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Global Analysis and Indeterminacy in a Two-sector Growth Model with Human Capital[☆]

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**Abstract**

The purpose of the present paper is to highlight some features of global dynamics of the two-sector growth model with accumulation of human and physical capital analyzed by Brito and Venditti [14], which is a specification of the model proposed by Mulligan and Sala-i-Martin [28]. In particular, our analysis focuses on the context in which Brito-Venditti system admits two balanced growth paths each of them corresponding, after a change of variables, to an equilibrium point of a 3-dimensional system, and proves the possible existence of points $\overline{P}$ such that in any neighborhood of $\overline{P}$ lying on the plane corresponding to a fixed value of the state variable there exist points $\overline{Q}$ whose positive trajectories tend to either equilibrium point. This implies that equilibrium selection in Brito-Venditti system may depend on expectations of economic agents rather than on the history of the economy. That is, economies with identical technologies and preferences, starting from the same initial values of the state variables (history), may follow rather different equilibrium trajectories according to the economic agents' choices of the initial values of the jumping variables (expectations). Moreover we prove that the basins of attraction (two or three dimensional) of locally indeterminate equilibrium points may be very large, as they may extend up to the boundary of the system phase space.

*Keywords:* global and local indeterminacy; two-sector model; endogenous growth; poverty trap; global analysis
*JEL:* C62, E32, O41

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## 1 Introduction

Equilibrium selection in dynamic optimization models with externalities may depend on expectations of economic agents rather than on the history of the economy, as Krugman [19] and Matsuyama [21] pointed out in their seminal papers. Economies with identical technologies and preferences, starting from the same initial values of the state variables (history), may follow rather different equilibrium trajectories according to the economic agents' choices of the initial values of the jumping variables (expectations). A well known context in which expectations matter is that in which the dynamic system describing the evolution of the economy admits a locally attracting equilibrium point (which may correspond to a balanced growth path). In such a case, if the initial values of the state variables are close enough to the equilibrium values, the transition dynamics depend on the initial choice of the jumping variables and so there exists a continuum of equilibrium trajectories that the economy may follow to approach the equilibrium point. There exists an enormous literature about this type of indeterminacy, which is known with the term "local indeterminacy".[1]. The analysis of the linearization of a dynamic system around an equilibrium point gives all information required to detect local indeterminacy (if the equilibrium point is hyperbolic).[2] The relative simplicity of local analysis explains why a great amount of works in literature focuses on local indeterminacy issues. However a fast growing number of contributions suggests caution in drawing predictions on the future evolution of the economy based exclusively on local analysis; in fact, local stability analysis refers to a *small* neighborhood of an equilibrium point, whereas the initial values of the jumping variables do not have to belong to such a neighborhood (see, among the others, [21, 33, 11, 13, 10, 16, 24]). According to such works, global analysis of dynamic systems is necessary to get satisfactory information about the equilibrium selection process. Global analysis allows us to highlight more complex contexts in which equilibrium selection is not univocally determined by the initial values of the state variables. The indeterminacy, in such contexts, is called "global". There is not a unique definition of "global indeterminacy" in economic literature, differently from the case of local indeterminacy. Some authors (see, among the others, [13, 24])[3] use the term *global indeterminacy* to refer to all the contexts in which, given the initial values of the state variables, there exists a continuum of equilibrium trajectories which lies outside a "small" neighborhood of an equilibrium point. By such a

[1] See [9]. Even if the main body of the literature on local indeterminacy concerns economies with increasing social returns (see, e.g. [6, 12]), a growing proportion of articles deals with models where indeterminacy is obtained under the assumption of social constant return technologies, see, e.g., [9, 25, 27].

[2] In particular, local indeterminacy occurs if the number of eigenvalues with negative real parts of the linearization matrix evaluated at the equilibrium point is greater than the number of state variables. So, in a 2-dimensional system, we have local indeterminacy if and only if the equilibrium point is a sink.

[3] In [24], it is simply stated: "If equilibrium is indeterminate for a reason different from the case of local indeterminacy, it is said that equilibrium is globally indeterminate".

definition, global indeterminacy occurs, for example, if there exists an attracting limit cycle around an equilibrium point (see, e.g., [22, 30, 32]). Therefore, according to it, global indeterminacy may be observed even if a unique equilibrium point exists. Another definition of global indeterminacy (implicitly given in [21] and explicitly stated, among the others, in [14, 34]) requires, instead, the existence of at least two equilibrium points. Hence *global indeterminacy* occurs if there exist multiple equilibrium trajectories, from a given initial condition, approaching different equilibrium points. The latter definition can be extended to take into account the scenario where the economy can follow equilibrium trajectories converging towards different $\omega$-limit sets, not necessarily coinciding with equilibrium points. For example, in [11, 24], the economy can approach either a locally determinate equilibrium point or an attracting homoclinic trajectory. Notice that, according to the latter definition of global indeterminacy, from given initial conditions the economy can follow equilibrium trajectories along which the long run behavior of the state variables is rather different, in that the trajectories converge to different $\omega$-limit sets. This may not happen when the equilibrium selection process is globally indeterminate according to the former definition (for example, all the trajectories approaching a unique limit cycle exhibit the same long run behavior). However, even in a context in which there exists a unique $\omega$-limit set, the long run behavior of trajectories can be different, as is the case, in particular, when the $\omega$-limit set is a chaotic attractor (see, e.g., [1, 2, 13]). The purpose of the present paper is to show examples proving the occurrence of global indeterminacy, in the two senses by which it is known in literature, in the two-sector growth model with accumulation of human and physical capital analyzed by Brito and Venditti [14], which is a particular specification of the more general model proposed by [28]. The Brito-Venditti 3-dimensional system can admit two balanced growth paths that can be either simultaneously locally indeterminate (one with a 2-dimensional stable manifold, the other with a 3-dimensional one) or only one indeterminate and the other determinate (i.e. with a 1-dimensional stable manifold or repelling). Therefore, the system offers a particularly rich environment where to apply global analysis techniques. Obviously, our analysis is not exhaustive; in fact, we limit ourselves to explore two cases where the Brito-Venditti system admits two balanced growth paths, each of them corresponding, after a change of variables, to an equilibrium point of a 3-dimensional system. In the former one, the two equilibrium points have, respectively, a 2-dimensional and a 1-dimensional stable manifold (i.e. they are, respectively, in the Brito-Venditti terminology, locally indeterminate of order 2 and determinate). In the latter case, instead, the stable manifolds of the two equilibria have, respectively, dimension two and three (i.e. they are locally indeterminate of order 2 and 3). In both cases we provide examples where we prove the existence of points $\overline{P}$ such that in any neighborhood of $\overline{P}$ lying on the plane corresponding to a fixed value of the state variable there exist points $Q$ whose positive trajectories tend to either equilibrium point (these results are illustrated in Figures 1, 2, 5, 6). In such a context, the 2-dimensional stable manifold of the order 2 locally indeterminate equilibrium, in the former case, and the basin of the attracting equilibrium, in the latter case, are both unbounded (i.e. they extend to the boundary of the originary phase-space).

The results concerning the former case are obtained assuming that the amount of externalities is the same in both sectors (i.e. $b_1 = b_2$ in the Brito-Venditti model). Under such assumption, there exists an invariant plane and the dynamics is completely described by a 2-dimensional system. In such a simplified context, it is also possible to prove that when the locally indeterminate equilibrium point becomes a repellor, a supercritical Hopf bifurcation occurs giving rise to an attracting (i.e. endowed with a 2-dimensional stable manifold) limit cycle. When this happens, global indeterminacy is observed (see Figure 1) in a context where no equilibrium point is locally indeterminate (an analogous result is obtained by [11, 16]).

In the latter case, the dimension of the Brito-Venditti system cannot be reduced and consequently global analysis of the system becomes more complex. In such a context, our result, i.e. the unboundedness, for suitable values of the parameters, of the basin of the attracting equilibrium, appears to contain more information than other global indeterminacy results, where the equilibrium is shown to be globally indeterminate in the interior of a two-dimensional invariant region enclosed by a periodic or homoclinic orbit (see, e.g., [11, 24]).

Very few authors have engaged in the investigation of global indeterminacy in two-sector models with human and physical capital. In a context in which a unique balanced growth path exists, [7] points out the possibility of a Hopf bifurcation in the Lucas model (see also [20, 22, 23]). In a context in which two balanced growth paths coexist, besides the cases in which the dynamics can be fully analyzed by imposing specific conditions on parameter values (see e.g. [26]), only [24] (to the best of our knowledge) uses global analysis techniques to prove the existence of global indeterminancy according to the two definitions given above. In particular, the authors analyze a model where physical capital is not an input in the production process of human capital and apply a theorem due to [18] to show that their dynamic system undergoes a homoclinic bifurcation.

The present paper has the following structure. Section 2 briefly presents the set-up of the model of Brito and Venditti and the associated dynamic system. Section 3 introduces a change of variables in the Brito-Venditti system and retrieves some local analysis results contained in their paper which are useful for global analysis. Sections 4, 5 deal with global analysis of the Brito-Venditti model. A mathematical appendix containing some proofs concludes the paper.

## 2 The Brito-Venditti model

Brito and Venditti (see [14]) have analyzed a two-sector endogenous growth model in which the representative agent solves the following optimization problem:

$$Max_{C(t),\ K_{11}(t),\ K_{21}(t),\ K_{12}(t),\ K_{22}(t)} \qquad \int_0^{+\infty} \frac{C(t)^{1-\sigma}-1}{1-\sigma} e^{-\rho t}\, dt$$

subject to:

$$\begin{aligned}
&\dot{K}_1(t) = Y_1(t) - C(t)\\
&\dot{K}_2(t) = Y_2(t)\\
&Y_j(t) = e_j(t)K_{1j}(t)^{\beta_{1j}}K_{2j}(t)^{\beta_{2j}}, \quad j = 1,2\\
&K_i(t) = K_{i1}(t) + K_{i2}(t), \quad i = 1,2\\
&K_j(0) > 0, \ \ \{e_j(t)\}_{t=0}^{+\infty}, \quad j = 1,2, \quad \text{given.}
\end{aligned} \tag{1}$$

where $K_1(t)$ and $K_2(t)$ represent physical and human capital, respectively; $K_{ij}(t)$ is the amount of capital good $i = 1,2$ used in sector $j = 1,2$; $\sigma > 0$ is the inverse of the elasticity of intertemporal substitution in consumption; $\rho > 0$ is the discount rate.

Each technology $Y_j(t)$ is characterized by constant returns at the private level, that is, $\sum_{i=1}^{2}\beta_{ij} = 1$, $j = 1,2$, $\beta_{ij} > 0$. $e_1(t)$ and $e_2(t)$ are productive externalities, assumed to be functions of physical capital by unit of efficient labor, that is:

$$e_j(t) = \overline{k}(t)^{b_j}, \quad j = 1,2 \tag{2}$$

where $\overline{k}(t) = \overline{K}_1(t)/\overline{K}_2(t)$, $\overline{K}_1(t)$ and $\overline{K}_2(t)$ are the economy-wide average stocks of physical and human capital, and $b_j \in [0,1]$. Therefore, Brito and Venditti assume external effects derived from a knowledge-based definition of physical capital.

