Endogenous labour supply, endogenous lifetime and economic growth: local and global indeterminacy

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Abstract

This paper develops an economic growth model with overlapping generations, endogenous labour supply and endogenous lifetime determined by the individual state of health, which can be improved by private and public health expenditures. The dynamics of the economy is characterised by a two-dimensional map describing the time evolution of capital and labour supply. It is shown that the link between private and public expenditures on health in an economy where labour supply decisions of individuals are endogenous, causes the existence of multiple (determinate or indeterminate) fixed points, endogenous fluctuations that may explain the observed persistent oscillations in economic and demographic variables and local and global indeterminacy. These phenomena make both the initial condition of a macro-economy and health tax rate of greater importance to determine long-term demo-economic outcomes, and small changes in one of them may determine very different dynamic behaviours. These events are impossible when labour supply is exogenous or when the government does not invest in health. The novelty of this study is to link the theoretical literature with endogenous labour supply and indeterminacy (by concentrating on global dynamics) with the theoretical literature on endogenous lifetime and economic growth.

Keywords Endogenous labour supply; Endogenous lifetime; Local and global indeterminacy; OLG model

JEL Classification C61; C62; J1, J22; O41

1 Introduction

Demographic and macroeconomic outcomes are recognised to be dramatically related to each other in the process of economic growth and development (de la Croix and Doepke, 2003, 2004; Galor, 2005, 2011; Acemoglu and Johnson, 2007; Weil, 2007; Hall and Jones, 2007; Lorentzen et al., 2008), and both demographers and macroeconomists are currently wondering whether are the former variables to influence the latter or viceversa, when inquiring into the causes of poverty or prosperity

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of nations. In addition, the well-known phenomenon of population ageing (Fogel, 2004; Livi-Bacci, 2006), experienced in several developed countries around the world, has called attention of governments to reform labour markets and pension systems to overcome this concern (Boeri et al. 2001, 2002; Blinder and Krueger 2004), especially because of the demographic shift – observed in Western countries – due to the steadily reducing number of young workers and the steadily increasing number of old and healthy pensioners.

Since Leibenstein (1957), Becker (1960) and the origin of the new home economics, the study of the interaction amongst population (fertility and longevity), income and other macroeconomic variables has become a pillar of the macroeconomic theory, while also representing a challenge with regard to empirical studies. The idea that fertility can be viewed as a result of a rational choice of individuals that compare benefits and costs of having children, has opened the route to causes for reflection that have given birth to several theoretical works for explaining the behaviour of fertility and income (e.g., the Demographic Transition, see Galor and Weil, 1999, 2000, Blackburn and Cipriani, 2002; Cervellati and Sunde, 2011). The way of modelling fertility as an endogenous variable of individuals has also become an important feature in this class of models. With regard to this issue, several contributions has followed the Becker's idea of considering fertility as a (normal) consumption good in the utility function of parents (Eckstein and Wolpin, 1985; Eckstein et al., 1988; Galor and Weil, 1996; van Groezen, 2003). This is called weak altruism towards children (Zhang and Zhang, 1998) because parents are selfish and directly derive utility by the number of children they have. This theory has been applied to explain the decline in fertility in developed countries that followed the increase in both real wages and opportunity costs of raising a child for families. Subsequently, other forms of altruism have been studied with several purposes. This is the case of pure and impure altruism towards children. The former literature is pioneered by the contributions of Barro (1974), Becker and Barro (1988), Barro and Becker (1989) and Becker et al. (1990) and it is characterised by the hypothesis that parents derive utility from the utility of their descendants. The latter approach markedly differ from the former one as it assumes that parents derive utility by both the quantity and quality (represented by an expenditure in child quality) of children (Andreoni, 1989, Strulik, 2004a, 2004b).

A subsequent step in the theoretical and empirical literature on economic growth and development has been the analysis of the effects of health of individuals and life expectancy on long-term demo-economic outcomes. As pointed out by Weil (2007, p. 1265), in fact, "People in poor countries are, on average, much less healthy than their counterparts in rich countries. How much of the gap in income between rich and poor countries is accounted for by this difference in health?" This question and others related to it are the object of a growing body of studies that is essentially motivated by the reason of explaining the relationship amongst mortality decline and economic growth and the historical patterns of the demographic transition, that suggests (in addition to the observed reduction in fertility rates) mortality declines and the rise in human capital and income (e.g., Ehrlich and Lui, 1991; Kalemli-Ozcan et al., 2000; Kalemli-Ozcan, 2002, 2008; Cervellati and Sunde, 2005, 2011; Lorentzen et al., 2008).

Particular emphasis was placed on the role that the individual state of health plays on life expectancy. With this regard, there exist studies that aim at evaluating the impact of health expenditure (disease-specific interventions, vaccination and programmes to prevent the HIV/AIDS epidemic) as a means of reducing ill-health (Dow et al., 1999; Royalty and Abraham, 2006) and improving the quality of in and out working hours. It is widely accepted that the increase in adult mortality has a negative impact on economic growth because the individual (life)time horizon reduces, fertility increases, and physical and human capital investments reduce (Lorentzen et al., 2008; Juhn et al., 2013). Improving health conditions, therefore, is an object of greater importance in economic studies, while also representing a challenge for several government around the world. From theoretical grounds, the overlapping generations (OLG) model with production à la Diamond (1965) has become a natural basis where studying this issue. There exist several studies that consider life expectancy of humans as an endogenous variable to try to give an answer to questions related to poverty or prosperity of countries. Endogenous life expectancy may take the form of private choices of individuals with regard to health investments that contribute to improve education, the state of health and labour productivity (Blackburn and Cipriani, 2002; Chakraborty and Das, 2005). The awareness of being much more educated improves the knowledge of the benefits of investing in health for individuals. The labour productivity then increases and this gives rise to an increase in human capital accumulation and a reduction in adult mortality that may determine the virtuous circle of escaping from poverty. Alternatively, the state of health of individuals can be improved by specific public interventions (Chakraborty, 2004; Fanti and Gori, 2014), to furnish health services to population that contribute to increase life expectancy, savings (individuals save more because they live longer) and economic growth. In addition, the individual state of health and life expectancy can be improved by both private and public health expenditures. To this purpose, Bhattacharya and Qiao (2007) (resp. Varvarigos and Zakaria, 2013) have analysed this topic in a traditional general equilibrium OLG context with exogenous (resp. endogenous) fertility. In particular, Bhattacharya and Qiao (2007) show that the existence of a health technology such that its elasticity with respect to the private input depends on the public input (which is improved by tax-financed health investments as in Chakraborty, 2004), may be a source of economic cycles in income and longevity. This because when the initial level of physical capital is relatively low, the wage rate, the government's tax revenues from public health spending and the public input in the longevity production function reduce. Given the relationship between the private input and the public input in the longevity production function, a reduction in private health expenditure causes a reduction in the public one. Then, longevity decreases through this channel. A reduction in the public expenditure on health also causes a reduction in life expectancy and an increase in saving and the capital-labour ratio. Then, longevity increases through this channel. The main message of the paper by Bhattacharya and Qiao (2007) is the existence of a mechanism that may explain fluctuations in income and longevity essentially because the elasticity of longevity with respect to private investments depends on public investments (and then on the size of the public health support as well). Their model is used to interpret the very different behaviour of countries such as South Korea and the Philippines with regard to the observed different paths of life expectancy and income, although initial conditions were similar in the 1960s. Varvarigos and Zakaria (2013) extends the model by Bhattacharya and Qiao (2007) by adding endogenous fertility decisions of individuals. This allows them to provide a further explanation of the fertility decline - caused by the interaction of public and private expenditures on health - along the process of economic growth and development of nations.

The aim of the present study is to extend the model by Bhattacharya and Qiao (2007) by including endogenous labour supply decisions of individuals and a Cobb-Douglas technology of production of longevity, to well capture the functioning of health systems in developed countries. Different from them, however, we use a longevity production function for which both arguments are complements for all the values of private investments in health.[1] This implies that the elasticity

[1] In fact, they assume a longevity production function such that for low values of private health spending, an increase in public health spending reduces the productivity of the private contribution to longevity. This is quite unsatisfactory especially if one wants to study the behaviour of adult mortality in developing countries.

of longevity with respect to private health expenditure is constant and independent from public investments in health. The additional assumption of endogenous labour supply causes several dynamic phenomena that cannot be observed when labour is inelastic. While the dynamics of the model by Bhattacharya and Qiao is characterised by a one-dimensional map describing the evolution of the capital-labour ratio over time, in the present paper the dynamics of the economy is characterised by a two-dimensional map describing the evolution of capital and labour supply over time. The study of the relationship between health-related quantity and quality of life and efficiency of labour is of great importance and it is an object of increasing interest in economic theory. Therefore, analysing an economic growth model that explicitly accounts for individual labour choices when lifetime is endogenous may be valuable, especially with regard to policy insights (Cervellati and Sunde, 2013).

The idea of studying models with overlapping generations that generate endogenous fluctuations dates back at least to Grandmont (1985), Farmer (1986) and Reichlin (1986). Subsequently, several other studies have dealt with problems related to stability of equilibria as well as local and global indeterminacy in OLG models with endogenous labour supply, by taking into account the hypothesis of gross substitutability between second-period consumption and first-period labour as in Woodford (1984) and Reichlin (1986) in competitive economies with externalities in production (Cazzavillan, 2001; Cazzavillan and Pintus, 2006) and without them (Nourry, 2001; Cazzavillan and Pintus, 2004; Nourry and Venditti, 2006).

This paper represents a first attempt of inquiring about nonlinear dynamics and indeterminacy in a general equilibrium OLG model with endogenous labour supply, by explicitly accounting for individual lifetime related to specific health investments. We also give some policy warnings regarding the interaction between private and public health spending on economic dynamics. In fact, while the present approach is theoretical, it is closely related to the political debate regarding whether transforming or not the public-based European welfare system (the public health expenditure is one of the pillars of the welfare state in Europe) to a private-based one, where health expenditure is mainly related to individual behaviours (see, World Health Statistics, 2010). The welfare state in several countries in Europe are currently experiencing some concerns because of the reduction in per capita GDP and adult mortality, and the improved healthy lifetime of older people. This paper wants to tackle this issue by studying a general equilibrium OLG model with endogenous labour supply to address the question of multiple equilibria and local and global indeterminacy in this context. It contributes to explain how the choices on how much spending on health determines different dynamic paths in income, longevity and labour supply in a macroeconomic model.

