition 12(d)

Set $\lambda = \frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}$; the denominator is positive as $c+f \geq 0$. The following holds.

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{h^2 f+(1-h)^2(h-c)}{c+f+(1-2h)(1-h)} \leq 0$ since $c \geq h+\frac{h^2}{(1-h)^2}f$ (=). Moreover, $\frac{\partial \mathcal{L}}{\partial x_{12}} = 2f+h^2+2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}\left(h(1-h)-f-c\right) = \frac{-(2-4h+h^2)c+h(2-3h)(1-h)+fh^2}{c+f-3h+2h^2+1} \leq \frac{-(2-4h+h^2)(h+\frac{h^2}{(1-h)^2}f)+h(2-3h)(1-h)+fh^2}{c+f-3h+2h^2+1} = -\frac{h^2(1-2h)(f+(1-h)^2)}{(1-h)^2(c+f-3h+2h^2+1)} \leq 0$ since $\mathbf{c} \geq \mathbf{h}+\frac{h^2}{(1-h)^2}\mathbf{f}$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_{12}}\}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2(1+f-h-c)}{c+f+(1-2h)(1-h)} \geq 0$. Moreover, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f+h(1-h)+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}\left(h^2+(1-h)^2-2f-2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{(1-2h)(c-ch+fh)}{c+f+(1-2h)(1-h)} > 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_2}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} > \frac{\partial \mathcal{L}}{\partial w_0}$ as $2f+(1-h)^2+2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}\left(h(1-h)-f-c\right) = f+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(1-h-f-c)$. Moreover, $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{(1-h)^2(1-c+f-h)}{c+f-3h+2h^2+1} \geq 0$.

From (i)-(iii) it follows it is optimal to set $x_0=1$, $y_2=1$, $z_1=1$, and $w_1 = w_2 > 0$, $w_{12} = 1-2w_1$ such that $u_{1E}=0$, that is $\alpha(1-\alpha)(1-h-c-f)+(1-\alpha)^2\left(w_1(1-h-c-f)+(1-2w_1)(h(1-h)-c-f)\right)=0$, which holds if and only if w_1 is as in S_D (=). The condition $w_1 \in [0,\frac{1}{2}]$ is equivalent to $\mathbf{c}_D-\mathbf{f} \leq \mathbf{c} \leq \mathbf{1}-\mathbf{h}-\mathbf{f}$ (=). Finally, $u_{1NE} = \alpha^2(h-c-f)+\alpha(1-\alpha)(h^2-c-f)+(1-\alpha)^2 w_2(h(1-h)-c-f) < 0$ since $c \geq h+\frac{h^2}{(1-h)^2}f$.

[23] If $f < 0$, then $c(1-h)+fh \geq \left((\alpha+h-2\alpha h)(1-\alpha-h+2\alpha h)-f\right)(1-h)+fh = -(1-2h)f+(1-h)(h+\alpha-2h\alpha)(1-h-\alpha+2h\alpha) > 0$.

[24] If instead $f < 0$, then this may be positive: case of $\alpha = 0.6$, $h = 0.35$, $f = -\frac{1}{50}$, $c = 0.3301$, and then in such case $x_0 = 1, w_{12} = 1, y_2 = \frac{3908}{12\,453}$, and $-(0.6)(1-0.6)\frac{3908}{12\,453}\left(0.3301-0.02-(.35)^2\right)-(0.6)^2(0.3301-0.02-0.35) = 2.3457\times 10^{-4}$.

7.6.5 Proof of Proposition 12(e)

Set $\lambda = \frac{2f+h^2}{2c+2f-2h(1-h)}$; the denominator is positive as $c + f \geq c_A \geq h(1-h)$**.** Then the following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial \mathcal{L}}{\partial x_0} = 0 \geq \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{h^2(f+h-c)}{2(c+f-h+h^2)}$ as $\mathbf{c} \geq \mathbf{h} + \mathbf{f}$. In order for $x_1 = x_2 = 0$ to be optimal it takes c high enough because $x_1 > 0$, $x_2 > 0$ have an additional benefit, revenue equal to f. In order to have $x_1 = x_2 > 0$ when $x_{12} < 1$ it is not necessary that $c < h$ but $c < h + f$ or $c < h + \frac{h^2}{(1-h)^2} f$ is enough because now f is in revenue.f
- (ii) $\frac{\partial \mathcal{L}}{\partial y_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_2}\}$ since $\frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{(1-2h)(2f(1-h)+2ch-h^2)}{2(c+f-h+h^2)}$, $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(h^2-4h+2)f+h^2(1-c-h)}{2(c+f-h+h^2)} \geq 0$ and $\frac{\partial \mathcal{L}}{\partial y_{12}} - \frac{\partial \mathcal{L}}{\partial y_2} = \frac{h(2-3h)(c-f-h)}{2(c+f-h+h^2)} \geq 0$ as $\mathbf{c} \geq \mathbf{h} + \mathbf{f}$.[25] A similar argument applies to prove $\frac{\partial \mathcal{L}}{\partial z_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_1}, \frac{\partial \mathcal{L}}{\partial z_2}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_0}, \frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = 1-2h > \frac{\partial \mathcal{L}}{\partial w_0} = 0$ and $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = \frac{(h^2-4h+2)f+h^2-ch^2-h^3}{2(c+f-h+h^2)} \geq 0$, $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(h^2-4h+2)c+5h^2-fh^2-2h-3h^3}{2(c+f-h+h^2)}$.