The representative agent considers $\overline{K}_1(t)$ and $\overline{K}_2(t)$ as exogenously determined; however, along the equilibrium trajectories, $\overline{K}_i = K_i$ and $\overline{k}(t) = k(t) = K_1(t)/K_2(t)$ hold and the technologies $Y_1(t)$ and $Y_2(t)$ at the social level are:

$$\begin{aligned}
Y_1(t) &= K_{11}(t)^{\beta_{11}}K_{21}(t)^{\beta_{21}}k(t)^{b_1} = K_{11}(t)^{\beta_{11}}K_{21}(t)^{\beta_{21}}\left(\frac{K_{11}(t)+K_{12}(t)}{K_{21}(t)+K_{22}(t)}\right)^{b_1}\\
Y_2(t) &= K_{12}(t)^{\beta_{12}}K_{22}(t)^{\beta_{22}}k(t)^{b_2} = K_{12}(t)^{\beta_{12}}K_{22}(t)^{\beta_{22}}\left(\frac{K_{11}(t)+K_{12}(t)}{K_{21}(t)+K_{22}(t)}\right)^{b_2}
\end{aligned}$$

Notice that $Y_1(t)$ and $Y_2(t)$ represent constant returns technologies. Therefore, the economy-wide external effects are formulated in such a way that the return to scale in both sectors are constant at the private and social levels. This assumption meets the empirical findings of [5] about the aggregate returns to scale in the US production and avoids the existence of private positive profits, which would stimulate entry of new firms (see [8], p. 69).

It is worth to stress that $K_1(t)$ and $K_2(t)$ could be interpreted as other forms of capital. The key distinction between these capital goods is that $K_1(t)$ is a perfect substitute for consumption while this is not the case for $K_2(t)$ (see [28] p. 742). Furthermore, notice that, in the general model proposed by [28], constant returns to scale at the private and social levels can be obtained only by posing $e_j(t) = \left(\frac{\overline{K}_1(t)}{\overline{K}_2(t)}\right)^{b_j}$ or $e_j(t) = \left(\frac{\overline{K}_2(t)}{\overline{K}_1(t)}\right)^{b_j}$. That is, it is necessary to assume some type of "congestion effect" produced by one capital good on the other, as done by Brito and Venditti.

The Hamiltonian and Lagrangian in current value associated to problem (1) are respectively:

$$\aleph = \frac{C(t)^{1-\sigma}-1}{1-\sigma} + P_1\left(Y_1 - C\right) + P_2 Y_2$$

$$\mathcal{L} = \aleph + R_1\left(K_1 - K_{11} - K_{12}\right) + R_2\left(K_2 - K_{21} - K_{22}\right)$$

where $P_i$ is the utility price and $R_i$ the rental rate of good $i = 1, 2$.

Applying the Pontryagin maximum principle and using the normalization of variables introduced by [15]:

$$\begin{aligned} k_1(t) &: = K_1(t)e^{-\gamma t} \\ k_2(t) &: = K_2(t)e^{-\gamma t} \\ c(t) &: = C(t)e^{-\gamma t} \\ p_1(t) &: = P_1(t)e^{-\gamma_p t} \\ p_2(t) &: = P_2(t)e^{-\gamma_p t} \end{aligned}$$

where $\gamma > 0$ and $\gamma_p = -\sigma\gamma < 0$ represent, respectively, the (constant) rate of growth of $K_1(t)$, $K_2(t)$, $C(t)$ and the rate of decrease of $P_1(t)$, $P_2(t)$ along a balanced growth path, Brito and Venditti obtain the 4-dimensional dynamic system:

$$\begin{cases} \dot{p}_1 = p_1\left(\rho + \sigma\gamma - r_1(\pi, k)\right) \\ \dot{p}_2 = p_2\left(\rho + \sigma\gamma - r_2(\pi, k)\right) \\ \dot{k}_1 = (\alpha_{11}(\pi, k) - \gamma)k_1 + \alpha_{12}(\pi, k)k_2 - p_1^{-\frac{1}{\sigma}} \\ \dot{k}_2 = \alpha_{21}(\pi, k)k_1 + \alpha_{22}(\pi, k)k_2 - \gamma k_2 \end{cases} \tag{3}$$

where $k_1$ and $k_2$ are the state variables while $p_1$ and $p_2$ are the jumping variables, with $\pi := \frac{p_2}{p_1}, k := \frac{k_1}{k_2}$. The transversality conditions are:

$$\lim_{t\to+\infty} p_1\left(t\right)k_1\left(t\right)e^{[\gamma(1-\sigma)-\rho]t} = \lim_{t\to+\infty} p_2\left(t\right)k_2\left(t\right)e^{[\gamma(1-\sigma)-\rho]t} = 0 \tag{4}$$

with the assumption $\gamma\left(1-\sigma\right) - \rho < 0$. Any solution $(k_1(t), k_2(t), p_1(t), p_2(t))$ of system (3) satisfying the transversality conditions (4) and initial conditions $(k_1(0), k_2(0)) = (k_1^0, k_2^0)$ is an optimal solution of problem (1) in that problem (1) satisfies the Arrow's condition (see [14]).

At an equilibrium point of (3) it holds, in particular, $r_1(\pi, k) = r_2(\pi, k) = r(\pi, k)$ and thus $\gamma = \frac{r(\pi,k)-\rho}{\sigma}$. The transversality conditions imply $0 < \gamma < r$

Furthermore $r_1(\pi, k) := c_1\pi^{\psi_{21}}k^{b_1\psi_{11}+b_2\psi_{21}}$, $r_2(\pi, k) := c_2\pi^{-\psi_{12}}k^{b_1\psi_{12}+b_2\psi_{22}}$, $\alpha_{ij}(\pi, k) := \psi_{ij}r_j(\pi, k)\pi^{j-i}$, $c_i := \left(\beta_i^*\right)^{\psi_{ii}}\left(\beta_j^*\right)^{\psi_{ji}}$ $i \neq j$, $\beta_i^* := \beta_{1i}^{\beta_{1i}}\beta_{2i}^{\beta_{2i}}$, $b_1, b_2 \in$

$[0,1]$.[4] The coefficients $\psi_{ij}$ are the entries of the matrix:

$$\Psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix} = \frac{1}{\beta_{11}-\beta_{12}} \begin{pmatrix} \beta_{22} & -\beta_{12} \\ -\beta_{21} & \beta_{11} \end{pmatrix} = B^{-1}$$

where:

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

is the matrix of private Cobb-Douglas coefficients satisfying $\beta_{11} + \beta_{21} = \beta_{12} + \beta_{22} = 1$, $\beta_{11} - \beta_{12} \neq 0$. Consequently, the entries of $\Psi$ satisfy the conditions $\psi_{11} + \psi_{21} = \psi_{12} + \psi_{22} = 1$, $\psi_{11} \cdot \psi_{22} > 0$, $\psi_{ii} \cdot \psi_{ij} < 0$ for $i \neq j$. Furthermore $\psi_{12}, \psi_{21} > 0 \iff \beta_{11} < \beta_{12}$, $\psi_{12} = \psi_{21} \iff \beta_{12} = \beta_{21}$ and $\psi_{11} = \psi_{22} \iff \beta_{11} = \beta_{22}$.

## 3 A change of variables in the Brito-Venditti system

By posing $\pi = e^u$, $k = e^v$, $p_1^{-\frac{1}{\sigma}} k_2^{-1} = e^w$ (i.e. $u = \ln \pi = \ln \frac{p_2}{p_1} = \ln \frac{P_2}{P_1}$, $v = \ln k = \ln \frac{k_1}{k_2} = \ln \frac{K_1}{K_2}$, $w = \ln \left(p_1^{\frac{1}{\sigma}} k_2\right)^{-1} = \ln \left(P_1^{\frac{1}{\sigma}} K_2\right)^{-1}$), we obtain, after multiplying the equations by $e^v$ (change of time), a 3-dimensional system defined in $\mathbb{R}^3$, whose trajectories generate those of (3). Namely:

$$\begin{cases} \dot{u} = e^v \big(r_1(u,v) - r_2(u,v)\big) = f(u,v) \\ \dot{v} = e^v \big(\psi_{11} r_1(u,v) - \psi_{22} r_2(u,v) + \psi_{12} r_2(u,v) e^{u-v} + \\ \quad -\psi_{21} r_1(u,v) e^{v-u}\big) - e^w = g(u,v) - e^w \\ \dot{w} = e^v \big( -\frac{\rho}{\sigma} + \frac{r_1(u,v)}{\sigma} - \psi_{22} r_2(u,v) - \psi_{21} r_1(u,v) e^{v-u}\big) = h(u,v) \end{cases} \tag{5}$$

where, by an abuse of language, $r_i(u,v) := r_i(e^u, e^v)$.

An equilibrium point $(\overline{u}, \overline{v}, \overline{w})$ of system (5) corresponds to a 1-dimensional manifold of equilibrium points of the Brito-Venditti system (3) defined, in the space $(\gamma, p_1, p_2, k_1, k_2)$, via the equations:

$$\begin{aligned} \gamma &= \frac{r(\overline{u}, \overline{v}) - \rho}{\sigma} \\ p_1 &= \left(e^{\overline{w}} k_2\right)^{-\sigma} \\ p_2 &= \overline{\pi} p_1 = e^{\overline{u}} p_1 = e^{\overline{u}} \left(e^{\overline{w}} k_2\right)^{-\sigma} \\ k_1 &= \overline{k} k_2 = e^{\overline{v}} k_2 \end{aligned}$$

The local analysis results of [14] can be retrieved by analyzing (5). In the remaining part of this section we focus on those on which our global analysis is built.

[4] Where $r_i$ represents the equilibrium rental rate $R_i/P_i$, $i = 1, 2$.

Pose:

$$\begin{aligned} \tau &:= \frac{b_1\psi_{12}+b_2\psi_{21}}{\psi_{12}+\psi_{21}} \\ \delta &:= \frac{(b_1-b_2)(\psi_{12}+\psi_{21}-1)}{\psi_{12}+\psi_{21}} \end{aligned} \tag{6}$$

implying $0 \leq \tau \leq 1$, $sgn(\delta) = sgn(b_1 - b_2)$. Since $\tau = 0 \Longleftrightarrow b_1 = b_2 = 0$, we assume in the following $\tau > 0$.

Then it is easily computed that the possible equilibrium points of (5) lie on the plane $u = \delta v + \overline{d}$, with $\overline{d} := (\psi_{12}+\psi_{21})^{-1} \ln \frac{c_2}{c_1}$. Moreover:

$$r_1(\delta v + \overline{d}, v) = r_2(\delta v + \overline{d}, v) = r(v) = ce^{\tau v}, \ c > 0 \tag{7}$$

It follows from straightforward computations that (5) has at most two equilibria if and only if one of the following cases occur:

1. $\psi_{12}, \psi_{21} > 0$ (implying $\psi_{12}, \psi_{21} > 1$ and therefore $|\delta| < 1$)
2. $\psi_{12}, \psi_{21} < 0$, $\delta > 1 + \tau$, $\sigma^{-1} - \psi_{22} > 0$
3. $\psi_{12}, \psi_{21} < 0$, $1 < \delta < 1 + \tau$, $\sigma^{-1} - \psi_{22} < 0$.

(5) has at most one equilibrium in all the other cases except when $\delta = 1+\tau$ and $\psi_{21}c + \frac{\rho}{\sigma} = 0$ or $\delta = 1$ and $\sigma^{-1} - \psi_{22} - \psi_{21}c \leq 0$. In the latter cases (5) has no equilibrium, except for $\delta = 1 + \tau$ and $\psi_{21}c - \frac{\rho}{\sigma} = \sigma^{-1} - \psi_{22} = 0$, when (5) has infinite equilibria.