Since one of the main objective of this paper is to study local and global phenomena in a context with endogenous lifetime, it is now useful to clarify the differences between local and global indeterminacy from a mathematical point of view. Indeed, the importance of the global analysis for economic models is recognised by the fact that studying just the local behaviour of a map does not give information with regard to the structure of the basins of attraction and their qualitative changes when the key parameters of the model are varied. Since in economic models it is also important to understand how variables behave in the long term given the initial conditions, analysing global phenomena is important if one wants to explain the occurrence of events by starting from initial conditions far away from a fixed point or an attracting set. A fixed point is said to be locally indeterminate if for every arbitrarily small neighbourhood of it and for a given value of the state variable close enough to its coordinate value at the stationary state, there exists a continuum of values of the control variable for which equilibrium trajectories converge towards the fixed point. Differently, the system is globally indeterminate when there exist values of the state variable such that different choices on the control variable lead to different invariant sets. In this case, determining the initial condition of the state variable is not enough to define the long-term dynamics of the system. We find that: 1) coexistence of attractors and local and global indeterminacy may occur in our model, and 2) the health policy plays a preeminent role in determining the long-term outcomes of the economy. In addition, global indeterminacy holds also when the public input in the longevity function is a concave function of public health investments. This is of importance especially for policy applications.

The rest of the paper is organised as follows. Sections 2 outlines the model with generic utility and production functions and Section 3 gives the conditions for the existence and local stability of a normalised fixed point. Section 4 studies the particular case of a CIES (Constant Inter-temporal Elasticity of Substitution) utility function and Cobb-Douglas production function. It analyses the conditions for the existence of fixed points of a two-dimensional map and analyses local bifurcations and stability. Section 4 describes the global properties of the map of the CIES-Cobb-Douglas economy and characterises the global dynamics of both capital and labour supply in an economy with endogenous lifetime. Conclusions are drawn in Section 5.

2 The model

2.1 Individuals

Consider an OLG closed economy comprised of a continuum of (two-period lived) rational and identical individuals of measure one per generation. In every period two generations are alive (Diamond, 1965): the young and the old. Each generation overlaps for one period with the previous generation and then overlaps for one period with the next generation. Time is discrete and indexed by $t = 0, 1, 2, ...$ Life of the typical agent born at time t is divided between youth and old age. In the first period of life (youth), the individual of generation t is endowed with $\widehat{l}$ units of time and supplies the share $l_t \in [0, \widehat{l}]$ to firms in exchange for wage w_t per unit of labour. The remaining share $\ell_t = \widehat{l} - l_t$ is used for leisure activities. He also chooses the amount of resources that should be allocated between (private) health investments and saving. When old, an individual retires and consumes on the basis of the resources saved when young (Woodford, 1984; Reichlin, 1986; Galor and Weil, 1996; Grandmont et al., 1998; Antoci and Sodini, 2009; Gardini et al., 2009). In addition, we assume that an individual survives at the onset of old age with certainty, and he is alive only for a fraction $\theta \in (0, 1]$ of the second period of his lifetime. Then, $1+\theta$ represents a measure of individual longevity. The probability of surviving when old is endogenous and determined by the individual state of health. As in Bhattacharya and Qiao (2007), we assume that an agent can increase his lifetime when old (longevity) by incurring private investments in health when young, which are accompanied by tax-financed health expenditure. This structure can indeed well captures health systems in several actual European economies, where both public and private components coexist and the former component represents a relevant portion of total expenditure on health over per capita GDP (see World Health Statistics, 2010). Therefore, population is endogenous because the length of life of the typical agent when old varies as long as adult mortality varies due to changes in public and/or private health spending. As a consequence, the lifetime of an individual when old depends upon his health status when young, which is augmented through private investments in health and tax-financed public investments in health. The former are represented by private effort to better health and longevity directly provided by individuals, which can reasonably be represented by "annual diagnostic health screening, opportunity cost of regular exercise, taking vitamins, nutrients, and other supplements, eating organically grown food, health benefits from quitting unhealthy habits such as smoking, etc." (see Bhattacharya and Qiao, 2007, p. 2520). The latter ones can be summarised, following Chakraborty (2004, p. 121), by the provision of "clinical facilities, sanitation, inoculation and disease control programs", or being represented by policies to promote healthy environments. Therefore, adult mortality can be reduced through the rise in health spending. In this respect, we strictly follow Bhattacharya and Qiao (2007) and assume that the survival probability when old of the typical agent of generation t is determined by $\theta_t = \theta(x_t, \eta_t)$, where θ is the so called longevity production function, with x_t being the private input (private investments in health) and η_t the public input (public investments in health). Different from Bhattacharya and Qiao (2007), we assume that the private input and the public input in the longevity production function can generally be viewed as complements, that is we assume that an increase in public investments in health always acts as an incentive to increase private investments in health. In other words, the public expenditure on health increases the marginal productivity of the private one. This to capture the interrelationship between private and public inputs for developed (rather than developing) countries. This assumption represents a difference with respect to the longevity production function used by Bhattacharya and Qiao (2007): $\theta_t = b\eta_t x_t^{b\eta_t}$, where $b > 0$. In fact, with this formulation the expression $\theta''_{x_t,\eta_t} = b^2\eta_t x_t^{b\eta_t}\left(\frac{2+b\eta_t \ln(x_t)}{x_t}\right)$ is negative for low values of x_t. This means that an increase in public health spending reduces the productivity of the private contribution to longevity when the private health spending is sufficiently low: a realistic scenario for underdeveloped and developing countries, where individuals earn low wages and cannot allocate adequate private resources for health care.

Assumption A.1 $\theta(x_t, \eta_t) : D \to [0, 1]$, where $D = [0 + \infty) \times [0 + \infty)$. It is C^n on the set $int(D)$ with n sufficiently high, and $\theta'_{x_t}(x_t, \eta_t) > 0$, $\theta'_{\eta_t}(x_t, \eta_t) > 0$, $\theta''_{x_t}(x_t, \eta_t) < 0$, $\theta''_{\eta_t}(x_t, \eta_t) < 0$, $\theta''_{x_t,\eta_t}(x_t, \eta_t) > 0$, $\theta(0, \eta_t) = 0\ \forall \eta_t \geq 0$, $\theta(x_t, 0) = \underline{\theta} > 0\ \forall x_t > 0$, $\lim_{x_t \to 0} \theta(x_t, \eta_t) = +\infty$.

The budget constraint of a young individual of generation t is $s_t + x_t = (1 - \tau)w_t l_t$, where $0 < \tau < 1$ is a constant labour income tax rate. This constraint implies that labour income is divided between saving, s_t, and private health expenditure, x_t. When old, consumption, C_{t+1}, is constrained by the amount of resources saved when young plus expected interest accrued from t time to time $t + 1$, so that $C_{t+1} = R^e_{t+1} s_t$, where R^e_{t+1} is the expected interest factor, which will become the realised interest factor at time $t + 1$. Therefore, the lifetime budget constraint of an individual of generation t can be written as follows:

$$C_{t+1} = R^e_{t+1}[w_t l_t(1 - \tau) - x_t]. \tag{1}$$

The individual representative of generation t has preferences towards leisure when young and consumption when old, described by the following expected utility function:

$$U(\widehat{l} - l_t, x_t, C_{t+1}) := v(\widehat{l} - l_t) + \theta(x_t, \eta_t)u\left(C_{t+1}/B\right), \tag{2}$$

where $B > 0$ is a scaling parameter. Decisions on how much to consume when old and how much time to devote to labour activities and health spending when young determine saving behaviour.

Assumption A.2 $v(\ell_t)$ and $u\left(C_{t+1}/B\right)$ are defined and continuous on the sets $0 \leq \ell_t \leq \widehat{l}$, R_+, respectively. They are C^n on the set R_{++} for n sufficiently high, with $v'(\ell_t) > 0$, $u'\left(C_{t+1}/B\right) >$

0, $v''(\ell_t) < 0$, $u''\left(C_{t+1}/B\right) < 0$, and

$$\lim_{\ell_t \to 0} v'(\ell_t) = \lim_{C_{t+1} \to 0} u'(C_{t+1}) = +\infty, -(C_{t+1}/B) \cdot \frac{u''\left(C_{t+1}/B\right)}{u'\left(C_{t+1}/B\right)} < 1.$$

Proposition 1 *Under Assumption A.2, expected utility function U is strictly concave if $\theta v \theta''_x v'' - \left(\theta'_x v'\right)^2 > 0$.*

Proof. The proof follows from the study of the principal minors of the Hessian matrix

$$H(U(\widehat{l} - l_t, x_t, C_{t+1})) = \begin{pmatrix} u'' & 0 & 0 \\ 0 & \theta''_x v & \theta'_x v' \\ 0 & \theta'_x v' & \theta v'' \end{pmatrix},$$

associated to (2). ■

Given the public input in the longevity production function η_t, the health tax rate τ and factor prices w_t and R^e_{t+1}, the individual representative of generation t maximises utility function (2) with respect to choice variables l_t, x_t and C_{t+1} subject to the lifetime budget constraint (1) and $0 \le l_t \le \widehat{l}$, $C_{t+1} \ge 0$, $x_t \ge 0$.

By substituting (1) for C_{t+1} into utility function (2), the optimisation programme reduces to:

$$\max_{x_t, l_t} \left\{ \widetilde{U}(l_t, x_t) := v(\widehat{l} - l_t) + \theta(x_t, \eta_t) u \left(\frac{R^e_{t+1}[w_t l_t (1-\tau) - x_t]}{B} \right) \right\}, \tag{3}$$

with $0 \le x_t \le w_t l_t (1-\tau)$, and $0 \le l_t \le \widehat{l}$.