From (i)-(iii) it is optimal to set $x_0 > 0$, $x_{12} = 1 - x_0$, $y_{12} = 1$, $z_{12} = 1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is $\alpha^2(1-x_0)(h(1-h) - c - f) + \alpha(1-\alpha)\left(h^2 + (1-h)^2 - 2c - 2f\right) + (1-\alpha)^2 (h(1-h) - c - f) = 0$, which holds if and only if x_0 is as in S_E^f (=). The condition $x_0 \in [0,1]$ is equivalent to $\mathbf{c}_A - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_C - \mathbf{f}$ (=). Finally, $u_{1NE} = -\alpha^2 x_0 (c + f - h) \leq 0$ since $c \geq h + f$.

7.6.6 Proof of Proposition 12(f)

Set $\lambda = \frac{f+h^2}{c+f-h+2h^2}$; the denominator is positive as $c + f \geq c_B > h(1-2h)$. The following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} \geq \frac{\partial \mathcal{L}}{\partial x_0} = 0$ as $2f + h^2 + 2\frac{f+h^2}{c+f-h+2h^2}(h(1-h) - f - c) = f + \frac{f+h^2}{c+f-h+2h^2}(h-c-f)$ and $\frac{\partial \mathcal{L}}{\partial x_1} = h^2 \frac{f+h-c}{c+f-h+2h^2} \geq 0$ as $\mathbf{c} \leq \mathbf{h} + \mathbf{f}$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_{12}}\}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = f + \frac{f+h^2}{c+f-h+2h^2}(1-h-f-c) \geq 0$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f + h(1-h) + \frac{f+h^2}{c+f-h+2h^2}\left(h^2 + (1-h)^2 - 2f - 2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{h(1-2h)(f+h-c)}{c+f-h+2h^2} \geq 0$ as $c \leq h + f$. A similar argument applies to prove $\frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_1}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_0}, \frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}\}$ as $\frac{\partial \mathcal{L}}{\partial w_{12}} = 2f + (1-h)^2 + 2\frac{f+h^2}{c+f-h+2h^2}(h(1-h) - f - c)$ and $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} = f + \frac{f+h^2}{c+f-h+2h^2}(1-h-f-c) \geq 0$, $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(1-2h)(c-h+h^2)}{c+f-h+2h^2} \geq 0$ since $\mathbf{c} \geq \mathbf{h} - \mathbf{h}^2$.

From (i)-(iii) it is optimal to set $x_1 = x_2 > 0$, $x_{12} = 1 - 2x_1$, $y_2 = 1$, $z_1 = 1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is $\left[\alpha^2 (x_1(h-c-f) + (1-2x_1)(h(1-h) - c - f)) + \alpha(1-\alpha)(1-h-c-f) + (1-\alpha)^2 (h(1-h) - c - f)\right] = 0$, which holds if and only if x_1 is as in S_F^f (=). The condition $x_1 \in [0, \frac{1}{2}]$ is equivalent to $\mathbf{c}_B - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_E - \mathbf{f}$ (=). Finally, $u_{1NE} = \alpha^2 x_2 (h(1-h) - c - f) + \alpha(1-\alpha)(h^2 - c - f) < 0$ as $c \geq h - h^2$.

7.6.7 Proof of Proposition 12(g)

Set $\lambda = \frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}$; the denominator is positive as $c + f \geq 0$. The following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = f + \frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(h - f - c) = \frac{h^2 f + (1-h)^2(h-c)}{c+f+(1-2h)(1-h)} \geq 0$ as $\mathbf{c} \leq \mathbf{h} + \frac{h^2}{(1-h)^2}\mathbf{f}$, $\frac{\partial \mathcal{L}}{\partial x_{12}} = 2f + h^2 + 2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(h(1-h) - f - c)$ and $\frac{\partial \mathcal{L}}{\partial x_1} - \frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{(1-2h)(c-h+h^2)}{c+f-3h+2h^2+1} \geq 0$ as $\mathbf{c} \geq \mathbf{h} - \mathbf{h}^2$.

[25] If $f < 0$ then $2f(1-h) + 2ch - h^2 \geq 2f(1-h) + 2(c_A - f)h - h^2 = 2f(1-h) + 2((\alpha + h - 2\alpha h)(1 - \alpha - h + 2\alpha h) - f)h - h^2 = (2-4h)f + h(1-2h)\left(h + 2\alpha - 4h\alpha - 2\alpha^2 + 4h\alpha^2\right) > 0$ and the expression is greater than $(2-4h)(-(1-h)^2) + h(1-2h)\left(h + 2\alpha - 4h\alpha - 2\alpha^2 + 4h\alpha^2\right) = (1-2h)\left(2h(1-2h)\alpha^2 + \left(4h^2 - 2h\right)\alpha + \left(h^2 - 4h + 2\right)\right) > 0$.

- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_{12}}\}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2(1-c+f-h)}{c+f-3h+2h^2+1} > 0$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f+h(1-h)+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}\left(h^2+(1-h)^2-\right.$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{(1-2h)(c-ch+fh)}{c+f-3h+2h^2+1} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_1}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} > \frac{\partial \mathcal{L}}{\partial w_0}$ as $2f+(1-h)^2+2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(h(1-h)-f-c) = f+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(1-h-f-c)$ and $\frac{\partial \mathcal{L}}{\partial w_1} \geq 0$.

From (i)-(iii) it is optimal to set $x_1 = x_2 = \frac{1}{2}$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 > 0$, $w_{12} = 1-2w_1$ such that $u_{1E} = 0$, that is $\alpha^2\frac{1}{2}(h-c-f)+\alpha(1-\alpha)(1-h-c-f)+(1-\alpha)^2\left(w_1(1-h-c-f)+(1-2w_1)(h(1-h)-c-f)\right) = 0$, which holds if and only if w_1 is as in S_G^f (=). The condition $w_1 \in [0, \frac{1}{2}]$ is equivalent to $\mathbf{c}_E - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_G - \mathbf{f}$ (=). Finally, $u_{1NE} = \alpha(1-\alpha)(h^2-f-c)+(1-\alpha)^2 w_2(h-h^2-f-c) < 0$ as $c \geq h-h^2$.

7.6.8 Proof of Proposition 12(h)

Set $\lambda = \frac{f}{c+f-h}$; the denominator is positive since $h-f < c_G - f \leq c$ Then the following holds.

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = 0 = \frac{\partial \mathcal{L}}{\partial x_0}$, $\frac{\partial \mathcal{L}}{\partial x_{12}} = 2f+h^2+2\frac{f}{c+f-h}(h(1-h)-f-c) = -h^2\frac{f+h-c}{c+f-h} \leq 0$ as $\mathbf{c} \leq \mathbf{h}+\frac{h^2}{(1-h)^2}\mathbf{f}$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} = f+\lambda(1-h-f-c) \geq 0$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f+h(1-h)+\lambda\left(h^2+(1-h)^2-2f-2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{h(1-h)(f+h-c)}{c+f-h} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_1}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = 2f+(1-h)^2+2\frac{f}{c+f-h}(h(1-h)-f-c)$, $\frac{\partial \mathcal{L}}{\partial w_1} = f+\frac{f}{c+f-h}(1-h-f-c) \geq 0$ and $\frac{\partial \mathcal{L}}{\partial w_1} - \frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{h^2 f+(h-1)^2(h-c)}{c+f-h} \geq 0$ as $c \leq h+\frac{h^2}{(1-h)^2}f$.

From (i)-(iii) it is optimal to set $x_1 = x_2 < \frac{1}{2}, x_0 = 1-2x_1, y_2 = 1$, $z_1 = 1$, $w_1 = \frac{1}{2}, w_2 = \frac{1}{2}$ such that $u_{1E} = 0$, that is $\alpha^2 x_1(h-f-c)+\alpha(1-\alpha)(1-h-f-c)+(1-\alpha)^2\frac{1}{2}(1-h-f-c) = 0$, which holds if and only if x_1 is as in S_H^f (=). The condition $x_1 \leq \frac{1}{2}$ is equivalent to $\mathbf{c}_G - \mathbf{f} \leq \mathbf{c}$ (=). Finally, $u_{1NE} = \alpha^2 x_2(h(1-h)-f-c)+\alpha(1-\alpha)(h^2-f-c)+(1-\alpha)^2\frac{1}{2}(h(1-h)-f-c) < 0$ as $c \geq c_G - f > h-f$.

7.6.9 Proof of Proposition 12(i)

Set $\lambda = \frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}$; the denominator is positive as $c+f \geq 0$. The following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{h^2 f+(1-h)^2(h-c)}{c+f+(1-2h)(1-h)} \geq 0$ as $c \leq h-h^2$, $\frac{\partial \mathcal{L}}{\partial x_{12}} = 2f+h^2+2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(h(1-h)-f-c)$ and $\frac{\partial \mathcal{L}}{\partial x_{12}} - \frac{\partial \mathcal{L}}{\partial x_1} = \frac{(1-2h)(h-h^2-c)}{c+f-3h+2h^2+1} \geq 0$ as $\mathbf{c} \leq \mathbf{h}-\mathbf{h}^2$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_{12}}\}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2(1+f-h-c)}{c+f+(1-2h)(1-h)} \geq 0$. Moreover, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f+h(1-h)+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}\left(h^2+(1-h)^2-2f-2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{(1-2h)(c-ch+fh)}{c+f+(1-2h)(1-h)} \geq 0$. A similar argument establishes $\frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_2}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} > \frac{\partial \mathcal{L}}{\partial w_0}$ as $2f+(1-h)^2+2\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(h(1-h)-f-c) = f+\frac{f+(1-h)^2}{c+f+(1-2h)(1-h)}(1-h-f-c)$ and $\frac{\partial \mathcal{L}}{\partial w_1} \geq 0$.