Remember that $\psi_{12}, \psi_{21} > 0 \iff \beta_{11} < \beta_{12}$, where $\beta_{11}$ and $\beta_{12}$ measure, respectively, the physical capital intensity in sectors 1 (final good sector) and 2 (human capital sector). Then the above results show that, as stressed by Brito and Venditti, multiple equilibrium points (i.e. multiple balanced growth paths) can arise in both contexts $\beta_{11} < \beta_{12}$ (i.e. the final good is intensive in human capital at the private level) and $\beta_{11} > \beta_{12}$ (i.e. the final good is intensive in physical capital at the private level).[5]

Now let $P_0 = (u_0, v_0, w_0)$ be an equilibrium point of (5) and pose $r(v_0) = r_0$. Then its Jacobian matrix is:

$$J(P_0) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & 0 \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & -e^{w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & 0 \end{pmatrix} (P_0) \tag{8}$$

where $\frac{\partial f}{\partial u} = e^{v_0} r_0 (\psi_{12}+\psi_{21})$, $\frac{\partial f}{\partial v} = -\delta e^{v_0} r_0 (\psi_{12}+\psi_{21})$, while $\frac{\partial g}{\partial u} < 0$. Then set $\widetilde{h}(v) := h(\delta v + \overline{d}, v)$. It easily follows that:

$$sgn\left[\det J(P_0)\right] = sgn\left[\widetilde{h}'(v_0)(\psi_{12}+\psi_{21})\right] \tag{9}$$

[5]The relevance, with respect to the existing literature, of the local analysis results illustrated in this section is exhaustively discussed in Brito and Venditti's article.

In particular, assume $\psi_{12}, \psi_{21} > 0$ and two equilibria exist, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$. Then $\det J(P_1) > 0 > \det J(P_2)$.

Vice-versa, suppose $\psi_{12}, \psi_{21} < 0$ and $\delta \leq 1$. In this case at most one equilibrium $P_0$ exists, where $\det J(P_0) < 0$.

The following Proposition rephrases one of the Brito-Venditti results:

**Proposition 1.** *Let $P$ be one of the equilibria of (5). Then $\delta \geq 0$ (i.e. $b_1 \geq b_2$) implies trace $[J(P)] > 0$.*

*Proof.* See Appendix 6.1 □

In particular, if $\delta \geq 0$ (i.e. $b_1 \geq b_2$: the amount of externalities in the final good sector is greater than that in the human capital sector), $P$ cannot be an attractor. Hence, as underlined in Brito-Venditti's article, the coexistence of two local indeterminate equilibria (of order, respectively, two and three) can occur only if $b_1 < b_2$ and $\psi_{12}, \psi_{21} > 0$ (thus $> 1$). Finally the following Proposition reformulates results stated in Theorem 5 of [14], illustrating the local stability results relative to the above Cases **2** and **3**, when two equilibria exist.

**Proposition 2.** *Suppose in the above Cases **2** or **3** that two equilibria exist, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$. Then in Case **2** $P_1$ is a repellor, while $P_2$ is a saddle with a one-dimensional stable manifold. Vice-versa, in Case **3** $P_1$ is a saddle with a one-dimensional stable manifold, while $P_2$ can be either repelling or locally indeterminate of order two (i.e. its stable manifold can be two-dimensional).*

*Proof.* See Appendix 6.2 □

**Example 1.** *Let in system (5) $c_1 = c_2 = 1$ (this can be always obtained by a suitable translation of $(u, v, w)$ and a rescaling of the parameter $\rho$ and the time variable $t$). Pose $\psi_{21} = -\varepsilon - \varepsilon^3$, $\psi_{12} = -\varepsilon^2$, $\sigma^{-1} = 1 - \varepsilon^2$, $\rho = 2\exp(\tau v_2)\sigma\varepsilon^4$, $b_1 = 1$, $b_1 - b_2 = \varepsilon(1+\varepsilon)(1+\varepsilon+\varepsilon^2)/(1+\varepsilon+\varepsilon^2+\varepsilon^3)$, where $\varepsilon > 0$ is sufficiently small. Then the conditions of Case **3** are satisfied and there exist two equilibria, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$ and $\exp(v_2 - u_2) = 2\varepsilon$. Hence it is easily checked that $P_1$ is a saddle with a one-dimensional stable manifold, while $P_2$ is a saddle with a two-dimensional stable manifold.*

## 4 Global analysis in a context with indeterminacy of order 2

Our aim is to show, via global analysis of system (5), examples proving the occurrence of global indeterminacy in the two senses by which it is known in literature. In fact we will consider two cases where system (5) exhibits two equilibrium points. In the former one, object of the present section, the two equilibria will have, respectively, a 2-dimensional and a 1-dimensional stable manifold (i.e. they will be, in the Brito-Venditti terminology, locally indeterminate of order 2 and determinate)[6]. In the latter case, instead, the stable

[6]Notice that, in system (5), $v$ is a state variable while $u$ and $w$ are jump variables. So, an equilibrium point is locally determinate if it has a 1-dimensional stable manifold or is repelling

manifolds of the two equilibria will have, respectively, dimension two and three (i.e. one equilibrium will be attracting; in the Brito-Venditti terminology the equilibria will be locally indeterminate of order 2 and 3). In both cases we will prove, for suitable values of the parameters, the existence of points $\overline{P} = (\overline{u}, \overline{v}, \overline{w})$ such that in any neighborhood of $\overline{P}$ lying on the plane $v = \overline{v}$ (corresponding to a fixed value of the state variable $k = k_1/k_2 = K_1/K_2)$[7] there exist points $Q$ whose positive trajectories tend to either equilibrium. Moreover we will prove that the 2-dimensional stable manifold of the order 2 locally indeterminate equilibrium, in the former case, and the basin of the attracting equilibrium, in the latter case, can be both unbounded.

We start by stating the following result.

**Proposition 3.** *When $\delta = 0$, the plane $u = \overline{d}$ (recall $\overline{d} = (\psi_{12} + \psi_{21})^{-1} \ln \frac{c_2}{c_1}$) is invariant.*

*Proof.* Recall that $u = \delta v + \overline{d}$ implies $r_1(\delta v + \overline{d}, v) = r_2(\delta v + \overline{d}, v)$ and thus (see system (5)) $\dot{u} = 0$. Hence, when $\delta = 0$, $u = \overline{d}$ is invariant. □

Therefore we first assume $\delta = 0$ (i.e. $b_1 = b_2$: the amount of externalities is the same in both sectors). In such a context, if $\psi_{12}, \psi_{21} < 0$ (i.e. $\beta_{11} > \beta_{12}$: the final good sector is physical capital intensive at the private level), there exists at most one equilibrium $P_0$, lying on $u = \overline{d}$, such that $\det J(P_0) < 0 < trace\,[J(P_0)]$. Hence $P_0$ is locally determinate. If, instead, $\psi_{12}, \psi_{21} > 0$ (i.e. $\beta_{11} < \beta_{12}$: the final good sector is human capital intensive at the private level), there can exist up to two equilibria lying on the invariant plane $u = \overline{d}$. Suppose this is the case and denote the two equilibria as $P_1 = (\overline{d}, v_1, w_1)$ and $P_2 = (\overline{d}, v_2, w_2)$, with $v_1 < v_2$ (note that, by (7), the growth rate $\gamma$ associated to $P_2$ is higher than that associated to $P_1$). Then $\det J(P_1) > 0 > \det J(P_2)$, while $trace\,[J(P_1)], trace\,[J(P_2)] > 0$. Therefore $P_2$ is locally determinate, whereas $P_1$ can be either repelling or locally indeterminate of order 2. As a matter of fact, the system on the invariant plane $u = \overline{d}$ reduces to:

$$\begin{cases} \dot{v} = \widetilde{g}(v) - e^{w} \\ \dot{w} = \widetilde{h}(v) \end{cases} \tag{10}$$

where $\widetilde{g}(v) = g(\delta v + \overline{d}, v)$, $\widetilde{h}(v) = h(\delta v + \overline{d}, v)$. So, being $\delta = 0$, it follows that, on the plane $u = \overline{d}$, $\widetilde{g}'(v) = \frac{\partial g}{\partial v}$ and $\widetilde{h}'(v) = \frac{\partial h}{\partial v}$. Therefore $P_1$ is locally indeterminate of order 2 if and only if $\frac{\partial g}{\partial v}(\overline{d}, v_1) < 0$.

We refer to system (10), defined on the plane $u = \overline{d}$. It is easily computed that:

$$\begin{aligned} \widetilde{g}(v) &= r(v)(1 + e^{v-\overline{d}})(\psi_{12} e^{\overline{d}} - \psi_{21} e^{v}) \\ \widetilde{h}(v) &= e^{v} \left[ -\frac{\rho}{\sigma} + r(v)(\frac{1}{\sigma} + \psi_{12} - 1 - \psi_{21} e^{v-\overline{d}}) \right] \end{aligned}$$

[7] Remember that $v = \ln k = \ln \frac{k_1}{k_2} = \ln \frac{K_1}{K_2}$.

where $r(v) = ce^{\tau v}$, $\tau = b_1 = b_2$. Assuming $\psi_{12}, \psi_{21} > 0$ (and thus $> 1$), it easily follows that $\widetilde{h}(v)$ has two zeros, $v_1 < v_2$, if and only if $\widetilde{h}(v^*) > 0$, where $v^* = \overline{d} + \ln \frac{\tau\left(\frac{1}{\sigma} + \psi_{12} - 1\right)}{(1+\tau)\psi_{21}}$. On the other hand the function $w = \ln \widetilde{g}(v)$ is defined for $v < \overline{v} = d + \ln \frac{\psi_{12}}{\psi_{21}}$ and has a maximum at the point $v_0$, where $e^{v_0}$ is the positive solution of the equation $\psi_{21} e^{-\overline{d}} (2+\tau) x^2 - [\psi_{12}(1+\tau) - \psi_{21}] x - \tau \psi_{12} e^{\overline{d}} = 0$. Hence two equilibria exist if and only if there exist $v_1 < v_2$ such that $\widetilde{h}(v_1) = \widetilde{h}(v_2) = 0$ and $v_2 < \overline{v}$. Moreover $P_1 = \left(\overline{d}, v_1, w_1\right)$ has a two-dimensional stable manifold if and only if $v_0 < v_1$.

**Remark 1.** *Suppose all the previous conditions are satisfied. Then, by observing the phase portrait of system* (10)*, defined on $u = \overline{d}$, it easily follows that $\widetilde{P_1} = (v_1, w_1)$ is an attractor (in the plane $u = \overline{d}$), $\widetilde{P_2} = (v_2, w_2)$ is a saddle and, moreover, there is a repellor at the boundary point $v = +\infty, w = +\infty$ and an attractor at the boundary point $v = \overline{v}, w = -\infty$. Consequently, if for suitable values of the parameters* (10) *has no limit cycle, then the basin of attraction of $\widetilde{P_1}$ (i.e. the two-dimensional stable manifold of $P_1$) is limited by the stable manifold of $\widetilde{P_2}$, connecting $\widetilde{P_2}$ to the repellor $(+\infty, +\infty)$, and thus is unbounded. In the following we provide an example where that occurs.*

First of all, for sake of simplicity, we assume $\psi_{12} = \psi_{21} = \psi > 1$ (and thus $\psi_{11} = \psi_{22} = 1 - \psi$).[8] As a consequence $c_1 = c_2$ and therefore $\overline{d} = 0$ and $\overline{v} = 0$ (i.e. $e^{\overline{v}} = 1$). Moreover $\widetilde{g}'(v_0) = 0$ implies $e^{2v_0} = \frac{\tau}{2+\tau}$. It follows that there exist two equilibria $\widetilde{P_1}$ and $\widetilde{P_2}$ of system (10), where the former is an attractor and the latter a saddle, if and only if $\widetilde{h}(v_0), \widetilde{h}(0) < 0$ while $\widetilde{h}(v^*) > 0$, with $e^{v^*} = \frac{\tau\left(\frac{1}{\sigma} + \psi_{12} - 1\right)}{(1+\tau)\psi_{21}} < 1$. Denote, as above, by $v_1 < v_2$ the zeros of $\widetilde{h}(v)$ for $v \in (v_0, 0)$. By suitably varying $\rho$ and $\sigma$ we can have $v_1$ coincide with $v_0$, causing (generically) a Hopf bifurcation to occur. The following Proposition holds

**Proposition 4.** *Under our assumptions the Hopf bifurcation occurs and is supercritical (i.e. an attracting limit cycle arises around $\widetilde{P_1}$ when it becomes a repellor).*

*Proof.* See Appendix 6.3 □

Notice that, according to such a Proposition, the two coexisting $\omega$-limit sets, $\widetilde{P_2}$ and the limit cycle around $\widetilde{P_1}$, have respectively 1-dimensional and 2-dimensional stable manifolds lying in the plane $u = \overline{d}$. It is worth to note that this globally indeterminacy scenario occurs in a context in which $\widetilde{P_1}$ is a repellor and $\widetilde{P_2}$ is locally determinate, that is, in a context in which no equilibrium point is locally indeterminate (a similar result is obtained by [11,

[8] Remember that $\psi_{12} = \psi_{21} \iff \beta_{12} = \beta_{21}$ and $\psi_{11} = \psi_{22} \iff \beta_{11} = \beta_{22}$.