Assumptions A.1 and A.2 avoid corner solutions of problem (3), from which the first order conditions

$$-v'(\widehat{l} - l_t) + \theta(x_t, \eta_t) u' \left(\frac{R^e_{t+1}[w_t l_t (1-\tau) - x_t]}{B} \right) \frac{R^e_{t+1} w_t (1-\tau)}{B} = 0, \tag{4}$$

$$\theta'_{x_t}(x_t, \eta_t) u \left(\frac{R^e_{t+1}[w_t l_t (1-\tau) - x_t]}{B} \right) - \theta(x_t, \eta_t) u' \left(\frac{R^e_{t+1}[w_t l_t (1-\tau) - x_t]}{B} \right) \frac{R^e_{t+1}}{B} = 0, \tag{5}$$

characterise the solution of the optimal programme.

Proposition 2 *Conditions (4) and (5) define a couple of functions*

$$\begin{aligned} x_t &= x(w_t(1-\tau), R^e_{t+1}, \eta_t, B), & (6) \\ l_t &= l(w_t(1-\tau), R^e_{t+1}, \eta_t, B), & (7) \end{aligned}$$

differentiable for $(w_t(1-\tau), R^e_{t+1}, \eta_t) \in R_{++} \times R_{++} \times R_{+}$.

Proof. The Hessian matrix of $\widetilde{U}$ is

$$H(\widetilde{U}(l_t, x_t)) = \left(\begin{array}{cc} u'' + \theta v'' (wR^e)^2 & \theta'_x v' wR^e - \theta v'' w(R^e)^2 \\ \theta'_x v' wR^e - \theta v'' w(R^e)^2 & \theta''_x v - v'\theta'_x R^e - \theta'_x v' R^e + \theta v''(R^e)^2 \end{array} \right).$$

From the properties of U and from the linearity of the budget constraint we have the following inequalities

$$u'' + \theta v'' (wR^e)^2 < 0,$$

$$(u'' + \theta v'' (wR^e)^2) \times (\theta''_x v - v'\theta'_x R^e - \theta'_x v' R^e + \theta v''(R^e)^2) - (\theta'_x v' wR^e - \theta v'' w(R^e)^2)^2 > 0.$$

In particular from the second one we have the result. ■

Assumption A.3 $\frac{\partial l_t}{\partial w_t} > 0$, $\frac{\partial l_t}{\partial R^e_{t+1}} > 0$, $\frac{\partial x_t}{\partial w_t} > 0$, $\frac{\partial x_t}{\partial R^e_{t+1}} > 0$.

Assumption A.3 guarantees standard behaviours of both the labour supply and private expenditure on health (which is a normal good as in Bhattacharya and Qiao, 2007). In a model similar than ours, Nourry (2001) has assumed that $\frac{\partial (w_t l_t)}{\partial w_t} > 0$. In the present work, we have preferred to characterise solutions in variables l_t and x_t (together with their properties) in order to highlight the main characteristics related to the agent's allocation problem with regard to private investments in health. In general, our hypotheses are not sufficient to guarantee the so-called "agent monotonicity". Nevertheless, in Section 4 we will introduce opportune functional forms for which this property is satisfied. It is important to note, however, that our results are not related to the violation of the assumptions stated above but they are essentially concerned with the joint existence of both private and public investments in health.

2.2 Production and government

We assume that at time t identical and competitive firms produce a homogeneous good, Y_t, by combining capital, K_t, and labour, L_t, through a constant returns to scale technology, that is $Y_t = AF(K_t, L_t)$, where $A > 0$ is a scaling parameter. The production function satisfies the following properties.

Assumption A.4 $F(K, L)$ is defined, continuous, strictly concave on set R^2_{++}, and it is homogeneous of degree one, i.e. $F(K, L) = Lf(k)$, with $k = K/L$. Moreover, $f'(k) > 0$ and $f''(k) < 0$, for all $k > 0$. It follows that the marginal productivity of capital and the marginal productivity of labour are respectively given by

$$R(k) = Af'(k) > 0,$$

and

$$w(k) = A(f(k) - kf'(k)) = A\Omega(k) > 0,$$

with $R'(k) < 0$ and $w'(k) > 0$.

We assume that the public input in the longevity production function is determined by public investments in health per young person at time t, p_t, which are provided by the government at a balanced budget by levying labour income taxes at the constant rate $0 < \tau < 1$ (Chakraborty, 2004; Bhattacharya and Qiao, 2007; Fanti and Gori, 2014), that is $\eta_t = \eta(p_t)$, where

$$p_t = \tau w_t l_t. \tag{8}$$

Assumption A.5 $\eta : R_+ \rightarrow [\underline{\eta}, +\infty)$, where $\underline{\eta} > 0$. It is C^n on R_{++} with n sufficiently high, and $\eta'(p_t) > 0$.

Remark 3 *Function η can be non-concave. This implies that an increase in public expenditure on health p_t (through an increase in health tax rate τ) produces a more than proportional increase in the public input that contributes to produce longevity through the health technology. This is reasonable at least for low values of the public health spending.*

Below we define an intertemporal competitive equilibrium with perfect foresight for our economy.

Definition 4 *An intertemporal competitive equilibrium with perfect foresight is a sequence of prices $\{w_t, R_t\}_{t=0}^{+\infty}$ and allocations $\{k_t, l_t, x_t, C_{t+1}\}_{t=0}^{+\infty}$ such that, for all $t \geq 0$:*
(i) given $(w_t, \eta_t, R^e_{t+1}, \tau)$, the triple (l_t, x_t, C_{t+1}) solves the problem of the representative agent;
(ii) $R^e_{t+1} = R_{t+1}$;
(iii) given (w_t, R_t), the couple (k_t, L_t) solves the profit maximisation problem of the representative firm, and $L_t = l_t$ is a condition that determines the intra-temporal equilibrium in the labour market;
(iv) $\eta_t = \eta(p_t)$ and $p_t = \tau w_t l_t$;
(v) $s_t = k_{t+1} l_{t+1}$.

From Definition 4 it follows that equilibrium dynamics are characterised by the following two equations.

$$-v'(\widehat{l} - l_t) + \theta(x_t, \eta(\tau A\Omega(k_t)l_t))u'\left(\frac{Af'(k_{t+1})[A\Omega(k_t)l_t(1-\tau) - x_t]}{B}\right) \times \\ \times \frac{A^2 f'(k_{t+1})\Omega(k_t)(1-\tau)}{B} = 0 \tag{9}$$

$$\theta'_{x_t}(x_t, \eta(\tau A\Omega(k_t)l_t))u\left(\frac{Af'(k_{t+1})[A\Omega(k_t)l_t(1-\tau) - x_t]}{B}\right) + \\ -\theta(x_t, \eta(\tau A\Omega(k_t)l_t))u'\left(\frac{Af'(k_{t+1})[A\Omega(k_t)l_t(1-\tau) - x_t]}{B}\right)\frac{Af'(k_{t+1})}{B} = 0 \tag{10}$$

We note that, under some suitable hypotheses, once the dependency of η_t, w_t, R^e_{t+1} on k_t, k_{t+1} and l_t is taken into account, the equations that determine individual allocations (6) and (7) implicitly define the allocations at the macroeconomic level of both the private health expenditure and labour as functions of k_t and k_{t+1}, that is $x_t = X(k_t, k_{t+1})$ and $l_t = \Lambda(k_t, k_{t+1})$. More specifically, we have the following proposition.

Proposition 5 *Let*
(i) k_t and k_{t+1} be two values of the stock of capital consistent with Definition 4, and
(ii) l_t and x_t be two allocations of the individual labour supply and private health expenditure, respectively, consistent with Definition 4.
If

$$l'_{l_t}(\cdot) := l'_\eta[A\Omega(k_t)(1-\tau), Af'(k_{t+1}), \eta[\tau A\Omega(k_t)l_t], B] \times \eta'_p[\tau A\Omega(k_t)l_t] \times \tau A\Omega(k_t) \neq 1,$$

then it is possible to define an aggregate level of labour supply Λ_t and an aggregate level of private health expenditure X_t as functions of k_t and k_{t+1}.

Proof. From equilibrium conditions on the labour market, the capital market and the government budget constraint, we have the following expressions:

$$x_t = x[A\Omega(k_t)(1-\tau), Af'(k_{t+1}), \eta[\tau A\Omega(k_t)l_t], B], \tag{11}$$

$$l_t = l[A\Omega(k_t)(1-\tau), Af'(k_{t+1}), \eta[\tau A\Omega(k_t)l_t], B]. \tag{12}$$

In order to apply the implicit function theorem, from (12) we have to impose that

$$l'_{l_t}(\cdot) := l'_\eta[A\Omega(k_t)(1-\tau), Af'(k_{t+1}), \eta[\tau A\Omega(k_t)l_t], B] \times \eta'_p[\tau A\Omega(k_t)l_t] \times \tau A\Omega(k_t) \neq 1.$$

It implies the existence of a functional dependence of l_t on k_t and k_{t+1}, that is $l_t = \Lambda(k_t, k_{t+1})$. By substituting $\Lambda(k_t, k_{t+1})$ for l_t in (11) we have the result. ■

Lemma 6 $\frac{\partial l_t}{\partial k_t} = \frac{l'_{\widetilde{w}} A\Omega'(k_t)(1-\tau) + l'_\eta \eta'_p \tau A\Omega'(k_t) l_t}{1 - l'_{l_t}(\cdot)}$; $\frac{\partial l_t}{\partial k_{t+1}} = \frac{l'_R A f''(k_{t+1})}{1 - l'_{l_t}(\cdot)}$;
$\frac{\partial x_t}{\partial k_t} = x'_{\widetilde{w}} A\Omega'(k_t)(1-\tau) + x'_\eta \eta'(p)\tau A \left[\Omega'(k_t) l_t + \frac{\partial l_t}{\partial k_t}\Omega(k_t)\right]$;
$\frac{\partial x_t}{\partial k_{t+1}} = x'_R A f''(k_{t+1}) + x'_\eta \eta'(p)\tau A\Omega(k_t)\frac{\partial l_t}{\partial k_{t+1}}$.