From (i)-(iii) it is optimal to set $x_{12} = 1$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 > 0$, $w_{12} = 1-2w_1$ such that $u_{1E} = 0$, that is $\alpha^2(h(1-h)-f-c)+\alpha(1-\alpha)(1-h-f-c)+(1-\alpha)^2\left(w_1(1-h-f-c)+(1-2w_1)((1-h)h-f-c)\right) = 0$, which holds if and only if w_1 is an in S_I^f. The condition $0 \leq w_1 \leq \frac{1}{2}$ is equivalent to $\mathbf{c}_B - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_F - \mathbf{f}$ (=). Finally, $u_{1NE} = \alpha(1-\alpha)(h^2-f-c)+(1-\alpha)^2 w_2(h(1-h)-f-c)$ because $c \geq c_B - f > h(1-h)-f$

7.6.10 Proof of Proposition 12(j)

Set $\lambda = \frac{f+h^2}{c+f-h+2h^2}$; the denominator is positive as $c+f \geq c_F > h(1-2h)$. The following holds

- (i) $\frac{\partial \mathcal{L}}{\partial x_{12}} = \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} \geq \frac{\partial \mathcal{L}}{\partial x_0} = 0$ as $2f + h^2 + 2\frac{f+h^2}{c+f-h+2h^2}\left(h(1-h) - f - c\right) = f + \frac{f+h^2}{c+f-h+2h^2}(h-c-f)$ and $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{h^2(f+h-c)}{c+f-h+2h^2} \geq 0$ as $c \leq h - h^2$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial y_0}, \frac{\partial \mathcal{L}}{\partial y_1}, \frac{\partial \mathcal{L}}{\partial y_{12}}\}$ since $\frac{\partial \mathcal{L}}{\partial y_2} = \frac{(1-h)^2 f + h^2(1-c-h)}{c+f-h+2h^2} \geq 0$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f + h(1-h) + \frac{f+h^2}{c+f-h+2h^2}\left(h^2 + (1-h)^2 - 2f\right.$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{h(1-2h)(f+h-c)}{c+f-h+2h^2} \geq 0$ as $c \leq h - h^2$. A similar argument applies to prove $\frac{\partial \mathcal{L}}{\partial z_1} \geq \max\{\frac{\partial \mathcal{L}}{\partial z_0}, \frac{\partial \mathcal{L}}{\partial z_2}, \frac{\partial \mathcal{L}}{\partial z_{12}}\}$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_1} = \frac{\partial \mathcal{L}}{\partial w_2} \geq \max\{\frac{\partial \mathcal{L}}{\partial w_0}, \frac{\partial \mathcal{L}}{\partial w_{12}}\}$ as $\frac{\partial \mathcal{L}}{\partial w_1} = f + \frac{f+h^2}{c+f-h+2h^2}(1-h-f-c) \geq 0$ and $\frac{\partial \mathcal{L}}{\partial w_{12}} = 2f + (1-h)^2 + 2\frac{f+h^2}{c+f-h+2h^2}\left(h(1-h) - f - c\right)$, $\frac{\partial \mathcal{L}}{\partial w_1} - \frac{\partial \mathcal{L}}{\partial w_{12}} = \frac{(1-2h)(h-h^2-c)}{c+f-h+2h^2} \geq 0$, since $\mathbf{c} \leq \mathbf{h} - \mathbf{h}^2$.

From (i)-(iii) it is optimal to set $x_1 = x_2$, $x_{12} = 1 - 2x_1$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 = \frac{1}{2}$ such that $u_{1E} = 0$, that is $\alpha^2\left(x_1(h-f-c) + (1-2x_1)(h(1-h)-f-c)\right) + \alpha(1-\alpha)(1-h-f-c) + (1-\alpha)^2\frac{1}{2}(1-h-f-c) = 0$ which holds if and only if x_1 is as in S_J^f. The condition $0 \leq x_1 \leq \frac{1}{2}$ is equivalent to $\mathbf{c}_F - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_G - \mathbf{f}$. Finally, $u_{1NE} = \alpha^2 x_2(h(1-h) - f - c) + \alpha(1-\alpha)\left(h^2 - f - c\right) + (1-\alpha)^2\frac{1}{2}(h(1-h) - f - c) < 0$ as $c + f \geq c_F > h(1-h)$.

7.6.11 Proof of Proposition 12(k)

Set $\lambda = \frac{f}{c+f-h}$; the denominator is positive as $c \geq h + \frac{h^2}{(1-h)^2}f$ implies $c + f - h > 0$. The following holds.