24, 16]). Figure 1 shows a numerical simulation of the phase portrait of system (10); observe that there exists an interval (which is, in fact, unbounded) of values of the predetermined variable $v$ from which the economy can approach either $\widetilde{P_2}$ or the limit cycle around $\widetilde{P_1}$, according to the initial choice of the jumping variable $w$ (the initial value of the other jumping variable $u$ is fixed at the value $u = \overline{d}$). In $\widetilde{P_1}$ the value of $v$ (and consequently, by (7), the value of the growth rate $\gamma$) is lower than in $\widetilde{P_2}$. However, even if the equilibrium $\widetilde{P_1}$ is not (generically) reachable by the economy, there exist a continuum of equilibrium growth trajectories approaching the cycle around $\widetilde{P_1}$. The basin of attraction of the cycle is limited by the 1-dimensional stable manifold of the locally determinate point $\widetilde{P_2}$. In particular, if the initial value $v_0$ of the predetermined variable $v$ is high enough, then there always exists an interval of initial values $w_0$ of the jumping variable $w$ such that the trajectory starting from $(v_0, w_0)$ approaches the limit cycle and there exist two values $w_1, w_2$ of $w$ such that the points $(v_0, w_1)$ and $(v_0, w_2)$ belong to the stable manifold of $\widetilde{P_2}$. Notice that, in such a context, the economy may approach the locally determinate point $\widetilde{P_2}$ by following rather different transition paths according to the initial choice ($w_1$ or $w_2$) of $w$ (a similar result is obtained in [11, 24], where the existence of a homoclinic trajectory is proven).

Now we want to produce an example where $\widetilde{P_1}$ is an attractor of (10) with an unbounded basin.

First of all we observe that system (10) can be regarded as a Liènard system when $v \in (-\infty, v_2)$. To fix the ideas, let us take $\tau = 0.5$. Then $v_0 = -\frac{1}{2}\ln 5$. If $\widetilde{h}(v_0) < 0$ and the parameters $\rho, \sigma, \psi$ are suitably chosen, an important Theorem on the uniqueness of limit cycles for Liènard systems (see [35]) can be applied. Precisely, consider the new variables $x = v - v_1$, $y = w - w_1$ and change $t$ into $-t$. Then the following Liènard system is defined in the strip $-\infty < x < \overline{x}$, where $\overline{x} = v_2 - v_1$.

$$\begin{cases} \dot{x} = \lambda\,(y) - \Phi(x) \\ \dot{y} = -\gamma(x) \end{cases} \tag{11}$$

where $\lambda\,(y) = e^{w_1}(e^y - 1)$, $\Phi(x) = \widetilde{g}(v_1 + x) - e^{w_1}$, $\gamma(x) = \widetilde{h}(v_1 + x)$. Then, posed $\overline{x} = v_2 - v_1$, $x_0 = v_0 - v_1 < 0$, $\varphi\,(x) = \Phi'(x)$, $\Gamma\,(x) = \int\limits_0^x \gamma(z)dz$, it is easily checked that the smooth system (11), defined in the strip $x \in (-\infty, \overline{x})$, satisfies:

1. $\lambda\,(y)$ is increasing and $y \cdot \lambda\,(y) > 0$ when $y \neq 0$
2. $(x - x_0) \cdot \varphi\,(x) < 0$ when $x \neq x_0$
3. $x \cdot \gamma\,(x) > 0$ when $x \neq 0$.

Moreover, by Theorem 3 of [35], if the further two conditions are met:

4 $\frac{\varphi(x)}{\gamma(x)}$ is non-decreasing in $(-\infty, b)$, where $b \in (-\infty, x_0)$ is defined by $\Phi(b) = 0$ (i.e. $\widetilde{g}(v_1 + b) = \widetilde{g}(v_1)$)

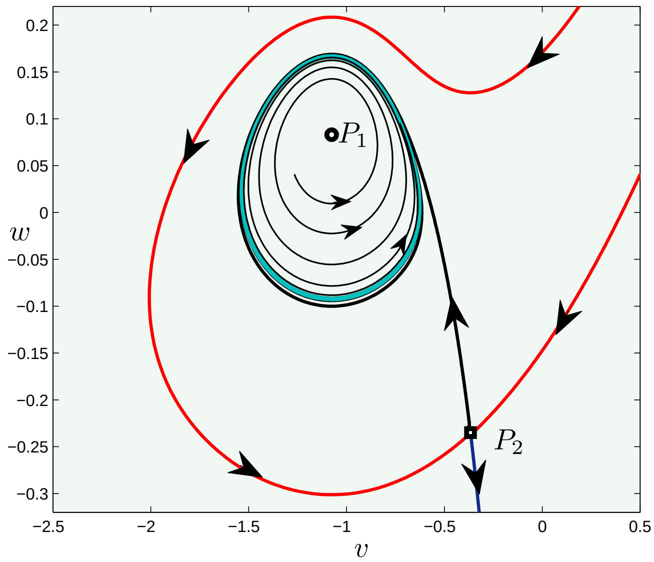


Figure 1: Global indeterminacy scenario in which two $\omega$-limit sets exist, $\widetilde{P_2}$ and the limit cycle around $\widetilde{P_1}$, having respectively 1-dimensional and 2-dimensional stable manifolds lying in the plane $u = \overline{d}$. The (unbounded) basin of attraction of the limit cycle is limited by the 1-dimensional stable manifold of the locally determinate point $\widetilde{P_2}$. Parameter values: $\tau = 0.3, c = 1, \psi = 1.698, \sigma = \dfrac{1}{3}$, $\rho = 0.752877378571337$.

5 the system of equations $\Phi(x) = \Phi(z)$, $\Gamma(x) = \Gamma(z)$ has at most one solution for $x \in (-\infty, b)$, $z \in (0, \overline{x})$

then (11) has at most one limit cycle, which, if it exists, is simple (hence it does not *generate* several limit cycles).

**Example 2.** *Let, in system* (5)*,* $\delta = 0$*,* $\tau = 0.5$*,* $\frac{\rho}{c} = \frac{1}{\sqrt[4]{5}}$*,* $\sigma = \frac{1}{3}$*,* $\psi_{12} = \psi_{21} = 1.698$*. Then* (5) *has two equilibria,* $P_1$ *and* $P_2$*, lying on the invariant plane* $u = 0$ *and the planar system* (11) *satisfies the above conditions 1-5.*

The following Theorem builds on Theorem 3 of [35] and gives sufficient conditions under which system (10) does not admit limit cycles, and therefore, for what we have said in Remark 1, the basin of attraction of $P_1$ is unbounded.

**Theorem 1.** *Assume system* (5) *has parameters* $\delta = 0$*,* $\tau = 0.5$*,* $\psi_{12} = \psi_{21} = \psi > 0$ *(and thus* $> 1$*). Assume there exist two equilibrium points* $P_1$ *and* $P_2$ *being, for the system* (10) *defined on the invariant plane* $u = 0$*, respectively a sink and a saddle. Then, if the planar system* (11) *satisfies conditions 1-5, the basin of attraction of* $P_1$*, on the plane* $u = 0$*, is unbounded and there exists a trajectory leaving from* $P_2$ *(as* $t \to -\infty$*) and converging to* $P_1$ *(as* $t \to +\infty$*) (see Figure 2).*

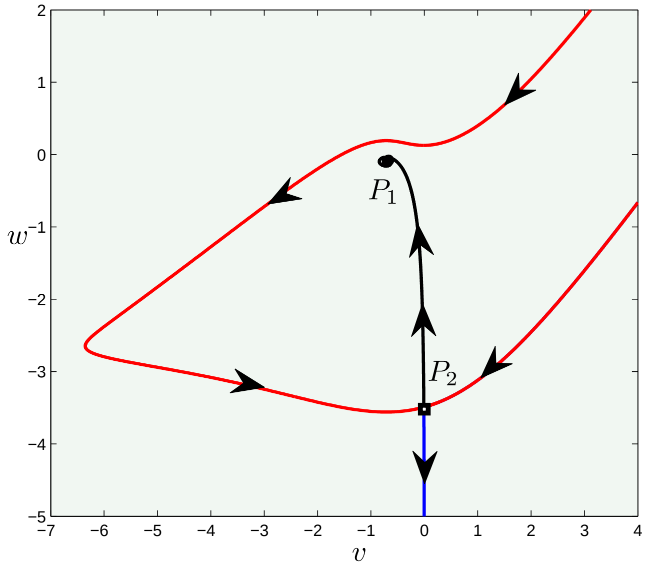


Figure 2: A numerical simulation of the phase portrait of system (10) with parameter values satisfying the conditions of Theorem 1. The unbounded basin of attraction of the attractive equilibrium $P_1$ (which is a poverty trap) is limited by the 1-dimensional stable manifold of the determinate equilibrium $P_2$. Parameter values: $\psi = 1.698$, $\sigma = \frac{1}{3}$, $\tau = 0.5$, $\rho = \frac{1}{\sqrt[4]{5}}$.

*Proof.* See Appendix 6.4 □

Figure 2 shows a numerical simulation of the phase portrait of system(10) with parameter values satisfying the conditions of Theorem 1. The unbounded basin of attraction of the attracting equilibrium $P_1$ (which is a poverty trap) is limited by the 1-dimensional stable manifold of $P_2$. Notice that, if the initial value $v_0$ of the predetermined variable $v$ is high enough, there exists a continuum of initial values $w_0$ of the jump variable $w$ such that the trajectory starting from $(v_0, w_0)$ approaches $P_1$ while the stable manifold of $P_2$ can be selected by choosing two different initial values of $w$. This is an interesting example of indeterminacy because, given the initial value of $v$, the economy can approach the locally determinate equilibrium $P_2$ by following rather different transition paths. Observe that in this case we possess a full description of the unbounded basin of $P_1$ (on the plane $u = 0$) and therefore of the global indeterminacy. Finally, notice that, as in [21, 3], the poverty trap $P_1$ can be reached even if the initial value $v(0)$ coincides with the value assumed by the predetermined variable $v$ at the locally determined equilibrium $P_2$; symmetrically, $P_2$ can be reached even if the economy starts with an initial value of $v$ coinciding with that of the poverty trap $P_1$.