By introducing the following notation and identities:

$$\alpha := \frac{f'(k)k}{f(k)};\ \sigma := -\left[\frac{f(k) - f'(k)k}{f(k)}\right]\frac{f'(k)}{kf''(k)};\ \varepsilon_R := \frac{R'(k)k}{R(k)} = \frac{f''(k)k}{f'(k)} = -\frac{1-\alpha}{\sigma};$$

$$\varepsilon_\Omega := \frac{\Omega'(k)k}{\Omega(k)} = -\frac{f''(k)k^2}{f(k) - f'(k)k} = \frac{\alpha}{\sigma};\ \varepsilon_{l,\widetilde{w}} := \frac{l'_{\widetilde{w}}\widetilde{w}}{l};\ \varepsilon_{l,\eta} := \frac{l'_\eta \eta}{l};\ \varepsilon_{\eta,p} := \frac{\eta'_p p}{\eta},$$

it is possible to express the results of Lemma 6 in the following way:[2]

Proposition 7 $\frac{\partial l_t}{\partial k_t} = \frac{\varepsilon_{l,\widetilde{w}} + \varepsilon_{l,\eta}\varepsilon_{\eta,p}}{1 - \varepsilon_{l,\eta}\varepsilon_{\eta,p}}\frac{l}{k}\frac{\alpha}{\sigma}$; $\frac{\partial l_t}{\partial k_{t+1}} = -\varepsilon_{l,R}\frac{1-\alpha}{\sigma}\frac{l}{k}$; $\frac{\partial x_t}{\partial k_t} = \frac{\alpha}{\sigma}\frac{x}{k}\left[\varepsilon_{x,\widetilde{w}} + \varepsilon_{x,\eta}\varepsilon_{\eta,p}\frac{1 + \varepsilon_{l,\widetilde{w}}}{1 - \varepsilon_{l,\eta}\varepsilon_{\eta,p}}\right]$;
$\frac{\partial x_t}{\partial k_{t+1}} = -\frac{1-\alpha}{\sigma}\frac{x}{k}(\varepsilon_{x,R} + \varepsilon_{x,\eta}\varepsilon_{\eta,p}\varepsilon_{l,R})$.

At the macroeconomic level, the behaviour of both l_t and x_t can be explained on the basis of the interaction of several elasticities whose sign depend on their relative magnitudes. In particular, from the hypotheses stated above only the sign of $\frac{\partial l_t}{\partial k_{t+1}} < 0$ is unambiguous for any values of the parameters of the model. The remaining derivatives are ambiguous because their sign depend on both the responses of agent's control variables with respect to the parameters of the model and how technologies react when their inputs are changed.

[2] Note that we have deleted the time subscript.

3 Local dynamics

In this section we study the dynamics of the economy around a fixed point. In general, in overlapping generations models the existence of non-trivial fixed points is not guaranteed. Specifically, in our model the joint existence of both private investments in health and the externality caused by public investments in health (that directly enter individual utility) makes impossible to find a non-trivial fixed point in closed form. The main difference between the present model and the model developed by Nourry (2001) - where consumption when young, consumption when old and leisure are control variables - is that in our model it is not possible to define (in general) a labour supply function at the macroeconomic level for every couple (k_t, k_{t+1}).

In order to simplify the analysis, we will follow the procedure adopted by Cazzavillan et al. (1998) and use the scaling parameters A and B to give conditions for the existence of a normalised fixed point. In this model-economy, a normalised steady-state equilibrium is defined as a stationary sequence $(k_t, l_t) = (1, 1)$ that satisfies (9) and (10) for any $t \geq 0$.

Proposition 8 *If*

$$\lim_{A \to +\infty} 1 - A\Omega(1)(1-\tau) + x_1(A, B_A) < 0,$$

and

$$l'_{l_t}[A\Omega(1)(1-\tau), Af'(1), \eta_t[\tau A\Omega(1)], B] \neq 1,$$

then the point $(k^*, l^*) = (1, 1)$ *is a fixed point of the dynamic system defined by (9) and (10), where* $x_1(A, B_A)$ *is the expression of function* x *evaluated at* $(1, 1)$ *and* B_A *is defined in the proof.*

Proof. In order to show the conditions under which a normalised fixed point does exist, we note from Proposition (5) that it is possible to obtain the expression of x_t corresponding to $(1, 1)$, that is:

$$x_1(A, B) := x_1[A\Omega(1)(1-\tau), Af'(1), \eta_t[\tau A\Omega(1)], B]. \tag{13}$$

From the hypothesis

$$l'_{l_t}[A\Omega(1)(1-\tau), Af'(1), \eta[\tau A\Omega(1)], B] \neq 1,$$

it follows that this expression is well defined for non-negative values of A and B. By substituting out (13) into (9) and taking into account the dynamics of capital referred to point (v) in Definition (4), we can focus on solutions for A and B of the following system:

$$-v'(\widehat{l} - 1) + \theta(x_1(A, B), \eta[\tau A\Omega(1)]) \times \tag{14}$$
$$\times u' \left(\frac{Af'(1)\,[A\Omega(1)(1-\tau) - x_1(A, B)]}{B} \right) \frac{A^2 f'(1)\Omega(1)(1-\tau)}{B} = 0$$

$$1 - A\Omega(1)(1-\tau) + x_1(A, B) = 0. \tag{15}$$

We can now solve (15) for $x_1(A, B)$. Then by substituting it in (14) we obtain:

$$-v'(\widehat{l} - 1) + \theta(A\Omega(1)(1-\tau) - 1, \eta[\tau A\Omega(1))] \times \tag{16}$$
$$\times u' \left(\frac{Af'(1)}{B} \right) \frac{A^2 R(1)\Omega(1)(1-\tau)}{B} = 0,$$
$$1 - A\Omega(1)(1-\tau) + x_1(A, B) = 0. \tag{17}$$

From the hypotheses in Assumption A.2, we get:

$$B_A = B(A) := W^{-1}\left(\frac{v'(\widehat{l}-1)}{AR(1)\Omega(1)(1-\tau)\theta[A\Omega(1)(1-\tau)-1,\eta[\tau A\Omega(1)]]}\right).$$

By substituting out B_A in (17) we get the result. ■

We can now concentrate on the local dynamics around the normalised fixed point $(1,1)$. In order to make the outcomes of this study more readable, it is convenient to rewrite (9) and (10) in a different manner. In particular, by using expressions of l_t and x_t at the macroeconomic level, from the budget constraint of a young individual of generation t and point (v) in Definition 4, it is possible to redefine equilibrium dynamics through the following second order nonlinear difference equation:

$$k_{t+1}\Lambda(k_{t+1},k_{t+2}) = (1-\tau)A^*\Omega(k_t)\Lambda(k_t,k_{t+1}) - X(k_t,k_{t+1}).$$

By using the change of variable $k_{t+1} = y_t$, we definitely obtain the following system:

$$\begin{array}{rcl} \Lambda(y_t,y_{t+1})y_t & = & (1-\tau)A^*\Omega(k_t)\Lambda(k_t,y_t) - X(k_t,y_t) \\ k_{t+1} & = & y_t \end{array}$$

We note that the relationship

$$A^* := \frac{1+x(1,1)}{(1-\tau)\Omega(1)},$$

holds and that in general $x(1,1)$ depends on A^* and B^*. This does not allow us to use the geometrical method to study local dynamics. The Jacobian matrix associated to the normalised fixed point is the following:

$$J(1,1) := \begin{pmatrix} \frac{1+l'_{k_t}+x'_{k_{t+1}}-[1+x(1,1)]l'_{k_{t+1}}}{-l'_{k_{t+1}}} & \frac{x'_{k_t}-[1+x(1,1)](\varepsilon_\Omega+l'_{k_t})}{-l'_{k_{t+1}}} \\ 1 & 0 \end{pmatrix},$$

and its characteristic polynomial is given by:

$$P(\lambda) = \lambda^2 + \left(\frac{1+l'_{k_t}+x'_{k_{t+1}}-[1+x(1,1)]\,l'_{k_{t+1}}}{l'_{k_{t+1}}}\right)\lambda + \frac{x'_{k_t}-[1+x(1,1)](\varepsilon_\Omega+l'_{k_t})}{l'_{k_{t+1}}}.$$

Starting from the characteristic polynomial it is simple to characterise the local dynamics around the normalised fixed point. Fixed point $(1,1)$ is locally indeterminate if and only if $P(1) > 0$, $P(-1) > 0$ and $P(0) < 1$; it is a saddle if and only if $P(1)P(-1) < 0$; it is a source if and only if $P(1)P(-1) > 0$ and $|P(0)| > 1$. Since these expressions are difficult to be handled in a neat analytical form, the following propositions provide sufficient conditions to have or not a normalised indeterminate fixed point.

Proposition 9 *From the characteristic polynomial we find a sufficient condition that rules out local indeterminacy for the normalised fixed point. Since* $l'_{k_{t+1}} < 0$, *this condition is given by* $x'_{k_t} < l'_{k_{t+1}} + [1 + x(1,1)](\varepsilon_\Omega + l'_{k_t})$ *or, alternatively, by* $x'_{k_t} > -l'_{k_{t+1}} + [1 + x(1,1)](\varepsilon_\Omega + l'_{k_t})$.

Proposition 10 *If*

$$l'_{k_{t+1}} + [1 + x(1,1)](\varepsilon_\Omega + l'_{k_t}) < x'_{k_t} < [1 + x(1,1)](\varepsilon_\Omega + l'_{k_t}), \tag{18}$$

and

$$[2 + x(1,1)]\, l'_{k_{t+1}} - (1 + l'_{k_t}) < x'_{k_{t+1}} < x(1,1) l'_{k_{t+1}} - (1 + l'_{k_t}), \tag{19}$$

then the normalised fixed point is indeterminate.