- (i) $\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = 0 = \frac{\partial \mathcal{L}}{\partial x_0}$, $\frac{\partial \mathcal{L}}{\partial x_{12}} = -\frac{h^2(f+h-c)}{c+f-h} \leq 0$ as $c \leq h + f$.
- (ii) $\frac{\partial \mathcal{L}}{\partial y_2} = f + \frac{f}{c+f-h}(1-h-f-c) \geq 0$, $\frac{\partial \mathcal{L}}{\partial y_{12}} = 2f + h(1-h) + \frac{f}{c+f-h}\left(h^2 + (1-h)^2 - 2f - 2c\right)$ and $\frac{\partial \mathcal{L}}{\partial y_2} - \frac{\partial \mathcal{L}}{\partial y_{12}} = \frac{h(1-h)(f+h-c)}{c+f-h} \geq 0$.
- (iii) $\frac{\partial \mathcal{L}}{\partial w_{12}} = 2f + (1-h)^2 + 2\frac{f}{c+f-h}\left(h(1-h) - f - c\right)$, $\frac{\partial \mathcal{L}}{\partial w_1} = f + \frac{f}{c+f-h}(1-h-f-c) \geq 0$ and $\frac{\partial \mathcal{L}}{\partial w_{12}} - \frac{\partial \mathcal{L}}{\partial w_1} = \frac{(1-h)^2(c-h) - h^2 f}{c+f-h} \geq 0$ as $\mathbf{c} \geq \mathbf{h} + \frac{h^2}{(1-h)^2}\mathbf{f}$.

From (i)-(iii) it is optimal to set $x_1 = x_2 < \frac{1}{2}, x_0 = 1 - 2x_1, y_2 = 1$, $z_1 = 1$, $w_{12} = 1$ such that $u_{1E} = 0$, that is $\alpha^2 x_1\left(h - f - c\right) + \alpha(1-\alpha)(1-h-f-c) + (1-\alpha)^2((1-h)h - f - c) = 0$, which holds if and only if x_1 is as in S_K^f (=). The condition $0 \leq x_1 \leq \frac{1}{2}$ is equivalent to $\mathbf{c}_E - \mathbf{f} \leq \mathbf{c} \leq \mathbf{c}_D - \mathbf{f}$ (=). Finally, $u_{1NE} = \alpha^2\left(x_2(h(1-h) - f - c) + (1-2x_2)(h-f-c)\right) + \alpha(1-\alpha)\left(h^2 - f - c\right) < 0$ as $c \geq h + \frac{h^2}{(1-h)^2}f$.

7.7 Proof of Proposition 13

When $\alpha = \frac{1}{2}$ and $c \geq \frac{1}{2}$, from Proposition 12 it follows that the optimal information structure is S_D^f if $f \leq \min\{(c-h)\frac{h^2}{(1-h)^2}, 1-h-c\}$ (since $c_D - c < 0$), is S_G^f if $(c-h)\frac{(1-h)^2}{h^2} \leq f \leq c_G - c$ (since $c_E - c < 0$), is S_H^f if $\max\{(c-h)\frac{(1-h)^2}{h^2}, c_G - c\} < f \leq 1 - h - c$. In each of these cases, the revenue is increasing in f, hence the optimal f is $1 - h - c$. In particular, the revenue under S_D^f is

$$
\begin{aligned}
& 2\alpha(1-\alpha)f + (1-\alpha)^2\left((1-2w_1)(2f + (1-h)^2) + 2w_1 f\right) \\
= \; & (1-\alpha)^2(1-h)^2 + 2(1-\alpha)f - 2(1-\alpha)\left((1-h)^2 + f\right)\frac{c + f - (1-h)(h + \alpha - h\alpha)}{c + f + (1-2h)(1-h)}
\end{aligned}
$$

and has derivative $2(1-\alpha)(\alpha+1)(1-h)^2\frac{c-h+h^2}{(c-3h+x+2h^2+1)^2}$, which is positive as $c \geq \frac{1}{2}$ implies $c > h - h^2$.

The revenue under S_H^f is

$$\alpha^2\left(2\cdot\frac{(1-\alpha^2)(1-h-c-f)}{2\alpha^2(c+f-h)}f\right)+2\alpha(1-\alpha)f+(1-\alpha)^2 f$$

with derivative $(1-\alpha)(\alpha+1)(1-2h)\frac{c-h}{(c-h+f)^2}>0$ as $c\geq\frac{1}{2}$ implies $c>h$.

The revenue under S_G^f is

$$\alpha^2 f+\alpha(1-\alpha)f+(1-\alpha)\alpha f+(1-\alpha)^2\left((1-2w_1)(2f+(1-h)^2)+w_1 f+w_1 f\right)$$

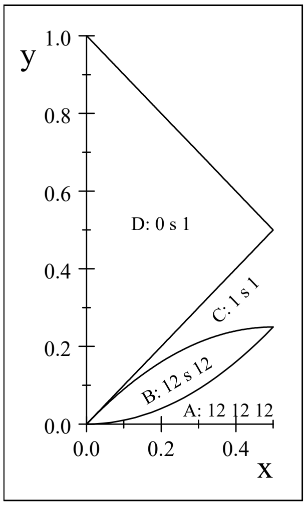
given $w_1=\frac{(\alpha^2-2\alpha+2)(c+f-\frac{2\alpha-2h^2\alpha^2+4h^2\alpha-2h^2+5h\alpha^2-6h\alpha+2h-2\alpha^2}{2-2\alpha+\alpha^2})}{2(1-\alpha)^2(c+f+(1-2h)(1-h))}$, with derivative $\frac{\left(7-14h+8h^2\right)(c-h+h^2)}{4(c+f-3h+2h^2+1)^2}>0$.