## 5 Global analysis in a context with indeterminacy of order 3

The above discussion shows that such a situation can take place only if two equilibria exist with $\psi_{12}, \psi_{21} > 0$ (thus $\psi_{11}, \psi_{22} < 0$) and $\delta < 0$.[9] Thus it may happen that the equilibria $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, $u_1 > u_2$ and $v_1 < v_2$, are respectively a saddle endowed with a two-dimensional stable manifold and a sink. We will illustrate a case of this type, starting from a bifurcation where $P_1 = P_2 = P_0$ and $P_0$ has one zero eigenvalue and two complex conjugate eigenvalues with negative real part. Then we will prove that there exists an open, unbounded region[10] constituted by trajectories converging to $P_0$ (as $t \to +\infty$). Consequently, when $P_1$ is slightly separated from $P_2$, the above situation persists, i.e. the basin of attraction of $P_2$ is unbounded; moreover, when $\overline{v} \in (v_1, v_2)$, $v_2 - v_1$ being sufficiently small, there exists an open interval $I$ contained in the line $\{u = \delta\overline{v} + \overline{d}, v = \overline{v}\}$ whose trajectories converge to $P_2$ (as $t \to +\infty$), while the trajectory starting at one extreme of $I$ tends to $P_1$ (as $t \to +\infty$). Hence a global indeterminacy scenario occurs: starting from any initial value $v(0) = \overline{v}$ of the state variable $v$ belonging to the interval $(v_1, v_2)$, the economy may approach either the poverty trap[11] $P_1$ or the equilibrium point $P_2$, according to the choice of the initial value of the jumping variable $w$.

Hence assume $\psi_{12}, \psi_{21} > 0$, $\delta < 0$ and two equilibria $P_1$ and $P_2$, defined as above, exist, lying on the plane $u = \delta v + \overline{d}$, $\overline{d} = (\psi_{12} + \psi_{21})^{-1} \ln \frac{c_2}{c_1}$. In Appendix 6.5 we provide a description of of the system in a neighborhood of such a plane, which helps in understanding the proofs of Theorems 5.1 and 5.2.

Now we consider the following configuration:

$$\begin{cases} P_1 = P_2 = P_0 \\ \textit{The Jacobian matrix } J(P_0) = J_0 \textit{ has one zero} \\ \textit{and two complex with negative real part eigenvalues} \end{cases} \tag{12}$$

**Example 3.** *Let us take a system* (5) *where* $\psi_{12} = 1.1, \psi_{22} = -0.1, \psi_{21} = 2, \psi_{11} = -1$. *Assume* $P_0 = (u_0, v_0, w_0)$ *to be an equilibrium of such a system. Hence* $r_1(u_0, v_0) = r_2(u_0, v_0) = r_0$. *By sake of simplicity let* $\rho = r_0$ *(the transversality conditions require* $\rho < r_0$*). Then, if* $\psi_{21} e^{v_0 - u_0} = \psi_{12} - 1$, $\sigma^{-1} = \frac{(\psi_{12}-1)(1-\delta)}{\tau}$, $\delta = -0.615$, $\tau = 0.645$, *it is easily checked that* $b_1, b_2 \in (0, 1)$, *that* $P_0$ *is the unique equilibrium of* (5) *and, finally, that* $J_0$ *satisfies the conditions* (12).

**Theorem 2.** *Assume a system* (5) *with* $\psi_{12}, \psi_{21} > 0$ *and* $\delta < 0$ *has two* coinciding *equilibria and satisfies* (12). *Then there exists a two-dimensional smooth manifold through* $P_0$, *whose trajectories converge to* $P_0$, *separating a region* $R_1$ *constituted by trajectories tending to* $P_0$ *(as* $t \to +\infty$*) from a region* $R_2$ *constituted by trajectories leaving from* $P_0$ *(as* $t \to -\infty$*). Moreover* $R_1$ *is unbounded.*

[9] Remember that $\psi_{12}, \psi_{21} > 0$ if and only if $\beta_{11} < \beta_{12}$, that is, the final good is intensive in human capital at the private level.

[10] By region we mean an open connected subset of $\mathbb{R}^3$.

[11] Remember that, by (7), the growth rate $\gamma$ associated to each equilibrium point is positively correlated with the equilibrium value of $v$.

*Proof.* First of all, the existence of *two coinciding equilibria* in the point $P_0 = (u_0, v_0, w_0)$ implies $u_0 = \delta v_0 + \overline{d}$ and, posed $\widetilde{h}(v) = h(\delta v + \overline{d}, v)$, $\widetilde{h}(v_0) = \widetilde{h}'(v_0) = 0$, while $\widetilde{h}''(v_0) < 0$ (as it is easily computed). Moreover, referring to the expression (8) of $J(P_0)$, we have $\frac{\frac{\partial h}{\partial u}}{\frac{\partial f}{\partial u}}(P_0) = \frac{\frac{\partial h}{\partial v}}{\frac{\partial f}{\partial v}}(P_0) = m$. Consider, then, the change of coordinates:

$$x = u - u_0,\ y = v - v_0,\ z = w - w_0 - m(u - u_0) \tag{13}$$

Therefore, in the new coordinates, $P_0 = O = (0,0,0)$ and

$$J(O) = \begin{pmatrix} a & b & 0 \\ -c & -d & -l \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

where $a, b, c, d, e > 0$, $a < d$ and $(d-a)^2 < 4(bc - ad)$. In fact, multiplying the vector field of the system, in the new coordinates, by $e^{-mx}$, we obtain a system similar to (5), i.e.

$$\begin{cases} \dot{x} = p(x,y) \\ \dot{y} = q(x,y) - le^z \\ \dot{z} = s(x,y) \end{cases} \tag{15}$$

where $O = (0,0,0)$ is the unique equilibrium, $\frac{\partial s}{\partial x}(0,0) = \frac{\partial s}{\partial y}(0,0) = 0$ and, being $\widetilde{h}''(v_0) < 0$,

$$\left(\frac{\partial^2 s}{\partial x^2}\delta^2 + 2\frac{\partial^2 s}{\partial x \partial y}\delta + \frac{\partial^2 s}{\partial y^2}\right)(0,0) < 0 \tag{16}$$

Moreover, called $z = -\ln l + \ln q(\delta y, y) = \varphi(y)$, it can be easily checked that $\varphi'(0) > 0$ and $\varphi''(0) < 0$.

From straightforward computations it follows that the eigen-line associated to the zero eigenvalue of $J(O)$ is given by $\{x = \delta y, z = \varphi'(0)\, y\}$, while the eigen-plane associated to the complex conjugate eigenvalues of $J(O)$ is $z = 0$. Take now a sufficiently small neighborhood $N$ of $O$. From the previous considerations it follows that there exists a two-dimensional smooth manifold $S$, whose trajectories converge to $O$, which separates $N$ into two disjoint open subsets $A_1$ and $A_2$, containing, say, respectively the intersections of $N$ with the positive and negative $z$-semiaxis. Therefore the intersection with $N$ of a central manifold at $O$ of (15), tangent to $L = \{x = \delta y, z = \varphi'(0)\, y\}$ in $O$, can be written as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{O\}$, $\Gamma_1 \subset A_1$, $\Gamma_2 \subset A_2$. Besides, straightforward calculations show that, if $N$ is small enough, the coordinates of $\Gamma$ satisfy:

$$\begin{array}{l} x = \delta y + \eta y^2 + h.o.t. \\ z = \varphi'(0)\, y + h.o.t. \end{array} \tag{17}$$

where $\eta > 0$ and $h.o.t. =$ *higher order terms*. More precisely it can be shown that for a sufficiently small $N$ the equations of a central manifold $\Gamma$ (i.e. of

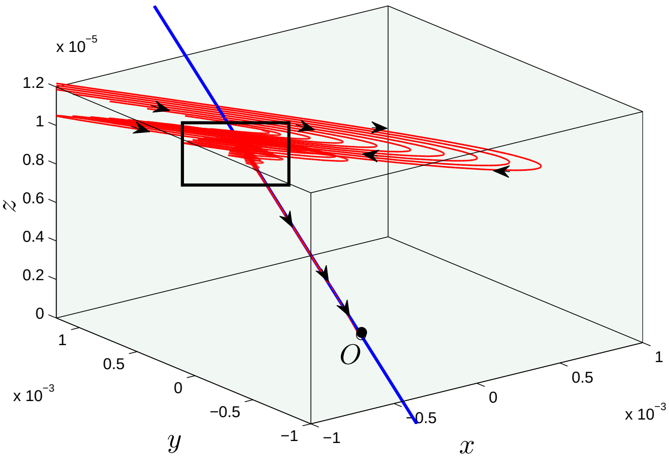


Figure 3: The figure shows, by utilizing the parameter values of Example 3, the dynamics of trajectories converging to $P_0$ in the half-space $z > 0$. Parameter values: $\delta = 0.6$, $\tau = 0.645$, $\rho = .1694098593$, $\psi_{12} = 1.1$, $\psi_{11} = -1$, $\psi_{22} = -0.1$, $\psi_{21} = 2$.

an invariant manifold tangent in $O$ to the line $L$) are of the type $x = \chi(y)$, $z = \zeta(y)$, with $\chi(y)$ and $\zeta(y)$ smooth in a neighborhood of $y = 0$. Moreover the central manifold is proven to be unique (see Appendix 6.6).

It follows that along $\Gamma_1 \cup \Gamma_2$ $x(t)$ increases, while $y(t)$ and $z(t)$ decrease (recall (16)).

Consider now a point $Q = (\overline{x}, \overline{y}, \overline{z}) \in \Gamma_1$ and a sufficiently small disc $D$ centered in $Q$ and lying in $z = \overline{z}$. From what we have seen and from the Central Manifold Theorem (see [17]) it follows that all the trajectories starting in $D$ converge to $O$ and those from $D - \{Q\}$ do so spiralling. In particular along them $\dot{x}(t)$ changes sign infinitely many times and thus they intersect infinitely many times the plane $x = \delta y$ (corresponding to $\dot{x} = 0$). Moreover all the trajectories in $A_1$ converge to $O$ (if $N$ is small enough), as they cross $x = \delta y$ alternately on each *side* of the line $L$ and therefore eventually wind around $\Gamma_1$ and so spiral toward $O$.

Our final step is to prove that along the negative trajectory starting from a point of $\Gamma_1$ $x\left(\widetilde{t}\right)$ decreases, where $\widetilde{t} = -t$. Suppose, by contradiction, this is not the case. Then there should exist a first point $R = \left(x\left(\widetilde{t}^*\right),\ y\left(\widetilde{t}^*\right),\ z\left(\widetilde{t}^*\right)\right)$ $= (x^*,\ y^*,\ z^*)$ on the above mentioned trajectory such that $x^* = \delta y^*$ (i.e. $\dot{x}\left(\widetilde{t}^*\right) = 0$) and $\dot{x}\left(\widetilde{t}\right) > 0$ for $\widetilde{t}$ being in a right neighborhood of $\widetilde{t}^*$. Therefore, for what we have seen, it should be $z^* \leq \varphi(y^*)$. Suppose $z^* < \varphi(y^*)$. Then, by the continuous dependence of trajectories on initial conditions, there should exist a small disc $\widetilde{D}$ centered in $R$ and contained in the planar region $\{x = \delta y,\ z < \varphi(y)\}$, such that all the positive trajectories starting from

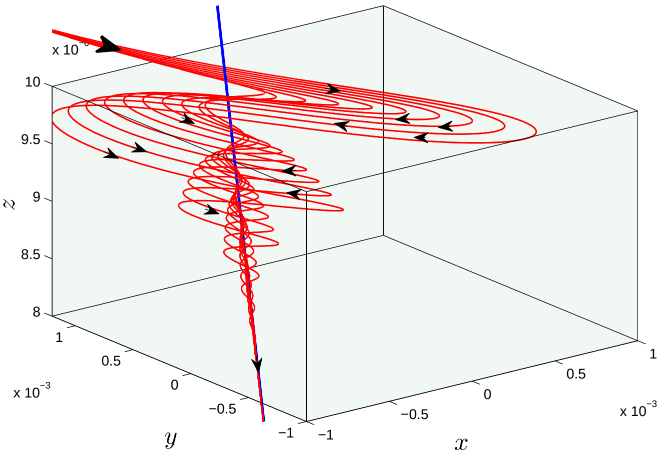


Figure 4: Zoom of the small region indicated in Figure 3, highlighting as the generic trajectory converging to $P_0$ winds around the central manifold.