From Proposition 9 it follows that a necessary condition for the existence of a normalised fixed point is that the reactivity of x_t when k_t varies should not be too high. If this holds, a low reactivity of x_t with respect to k_{t+1} implies that the normalised fixed point if indeterminate. To give much more policy insights, the next section deals with the local and global properties of a two-dynamical system by assuming specific functional forms with regard to individual preferences, and production of both longevity and final output.

4 CIES-Cobb-Douglas economy

The importance of global analysis for economic models is recognised by the fact that studying just the local behaviour of a map does not give information with regard to the structure of the basins of attraction and their qualitative changes when some parameters vary. Since in economic models it is also important to understand the long-term behaviour of variables given initial conditions, a characterisation of the basins of attraction is required if one wants to explain phenomena that occur by starting from initial conditions far away from a fixed point or an attracting set. To this purpose, in this section we consider specific functional forms with regard to utility, longevity and production functions. In particular, we will use the Constant Inter-temporal Elasticity of Substitution (CIES) formulation to describe individual preferences and Cobb-Douglas technologies to produce longevity and final output. Under these assumptions, we get expressions of x_t and l_t (solutions to the individual problem on how allocate resources) such that their ratio is independent from the interest factor R^e_{t+1}. This allows us to obtain a map in explicit form in variables K_t and l_t to describe the equilibrium dynamics of the economy.

In this section, therefore, we show how the study of: (i) the dynamics around the normalised fixed point $(1, 1)$, and (ii) the global structure of the two-dimensional map permit us to explain events related to endogenous fluctuations in macroeconomic and demographic variables that cannot be investigated with the local analysis (Pintus et al., 2000). First of all, we characterise the behaviour of economic agents in this economy.

Longevity. As stressed by Bhattacharya and Qiao (2007), complementarity between public and private inputs in the longevity production function means that a marginal increase in private effort to better health is more efficient as a means of higher longevity when public investments in health are high (Dow et al., 1999). In their paper, therefore, complementarity between the two inputs is captured by assuming a longevity production function such that the elasticity of longevity with respect to private investments in health depends on public investments in health. Different from Bhattacharya and Qiao (2007), we take into account a longevity production function where both the private and public inputs are complements for every x_t. This implies that an increase in public health investments increases the marginal productivity of private health investments, that is $\theta''_{x_t,\eta_t}(x_t,\eta_t) > 0$.[3] These assumptions are captured by the following Cobb-Douglas longevity production function:

$$\theta(x_t,\eta_t) := \frac{x_t^{\rho}\eta_t^{1-\rho}}{Z}, \tag{20}$$

where $\rho \in (0,1)$ is the (constant and independent from η) elasticity of longevity with respect to private investments in health and Z is a positive parameter that will appropriately be fixed to get well defined economic dynamics ($\theta(x_t,\eta_t) \in (0,1]$). We note that Z is a scaled constant that has the same unit of measurement of x_t and η_t. Before setting up the maximisation problem of expected utility by the typical agent and the profit maximisation problem by the typical firm, we additionally require that the relationship between the public input in the longevity function and public expenditure $\eta(p_t)$ has the form:

$$\eta(p_t) := \underline{\eta} + p_t^{\delta}, \tag{21}$$

that satisfies the following properties: if $\tau = 0$, $\eta(0) = \underline{\eta} > 0$; if $\tau > 0$, $\eta(p_t) > \underline{\eta}$, and where $\delta > 0$ is a parameter that weights the intensity of the effects of public health expenditure as a means of higher longevity. If $\delta \leq 1$ (resp. $\delta > 1$), η is a concave (resp. convex) function. The functional form of the public input in the longevity function given by (21) implies that if the government does not invest in health, a minimum level of public health support is in existence.

Individuals. The individual representative of generation t is assumed to have preferences (over the lifetime) towards leisure, private health spending and consumption described by the following CIES expected utility function:

$$U(2-l_t,x_t,C_{t+1}) = \frac{(2-l_t)^{1-\gamma}}{1-\gamma} + \theta(x_t,\eta_t)\frac{(C_{t+1}/B)^{1-\mu}}{1-\mu}, \tag{22}$$

where $\widehat{l} = 2$, $v(2-l_t) = \frac{(2-l_t)^{1-\gamma}}{1-\gamma}$, $u\left(C_{t+1}/B\right) = \frac{(C_{t+1}/B)^{1-\mu}}{1-\mu}$, $\gamma > 0$ ($\gamma \neq 1$) is a measure of the constant elasticity of utility with respect to leisure time, while the parameter $\mu \in (0,1)$, which contributes to determine the constant elasticity of utility with respect to consumption, is fixed between zero and one to avoid paradoxical effects of longevity on utility (Hall and Jones, 2007). This functional form for expected utility (general iso-elastic specification) is aimed for generality. It is important to note that expected utility function (22) is concave if and only if:

Assumption A.6 $\rho < \mu$.

[3] This hypothesis well captures health systems in developed countries, where public support to better health is already well-developed and individuals decide to privately spend in health as a substitute to public investments provided by governments.

Assumption A.6 always holds in what follows. The typical agent takes as given η_t, τ, and w_t and R^e_{t+1}. Then, by substituting out the lifetime budget constraint (1) for C_{t+1} into utility function (22) and maximising it with respect to l_t and x_t, the first order conditions for the optimisation problem of the typical consumer can be expressed as follows:

$$\frac{\eta_t^{1-\rho} x^{\rho} \left[(w_t(1-\tau)l_t - x_t)R^e_{t+1}/B\right]^{1-\mu} w_t(1-\tau)}{Z[w_t(1-\tau)l_t - x_t]} - (2-l_t)^{-\gamma} = 0, \tag{23}$$

$$\frac{\eta_t^{1-\rho} x_t^{\rho} \rho \left[(w_t(1-\tau)l_t - x_t)R^e_{t+1}/B\right]^{1-\mu}}{x_t Z(1-\sigma)} - \frac{\eta_t^{1-\rho} x_t^{\rho} \left[(w_t(1-\tau)l_t - x_t)R^e_{t+1}/B\right]^{1-\mu}}{Z[w_t(1-\tau)l_t - x_t]} = 0. \tag{24}$$

From (23) and (24), we find that the relationship

$$\frac{x_t}{l_t} = \frac{\rho w_t(1-\tau)}{\rho + 1 - \mu}, \tag{25}$$

holds at the optimum. This implies that the ratio between the optimal levels of private health expenditure and labour supply is independent from both the interest factor and public investments in health (this happens because, different from Bhattacharya and Qiao, 2007, the elasticity of longevity with respect to private investment in health is independent from public investments in health). In addition, by the implicit function theorem it follows that $\frac{\partial l_t}{\partial w_t} > 0$, $\frac{\partial l_t}{\partial R^e_{t+1}} > 0$ and $\frac{\partial l_t}{\partial \eta_t} > 0$. From (25) and the previous results we also find that $\frac{\partial x_t}{\partial w_t} > 0$, $\frac{\partial x_t}{\partial R^e_{t+1}} > 0$ and $\frac{\partial x_t}{\partial \eta_t} > 0$. As a direct consequence, when individuals experience an increase in labour income (wage), they increase the proportion of private health spending with respect to labour, while also increasing the proportion of private health spending with respect to leisure. This means that private health investments play an important role in countries where wages are sufficiently high, that is when the wage increases agents increase private health spending more than leisure in their optimal consumption bundles. Furthermore, the following result holds:

Proposition 11 $\frac{\partial s_t}{\partial w_t} > 0$.

Proof. Given the individual budget constraint when young $s_t = (1-\tau)w_t l_t - x_t$, we have that

$$\frac{\partial s_t}{\partial w_t} = (l_t + w_t l_t') \left[1 - \frac{\rho(1-\tau)}{\rho + 1 - \sigma}\right] > 0.$$

■

Firms. We assume that at time t identical and competitive firms produce a homogeneous good, Y_t, by combining capital, K_t, and labour, L_t, through the Cobb-Douglas technology:

$$Y_t = AF(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}, \tag{26}$$

where $A > 0$ and $0 < \alpha < 1$ are a production scaling parameter and the capital share, respectively. The temporary equilibrium condition on the market for labour at time t is $L_t = l_t$. Then, by assuming that capital fully depreciates at the end of every period and output is sold at unit price, maximisation of profits $AF(K_t, l_t) - w_t l_t - R_t K_t$ (by taking the wage and the interest factor as given) implies that the typical firm equals the marginal productivity of labour (resp. capital) to the wage rate (resp. the interest factor), that is:

$$w_t = (1-\alpha)AK_t^{\alpha}l^{-\alpha}, \tag{27}$$
$$R_t = \alpha AK_t^{\alpha-1}l^{1-\alpha}. \tag{28}$$

Equilibrium and dynamics. The market-clearing condition in the capital market can be expressed as $K_{t+1} = s_t = w_t l_t(1-\tau) - x_t$. Then, by using the consumer's first order conditions (23) and (24), and firm's ones (27) and (28), and knowing that individuals have perfect foresight, we are able to explicit the forward equilibrium dynamics in variables K and l as follows:[4]

$$K' = \frac{AK^{\alpha}l^{1-\alpha}(1-\tau)(1-\mu)(1-\alpha)}{\rho+1-\mu}, \tag{29}$$

$$l' = \left(\frac{\left(\frac{(\rho+1-\mu)Bl^{\alpha-1}K^{-\alpha}}{\alpha(1-\mu)(1-\tau)(1-\alpha)A^2}\right)^{1-\mu} \left(\frac{l(1-\mu)Z}{(\rho+1-\mu)\eta^{1-\rho}}\right)(K')^{(1-\alpha)(1-\mu)}}{(2-l)^{\gamma}\left(\frac{\rho AK^{\alpha}l^{1-\alpha}(1-\tau)(1-\alpha)}{\rho+1-\mu}\right)^{\rho}} \right)^{\frac{1}{(1-\alpha)(1-\mu)}}, \tag{30}$$

where we have omitted the time subscript and $'$ is represents the unit-time advancement operator of variables K and l. The dynamic system described by (29) and (30) defines variables K_{t+1} and l_{t+1} as functions of K_t and l_t. The next section studies existence and stability properties of the fixed point of the system from both local and global perspectives. With regard to the local analysis, we will apply the geometrical-graphical method developed by Grandmont et al. (1998) and used, amongst others, by Cazzavillan et al. (1998).