7.8 Proof of Proposition 14

We first deal with the socially optimal entry. In state ww, social welfare is maximized by (i) $x_{12}=1$ if $c<h-h^2$; (ii) $x_1=x_2=\frac{1}{2}$ if $h-h^2<c<h$; (i) $x_0=1$ if $h<c$. In state ws (sw) social welfare is maximized by (i) $y_{12}=1$ ($z_{12}=1$) if $c<h^2$; (ii) $y_2=1$ ($z_1=1$) if $h^2<c$. In state ss, social welfare is maximized by (i) $w_{12}=1$ if $c<h-h^2$; (ii) $w_1=w_2=\frac{1}{2}$ if $h-h^2<c$. Figure 4 below summarizes the socially optimal entry in the different states of the world. Precisely, it partitions the space (h,c), in four regions denoted A,B,C,D, and for each region 0 indicates that no entry is optimal, 1 indicates that it is optimal that only bidder 1 enters (or equivalently only bidder 2), s indicates that only the strong bidder enters in the social optimum, 12 indicates that entry of both bidders maximizes social welfare. For instance, in region B it is optimal that both bidders enter in state ww, that only the strong bidder enters in states ws,sw, that both bidders enter in state ss:

Figure 4: Socially optimal entry depending on the state of the world

$ww, ws\&sw, ss$:



Let $r=1-\varepsilon_r$ and $f=-c+\varepsilon_f$ with $\varepsilon_r,\varepsilon_f$ close to 0. Then

$$\begin{aligned}u_{1E} &= \alpha^2\left(x_1(h\varepsilon_r-\varepsilon_f)+x_{12}(h(1-h)\varepsilon_r-\varepsilon_f)\right)+\alpha(1-\alpha)\left(y_1(h\varepsilon_r-\varepsilon_f)+y_{12}(h^2\varepsilon_r-\varepsilon_f)\right)\\&+\alpha(1-\alpha)\left(z_1((1-h)\varepsilon_r-\varepsilon_f)+z_{12}((1-h)^2\varepsilon_r-\varepsilon_f\right)+(1-\alpha)^2\left(w_1((1-h)\varepsilon_r-\varepsilon_f)+w_{12}(h(1-h)\varepsilon_r-\varepsilon_f)\right)\end{aligned}$$

$$\begin{aligned}u_{1NE} &= \alpha^2\left(x_2(h(1-h)\varepsilon_r-\varepsilon_f+x_0(h\varepsilon_r-\varepsilon_f)\right)+\alpha(1-\alpha)\left(y_2(h^2\varepsilon_r-\varepsilon_f)+y_0(h\varepsilon_r-\varepsilon_f)\right)\\&+\alpha(1-\alpha)\left(z_2((1-h)^2\varepsilon_r-\varepsilon_f)+z_0((1-h)\varepsilon_r-\varepsilon_f)\right)+(1-\alpha)^2\left(w_2(h(1-h)\varepsilon_r-\varepsilon_f)+w_0((1-h)\varepsilon_r-\varepsilon_f)\right)\end{aligned}$$

Revenue when two bidders enter is 1 if both have value 1, $1-\varepsilon_r$ if only 1 has value 1, 0 if both have value 0, all minus $2c-2\varepsilon_f$, that is it is social welfare, Revenue if only one enters is $1-\varepsilon_r$ if the bidder has value 1, 0 if the bidder has value 0, minus $c-\varepsilon_f$, just like social welfare. We only need to verify that IC_{1E} and IC_{1NE} hold.

- In region A, the socially optimal information structure is $x_{12}=1$, $y_{12}=1$, $z_{12}=1$, $w_{12}=1$. Then

$$u_{1E} = c_A\varepsilon_r - \varepsilon_f$$

and u_{1NE} is undefined as signal NE is never sent. Hence the desired entry is achieved as long as $c_A\varepsilon_r > \varepsilon_f$.

- In region B, the socially optimal structure is $x_{12}=1$, $y_2=1$, $z_1=1$, $w_{12}=1$. Then

$$\begin{aligned} u_{1E} &= (1-h)\left(h+\alpha(1-\alpha)(1-2h)\right)\varepsilon_r - \varepsilon_f\left(1-\alpha+\alpha^2\right) \\ u_{1NE} &= \alpha(1-\alpha)\left(h^2\varepsilon_r - \varepsilon_f\right) \end{aligned}$$

Hence the desired entry is achieved as long as

$$\frac{(1-h)\left(h+\alpha(1-\alpha)(1-2h)\right)}{1-\alpha+\alpha^2}\varepsilon_r > \varepsilon_f > h^2\varepsilon_r \tag{28}$$

There exist $\varepsilon_r > 0$, $\varepsilon_r > 0$ which satisfy (28) as $\frac{(1-h)(h+\alpha(1-\alpha)(1-2h))}{1-\alpha+\alpha^2} - h^2 = \frac{-(2-3\alpha+3\alpha^2)h^2+(3\alpha^2-3\alpha+1)h+\alpha-\alpha^2}{1-\alpha+\alpha^2}$ is positive.