$\widetilde{D}$ would enter into $A_1$ and then converge to $O$. Besides, all the positive trajectories from $\widetilde{D} - \{R\}$ would cross again $x = \delta y$ for the first time at some positive value of $t$. This way we can define a map $\xi$ from $\widetilde{D} - \{R\}$ into the plane $x = \delta y$, which can be extended to $R$ setting $\xi(R) = O$. Therefore $\xi$ should be a homeomorphism mapping $\widetilde{D}$ onto an open neighborhood of $O$, which is clearly impossible, since in any neighborhood of $O$ on the plane $x = \delta y$ there exist points (with $z < 0$) whose orbits move away from $O$. Hence $z^* = \varphi(y^*)$. Therefore, being $\dot{x}\left(\widetilde{t^*}\right) = \dot{y}\left(\widetilde{t^*}\right) = 0$, it follows $\ddot{x}\left(\widetilde{t^*}\right) = 0$, while

$$\dddot{x}\left(\widetilde{t^*}\right) = -l\frac{\partial p}{\partial y}(x^*, y^*)\, e^{z^*}\dot{z}\left(\widetilde{t^*}\right) = l\frac{\partial p}{\partial y}(x^*, y^*)\, e^{z^*} s(x^*, y^*) < 0 \tag{18}$$

Hence $\dot{x}\left(\widetilde{t}\right) < 0$ both in a left and a right neighborhood of $\widetilde{t}$, which leads to a contradiction. Consequently it can be proven (see Appendix 6.7) that along the above trajectory (say the *continuation* of $\Gamma_1$) $x$, $y$ and $z$ are all unbounded: precisely, coming back to the original time $t$, $\lim_{t\to-\infty} x(t) = -\infty$, $\lim_{t\to-\infty} y(t) = \lim_{t\to-\infty} z(t) = +\infty$. This completes the proof of the Theorem. $\square$

Figures 3 and 4 show, by utilizing the parameter values of Example 3, the dynamics of trajectories converging to $P_0$ in the half-space $z > 0$. Actually Figure 4 zooms a small indicated region in the previous Figure, highlighting as the generic orbit converging to $P_0$ winds around the central manifold.

The following Theorem is in fact a Corollary of the previous one.

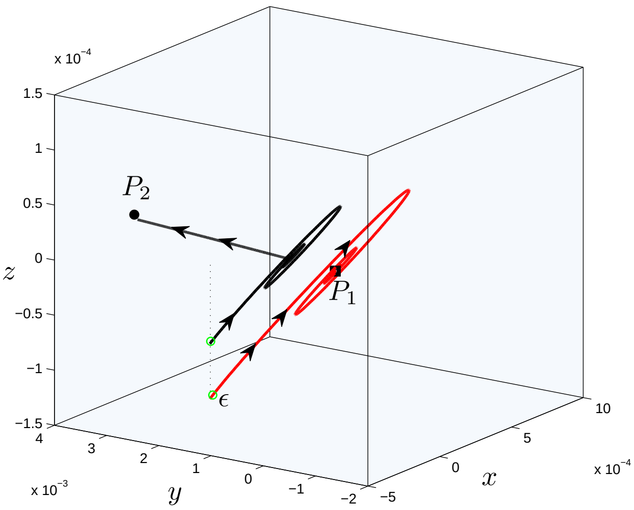


Figure 5: Two trajectories starting from the same initial value of the state variable $y = \overline{y} = 0.0024$ (remember that $y = v - v_1$). The red trajectory starting from $(x, y, z) = (0, \overline{y}, \epsilon)$, with $\epsilon = -0.00884572$, approaches the determinate equilibrium $P_1$; the black trajectory, converging to the locally attractive equilibrium $P_2$, starts from $(x, y, z) = (0, \overline{y}, \epsilon)$, with $\epsilon = -0.00013805701$. The parameter values are those given in the Example 4 with $\rho = 0.3022682$.

**Theorem 3.** *Assume, in system* (5)*, that* $\psi_{12}, \psi_{21} > 0$*,* $\delta < 0$ *and there exist two equilibria,* $P_1 = (u_1, v_1, w_1)$ *and* $P_2 = (u_2, v_2, w_2)$*, with* $v_1 < v_2$*. Moreover, suppose that* $P_1$ *has a two-dimensional stable manifold,* $P_2$ *is a sink and both the Jacobian matrices* $J(P_1)$ *and* $J(P_2)$ *have two complex conjugate eigenvalues. Then, if* $v_2 - v_1$ *is sufficiently small, there exists on every line* $\left\{u = \delta\overline{v} + \overline{d}, v = \overline{v}\right\}$*,* $v_1 < \overline{v} < v_2$*, an interval* $I = (A, B)$ *such that all the trajectories starting from* $I$ *converge to* $P_2$*, while the trajectory starting from either* $A$ *or* $B$ *converges to* $P_1$*. Besides, the basin of attraction of* $P_2$ *is unbounded.*

*Proof.* See Appendix 6.8 □

The above Theorem proves the occurrence of global indeterminacy, in the two senses by which it is known in literature, when $\psi_{12}, \psi_{21} > 0$, $\delta < 0$ and two locally indeterminate (of order two and three, respectively) equilibrium points, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$, exist. According to such result, if $v_2 - v_1$ is sufficiently small (i.e. if $P_1$ and $P_1$ are close enough), for every initial value $\overline{v} \in (v_1, v_2)$ of the state variable $v$, there exists a continuum of initial values $w \in (a, b)$ of the jumping variable $w$ such that the trajectory starting from $(u, v, w) = \left(\delta\overline{v} + \overline{d}, \overline{v}, w\right)$ approaches $P_2$ while the trajectory starting from either $(u, v, w) = \left(\delta\overline{v} + \overline{d}, \overline{v}, a\right)$ or $(u, v, w) = \left(\delta\overline{v} + \overline{d}, \overline{v}, b\right)$ converges to $P_1$.

Notice that the value of $v$ (and consequently, by (7), the value of the growth

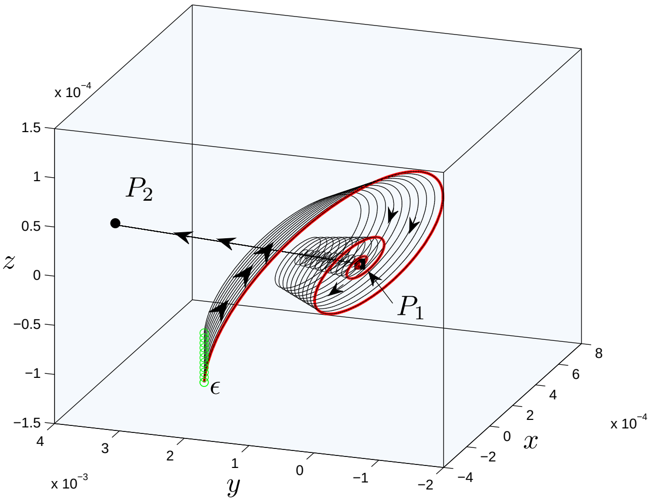


Figure 6: Phase portrait of system (5) obtained with the same parameter values of the simulation in Figure 5. Only the red trajectory (the same illustrated in Figure 5) approaches the determinate equilibrium $P_1$; the black trajectories, starting from $(x, y, z) = (0, \overline{y}, \epsilon)$ with $\epsilon \in (-0.00884572, -0.00013805701)$, belongs to the basin of attraction of the equilibrium $P_2$.

rate $\gamma$) in $P_2$ is higher than in $P_1$. Besides, the basin of attraction of $P_2$ is unbounded; in particular, as the proof of the above Theorem shows, there exists a continuum of trajectories approaching the *virtuous* equilibrium $P_2$ if the initial value of the predetermined variable $v$ (remember that $v = \ln \frac{k_1}{k_2} = \ln \frac{K_1}{K_2}$) is high enough, that is if the initial ratio between physical capital $K_1$ and human capital $K_2$ is high enough.

**Example 4.** *Consider the system* (5) *with* $\psi_{12} = 1.1, \psi_{22} = -0.1$, $\psi_{22} = -0.1$, $\psi_{21} = 2$, $\psi_{11} = -1$, $\sigma^{-1} = \frac{(\psi_{12}-1)(1-\delta)}{\tau}$, $\delta = -0.615$, $\tau = 0.645$. *By a suitable translation of* $u, v$ *and a rescaling of* $t$, $\rho$ *we can assume* $c_1 = c_2 = 1$. *Take* $\rho < r\,(u_0, v_0) = r_0$, *where* $(u_0, v_0)$ *satisfies* $u_0 = \delta v_0$, $u_0 - v_0 = \ln \frac{\psi_{21}}{\psi_{12}-1} = \ln 20$.

*Then, if* $r_0 - \rho$ *is sufficiently small the system has two equilibrium points* $P_1 = (u_1, v_1, w_1)$ *and* $P_2 = (u_2, v_2, w_2)$, *with* $v_1 < v_2$, *satisfying the conditions of Theorem 3. Precisely* $P_1$ *is a saddle with a two-dimensional stable manifold and* $P_2$ *is a sink.*

*We can consider a further linear change of coordinates, namely* $x = u - \delta v$, $y = v - v_1$, $z = w - w_1 - m\,(u - u_1)$, $m = \frac{\frac{\partial h}{\partial u}}{\frac{\partial f}{\partial u}}\,(u_1, v_1)$. *This way* $P_1$ *is translated into the origin and* $P_1$, $P_2$ *lie on* $x = 0$. *On such a plane a line* $y = \overline{y}$ *represents a fixed choice of the state variable. Then let* $z$ *vary on a line* $\{x = 0, y = \overline{y},$ *with* $0 < \overline{y} < v_2 - v_1\}$: *for a suitable value of* $z$ *close to* 0, *say* $z = \varepsilon$, *the trajectory starting at* $(0, \overline{y}, \varepsilon)$ *spirals toward* $P_1$, *while the trajectories starting from points of the line with* $z > \varepsilon$, *up to a certain value of* $z$, *converge* *to $P_2$ (see Figures 5-6).*

Figures 5 and 6 illustrate the phase portrait of system (5) with the parameter values suggested in the above example. Figure 5 shows two trajectories starting from the same initial value of the state variable $y$ (remember that $y = v - v_1$), one approaching $P_1$ and the other converging to $P_2$. Figure 6 is obtained with the same parameter values; however more trajectories are plotted, all starting from the same value of the state variable $y$. Notice that only one trajectory approaches $P_1$ while the others belong to the basin of attraction of the virtuous equilibrium $P_2$.