Straightforward calculations allow us to show that there exists one and only one couple $(A,B) = (A^*,B^*)$ such that the normalised fixed point $(1,1)$ always exists, where

$$A^* \; : \; = \frac{\rho+1-\mu}{(1-\mu)(1-\alpha)(1-\tau)}, \tag{31}$$

$$B^* \; : \; = \frac{A^*\left[\frac{\rho^{\rho}\alpha^{1-\mu}(\rho+1-\mu)^{2-\mu}}{Z(1-\mu)^{1+\rho}}\left(1+\left(\frac{\tau(\rho+1-\mu)}{(1-\tau)(1-\mu)}\right)^{\delta}\right)^{1-\rho}\right]^{\frac{1}{1-\mu}}}{\rho+1-\mu}. \tag{32}$$

By substituting out (31) and (32) in (29) and (30) we get the two-dimensional map that describes the dynamics of the economy:

$$M : \begin{cases} K' = V_1(K,l) := K^{\alpha}l^{1-\alpha} \\ l' = V_2(K,l) := P(2-l)^{\frac{-\gamma}{(1-\alpha)(1-\mu)}} l^{\frac{1-(1-\alpha)[\rho+\alpha(1-\mu)]}{(1-\alpha)(1-\mu)}} K^{\frac{-\alpha[\rho+\alpha(1-\mu)]}{(1-\alpha)(1-\mu)}} \left[1+\frac{\tau^{\delta}(\rho+1-\mu)^{\delta}(K^{\alpha}l^{1-\alpha})^{\delta}}{(1-\tau)^{\delta}(1-\mu)^{\delta}}\right]^{\frac{\rho-1}{(1-\alpha)(1-\mu)}} \end{cases}, \tag{33}$$

[4] Other functional forms with regard utility, longevity and production functions do not allow to have a two-dimensional dynamic system in explict form. It also possible that forward dynamics may not be well defined for any t (Gardini et al., 2009).

where

$$P := \left[1 + \frac{\tau^\delta(\rho+1-\mu)^\delta}{(1-\tau)^\delta(1-\mu)^\delta}\right]^{\frac{1-\rho}{(1-\alpha)(1-\mu)}}.$$

We note that given the couple (K, l), it is possible to compute its subsequent iterate if and only if we start by a point in set

$$W_1 = \{(K, l) \in R^2 : K > 0, 0 < l < 2\}. \tag{34}$$

However, for economic reasons we restrict the study of map (33) to the following set

$$W_2 = \{(K, l) \in R^2 : 0 < K < 2, 0 < l < 2\} \subset W_1. \tag{35}$$

Nevertheless, feasible (and economically significant) trajectories lie in a set smaller than W_2, since by starting from an initial condition in W_2 it is possible to have an iterate from which the existence of the subsequent one is not guaranteed. Then, we introduce the set of feasible and economically significant trajectories, which is given by

$$W_3 = \{(K, l) \in R^2 : M^n(K, l) \in W_2, \forall n > 0\}, \tag{36}$$

where $M^n(K, l)$ is the nth iterate of the map applied to point (K, l).

Since one the main objective of this paper is the study of global dynamics of map M, it is now important to define a threshold value of the scale parameter Z such that $\theta \in (0, 1]$ for every iterate of the map. Generally speaking, from a mathematical point of view this procedure would not be applicable due to the existence of basins of attraction for which K may take values without bounds. However, from the capital accumulation equation we get $K' = K^\alpha l^{1-\alpha} \leq K^\alpha 2^{1-\alpha}$ (where the right-hand side of the inequality gives the accumulation of capital when individuals work for all their time endowment), and every feasible trajectory that starts from that region cannot take values of the stock of capital such that $K' < 2 = K_{\max}$, where $K_{\max}$ represents the stationary-state value of K such that $l = 2$ (unbounded trajectories do not exist). Then, we can restrict the study of map M to trajectories that start from values of the capital stock smaller than $K = K_{\max}$. Now, by substituting out $K = 2$ and $l = 2$ in (20) we get the following threshold value of Z

$$\widetilde{Z} := \left[1 + \left(\frac{2\tau(\rho+1-\mu)}{(1-\tau)(1-\mu)}\right)^\delta\right]^{1-\rho} \left(\frac{2\rho}{(1-\mu)}\right)^\rho, \tag{37}$$

such that $\theta(x, \eta) < 1$ for any $Z > \widetilde{Z}$ and t. The proposition with regard to the number of fixed points of map M now follows.

Proposition 12 *[Existence of fixed points]. (a) Map M generically admits an odd number of fixed points (at most three). (b) In particular, if*

$$\left[(1-\tau)^\delta(1-\mu)^\delta + \tau^\delta(\rho+1-\mu)^\delta\right](-\rho+\mu+\gamma) + \tau^\delta(\rho+1-\mu)^\delta(-1+\rho)\delta < 0,$$

there exist three fixed points with $(1, 1)$ being the intermediate one. Ceteris paribus, the latter inequality is verified when δ is sufficiently high.

Proof. Since $K = l$ always holds as a coordinate of a stationary state of map M, then stationary-state coordinate values of l are determined as solutions of $l = V_2(l, l)$ or they are equivalently obtained by solving the following equation:

$$g(l) := P(2-l)^{-\gamma} l^{\mu-\rho} \left[1 + \frac{\tau^\delta (\rho + 1 - \mu)^\delta}{(1-\tau)^\delta (1-\mu)^\delta} l^\delta \right]^{\rho-1} = 1. \tag{38}$$

We have that $\lim_{l\to 0} g(l) = 0$, $\lim_{l\to 2} g(l) = +\infty$ because Assumption A.6 holds. In order to determine the number of fixed points of map M, the use of (38) allows us (through cumbersome but straightforward calculations) to characterise the maximum number of monotonic intervals of g. In particular,

$$sgn\{g\prime(l)\} = sgn\{H(l)\},$$

where

$$H(l) := (1 + \widetilde{N})^{1-\rho} l^{\mu-\rho} (1 + l^\delta \widetilde{\widetilde{N}})^{\delta(\rho-1)} (2-l)^{-\gamma},$$

with $\widetilde{N}$ and $\widetilde{\widetilde{N}}$ being opportune combinations of parameters. Now, $H\prime(l) = 0$ when

$$l := \frac{2(\delta - 1)(-\delta^2 + \delta^2\rho - \rho + \mu)}{(1+\delta)(-\delta^2 + \delta^2\rho - \rho + \sigma - \gamma)}.$$

By considering $\lim_{l\to 0} g(l)$, $\lim_{l\to 2} g(l)$ and knowing that $g\prime$ can change sign at most four times, we get Point (a). With regard to Point (b) it is sufficient to evaluate $g\prime(1)$. ■

Proposition 12 characterises the number of fixed points of map M (and then of long-term behaviours of the economy) depending of the relative interactions of the main parameters of the problem (that includes policy variables). In particular, if the relative weight of the effects of public health expenditure as a means of higher longevity is high, three fixed points can exist. This opens several questions with regard to how an economy may behave in the long term depending on initial conditions and stability properties of a fixed point at both local and global perspectives.

4.1 Local bifurcations and stability of the normalised fixed point

This section starts by analysing the local dynamics around the normalised fixed point $(1, 1)$. In the present model, the stock of capital K is a state variable, so that its initial value K_0 is given, while the supply of labour l is a control variable. It follows that individuals of the first generation $(t = 0)$ choose the initial value l_0. If the normalised fixed point is a saddle and the initial condition of K is close enough to 1, there exists a unique initial value of l (l_0) such that the orbit that passes through (K_0, l_0) approaches the fixed point. In contrast, when the fixed point is a sink, given the initial value of the state variable, there exists a continuum of initial values of the control variable such that the orbit that passes through (K_0, l_0) approaches the fixed point. As a consequence, the orbit that the economy will follow is "locally indeterminate" because it depends on the choice of l_0.

The Jacobian matrix of the map M evaluated at $(1, 1)$ is the following:

$$J = \begin{pmatrix} \alpha & 1-\alpha \\ J_{21} & J_{22} \end{pmatrix},$$

where

$$J_{21} := \frac{\alpha(\rho+1-\mu)^\delta\{\tau^\delta[(\delta-1)\rho-(1-\mu)\alpha-\delta]\}-(1-\tau)^\delta(1-\mu)^\delta[\rho+(1-\mu)\alpha]}{(1-\alpha)(1-\mu)[(1-\tau)^\delta(1-\mu)^\delta+\tau^\delta(\rho+1-\mu)^\delta]},$$

$$J_{22} := \frac{\tau^\delta(\rho+1-\mu)^\delta\{(1-\mu)\alpha^2+[(\delta-1)(1-\rho)+\mu]\alpha+(\delta-1)(1-\rho)+\gamma\}}{(1-\alpha)(1-\mu)[(1-\tau)^\delta(1-\mu)^\delta+\tau^\delta(\rho+1-\mu)^\delta]}+$$
$$+\frac{[(1-\mu)\alpha^2+(-1+\rho+\mu)\alpha+1+\gamma-\rho](1-\mu)^\delta(1-\tau)^\delta}{(1-\alpha)(1-\mu)[(1-\tau)^\delta(1-\mu)^\delta+\tau^\delta(\rho+1-\mu)^\delta]}.$$