- In region C, the socially optimal structure information structure is $x_1 = x_2 = \frac{1}{2}$, $y_2 = 1$, $z_1 = 1$, $w_1 = w_2 = \frac{1}{2}$. Then

$$\begin{aligned} u_{1E} &= \frac{1}{2}\left(1-h-\alpha^2+2h\alpha^2\right)\varepsilon_r - \frac{1}{2}\varepsilon_f \\ u_{1NE} &= \left(\frac{1}{2}h - 2h^2\alpha^2 + 2h^2\alpha - \frac{1}{2}h^2 + h\alpha^2 - h\alpha\right)\varepsilon_r - \frac{1}{2}\varepsilon_f \end{aligned}$$

Hence the desired entry is achieved as long as

$$\left(1-h-\alpha^2+2h\alpha^2\right)\varepsilon_r > \varepsilon_f > (h-4h^2\alpha^2+4h^2\alpha-h^2+2h\alpha^2-2h\alpha)\varepsilon_r \tag{29}$$

There exist $\varepsilon_r > 0$, $\varepsilon_r > 0$ which satisfy (29) as $\left(1-h-\alpha^2+2h\alpha^2\right) - (h-4h^2\alpha^2+4h^2\alpha-h^2+2h\alpha^2-2h\alpha) = (2\alpha-1)^2h^2 + (1-\alpha)(1-2h+\alpha)$ is positive.

- In region D, the socially optimal information structure is $x_0=1$, $y_2=1$, $z_1=1$, $w_1=w_2=\frac{1}{2}$. Then

$$\begin{aligned} u_{1E} &= \frac{1}{2}\left(1-\alpha^2\right)\left((1-h)\varepsilon_r - \varepsilon_f\right) \\ u_{1NE} &= \frac{1}{2}h\left(1-h-2\alpha+4h\alpha+3\alpha^2-3h\alpha^2\right)\varepsilon_r - \frac{1}{2}\left(\alpha^2+1\right)\varepsilon_f \end{aligned}$$

Hence the desired entry is achieved as long as

$$(1-h)\varepsilon_r > \varepsilon_f > \frac{h\left(1-h-2\alpha+4h\alpha+3\alpha^2-3h\alpha^2\right)}{\alpha^2+1}\varepsilon_r \tag{30}$$

There exist $\varepsilon_r > 0$, $\varepsilon_r > 0$ which satisfy (30) as $1-h-\frac{h\left(1-h-2\alpha+4h\alpha+3\alpha^2-3h\alpha^2\right)}{\alpha^2+1} = \frac{(1-3\alpha)(1-\alpha)h^2-\left(2-2\alpha+4\alpha^2\right)h+\alpha^2+1}{\alpha^2+1}$ is positive (the numerator is decreasing in h and has value $\frac{1}{4}(1-\alpha^2) > 0$ at $h=\frac{1}{2}$).