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## 6 Appendix

*6.1 Proof of Proposition 1*

*Proof.* Let $P_0 = (u_0, v_0, w_0)$ be an equilibrium of (5). Then $g(u_0, v_0) > 0 \Leftrightarrow \psi_{12}e^{u_0} - \psi_{21}e^{v_0} > 0$, as it is easily checked. From straightforward computations it follows that:

$$sgn\left(\frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}\right)(P_0) = sgn\left[\begin{array}{c} \psi_{12} + \psi_{21} + \psi_{11} - \psi_{22} - 2\psi_{21}e^{v_0-u_0}+ \\ \tau(\psi_{11} - \psi_{22} + \psi_{12}e^{u_0-v_0} - \psi_{21}e^{v_0-u_0})+ \\ \delta\left(-\psi_{11}\psi_{21} - \psi_{22}\psi_{12} + \psi_{12}^2e^{u_0-v_0} + \psi_{21}^2e^{v_0-u_0}\right) \end{array}\right]$$

Hence the coefficient of $\tau$ is positive, being equal to $g(u_0, v_0)\, r_0^{-1} e^{-v_0}$, and so is the coefficient of $\delta$, as $\psi_{11}\psi_{21}, \psi_{22}\psi_{12} < 0$. Moreover:

$$\psi_{12} + \psi_{21} + \psi_{11} - \psi_{22} - 2\psi_{21} e^{v_0 - u_0} = 2e^{-u_0}\left(\psi_{12} e^{u_0} - \psi_{21} e^{v_0}\right) > 0.$$

This proves the Proposition. □

### 6.2 *Proof of Proposition 2*

*Proof.* Let us assume that, in Case 2 or 3, two equilibrium points exist, $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$, with $v_1 < v_2$. Then, in Case 2 $\det J(P_1) > 0 > \det J(P_2)$ and the reverse holds in Case 3. While $\delta > 0$ implies in both cases $traceJ(P_i) > 0$, $i = 1, 2$, it follows that $P_2$ in Case 2 and $P_1$ in Case 3 are saddles with a one-dimensional stable manifold, hence locally determinate. When, instead, $\det(J) > 0$, writing the characteristic polynomial as $P(\lambda) = \lambda^3 - trace(J)\lambda^2 + \sigma(J)\lambda - \det(J)$, where, of course, $\sigma(J) = \det(J_{11}) + \det(J_{22}) + \det(J_{33})$, it follows from straightforward computations that two negative real part eigenvalues exist if and only if $P(trace(J)) = \sigma(J)\, trace(J) - \det(J) < 0$. In fact $P(trace(J))$ can be written as a function of $\delta$, i.e. $F(\delta) = a\delta^2 + b\delta + c$. Then it can be calculated that in Case 2, when $J = J(P_1)$, $F(\delta) > 0$ as $\delta \geq 1 + \tau$. Hence it follows that $P_1$ is a repellor. Vice-versa in Case 3 $P_2$ can be locally indeterminate of order two, as Example 1 shows. □

### 6.3 *Proof of Proposition 4*

*Proof.* We want to prove that the bifurcation of system (10), occurring as $\widetilde{h}(v_0) = \widetilde{g}'(v_0) = 0$, is Hopf supercritical. To this end , since $\widetilde{h}'(v_0) > 0$, let us replace $w$ by $kw$, where $k = \sqrt{\frac{\widetilde{h}'(v_0)}{\widetilde{g}(v_0)}}$, so that (10) becomes:

$$\begin{cases} \dot{v} = \widetilde{g}(v) - e^{kw} \\ \dot{w} = k^{-1}\widetilde{h}(v) \end{cases}$$

Then it follows ([17]) that the above bifurcation is Hopf supercritical if and only if:

$$\widetilde{g}'''(v_0) - \frac{k^{-1}}{\omega}\widetilde{g}''(v_0)\widetilde{h}''(v_0) < 0 \tag{19}$$

where $\omega = k^{-1}\widetilde{h}'(v_0)$.

As $u = 0$ and $\psi_{12} = \psi_{21} = \psi > 1$, we can write:

$$\widetilde{h}(v) = pe^v\left[-m + e^{\tau v}(n - \psi e^v)\right], \quad \widetilde{g}(v) = qe^{\tau v}\left(1 - e^{2v}\right) \tag{20}$$

where $p, m, n, q > 0$.

Hence, recalling $\widetilde{h}(v_0) = \widetilde{g}'(v_0) = 0$, the inequality (19) can be checked through straightforward computations. □

*6.4 Proof of Theorem 1*

*Proof.* If system (11) satisfies conditions 1-5, there exists at most one simple, repelling limit cycle surrounding $\widetilde{P}_1$. Suppose, by contradiction, this is the case and move $\widetilde{P}_1 = (v_1, \ln \widetilde{g}(v_1)$ toward $\widetilde{P}_0 = (v_0, \ln \widetilde{g}(v_0))$, where $\widetilde{g}'(v_1) < \widetilde{g}'(v_0) = 0$. More precisely, posed $v_\alpha = \alpha v_0 + (1-\alpha) v_1$, $0 \leq \alpha \leq 1$, we choose smooth functions $\tau(\alpha)$, $\rho(\alpha)$, $\sigma(\alpha)$, $\psi(\alpha)$, starting from the original parameters, as $\alpha = 0$, with $\tau(1) = \tau(0) = 0.5$, such that, for any $\alpha$, (10) has equilibria $\widetilde{P}_\alpha = (v_\alpha, \ln \widetilde{g}(v_\alpha))$ and $\widetilde{P_2} = (v_2, \ln \widetilde{g}(v_2))$. Moreover, as the original system ($\alpha = 0$) possesses a limit cycle, the trajectory $\gamma(t)$, from a point of the unstable manifold of $\widetilde{P_2}$ in the half-plane $v < v_2$, intersects, if it does, the line $v = v_2$ at a point $(v_2, w^*)$, $w^* < \ln \widetilde{g}(v_2)$. Therefore we can choose the functions $\tau(\alpha)$, $\rho(\alpha)$, $\sigma(\alpha)$, $\psi(\alpha)$ in such a way that, for any $\alpha \in (0,1]$, $\gamma_\alpha(t)$, defined analogously as $\gamma(t)$, intersects, possibly, the line $v = v_2$ at a point $(v_2, w_\alpha)$, $w_\alpha \leq \widehat{w} < \ln \widetilde{g}(v_2)$. Thus no saddle connection occurs as $\widetilde{P}_1$ moves to $\widetilde{P}_0$. As a consequence, for any $\alpha \in [0,1]$, system (10) has an odd number of limit cycles. In fact, consider an intermediate equilibrium point $\widetilde{P}_\alpha = (v_\alpha, \ln \widetilde{g}(v_\alpha))$ and the analytical Poincaré return map $f(w)$ defined on an open interval $(a,b)$ of the half-line $\{v = v_\alpha, w > \ln \widetilde{g}(v_\alpha)\}$, where $b < b_\alpha$, $(v_\alpha, b_\alpha)$ being the intersection of the half-line with the unstable manifold of $\widetilde{P}_2$. Hence, if a bifurcation occurs, there is some $\overline{w} \in (a,b)$, where $f(\overline{w}) = \overline{w}$ and $f(w) - w$ has the same sign, positive or negative, in a neighborhood of $\overline{w}$ for $w \neq \overline{w}$. Thus an even number of limit cycles is possibly generated or removed. Moreover, by posing $e^{\frac{1}{2}v} = x$, $e^w = y$, system (10) is equivalent to a polynomial system defined in the invariant half-plane $x > 0$, which has a finite number of limit cycles (see, e.g., [4]). Finally a further limit cycle is generated by the Hopf bifurcation when $v_0 - v_1 = \varepsilon > 0$ is sufficiently small. Hence system (10) must have an even number greater than zero of limit cycles when $\varepsilon$ is sufficiently small. In this case the two equilibria can be written as $\widetilde{P_1} = (v_0 - \varepsilon, w_1)$, $\widetilde{P_2} = (v_2, w_2)$, $v_0 < v_2$. Again, we observe that by the change of variables $x = v_1 - v$, $y = w - w_1$, system (10) gives rise to a Liènard system of the type (11) defined in $(\overline{x}, +\infty)$, where $\overline{x} = v_1 - v_2$. It follows that, when $\varepsilon$ is small enough, this system has at most one simple limit cycle if:

1. $\frac{\widetilde{g}'(v)}{\widetilde{h}(v)}$ is non-increasing in $(v_0, v_2)$;
2. the system of equations $\widetilde{g}(v) = \widetilde{g}(z)$, $\int\limits_{v_1}^{v} \widetilde{h}(s)ds = \int\limits_{v_1}^{z} \widetilde{h}(s)ds$ has at most one solution for $v \in (-\infty, v_0)$, $z \in (v_1, v_2)$

As straightforward, even if lengthy, calculations show the two conditions to be satisfied , we get a contradiction, implying that the original system (10) has no limit cycle. $\square$

*6.5 Description of the dynamics in a neighborhood of the plane $u = \delta v + \overline{d}$.*

On such a plane, corresponding to $\dot{u} = 0$, $\dot{v} = 0$ is given by the graph of a function $w = \varphi(v) = \ln g(\delta v + \overline{d}, v)$, defined in an interval $(-\infty, v^*)$,

$v_1 < v_2 < v^*$, which has exactly one point of maximum. On the other hand $\dot{w} = 0$ is the union, on the plane, of the two lines $v = v_1$ and $v = v_2$. It is easily checked that, on such a plane, $\dot{v} > 0$ and $\ddot{u} > 0$ in the region $\{v < v^*, w < \varphi(v)\}$, while $\dot{v} < 0$ and $\ddot{u} < 0$ in the region $\{v < v^*, w > \varphi(v)\} \cup \{v > v^*\}$; moreover $\dot{w} > 0$ ($< 0$) inside (outside) the strip $\{v_1 < v < v_2\}$. Next, consider a plane parallel to $u = \delta v + \overline{d}$, say $u = \delta v + d$, with $d > \overline{d}$. Being everywhere $\frac{\partial f}{\partial u} > 0$ and $\frac{\partial g}{\partial u} < 0$, on such a plane we have $\dot{u} > 0$, while $\dot{v} = 0$ is given by the graph of a function $w = \widetilde{\varphi}(v)$, defined in an interval $(-\infty, \widetilde{v})$, $\widetilde{v} < v^*$, such that, as $v < \widetilde{v}$, $\widetilde{\varphi}(v) < \varphi(v)$. Moreover, on this plane, $\dot{u} - \delta\dot{v} = 0$ corresponds to the graph of a function $w = \widetilde{\eta}(v)$, defined in an interval $(-\infty, \widehat{v})$, $\widetilde{v} < \widehat{v}$, such that $\widetilde{\varphi}(v) < \widetilde{\eta}(v)$ for every $v < \widetilde{v}$; lastly, $\left(\dot{u} - \delta\dot{v}\right)\dot{u} > 0$ if $v < \widehat{v}$ and $w < \widetilde{\eta}(v)$, while $\left(\dot{u} - \delta\dot{v}\right)\dot{u} < 0$ if $v < \widehat{v}$ and $w > \widetilde{\eta}(v)$ or $v > \widehat{v}$. Exactly the opposite takes place in a plane $u = \delta v + d$, with $d < \overline{d}$.

*6.6 Uniqueness of the central manifold in Theorem 2*

As we have seen, a central manifold in a neighborhood of $O = (0,0,0)$ can be represented as $x = \chi(y)$, $z = \zeta(y)$, with $\chi(0) = \zeta(0) = 0$, $\chi'(0) = \delta$, $\zeta'(0) = \varphi'(0) = \frac{-c\delta - d}{l} > 0$ (using the notations of (14)). First of all we check that $\chi(y), \zeta(y)$ are $C^\infty$ in a suitable interval $[-\overline{y}, \overline{y}]$, $\overline{y} > 0$. In fact, by induction, let:

$$\begin{aligned}
\chi(y) &= \sum_{i=1}^{k} \alpha_i y^i + \alpha_{k+1} y^{k+1} + o\left(y^{k+1}\right) \\
\zeta(y) &= \sum_{i=1}^{k} \beta_i y^i + \beta_{k+1} y^{k+1} + o\left(y^{k+1}\right)
\end{aligned}$$

$k \geq 1$. Then, differentiating, we have:

$$\begin{aligned}
\dot{x} &= \chi'(y)\dot{y} \\
\dot{z} &= \zeta'(y)\dot{y}
\end{aligned} \tag{21}$$

where (see (15)) $\dot{x} = p(x,y)$, $\dot{y} = q(x,y) - le^z$, $\dot{z} = s(x,y)$. Hence, after straightforward computations,

$$\begin{aligned}
(a + \delta c)\,\alpha_{k+1} + \delta l \beta_{k+1} &= \mu \\
\varphi'(0)\left(c\alpha_{k+1} + \delta l \beta_{k+1}\right) &= \nu
\end{aligned}$$

where $\mu, \nu$ are determined by $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k$. So also $\alpha_{k+1}, \beta_{k+1}$ are univocally determined.