The trace and determinant of J are given by the following expressions:

$$Tr(J) := \alpha + J_{22},$$

$$Det(J) := \frac{\alpha(1+\gamma)}{(1-\alpha)(1-\mu)}.$$

Ceteris paribus, when γ varies the point

$$(P_1, P_2) := \left(\frac{\gamma}{(1-\alpha)(1-\mu)}+\overline{P}_1, \frac{(1+\gamma)\alpha}{(1-\alpha)(1-\mu)}\right),$$

where

$$\overline{P}_1 := \frac{-\tau^\delta(\rho+1-\mu)^\delta[\rho(\alpha-1)(\delta-1)-\alpha\delta+\delta-1]+(1-\tau)^\delta(1-\mu)^\delta[\rho(\alpha-1)+1]}{(1-\alpha)(1-\mu)\,[(1-\tau)^\delta(1-\mu)^\delta+\tau^\delta(\rho+1-\mu)^\delta]},$$

drawn in the $(Tr(J), Det(J))$ plane, describes a half-line of slope α that starts from $(\overline{P}_1, \overline{P}_2)$, where

$$\overline{P}_2 := \frac{\alpha}{(1-\alpha)(1-\mu)}$$

when $\gamma = 0$ and $(P_1, P_2) \to (-\infty, +\infty)$ for $\gamma \to +\infty$. In turn, when τ varies the point $(\overline{P}_1, \overline{P}_2)$, drawn in the $(Tr(J), Det(J))$ plane, describes a segment line that starts from $\left(\frac{1+\rho(\alpha-1)}{(1-\alpha)(1-\mu)}, \frac{\alpha}{(1-\alpha)(1-\mu)}\right)$ (when $\rho = 0$) and moves towards $\left(\frac{1+\rho(\alpha-1)-\delta(\alpha-1)(\rho-1)}{(1-\alpha)(1-\mu)}, \frac{\alpha}{(1-\alpha)(1-\mu)}\right)$ when ρ increases. The latter point is reached when $\rho = 1$.

From the geometrical findings above and Proposition 12 we can now state the following proposition with regard to local bifurcations of the normalised fixed point.

Proposition 13 *[Local stability and bifurcations]. Depending on the relative values of μ, α and τ we have the following results.*

(1) Let $\mu < \overline{\mu}$, $\alpha > \overline{\alpha}$ and $\tau > \overline{\tau}$. Then, the fixed point $(1,1)$ is a saddle for $0 < \gamma < \gamma_{fl}$, a source for $\gamma_{fl} < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(2) Let $\mu < \overline{\mu}$, $\overline{\overline{\alpha}} < \alpha < \overline{\alpha}$ and $\tau > \overline{\tau}$. Then, the fixed point $(1,1)$ is a saddle for $0 < \gamma < \gamma_{fl}$, indeterminate for $\gamma_{fl} < \gamma < \gamma_{ns}$, a source for $\gamma_{ns} < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(3) Let $\mu < \overline{\mu}$, $0 < \alpha < \overline{\overline{\alpha}}$ and $\tau > \overline{\tau}$. Then, the fixed point $(1,1)$ is a saddle for $0 < \gamma < \gamma_{fl}$, a source for $\gamma_{fl} < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(4) Let $\mu < \overline{\mu}$, $\alpha > \overline{\overline{\alpha}}$ and $\overline{\overline{\tau}} < \tau < \overline{\tau}$. Then, the fixed point $(1,1)$ is indeterminate for $0 < \gamma < \gamma_{ns}$, a source for $\gamma_{ns} < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(5) Let $\mu < \overline{\mu}$, $\alpha < \overline{\overline{\alpha}}$ and $\overline{\overline{\tau}} < \tau < \overline{\tau}$. Then, the fixed point $(1,1)$ is a source for $0 < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(6) Let $\mu < \overline{\mu}$ and $\tau < \overline{\overline{\tau}}$. Then, the fixed point $(1,1)$ is a saddle for any $\gamma > 0$.

(7) Let $\mu > \overline{\mu}$ and $\tau < \overline{\tau}$. Then, the fixed point $(1,1)$ is a saddle for $0 < \gamma < \gamma_{fl}$, a source for $\gamma_{fl} < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(8) Let $\mu > \overline{\mu}$ and $\overline{\overline{\tau}} < \tau < \overline{\tau}$. Then, the fixed point $(1,1)$ is a source for $0 < \gamma < \gamma_{tc}$, a saddle for $\gamma > \gamma_{tc}$.

(9) Let $\mu > \overline{\mu}$ and $\tau < \overline{\overline{\tau}}$. Then, the fixed point $(1,1)$ is a saddle for any $\gamma > 0$, where

$$\gamma_{tc} := \frac{(1-\tau)^{\delta}(1-\mu)^{\delta}(\rho-\mu) - \tau^{\delta}(\rho+1-\mu)^{\delta}[(\delta-1)\rho-\delta+\mu]}{(1-\tau)^{\delta}(1-\mu)^{\delta}+\tau^{\delta}(\rho+1-\mu)^{\delta}}, \tag{39}$$

$$\begin{aligned}\gamma_{fl} := \frac{-(1-\tau)^{\delta}(1-\mu)^{\delta}[(\mu+\rho)\alpha-\rho+2-\mu]}{(1+\alpha)[(1-\tau)^{\delta}(1-\mu)^{\delta}+\tau^{\delta}(\rho+1-\mu)^{\delta}]}+ \\ +\frac{\tau^{\delta}(\rho+1-\mu)^{\delta}\{[(\delta-1)\rho-\delta-\mu]\alpha+(-\delta+1)\rho+\mu+\delta-2\}}{(1+\alpha)[(1-\tau)^{\delta}(1-\mu)^{\delta}+\tau^{\delta}(\rho+1-\mu)^{\delta}]},\end{aligned} \tag{40}$$

$$\gamma_{ns} := \frac{-2\alpha+1-\mu+\alpha\sigma}{\alpha}, \tag{41}$$

represent the intersection points of the straight line (P_1, P_2) with the straight lines $Det(J) - Tr(J) + 1 = 0$, $Det(J) + Tr(J) + 1 = 0$, $Det(J) - 1 = 0$, respectively, and

$$\overline{\tau} := \frac{1}{1+\frac{(\delta\alpha\rho-\alpha\rho-\delta\rho+\rho-\alpha\delta+\delta-2-\alpha\mu+\mu)^{\frac{1}{\delta}}(\rho+1-\mu)}{(\alpha\rho-\rho+2+\alpha\mu-\mu)^{\frac{1}{\delta}}(1-\mu)}},$$

$$\overline{\overline{\tau}} := \frac{1}{1+\frac{(\delta\rho-\rho-\delta+\mu)^{\frac{1}{\delta}}(\rho+1-\mu)}{(\rho-\mu)^{\frac{1}{\delta}}(1-\mu)}},$$

$$\overline{\mu} := \frac{2\alpha-1}{\alpha-1},$$

$$\overline{\alpha} := \frac{\left[(1-\tau)^{\delta}(1-\mu)^{\delta}+\tau^{\delta}(\rho+1-\mu)^{\delta}\right](-1+\mu)}{(-2\mu+(\delta-1)\rho-\delta+2)\tau^{\delta}(\rho+1-\mu)^{\delta}-(1-\tau)^{\delta}(1-\mu)^{\delta}(2\mu+\rho-2)},$$

$$\overline{\overline{\alpha}} := \frac{\left[(1-\tau)^{\delta}(1-\mu)^{\delta}+\tau^{\delta}(\rho+1-\mu)^{\delta}\right](-1+\mu)}{(2\mu+(\delta-1)\rho-\delta-2)\tau^{\delta}(\rho+1-\mu)^{\delta}-(1-\tau)^{\delta}(1-\mu)^{\delta}(-2\mu+\rho+2)}.$$

Proof. In order to find the bifurcation values of γ, we impose the condition that (P_1, P_2) belongs to: (i) the straight line $Det(J) - Tr(J) + 1 = 0$, to obtain the transcritical bifurcation value γ_{tc}, (ii) the straight line $Det(J) + Tr(J) + 1 = 0$, to obtain the flip bifurcation value γ_{fl}, and (iii) the straight line $Det(J) - 1 = 0$, to obtain the Neimark-Sacker bifurcation value γ_{ns}. Then, we identify cases 1-9 by considering the position of the starting point $(\overline{P}_1, \overline{P}_2)$ on the half-line defined by (P_1, P_2) with respect to the stability triangle delimited by $1 \pm Tr(J) + Det(J) = 0$ and $Det(J) = 1$ (see

Grandmont et al., 1998 for details), the slope of such a half-line and the value of the parameter γ. ■

The choice on the parameters μ, α and τ (instead of those related to technologies ρ and δ) to describe the curves in the $(Tr(J), Det(J))$ plane is made because it allows us to better characterise local dynamics around the normalised fixed point.

[INSERT FIGURE 1 ABOUT HERE]

Figure 1. Stability triangle and local indeterminacy. Inside the triangle the fixed point is indeterminate (sink).

4.2 Global analysis in a CIES-Cobb-Douglas economy

In this section we perform the global analysis of map M. First of all, we note that with the functional forms used in the previous section map M is invertible on the non-negative orthant, as shown in the following lemma. This property is shared with Agliari and Vachadze (2011), that study an OLG model with credit market imperfections, but not with Grandmont et al. (1998), where the map is invertible only in a neighbourhood of the fixed point.

Lemma 14 *Map M is invertible on W_3.*

Proof. Notice that it is not possible to have a closed-form expression of the inverse map of M , i.e. M^{-1}. However, after some algebraic manipulations it is possible to find that M^{-1} is solution of the following system:

$$M^{-1}: \left\{ \begin{array}{c} \frac{l}{(2-l)^{\gamma}} = \frac{l'(K')^{\frac{-(-\alpha+\alpha\mu-\rho)}{(1-\alpha)(1-\mu)}} \left[1+\left(\frac{\tau(\rho+1-\mu)K'}{(1-\tau)(1-\mu)}\right)^{\delta}\right]^{\frac{(\rho-1)}{(1-\alpha)(1-\mu)}}}{P^{(1-\alpha)(1-\sigma)}} \\ K = \left(\frac{K'}{l^{1-\alpha}}\right)^{\frac{1}{\alpha}} \end{array} \right. . \qquad (42)$$

■

The invertibility of a map is an important result when the global properties of a dynamic system are studied. For instance, it implies that the basins of attraction of any attracting set of a map are connected sets. Furthermore, by making use of the inverse map, we can obtain the boundary of the attracting sets and, more generally, the stable manifolds of saddle points. The importance of the study of stable manifolds in dynamic economies with control variables rely on the fact that they represent loci on which feasible trajectories lie upon.