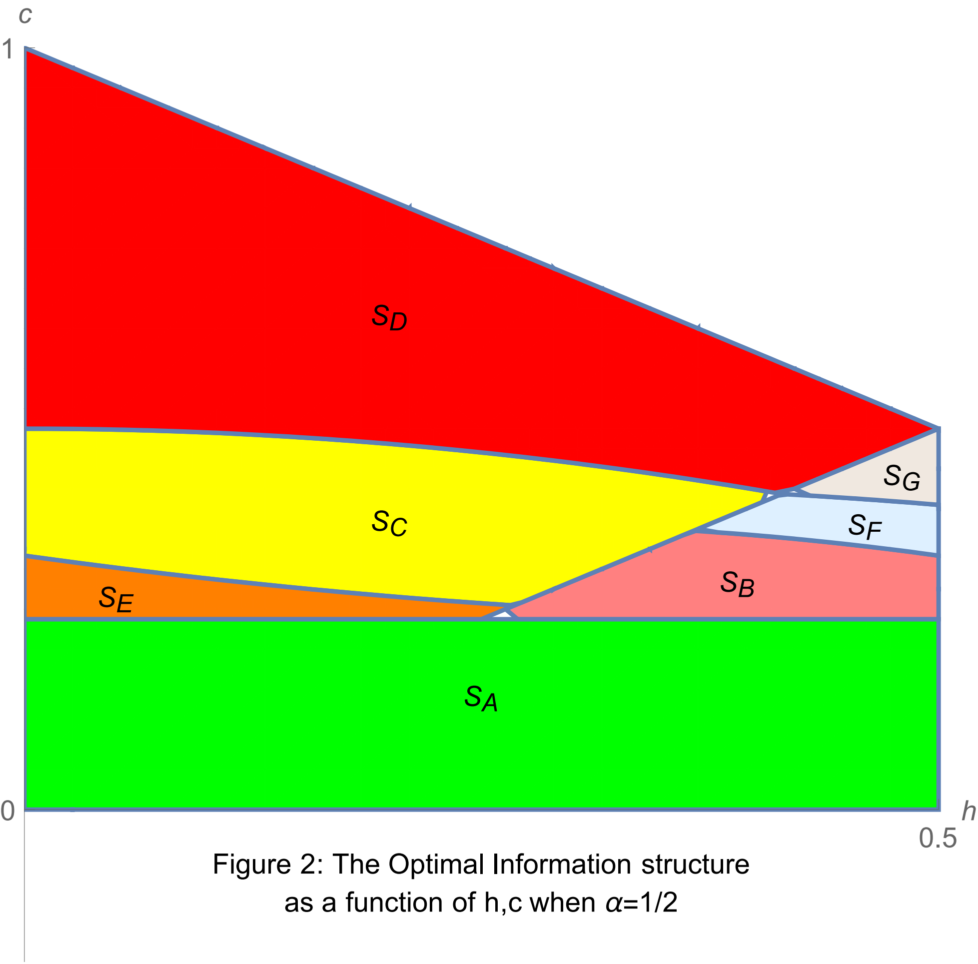


Figure 2: The Optimal Information structure
as a function of h,c when α=1/2

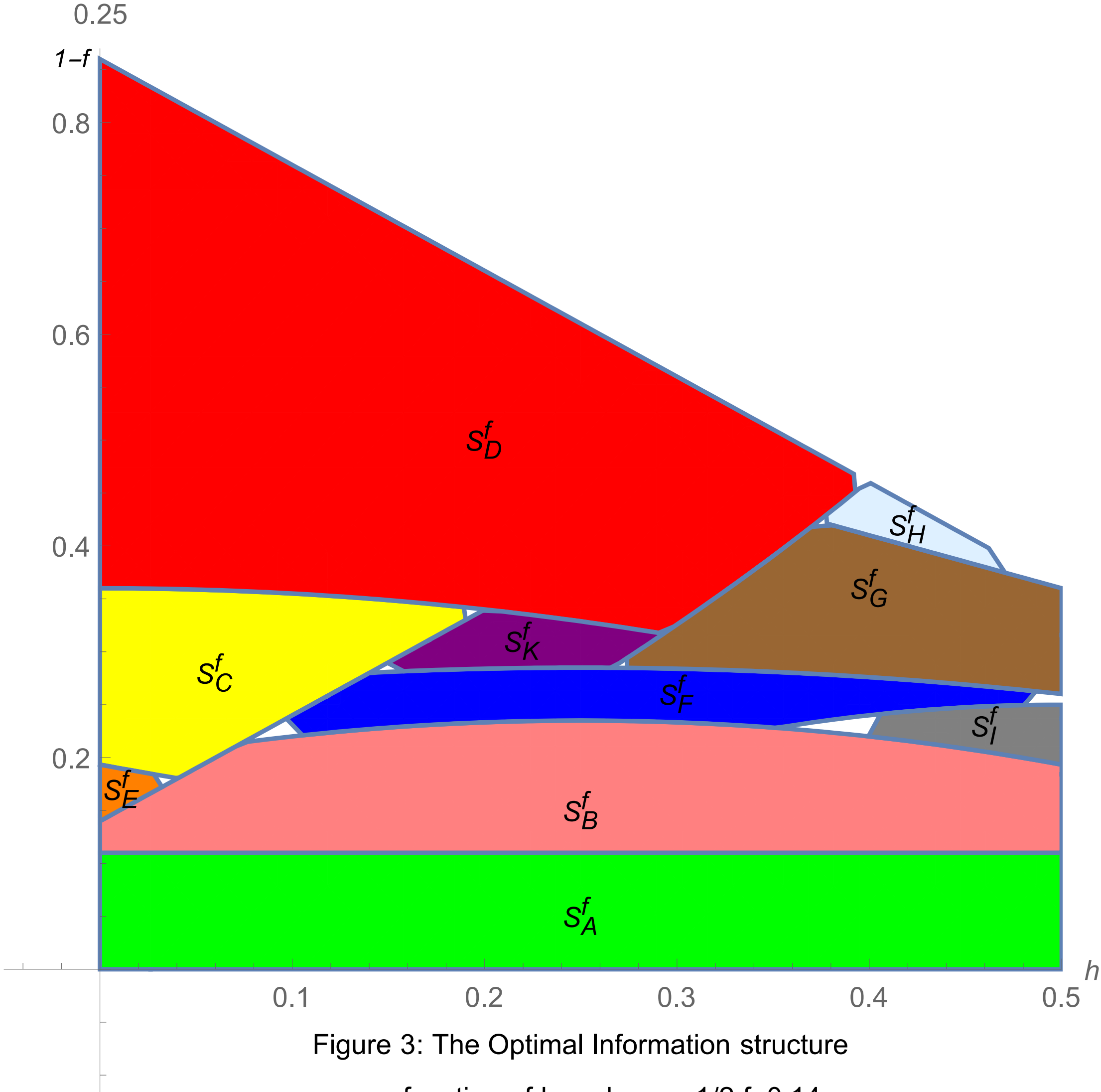


Figure 3: The Optimal Information structure

as a function of h,c when α=1/2,f=0.14