However this does not guarantee the central manifold to be analytic and thus unique.

Therefore we assume, by contradiction, that there exist infinitely many central manifolds. In fact we can bound ourselves to consider $y > 0$, as for $y < 0$ a trajectory lying on the central manifold tends to $O$ as $t \to -\infty$, which implies the central manifold in such half-space to be unique (see [31]).

Our first observation is that the *pencil* of central manifolds is bounded, i.e., when $y \in [0, \overline{y}]$, $\overline{y} > 0$ sufficiently small, all the central manifolds lie in a parallelepiped $[-\overline{x}, 0] \times [0, \overline{y}] \times [0, \overline{z}]$, $\overline{x}, \overline{z} > 0$. This follows from the fact that the trajectory starting at a point $Q$ of the half-plane $\{x = \delta y, y > 0\}$ sufficiently close to $O$ spirals toward $O$ (as $t \to +\infty$) crossing infinitely many times the plane $x = \delta y$ alternately on each side of the curve $\{x = \delta y, z = \varphi(y)\}$ and thus of the line $\{x = \delta y, z = \varphi'(0)\, y\}$. Hence the pencil of central manifolds *lies* inside these spirals.

Next we show that each central manifold $(y, x(y), z(y))$ is such to satisfy , in a suitable interval $[-\overline{y}, \overline{y}]$, a second-order differential equation $x'' = H(y, x, x')$. To this end, from (21) we derive:

$$\zeta'(y) = \frac{\dot{z}}{\dot{x}} \chi'(y) = \frac{p(y, \chi(y))}{s(y, \chi(y))} \chi'(y)$$

that is:

$$\zeta(y) = \int_0^y \frac{p(\widetilde{y}, \chi(\widetilde{y}))}{s(\widetilde{y}, \chi(\widetilde{y}))} \chi'(\widetilde{y}) d\widetilde{y} \tag{22}$$

On the other hand $\dot{x} = \chi'(y)\dot{y}$ yields $p(y, \chi(y)) = \chi'(y) \left( q(y, \chi(y)) - l e^{\zeta(y)} \right)$, from which we get $\zeta(y)$ as a function of $y, \chi(y), \chi'(y)$. Therefore, differentiating with respect to $y$, after easy steps we obtain:

$$p(y, \chi(y))\, \chi''(y) = R(y, \chi(y), \chi'(y))$$

Next we want to show that we can write:

$$\begin{array}{l} p(y, \chi(y)) = y^2 F(y, \chi(y), \chi'(y)) \\ R(y, \chi(y), \chi'(y)) = y^2 G(y, \chi(y), \chi'(y)) \end{array} \tag{23}$$

where $F(y, x, x')$ and $G(y, x, x')$ are smooth and $\neq 0$ in a neighborhood of $(0, 0, \delta)$. In fact, for any $k \geq 1$, let:

$$\chi(y) = \sum_{i=1}^{k} \alpha_i y^i + \alpha_{k+1} \chi_{k+1}(y)$$

where $\alpha_1, ..., \alpha_k$ are the same for any $\chi(y)$. From what we have observed, we can consider, in a suitable interval $[0, \overline{y}]$, the *lowest* central manifold with respect to $x$, i.e. $x = \chi^*(y)$ such that $\chi^*(y) \leq \chi(y)$ for any $\chi(y)$ when $y \in [0, \overline{y}]$. From the theory on central manifolds (see, [31]) it follows that there exist, for each $\chi(y)$, two constants, $\overline{c_1} > 0$ and $\overline{c_2} \geq 0$, such that:

$$\chi(y) - \chi^*(y) = e^{-\overline{c_1}/y} (\overline{c_2} - f(y, \chi(y))$$

where $0 \leq f(y, \chi(y) \leq \overline{c_2}$ and $f(y, \chi(y) = 0(\chi(y) - \chi^*(y))$. By differentiating with respect to $y$, we can calculate $\overline{c_1}$and $\overline{c_2}$ and in fact we can write:

$$\chi(y) - \chi^*(y) = e^{-c_1\left(y,\chi(y),\chi'(y)\right)/y} c_2(y, \chi(y), \chi'(y))$$

Analogously:

$$\chi'(y) - (\chi^*(y))' = \frac{e^{-d_1\left(y,\chi(y),\chi'(y)\right)/|y|} d_2(y, \chi(y), \chi'(y))}{y^2}$$

where the functions $c_1$, $d_1$ are $> 0$. Moreover, as $\chi(y)$ and $\chi'(y)$ are uniformly bounded when $y \in [0, \overline{y}]$, we can extend $c_1, c_2, d_1, d_2$ as functions of $(y, x, x')$ defined in a suitable neighborhood of $(0, 0, \delta)$. Clearly these functions may not be continuous in $(0, 0, \delta)$. However, for any $k \geq 1$,the functions defined as:

$$\frac{e^{-c_1\left(y,\chi(y),\chi'(y)\right)/y} c_2(y, \chi(y), \chi'(y))}{y^{k+1}} \text{ and } \frac{e^{-d_1\left(y,\chi(y),\chi'(y)\right)/y} d_2(y, \chi(y), \chi'(y))}{y^{k+1}}$$

when $y > 0$ and $0$ when $y \leq 0$ are smooth in a neighborhood of $(0, 0, \delta)$. Then, recalling $x''(0) = \eta > 0$, (23) follows from straightforward computations. Hence:

$$x'' = H(y, x, x') \tag{24}$$

where $H(y, x, x')$ is smooth in a neighborhood of $(0, 0, \delta)$. But this implies the existence of a unique solutions of (24) satisfying $x(0) = 0, x'(0) = \delta$, hence yielding a contradiction. Therefore the central manifold is unique.

### *6.7 Unbounded trajectory converging to $P_0$ in Theorem 2*

Let $Q \in \Gamma_1$, the intersection of the unique central manifold with the half-space $\{z > 0\}$. Exchanging $t$ with $\widetilde{t} = -t$, we have proved that the trajectory starting at $Q$ satisfies $\dot{x}\left(\widetilde{t}\right) < 0$ when $\widetilde{t} \in [0, \infty)$. Suppose, by contradiction, that $\lim\limits_{\widetilde{t} \to +\infty} x\left(\widetilde{t}\right) = \overline{x} > -\infty$. On the other hand, $\dot{x}\left(\widetilde{t}\right) < 0$ implies $r_1\left(\widetilde{t}\right) > r_2\left(\widetilde{t}\right)$, i.e. $y\left(\widetilde{t}\right) > \delta^{-1} x\left(\widetilde{t}\right)$, while, being $x\left(\widetilde{t}\right)$ bounded, for any $\varepsilon > 0$ there exists $\widetilde{t}(\varepsilon) > 0$ such that $y\left(\widetilde{t}\right) - \delta^{-1} x\left(\widetilde{t}\right) < \varepsilon$ as $\widetilde{t} \in \left(\widetilde{t}(\varepsilon), +\infty\right)$, except, possibly, in an interval of amplitude $o(\varepsilon)$. Consider, now, $\dot{w}\left(\widetilde{t}\right) = \dot{z}\left(\widetilde{t}\right) + m\dot{x}\left(\widetilde{t}\right)$. From straightforward calculations it follows that, when $\widetilde{t} > \widetilde{t}(\varepsilon)$, except possibly in an interval of amplitude $o(\varepsilon)$,

$$\dot{w}\left(\widetilde{t}\right) e^{-y\left(\widetilde{t}\right)} > \lambda(y, \varepsilon) = a + b(1 + 0(\varepsilon)) e^{\tau y} \left(m e^{(1+|\delta|)y} - n - p\right)$$

where $a, b, m, n, p > 0$ are suitably defined and, by our assumptions, $\lambda(0, 0) = 0$, $\frac{\partial \lambda(0,\varepsilon)}{\partial y} = 0$, $\frac{\partial \lambda(y,\varepsilon)}{\partial y} > 0$ when $y > 0$. Then, by taking $\varepsilon$ sufficiently small, it follows, for $\widetilde{t} > \widetilde{t}(\varepsilon)$, outside a possible interval of amplitude $o(\varepsilon)$, $\dot{w}\left(\widetilde{t}\right) > k > 0$ for a suitable $k$, implying $\lim\limits_{\widetilde{t} \to +\infty} w\left(\widetilde{t}\right) = +\infty$. Consequently $y\left(\widetilde{t}\right) \to +\infty$ and $x\left(\widetilde{t}\right) \to -\infty$, yielding a contradiction. Therefore, as $\widetilde{t} \to +\infty$, $x\left(\widetilde{t}\right) \to -\infty$, so that $y\left(\widetilde{t}\right) > \delta^{-1} x\left(\widetilde{t}\right)$ tends to $+\infty$ and the same does, as it can be easily seen, $z\left(\widetilde{t}\right)$.

*6.8 Proof of Theorem 3*

*Proof.* Let $v_2 - v_1 = \varepsilon > 0$ sufficiently small. Then we can assume the system to originate from a bifurcation where $P_1 = P_2 = P_0 = (u_2, v_2, w_2)$. Consider a point $Q \in \Gamma_1$, the intersection of the above central manifold with $\{v > v_2\}$, $Q$ being also in the attractive basin of $P_2$ in the bifurcated system. Then, if $\varepsilon$ is small enough, the negative trajectory of $Q$ remains *close* to $\Gamma_1$ for a sufficiently *long* time. More precisely, posed $s = -t$, we can take, for a sufficiently small $\varepsilon$, some $\overline{s} > 0$ so large that, called $(u(\overline{s}), v(\overline{s}), w(\overline{s})) = (\overline{u}, \overline{v}, \overline{w})$, the following holds:

1. $\overline{v} > \delta^{-1}\left(\overline{u} - \overline{d}\right)$
2. $e^{\overline{w}} > \max\limits_{v} g\left(\delta v + \overline{d}, v\right) = \max\limits_{v} \widetilde{g}(v)$ (recall $\lim\limits_{v \to -\infty} \widetilde{g}(v) = 0$ and $\widetilde{g}(v) < 0$ when $v$ is large enough)
3. $\dot{w}(\overline{s}) > 0$.

Then inequalities 1. and 2. imply, respectively, $\dot{u}(\overline{s}) < 0$ and $\dot{v}(\overline{s}) > 0$. Hence, for $s > \overline{s}$, $w(s)$ keeps increasing and because of 2. so does $v(s)$, while $u(s)$ decreases. It follows, by the same arguments used in Appendix 6.7, that such a trajectory is unbounded and, therefore, so is the basin of attraction of $P_2$.

Consider now the strip $\{v_1 \le v \le v_2\}$. Taking coordinates $x = u - u_1$, $y = v - v_1$, $z = w - w_1 - m(u - u_1)$, $m = \frac{\frac{\partial h}{\partial v}}{\frac{\partial f}{\partial v}}(P_1)$, it follows from the proof of Theorem 2 that the stable manifold of $P_1$ is tangent, at $P_1$, to a plane *close*, if $\varepsilon$ is small enough, to $z = 0$ in a neighborhood of $P_1$. Hence the manifold intersects each line $\{x = \delta y,\ y = \overline{y},\ 0 < \overline{y} < v_2 - v_1\}$. Therefore on such line there exists an interval with the properties described in the statement of Theorem 3. $\square$