Before performing the global analysis of map M, we recall the definitions of both the stable manifold

$$G^{s}(a) = \{x : M^{zn}(x) \rightarrow a \ as \ n \rightarrow +\infty\}$$

and unstable manifold

$$G^{u}(a) = \{x : M^{zn}(x) \rightarrow a \ as \ n \rightarrow -\infty\}$$

of a periodic point a of period z. If the periodic point $a \in R^2$ is a saddle, then the stable (resp. unstable) manifold is a smooth curve through a, tangent at a to the eigenvector of the Jacobian matrix evaluated at a corresponding to the eigenvalue λ with $|\lambda| < 1$ (resp. $|\lambda| > 1$), see, e.g., Guckenheimer and Holmes (1983). Outside the neighbourhood of a, the stable and unstable manifolds may even intersect each other with dramatic consequences on the global dynamics of the model (see Guckenheimer and Holmes, 1983, p. 22).[5] The global analysis of the map allows us to identify three phenomena (summarised in Case 1, Case 2 and Case 3) that are interesting from an economic point of view. We recall that a fixed point is locally indeterminate (resp. determinate) if for every arbitrarily small neighbourhood of it, and for a given value of the state variable close enough to its coordinate value at the stationary state, there exists a continuum of values (resp. a unique value) of the control variable for which an equilibrium trajectory converge towards the fixed point. From a mathematical point of view, the fixed point is a sink (resp. a saddle). Differently, the system is globally indeterminate when there exist values of the state variable such that different choices on the control variable lead to different invariant sets. In this case, the initial condition of the stock of capital is not sufficient to define the long-term dynamics of the system.

Case 1. Existence of quasiperiodic or chaotic long-term dynamics for the system. On the basis of the results of Proposition 13, it is possible to find numerical evidence of the existence either of flip bifurcations or Neimark-Sacker bifurcations around the normalised fixed point that create an attracting two-period cycle (which represents the first step towards the birth of a chaotic attractor, as shown in Figures 2.a and 2.b) and an attracting closed invariant curve (Figure 3.a), respectively.[6] In addition, we note that the birth of a closed invariant curve through a Neimark-Sacker bifurcation can occur around a non-normalised fixed point, as shown in Figure 3.b.

[INSERT FIGURES 2.a, 2.b, 3.a and 3.b ABOUT HERE]

Figure 2.a. Parameter set: $\alpha = 0.12$, $\delta = 4$, $\rho = 0.15$, $\mu = 0.3$ and $\tau = 0.6$. Bifurcation diagram for γ ($\gamma \in [0.7, 1.25]$). Flip bifurcation around the normalised fixed point.

Figure 2.b. Parameter set: $\alpha = 0.12$, $\delta = 4$, $\rho = 0.15$, $\mu = 0.3$, $\tau = 0.6$ and $\gamma = 0.78$. Basin of attraction.

Figure 3.a. Parameter set: $\alpha = 0.33$, $\delta = 7$, $\rho = 0.27$, $\mu = 0.31$ and $\tau = 0.35$. Bifurcation diagram for γ ($\gamma \in [0.38, 0.416]$). Neimark-Sacker bifurcation around the normalised fixed point.

Figure 3.b. Parameter set: $\alpha = 0.33$, $\delta = 4.2$, $\rho = 0.26$, $\mu = 0.27$, $\tau = 0.29$ and $\gamma = 0.285$. Basin of attraction (Neimark-Sacker around a non-normalised fixed point). The figure shows an attracting closed invariant curve (black-coloured) born through a Neimark-Sacker bifurcation. The gray-coloured region represents its basin of attraction bounded by the stable manifold of $(1, 1)$. The

[5] Non-trivial intersection points of stable and unstable manifolds of a unique saddle cycle, or non trivial intersection points between the stable manifold of one cycle and the unstable manifold of the other one are really important from a dynamic point of view because they sharply change the topological structure of the phase plane (global bifurcations). With regard to this phenomenon, which is not deepen in this paper, we refer to Agliari and Vachadze (2011).

[6] In Proposition 13 we did not analyse the stability properties of the bifurcations (to ascertain that they are subcritical or supercritical kind), because this study requires to take into account high order approximations of the map that in our case would result difficult to be handled in a neat analytical form.

figure also depicts the unstable manifold of $(1,1)$ (blue-coloured) that converges to the limit cycle. In the north-east region of the figure there exists another saddle point. None of the fixed points is attracting and nonetheless the system is globally indeterminate.

Case 2. Coexistence of attractors or other feasible trajectories for the system (saddle path stability). Depending on the initial condition, it is possible to converge towards different fixed points (path dependence). With regard to this case, numerical simulations plotted in Figure 4.a shows coexistence of two attractors: the normalised fixed point and a three-period cycle born through a saddle node bifurcation. Since l is a control variable, agents may coordinate either on the stable manifolds that define the basins of attraction of the saddles or on the saddle that lies in the south-west region of the phase plane (this last saddle is not reported in the figure). It is important to note that saddle node bifurcations can also be observed together with a sequence of flip bifurcations (around the normalised fixed point). In this case attractors of high periodicity coexist (Figure 4.b).

[INSERT FIGURES 4.a and 4.b ABOUT HERE]

Figure 4.a. Parameter set: $\alpha = 0.127$, $\delta = 4$, $\rho = 0.25$, $\mu = 0.35$, $\tau = 0.55$ and $\gamma = 0.9$. Coexistence of attractors and saddles.

Figure 4.b. Parameter set: $\alpha = 0.127$, $\delta = 4$, $\rho = 0.25$, $\mu = 0.35$ and $\tau = 0.55$. Bifurcation diagram for γ ($\gamma \in [0.5, 0.85]$).

Case 3. Global indeterminacy (expectations driven). A system is globally indeterminate when there exist values of the state variable such that different choices on the control variable lead to different invariant sets. The initial condition of the state variable is not sufficient to define the long-term dynamics of the system. We note that the simulations presented in Figures 2.b, 3.b and 4.a already showed the possibility of global indeterminacy of the economy. Most importantly, Figure 5 shows that global indeterminacy can be observed also when the longevity production function is concave in all its arguments, that is when the external effect created by public health investments is sufficiently low ($\delta < 1$). This result confirms the importance of performing a global analysis and the perils of focusing only on local stability (de Vilder, 1996; Agliari and Vachadze, 2011) for possible policy interventions of governments.

[INSERT FIGURE 5 ABOUT HERE]

Figure 5. Parameter set: $\alpha = 0.3$, $\delta = 0.8$, $\rho = 0.49$, $\mu = 0.501$, $\tau = 0.4$ and $\gamma = 0.4$. Global indeterminacy with a concave utility function and concave Cobb-Douglas technologies. A heteroclinic connection exists between $(1,1)$ and (K^*, l^*).

5 Conclusions

This paper has concerned with the study of a growth model that links the theoretical literature with endogenous labour supply and indeterminacy with the theoretical literature on endogenous lifetime and economic growth. In particular, the paper has introduced endogenous labour supply in a growth model à la Bhattacharya and Qiao (2007) with endogenous lifetime. This causes substantially different results with regard to the long-term behaviour of demo-economic outcomes than when labour supply is exogenous. The paper has provided a characterisation of the dynamics of the economy (existence and local stability of the fixed point) under general utility, production and longevity functions. Due to the externality of the health technology on the macro-economy, local indeterminacy becomes a plausible scenario with respect to several configurations of economic parameters (elasticities of substitution, capital shares and so on). In addition, in order to clarify the global properties of the system, we have assumed a Constant Inter-temporal Elasticity of Substitution utility function and Cobb-Douglas production and longevity functions. While Bhattacharya and Qiao (2007) has assumed a technology for the production of longevity whose elasticity with respect to the private input depends on the public input (which represents the main ingredient for observing endogenous fluctuations in their model), we have concentrated on a longevity function that generates a constant elasticity independent from public investments. This implies that the private input and the public input in the longevity technology are complements for any level of private health investments: an assumption close enough to capture the functioning of health systems in developed countries. Within this framework, we have found that global indeterminacy may characterise the behaviour of demo-economic variables. It is important to stress that this result has been obtained in spite of the existence of some restrictive hypotheses with regard to technologies, and can be found even when the production of longevity is concave in both arguments (i.e., private and public investments in health): a key ingredient for this finding being the endogenous labour supply. In fact, with exogenous labour supply the model collapses to the one proposed by Bhattacharya and Qiao (2007), in which one fixed point does exist and the non-convergent dynamics are possible only when the returns of public investment in health are sufficiently high (i.e., the public input in the longevity function is a convex function of public health expenditure). From a mathematical point of view, it is interesting to note that by relaxing the hypothesis of concavity of the returns of public investments as a means of higher longevity, global indeterminacy in our model may occur when all fixed points of the system are determinate (saddles and sources). This implies that the long-term behaviour of the economy can be strongly related to decisions regarding the magnitude of the public health policy.

Though our paper is essentially related to the behaviour of longevity and income in the long term, it is connected to the debate on the effects of the so called austerity measures in Europe on the health of people (that resulted in a sharp rise in infant mortality from 2008 to 2010 in Greece) with dramatic potential effect on longevity rates reversals. To this purpose, Kentikelenis et al. (2014) has reported the Greek public health tragedy due to the cuts in health services in Greece (see also Kentikelenis et al., 2011), including expenditures to prevent and treat illicit drug and infectious diseases (such as HIV/AIDS). This topic, therefore, deserves attention also in theoretical macroeconomic models, given the importance of how the health of people may affect demo-economic outcomes in both the short and long terms.

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